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Abstract

This paper analyzes the necessary and sufficient conditions for solving money-in-the-utility-function models when contemporaneous asset returns are uncertain. A unique solution to such models is shown to exist under certain measurability conditions. Stochastic Euler equations, whose existence is normally assumed in these models, are then formally derived. The regularity conditions are weak, and economically innocuous. The results apply to the broad range of discrete-time monetary and financial models that are special cases of the model used in this paper. The method is also applicable to other dynamic models that incorporate contemporaneous uncertainty.

Key words: money, asset pricing, dynamic programming, stochastic modelling, uncertainty
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1 Introduction

Many monetary and financial models—ranging from cash-in-advance to capital asset pricing models—are special cases of stochastic money-in-the-utility-function (MIUF) models.[1] Stochastic decision problems that include monetary or financial assets have been used to address a variety of topics including asset pricing [2, 15, 29, 30, 41, 51], price dynamics [23, 32, 40, 42, 43], intertemporal substitution [26], money demand [21, 33], currency substitution and exchange rates [6, 34], optimal monetary policy [16, 17, 18, 19, 20, 25], and monetary aggregation [3, 4, 5, 46]. In each case, the usefulness of the model depends on the derivation of the stochastic Euler equations that characterize the model's solution. Deriving stochastic Euler equations is not straightforward in stochastic settings. The difficulty of the derivation depends crucially on the specification of uncertainty. However, uncertainty is rarely explicitly modeled in this literature. Instead, stochastic Euler equations are just assumed to exist. Consequently, the validity and applicability of these models' results is difficult to ascertain.

The specific model used in this paper is based on Barnett, Liu, and Jensen's [3] discrete-time model.[2] The stochastic Euler equations derived from this model depend on a trade-off between an asset's rate of return, risk, and liquidity, instead of depending on just the trade-off between return and liquidity as is the norm in most of the monetary literature. The model also generalizes much of the voluminous asset-pricing literature in finance, where only the two-dimensional trade-off between risk and return is considered.[3] In particular, the consumption capital asset pricing model (CAPM), which ignores liquidity, is a special case of the model [3].[4] Stochastic MIUF models, therefore, have the capacity to integrate the monetary and financial approaches to asset pricing.

As the model is recursive, dynamic programming (DP) would seem to be the natural candidate for a solution method.[5] DP is especially appealing for stochastic problems where states of the dynamic system are uncertain, because the DP solution determines optimal control functions defined for all admissible states. But, for a stochastic model, care must be taken to ensure that DP is feasible. Bertsekas and Shreve [10, Chapter 1] and Stokey and Lucas [49, Chapter 9] discuss the requirements for implementing stochastic DP in more

[1] Deterministic MIUF models, cash-in-advance models, and other transaction cost models of money are functionally equivalent [28].
[2] Earlier versions of this model, both deterministic and stochastic, are in the papers collected in [4].
[3] An exception is [2], who use liquidity to explain [45]'s equity premium puzzle.
[4] See [37] for a textbook treatment of CAPM and a roadmap for the asset-pricing literature.
[5] The seminal paper is [7]. Reference texts for DP include [9, 39, 47, 49].

detail.

Probably, the simplest method to ensure that a solution to a stochastic problem can be found using DP is to specify the timing of the uncertainty. If all current state variables are known with perfect certainty, then further modeling of uncertainty is unnecessary. Although the timing restriction makes DP feasible, contemporaneous certainty is not always a reasonable assumption. Contemporaneous certainty is a particularly restrictive assumption for models containing assets; it implies that all asset returns are risk-free.

Other simple assumptions that support stochastic DP similarly impose strong restrictions on the applicability of the results. For example, the objective function or the stochastic processes can be restricted so that certainty equivalence holds. This requires either the strong assumption that the objective function is quadratic, or the strong assumption that all stochastic processes are i.i.d. or (conditionally) Gaussian. [6] DP is also feasible if the underlying probability space is finite or countable. Assuming countability is stronger than the norm in economics and finance, however, and does not meet Aliprantis and Border's [1] dictum that "[t]he study of financial markets requires models that are both stochastic and dynamic, so there is a double imperative for infinite dimensional models."

This paper takes a different tack. Following Bertsekas and Shreve [10], a rich infinite-dimensional model of uncertainty is developed that is similar to the measure-theoretic approach in [49]. Within this framework, the DP algorithm can be implemented for the stochastic problem and results can be obtained that are nearly as strong as those available for deterministic problems. In particular, the existence of a unique optimum that satisfies the principal of optimality can be shown. I further prove that the solution inherits differentiability. These results imply that the optimum can be characterized by stochastic Euler equations.

The main benefit of this approach is that applicability of the results is not restricted by strong assumptions on uncertainty. The regularity conditions—certain measurability requirements—are the least restrictive available, and are economically innocuous. As a result, the validity of many previous results, which depended on the existence of stochastic Euler equations, is broadly established. Although the method is specifically developed for stochastic MIUF models due to the central importance of risk in financial models, it should be clear that the method applies to other dynamic economic models that incorporate contemporaneous uncertainty, such as models with search costs or information restrictions.

The organization of this paper is as follows. Section 2 presents the dynamic

[6] This approach is commonly used in the finance literature [e.g. 16, 21]

decision problem. Section 3 lays out the measure-theoretic apparatus used to model expectations. Section 4 discusses measurable selection, which is necessary for the DP results. Section 5 reviews the stochastic DP approach to solving the decision problem. Conditions guaranteeing the existence of an optimal plan that satisfies the principal of optimality are developed. The optimal plan is shown to be stationary, non-random, and (semi-) Markov. Section 6 proves that differentiability of the solution follows from differentiability of the utility function. This result combined with the results in the previous section formally supports the derivation of the stochastic Euler equations. The last section provides a short conclusion.

2 Household Decision Problem

The model is based on the infinite-horizon stochastic household decision problem in [3]. The model is very general in that preferences are defined over an arbitrary (finite) number of assets and goods, but the form of the utility function is not specified. Results proven for this model, will hold for more restrictive stochastic MIUF models. Similar results also apply to finite horizon versions, except that the optimal policy would be time-varying.

The (representative) consumer attempts to maximize utility over an infinite horizon in discrete time. Define C to be the household's survival set: a subset of the n-dimensional nonnegative Euclidean orthant. Let $Y = \mathbb{R}^{k+1} \times C$ where $\mathbb{R}$ represents the real numbers. The complete consumption space is the product space $Y \times Y \times \ldots$

Let the consumption possibility set for any period $s \in \{t, t+1, \ldots, \infty\}$, $S(s)$, be defined as follows:

$$S(s) = \left\{ (\mathbf{a}_s, A_s, \mathbf{c}_s) \in Y \;\middle|\; \begin{array}{l} \sum\limits_{j=1}^{n} p_{js} c_{js} \leq \sum\limits_{i=1}^{k} \left[(1+\rho_{i,s-1}) p^*_{s-1} a_{i,s-1} - p^*_s a_{is} \right] \\ + (1+R_{s-1})\, p^*_{s-1} A_{s-1} - p^*_s A_s + I_s \end{array} \right\} \tag{1}$$

where, for each period s, $\mathbf{a}_s = (a_{1s}, \ldots, a_{ks})$ is a k-dimensional vector of planned real asset balances where each element a_{is} has a nominal holding-period yield of ρ_{is}, A_s is planned holdings of the benchmark asset which has an expected nominal holding period yield of R_s, $\mathbf{c}_s = (c_{1s}, \ldots, c_{ns})$ is a n-dimensional vector of planned real consumption of non-durable goods and services where each element c_{is} has a price of p_{is}, I_s is the nominal value of income from all other sources, which is nonnegative, and p^*_s is a true cost-of-living index defined as a function over some non-empty subset of the p_{is}.[7]

[7] The assumption that a true cost-of-living index exists is trivial, because, the

Prices, including the aggregate price, p_s^*, and rates of return are stochastic processes. Note that the construction of Y allows for short-selling.

For the stochastic processes, the information set must be specified. It is assumed that current prices and the benchmark rate of return are known at the beginning of each period and current interest on all other assets is realized at the end of each period. More specifically, for all i and s, p_{is}, p_s^*, R_s (and R_{s-1}) and $\rho_{i,s-1}$ are known at the beginning of period s, while ρ_{is} is not know until the end of period s. Since their returns are unknown in the current period, the assets, $\mathbf{a}_s$, are risky. Despite the uncertainty, the constraint contains only known variables in the current period, so the consumer can satisfy (1) with certainty.

The consumer is assumed to maximize a utility function over the complete consumption space, including all assets except for the benchmark asset. Consequently, the setup resembles a standard money-in-the-utility-function model. In the stochastic decision problem, however, the assets can be either monetary or financial assets.

Utility is assumed to be intertemporally additive; a standard assumption in expected utility models. In addition, although it is not necessary, preferences are assumed to be independent of time and depend on the distance from t only through a constant, subjective, rate of time preference, so that

$$U\left(\mathbf{a}_t, \mathbf{c}_t, \mathbf{a}_{t+1}, \mathbf{c}_{t+1}, \ldots\right) = \sum_{s=t}^{\infty} \left(\frac{1}{1+\xi}\right)^{s-t} u\left(\mathbf{a}_s, \mathbf{c}_s\right) \tag{2}$$

where $0 < \xi < \infty$ is the subjective rate of time preference. The period utility function $u\left(\ \right)$ inherits regularity conditions from the total utility function $U\left(\ \right)$.

The consumer, therefore, solves, at time t,

$$\sup \left\{ u\left(\mathbf{a}_t, \mathbf{c}_t\right) + E_t \left[\sum_{s=t+1}^{\infty} \left(\frac{1}{1+\xi}\right)^{s-t} u\left(\mathbf{a}_s, \mathbf{c}_s\right) \right] \right\} \tag{3}$$

subject to $\left(\mathbf{a}_s, \mathbf{c}_s, A_s\right) \in S\left(s\right)$ for all $s \in \{t, t+1, \ldots, \infty\}$ and the transversality condition

$$\lim_{s \to \infty} E_t \left[\left(\frac{1}{1+\xi}\right)^{s-t} A_s \right] = 0 \tag{4}$$

where the operator $E_t[\ ]$ denote expectations formed on the basis of the information available at time t. The transversality condition, which is standard in

limiting case is a singleton so that $p_s^* = p_{j^*s}$, where p_{j^*s} denotes the price of a numéraire good or service. Note that the previous literature denoted the vector of goods and services by $\mathbf{x}_t$.

infinite horizon decisions, rules out unbounded borrowing at the benchmark rate of return. The necessity of transversality conditions in stochastic problems is shown in [35, 36]. This condition implies that the constraint set is bounded.

It might appear that the problem given by (3) is deterministic in the current period. If this was actually the case, the model can already be solved by DP. But, allowing asset returns to be risky introduces contemporaneous uncertainty. As an alternative, contemporaneous uncertainty for goods could be specified, for example by assuming search costs. However, assuming that assets are exposed to risk is a more natural method of introducing such uncertainty. Explicitly defining how expectations are formed, is the subject of the next section.

3 Expectations

The decision problem defined in equation (3) is not fully specified, as the expectations operator $E_t[\ ]$ is not formally defined. Without some further structure, neither stochastic DP nor other solution techniques can be applied to the problem. Since the underlying probability space is assumed to be uncountable, the expectation operator becomes an integral against some probability measure.

A difficulty with the measure theory approach is that integration against a measure is not well-defined for all functions. [8] Therefore, the measurability of functions will be central to the ability to derive a stochastic DP solution. Furthermore, the measurable space cannot be arbitrary. The following standard definition will, therefore, be used repeatedly:

Definition 3.1 (Measurable function) *Let X and Y be topological spaces and let $\mathfrak{F}_X$ be any σ-algebra on X and let $\mathfrak{B}_Y$ be the Borel σ-algebra on Y. A function $f\colon X \to Y$ is $\mathfrak{F}$-measurable if $f^{-1}(B) \in \mathfrak{F}_X \quad \forall B \in \mathfrak{B}_Y$.*

Following [49], the disturbance spaces are assumed to be Borel spaces. This implies that the decision and constraint spaces for the infinite-horizon problem are also Borel spaces, because Euclidean spaces, Borel subsets of Borel spaces, and countable Cartesian products of Borel spaces are all Borel spaces.

[8] The outer integral, which is well-defined for any function, could be used. Not only is there no unique definition for the outer integral, but it is not a linear operator. Consequently, expectations would not be additive. Recursive methods, such as DP, require additivity so that the overall problem can be broken into smaller subproblems.

The formulation of the constraint set in [3] assumed compactness. Compactness along with the additional assumptions, not made by [3], that are needed for semicontinuous problems [10, pg. 208-210], or upper hemi-continuity [49, pg. 56], would allow the problem to be solved using only Borel measurability. As in [10], instead of assuming compactness and the necessary ancillary assumptions, a solution is sought within a richer set of functions: universally-measurable functions. Relaxing the measurability restrictions, which expands the class of possible solutions, is more appealing than imposing strong compactness restrictions. Section 5.3 discusses these trade-offs further.

Transition functions are commonly used to incorporate stochastic shocks into a functional equation [see 49, pg. 212]. The current treatment is similar, except instead of beginning with transition functions, stochastic kernels are employed.

Definition 3.2 (Stochastic Kernel) *Let X and Y be Borel spaces with $\mathfrak{B}_Y$ denoting the Borel σ-algebra. Let $P(Y)$ denote the space of probability measures on $(Y, \mathfrak{B}_Y)$. A stochastic kernel, $q(dy \mid x)$, on Y given X is a collection of probability measures in $P(Y)$ parameterized by $x \in X$. If $\mathfrak{F}$ is a σ-algebra on X, and $\gamma^{-1}(\mathfrak{B}_{P(Y)}) \subset \mathfrak{F}$ where $\gamma \colon X \to P(Y)$ is defined by $\gamma(x) = q(dy \mid x)$, then $q(dy \mid x)$ is $\mathfrak{F}$-measurable. If γ is continuous, $q(dy \mid x)$ is said to be continuous.*

A stochastic kernel is a special case of a regular conditional probability for a Markov process, defined in [48, Definition 6, pg. 226]. In abstract terms, if x is taken to represent the state of the system at time t and the system is Markovian, then, from [48, Theorem 3, pg. 226] and the properties of Markov processes, the conditional expectation operator applied to an element of Y can, formally be viewed as an integral against the stochastic kernel, i.e. $E_t\left[f\left(x,y\right)\right] = \int f(x,y)\, q(dy \mid x)$. As this definition requires measurability of the function, a measurability assumption is necessary to define expectations, even before a solution is sought. If a function, defined on the Cartesian product of X and Y, is Borel-measurable and the stochastic kernel is Borel-measurable, then $\int f(x,y)\, q(dy \mid x)$ is Borel-measurable [10, Proposition 7.29, pg. 144], i.e. the conditional expectation is Borel-measurable. Integration defined this way operates linearly and obeys classical convergence theorems. Also, the integral is equal to the appropriate iterated integral on product spaces. These statements will be clarified in the next section.

As the state space for the problem has not been defined, it may appear that the stochastic kernel is limited in that it only represents conditional expectations for Markovian systems. In fact, non-Markovian processes can always be reformulated as Markovian by expanding the state space. In the sequel, the state space for the problem will be formulated so that the process is Markovian.

4 Measurable Selection

Borel measurability, by itself, is not adequate to prove the existence of the solution. Bertsekas and Shreve [10] address this problem through a richer concept of measurability. Their apparatus includes upper semianalytic functions and measurability with regard to the universal σ-algebra. Definitions and relevant results are presented below; proofs can be found in [1, 10].[9] The key relation is that the universal σ-algebra includes the analytic σ-algebra, which in turn includes the Borel σ-algebra. This implies that all Borel-measurable functions are analytically measurable, and that all analytically measurable functions are universally measurable. Thus the move to universal measurability is, relative to the Borel model, relaxing a constraint, instead of imposing a restriction. By moving to universal-measurability, the information set is enriched and the measurability assumptions are technically relaxed. Such a relaxation does no damage to the economics behind the model. The results are standard and the proofs are omitted.

Borel measurability is not adequate for DP, because the orthogonal projection of a Borel set is not necessarily Borel measurable Specifically, if $f\colon X \times Y \to \mathbb{R}^*$, where $\mathbb{R}^*$ is the extended real numbers, is given and $f^*\colon X \to \mathbb{R}^*$ is defined by $f^*(x) = \sup_{y \in Y} f(x, y)$ then for each $c \in \mathbb{R}$, define the set

$$\{x \in X \mid f^*(x) > c\} = \text{proj}_X (\{(x, y) \in X \times Y \mid f(x, y) > c\}) \tag{5}$$

where $\text{proj}_X(\ )$ is the projection mapping from $X \times Y$ onto X. If $f(\ )$ is Borel-measurable then

$$\{(x, y) \in X \times Y \mid f(x, y) > c\} \tag{6}$$

is Borel-measurable, but $\{x \in X \mid f^*(x) > c\}$ may not be Borel-measurable ([11], [22, pg. 328-329], and [49, pg. 253]).

The DP algorithm repeatedly implements such projections, so the conditional expectation of functions like $f^*(\ )$ will need to be evaluated, requiring that the function is measurable. This leads to the definition of analytic sets, the analytic σ-algebra, and analytic measurability:

Definition 4.1 (Analytic sets) *A subset A of a Borel space X is analytic if there exists a Borel space Y and Borel subset B of $X \times Y$ such that $A = \text{proj}_X(B)$. The σ-algebra generated by the analytic sets of X is referred to as the analytic σ-algebra, denoted by $\mathcal{A}_X$, and functions that are measurable with respect to it are called analytically measurable.*

[9] Further references include [22, 24, 38, 48, 49].

Davidson [22, pg. 329] refers to analytic sets as "nearly" measurable because, for any measurable space and any measure, μ, on that space,the analytic sets are measurable under the completion of the measure. The completion of a space with respect to a measure involves setting $\mu(E) = \mu(A)$ for any set E such that $A \subset E \subset B$ whenever $\mu(A) = \mu(B)$. Effectively, this assigns measure zero to all subsets of measure zero sets [22, pg. 39].

Analytic sets address the problem of measurable selection within a dynamic program, because, if X and Y are Borel spaces, and, if $A \subset X$ is analytic and $f\colon X \to Y$ is Borel-measurable, then $f(A)$ is analytic. This implies that if $B \subset X \times Y$ is analytic then $\text{proj}_X(B)$ is also analytic. Analytic sets are the smallest groups of sets such that the projection of a Borel set is a member of the group [1]. Analytic sets are used to define upper semianalytic functions as follows:

Definition 4.2 *Let X be a Borel space and let $f\colon X \to \mathbb{R}^*$ be a function. Then $f(\ )$ is upper semianalytic if $\{x \in X \mid f(x) > c\}$ is analytic $\forall c \in \mathbb{R}$.*

The following result is key for the application of the DP algorithm:

Lemma 4.3 *Let X and Y be Borel spaces, and let $f\colon X \times Y \to \mathbb{R}^*$ be upper semianalytic, then $f^*\colon X \to \mathbb{R}^*$ defined by $f^*(x) = \sup_{y \in Y} f(x,y)$ is upper semianalytic.*

Two important properties of upper semianalytic functions are that the sum of such functions remains upper semianalytic, and if $f\colon X \to \mathbb{R}^*$ is upper semianalytic and $g : Y \to X$ is Borel measurable, then the composition $f \circ g$ is upper semianalytic. Most importantly, the integral of a bounded upper semianalytic function against a stochastic integral is upper semianalytic. This is stated as a lemma:

Lemma 4.4 *Let X and Y be Borel spaces and let $f\colon X \times Y \to \mathbb{R}^*$ be a upper semianalytic function either bounded above or bounded below. Let $q(dy \mid x)$ be a Borel- measurable stochastic kernel on Y given X. Then $g\colon X \to \mathbb{R}^*$ defined by $g(x) = \int_Y f(x,y)\, q(dy \mid x)$ is upper semianalytic.*

Semianalytic functions have one relevant limitation. If two functions are analytically measurable, their composition is not necessarily analytically measurable. This difficulty can be overcome moving to the richer universally measurable σ-algebra: [10]

Definition 4.5 (Universal σ-algebra) *Let X be a Borel space, $P(X)$ be the set of probability measures on X, and let $\mathfrak{B}_X(\mu)$ denote the completion of $\mathfrak{B}_X$*

[10] The slightly tighter, but less intutive, σ-algebra of limit measureable sets would be sufficient. Again, moving to the larger class does not impose any restrictions.

with respect to the probability measure $\mu \in P(X)$. The universal σ-algebra $\mathfrak{U}_X$ is defined by

$$\mathfrak{U}_X = \bigcap_{\mu \in P(X)} \mathfrak{B}_X(\mu). \tag{7}$$

If $A \in \mathfrak{U}_X$, A is called universally measurable, and functions that are measurable with respect to $\mathfrak{U}_X$ are called universally measurable.

The universally-measurable σ-algebra is the completion of the Borel σ-algebra with respect to every Borel measure. Consequently, it does not depend on any specific Borel measure. Note that every Borel subset of a Borel space X is also an analytic subset of X, which implies that the σ-algebra generated by the analytic sets is larger than the Borel σ-algebra. The fact that analytic sets are measurable under the completion of any measure implies that they are universally-measurable, so $\mathfrak{B}_x \subseteq \mathcal{A}_X \subseteq \mathfrak{U}_x$.

Universal measurability is the last type of measurability that will be needed to implement stochastic DP as universally measurable stochastic kernels will be used in the DP recursion. Of course, if a stochastic kernel is Borel-measurable, it is universally measurable. Integration against a universally measurable stochastic kernel operates linearly, obeys classical convergence theorems, and iterates on product spaces, as shown by the following theorem:

Theorem 4.6 *Let $X_1, X_2, \dots$ be a sequence of Borel spaces, $Y_n = X_1 \times \cdots \times X_n$, and $Y = X_1 \times X_2 \times \cdots$. Let $\mu \in P(X_1)$ be given and, for $n = 1, 2, \dots$, let $q_n(dx_{n+1} \mid y_n)$ be a universally measurable stochastic kernel on X_{n+1} given Y_n. Then for $n = 2, 3, \dots$, there exist unique probability measures $r_n \in P(Y_n)$ such that $\forall \underline{X}_1 \in \mathfrak{B}_{X_1}, \dots, \underline{X}_n \in \mathfrak{B}_{X_n}$*

$$r_n(\underline{X}_1 \cap \underline{X}_2 \cap \cdots \cap \underline{X}_n) = \int\limits_{\underline{X}_1} \int\limits_{\underline{X}_2} \cdots \int\limits_{\underline{X}_n} q_{n-1}(\underline{X}_n \mid x_1, \dots, x_{n-1}) \times q_{n-2}(dx_{n-1} \mid x_1, \dots, x_{n-2}) \dots \times q_1(dx_2 \mid x_1)\, \mu(dx_1) \tag{8}$$

If $f \colon Y_n \to \mathbb{R}^*$ is universally measurable, and the integral is well-defined,[11] *then*

$$\int\limits_{Y_n} f dr_n = \int\limits_{X_1} \int\limits_{X_2} \cdots \int\limits_{X_n} f(x_1, \dots, x_n)\, q_{n-1}(\underline{X}_n \mid x_1, \dots, x_{n-1}) \times q_{n-2}(dx_{n-1} \mid x_1, \dots, x_{n-2}) \times \cdots \times q_1(dx_2|x_1)\, \mu(dx_1). \tag{9}$$

There further exists a unique probability measure $r \in P(Y)$ such that for each n the marginal of r on Y_n is r_n.

The formal definition of the conditional expectations operator is, therefore,

[11] The integral is well-defined if either the positive or negative parts of the function are finite. Such a function will be called integrable.

the integral of the function versus r_n or r. This definition allows universally measurable selection:

Theorem 4.7 (Measurable Selection) *Let X and Y be Borel spaces, $D \in X \times Y$ be an analytic set such that $D_x = \{y \mid (x, y) \in D\}$, and $f\colon D \to \mathbb{R}^*$ be an upper semianalytic function. Define $f^*\colon \operatorname{proj}_X(D) \to \mathbb{R}^*$ by*

$$f^*(x) = \sup_{y \in D_x} f(x, y). \tag{10}$$

Then the set $I = \{x \in \operatorname{proj}_X(D) \mid$ for some $y_x \in D_x,\ f(x, y_x) = f^*(x)\}$ is universally measurable, and for every $\epsilon > 0$, there exists a universally measurable function $\phi\colon \operatorname{proj}_X(D) \to Y$ such that $Gr(\phi) \subset D$ and for all $x \in \operatorname{proj}_X(D)$ either

$$f[x, \phi(x)] = f^*(x) \quad \text{if } x \in I \tag{11}$$

or

$$f[x, \phi(x)] \geq \begin{cases} f^*(x) - \epsilon & \text{if } x \notin I \text{ and } f^*(x) < \infty \\ 1/\epsilon & \text{if } x \notin I \text{ and } f^*(x) = \infty \end{cases} \tag{12}$$

The selector obtained in Theorem 4.7, $f[x, \phi(x)]$ is universally measurable. If the function $\phi(\ )$ is restricted to be analytically measurable, then I is empty and (12) holds. In this case, the selector is not necessarily universally measurable. For Borel-measurable functions $\phi(\ )$, the analytic result does not hold uniformly in x. The strong result given by (11) is only available for universally measurable functions. Similarly, strong results are available for Borel-measurable functions if significantly stronger regularity assumptions are maintained.[12] The weaker regularity conditions are appealing, as they allow a solution without imposing restrictions on the economics of the problem.

5 Stochastic Dynamic Programming

There are several issues with simply applying DP to the stochastic MIUF problem. First, the functional form of the utility function is not specified, as doing so would restrict the applicability of the results. Second, as previously discussed, there are a number of technical difficulties in applying DP methods in a general stochastic setting. Section 3's measure theory was developed to overcome these difficulties.[13]

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[12] In particular, D must be assumed to be compact, and f must be upper semicontinuous. This is the approach taken in [49].

[13] For further references to DP in measure spaces, see [11, 12, 13, 27, 31, 50].

Three tasks are repeatedly performed in the DP recursion. First, a conditional expectation is evaluated. Second, the supremum of an extended real-valued function in two (vector-valued) variables, the state and the control, is found over the set of admissible control values. Finally, a selector which maps each state to a control that (nearly) achieves the supremum in the second step is chosen. Each of these steps involves mathematical challenges in the stochastic context. An especially important concern is making sure that the measurability assumptions are not destroyed by any of the three steps.

The first and second steps require that the expectation operator can be iterated and interchanged with the supremum operator. As shown in Section 4, these requirements are met by the integral definition of the expectations operator, for either the Borel- or universally-measurable specifications. Step two encounters a problem with measurability, because of the issue with projections of Borel sets also discussed in the previous section. Analytic-measurability is sufficient to address this particular problem, but such measurability is not necessarily preserved by the composition of two functions. By using semianalytic functions and assuming universal-measurability, not only is this problem solved, but measurable selection is also possible under mild regularity conditions. Consequently, all three steps of the DP algorithm can be implemented for the stochastic problem. Consequently, the existence of an optimal or nearly optimal program can be shown and the principal of optimality holds for the (near) optimal value function.

To show that these results are applicable to stochastic MIUF models, the problem laid out in (3) is mapped into Bertsekas and Shreve's general stochastic DP model. One adjustment is necessary as Bertsekas and Shreve define lower semianalytic functions rather than upper semianalytic functions, because their exposition addresses the finding the infimum of a function. This difference requires careful adjustment of their regularity conditions.

5.1 General Framework

Following Bertsekas and Shreve [10, pg. 188-189], the general infinite horizon model is defined as follows:

Definition 5.1 (Stochastic Optimal Control Model) *A infinite horizon stochastic optimal control model is an eight-tuple* $(X, Y, S, Z, q, f, \beta, g)$ *where:*

X State space: a non-empty Borel space;
Y Control space: a non-empty Borel space;
S Control constraint: a function from X to the set of non-empty subsets of Y. The set $\Gamma = \{(x, y) \mid x \in X, y \in S(x)\}$ is assumed to be analytic in $X \times Y$;
Z Disturbance space: a non-empty Borel space;

$q(dz|x,y)$ Disturbance kernel: a Borel-measurable stochastic kernel on Z given $X \times Y$;

f System function: a Borel-measurable function from $X \times Y \times Z$ to X;

β Discount factor: a positive real number; and

g One-stage value function: a upper semianalytic function from Γ to $\mathbb{R}^*$.

The filtered probability space used in the stochastic optimal control model consists four elements: 1) the (Cartesian) product of the disturbance space with the infinite product of the state and control spaces, $Z \times (\Pi_{i=t}^{\infty} (X \times Y)_i)$; 2) a σ-algebra (generally universally measurable) on that product space; 3) the probability measure defined in Theorem 4.6; and, 4) the filtration defined by the restriction to the product of the state and control spaces that have already occurred, $\left(\Pi_{i=t}^{s-1} (X_i \times Y_i)\right) \times X_s$ where it is understood that each subscripted space is a copy of the respective space.

Establishing the existence of a solution to a stochastic optimal control model means establishing the existence of an optimal policy for the problem. Specifically, the following definitions from [10] are used:

Definition 5.2 (Policy) *A policy is a sequence $\phi = (\phi_t, \phi_{t+1}, \ldots)$ such that, for each $s \in \{t, t+1, \ldots\}$,*

$$\phi_s(dy_s \mid x_t, y_t, \ldots, y_{s-1}, x_s) \tag{13}$$

is a universally measurable stochastic kernel on Y, given $X \times Y \times \cdots \times Y \times X$ satisfying

$$\phi_s(S(x_s) \mid x_t, y_t, \ldots, y_{s-1}, x_s) = 1, \tag{14}$$

for every $(x_t, y_t, \ldots, y_{s-1}, x_s)$. If for every s, ϕ_s is parameterized by only x_s, then ϕ_s is a Markov policy. Alternatively, if for every s, ϕ_s is parameterized by only (x_t, x_s), then ϕ_s is a semi-Markov policy. The set of all Markov policies, Φ, is contained in the set of all semi-Markov policies, Φ'. If for each s and $(x_t, y_t, \ldots, y_{s-1}, x_s)$, $\phi_s(dy_s \mid x_t, y_t, \ldots, y_{s-1}, x_s)$ assigns mass one to some element of Y, ϕ is non-randomized. If $\overline{\phi}$ is a Markov policy of the form $\overline{\phi} = (\phi_t, \phi_t, \phi_t, \ldots)$, it is called stationary.

Definition 5.3 (Value Function) *Suppose ϕ is a policy for the infinite horizon model. The (infinite horizon) value function corresponding to ϕ at $x \in X$ is*

$$\begin{aligned} V_\phi(x) &= \int \left[\sum_{k=0}^{\infty} \beta^k g(x_k, y_k)\right] dr(\phi, \mu_x) \\ &= \sum_{k=0}^{\infty} \left[\beta^k \int g(x_k, y_k)\, dr_{k+t}(\phi, \mu_x)\right] \end{aligned} \tag{15}$$

where, for each $\phi \in \Phi'$ and $\mu \in P(X)$, $r(\phi, \mu_x)$ is the unique probability measure defined in equation (4.6) and, for every k, the $r_{k+t}(\phi, \mu_x)$ is the ap-

propriate marginal measure.[14] *The (infinite horizon) optimal value function at* $x \in X$ *is* $V^*(x) = \sup_{\phi \in \Phi'} V_\phi(x)$.

Note that the optimal value function is defined over semi-Markov policies; this is without loss of generality. Furthermore, Bertsekas and Shreve [10, pg. 216] show that the optimal value can be reached by only considering Markov policies. The advantage of including semi-Markov policies is that the optimum may require a randomized Markov policy, but only need a non-randomized semi-Markov policy. Finally, the Jankov-von Neumann theorem guarantees the existence of at least one non-randomized Markov policy so Φ and Φ' are non-empty.

The following defines optimality for policies:

Definition 5.4 (Optimal policies) *If* $\epsilon > 0$, *the policy* ϕ *is* ϵ*-optimal if*

$$V_\phi(x) \geq \begin{cases} V^*(x) - \epsilon & \text{if } V^*(x) < \infty \\ 1/\epsilon & \text{if } V^*(x) = \infty \end{cases} \tag{16}$$

for every $x \in X$. *The policy* ϕ *is optimal if* $V_\phi(x) = V^*(x)$.

In the next subsection, equation (3) is restated in this optimal control framework. The last subsection addresses what conditions are need to guarantee the existence of a (nearly) optimal policy.

5.2 Restating the Household Problem

Embedding the household decision problem into this framework requires specifying the state and control spaces. Since the spaces in this problem are all finite Euclidean spaces, the state and control spaces will be Borel no matter how defined. For a utility problem, it is natural to generally define 'prices' as states and 'quantities' as controls, but there is no unique specification required for the DP algorithm. Also, if the utility function demonstrated habit persistence, as in Barnett and Wu [5], lagged consumption variables would naturally be state variables in the current stage.

Define the period s states by $\mathbf{x}_s = (\boldsymbol{\alpha}_s, \boldsymbol{\psi}_s)$, where $\boldsymbol{\alpha}_s = (\mathbf{a}_{s-1}, A_{s-1})$ and $\boldsymbol{\psi}_s$ denotes the vector of prices, interest rates and other income that were realized

[14] The interchange of the integral and the summation is justified by either the monotone or bounded convergence theorems.

at the beginning of period s, normalized by p_s^*,

$$\boldsymbol{\psi}_s = \left(\left(1+\rho_{1,s-1}\right)\frac{p_{s-1}^*}{p_s^*},\ldots,\left(1+\rho_{k,s-1}\right)\frac{p_{s-1}^*}{p_s^*},\right.$$
$$\left.\left(1+R_{s-1}\right)\frac{p_{s-1}^*}{p_s^*},\frac{p_{1,s}}{p_s^*},\ldots,\frac{p_{n,s}}{p_s^*},\frac{\mathbf{I}_s}{p_s^*}\right). \quad (17)$$

The period s controls are defined to be $\mathbf{y}_s = (\boldsymbol{\theta}_s, \mathbf{c}_s)$ where $\boldsymbol{\theta}_s = (\mathbf{a}_s, A_s)$. The state space X is $(2k+n+2)$–dimensional Euclidean space, and the control space Y is a subset of $(k+n+1)$–dimensional Euclidean space. This notation is useful in what follows. Again note that, for $s < \infty$, elements of a_s and A_s may be negative, so that short-selling is allowed.

The budget constraint can be used to eliminate one of the controls, because the constraint will hold exactly at every time-period for any optimal solution. Therefore, a redundant control has been specified and the set of admissible controls actually lie in a $(k+n)$-dimensional linear subspace of Y. Besides satisfying the budget constraint, the control variables need to be inside the survival set. By leaving in the redundant control, it is easier to explicitly specify this constraint. The controls set can be written as a function of only period s states and controls as

$$S(\mathbf{x}) = \left\{\mathbf{y} \in Y \;\middle|\; \sum_{i=1}^{k+1}\left(y_i - \psi_i x_i\right) + \sum_{j=k+2}^{k+1+n}\psi_j y_j - \psi_{k+n+2} \leq 0\right\} \quad (18)$$

where the period subscript s has been suppressed. The first key assumption is

Criterion 5.5 *Assume that* $\Gamma = \{(x,y) \mid x \in X, y \in S(x)\}$ *is analytic in* $X \times Y$.

The system function has a relatively simple form. It is defined by

$$\mathbf{x}_{s+1} = \left(\boldsymbol{\alpha}_{s+1}, \boldsymbol{\psi}_{s+1}\right) = f\left(\mathbf{x}_s, \mathbf{y}_s, \mathbf{z}_s\right) = \left(\boldsymbol{\theta}_s, \boldsymbol{\psi}_s + z_s\right) \quad (19)$$

In words, the first partition of the states evolves according to the simple rule $\alpha_{s+1} = \boldsymbol{\theta}_s$, and the second evolves as a state-dependent stochastic process, according to $\boldsymbol{\psi}_{s+1} = \boldsymbol{\psi}_s + \mathbf{z}_s$.[15] If $\mathbf{z}_s$was a pure white noise process, $\boldsymbol{\psi}_s$ would be a random walk.

The discount factor is defined by $\beta = 1/(1+\xi)$ and satisfies $0 < \beta < 1$. The one-stage value function $g(x_s, y_s)$ is simply the period utility function $(g(\mathbf{x}_s, \mathbf{y}_s) = g(\mathbf{y}_s) = u(\mathbf{a}_s, \mathbf{c}_s))$, so it is a function of only the controls.[16] The

[15] This would have to slightly modified to account for the model in [5] that includes habit persistence.
[16] Recall that there is a redundant control.

framework would allow $g(\ )$ to be time-varying or to depend on the states, however, this would complicate the derivation of stochastic Euler equations in Section 6. The remaining assumption, which completes the mapping of the stochastic utility problem into the DP model, is:

Criterion 5.6 *Assume that $g(y)$ is an upper semianalytic function from Γ to $\mathbb{R}^{+}$.*

5.3 Existence of a Solution and the Principal of Optimality

The existence of a solution to the household decision problem or equivalently the existence of a (nearly) optimal policy can now be proved. First, as it is easier, the optimal value function is shown to satisfy a stochastic version of Bellman's equation and the Principal of Optimality. The result is most cleanly stated using the following definition:

Definition 5.7 (State transition kernel) *The state transition kernel on X given $X \times Y$ is defined by*

$$t(B \mid x, y) = q(\{z \mid f(x, y, z) \in B\} \mid x, y) = q(f^{-1}(B)_{(x,y)} \mid x, y). \qquad (20)$$

Thus, $t(B \mid x, y)$ is the probability that the state at time $(s+1)$ is in B given that the state at time s is x and the s^{th} control is y. Note that $t(dx' \mid x, y)$ inherits the measurability properties of the stochastic kernel.

Then the following mapping helps to state results concisely:

Definition 5.8 *Let $V : X \to \mathbb{R}^{*}$ be universally measurable. Define the operator T by*

$$T(V) = \sup_{y \in S(x)} \left\{ g(y) + \beta \int_{X} V(x')\ t(dx' \mid x, y) \right\}. \qquad (21)$$

Several lemma's characterize the optimal policies. The following lemma shows that the optimal value function for the problem satisfies a functional recursion that is a stochastic version of Bellman's equation.

Lemma 5.9 *The optimal value function $V^{*}(x)$ satisfies $V^{*} = T(V^{*})$ for every $x \in X$.*

PROOF. Note that $g(y)$ is upper semianalytic and non-negative. This implies that $-g(y)$ is lower semianalytic and non-positive. Also

$$\widetilde{V}_{\phi}(x) = \int \left[\sum_{k=0}^{\infty} \beta^k \left(-g\left(x_{k+t}, y_{k+t} \right) \right) \right] dr\left(\phi, \mu_x \right) = -V_{\phi}(x) \tag{22}$$

and $\widetilde{V}^*(x) = \inf_{\phi \in \Phi} \widetilde{V}(x) = -V^*(x)$. Then from Proposition 9.8 in [10, pg. 225],

$$\widetilde{V}^*(x) = \inf_{y \in Y} \left\{ -g(y) + \beta \int_X \widetilde{V}^*(x') \, t(dx' \mid x, y) \right\} \tag{23}$$

since $-g(y)$ satisfies their assumption labeled (N) on page 214. Taking the negative of each side implies the result as the negation can be taken inside the integral. □

This necessity result implies that the optimal policy would be a fixed point of the mapping which is implicitly defined in the lemma. The following sufficiency result implies that a stochastic version of Bellman's principal of optimality holds for stationary policies.

Lemma 5.10 (Principal of Optimality) *Let $\overline{\phi} = (\overline{\phi}, \overline{\phi}, \ldots)$ be a stationary policy. Then the policy is optimal iff $V_{\overline{\phi}} = T\left(V_{\overline{\phi}}\right)$ for every $x \in X$.*

PROOF. Following the same argumentation as in the previous lemma, given the properties of $g(y)$ the result holds for $\widetilde{V}_{\overline{\phi}}(x)$ by Proposition 9.13 in [10, pg. 228], which implies the result. □

Before examining existence of optimal policies, note that the measurability assumptions already imply the existence of an ϵ-optimal policy. From Proposition 9.20 in [10, pg. 239], the non-negativity of the utility function is enough to assert the existence of an ϵ-optimal policy using similar arguments as in the previous lemmas.

Lemma 5.11 *For each $\epsilon > 0$, there exists an ϵ-optimal non-randomized semi-Markov policy for the infinite horizon problem. If for each $x \in X$ there exists a policy for the infinite horizon problem, which is optimal at x, then there exists a semi-Markov (randomized) optimal policy.*

The fact that existence of any optimal policy is sufficient for the existence of a semi-Markov randomized optimal policy is important. The primary concern is with the first period return or utility function. For the initial period, the semi-Markov ϵ-optimal policy is Markov as clearly $\phi_t(dy_s \mid x_t, x_t) = \phi_t(dy_s \mid x_t)$. If ϵ-optimality is judged to be sufficient, then simply use $g\left(\overline{\phi}_t^*\right)$ where $\overline{\phi}_t^*$ is the first element of the optimal policy. The principal of optimality would only hold approximately, however. Similarly, if an optimal policy does actually exist, the randomness is not an issue as the optimal policy is non-random in the first element. In that case, the principal of optimality may not hold as equation (3) is only guaranteed to hold for stationary policies. Consequently, minimal additional assumptions are useful.

Before making these additional assumptions, define the DP algorithm as follows:

Definition 5.12 (Dynamic Programming Algorithm) *The algorithm is defined recursively by*

$$V_0(x) = 0 \qquad \forall x \in X \tag{24}$$

$$V_{k+1}(x) = T(V_k(x)) \qquad \forall x \in X,\ k = 0, 1, \ldots \tag{25}$$

Proposition 9.14 of [10] implies that the algorithm converges for the problem as stated in the following lemma.

Lemma 5.13 $V_\infty = V^*$

Unfortunately, the convergence is not necessarily uniform in x. Additionally, it is not possible to synthesize the optimal policy from the algorithm, as is the case for deterministic problems, because V_k, while universally measurable, is not necessarily semianalytic for all k.

The regularity assumptions are strengthened by imposing a mild boundedness assumption.

Criterion 5.14 (Boundedness) *Assume that* $\forall i$ *and* $\forall s \in \{t, t+1, \ldots\}$, $\psi_{i,s} > 0$. *Further assume that the single stage utility function contains no points of global satiation.*

Assuming Criterion 5.14 leads to stronger results. First, under this boundedness condition, the DP algorithm converges uniformly for any initial upper semianalytic function not just zero. Furthermore, necessary and sufficient conditions for the existence of an optimal policy are available.

Lemma 5.15 *Assume Criterion 5.14 holds. Then for each* $\epsilon > 0$, *there exists an* ϵ*-optimal non-randomized stationary Markov policy. If for each* $x \in X$ *there exists a policy for the infinite horizon problem, which is optimal at* x, *then there exists an unique optimal non-randomized stationary policy. Furthermore, there is an optimal policy if and only if for each* $x \in X$ *the supremum in*

$$\sup_{y \in S(x)} \left\{ g(y) + \beta \int_X V^*(x')\ t(dx' \mid x, y) \right\} \tag{26}$$

is achieved.

PROOF. Criterion 5.14 and the Transversality condition given in (4) imply that $g(y)$ is bounded over Γ so that $\exists b$ such that $\forall (x, y) \in \Gamma$, $g(y) < b$. Since $g(y) \geq 0$, obviously $g(y) > -b$. Also recall that $\beta = 1/(1+\xi)$ so that $0 < \beta < 1$. This implies that the problem satisfies the assumption labeled (D) in [10, pg. 214]. The lemma follows from Proposition 9.19, Proposition 9.12 and Corollary 9.12.1 in [10, pg. 228]. □

Combining this lemma with lemma 5.10 implies that there exists an optimal policy if and only if there exists a stationary policy such that $V_{\overline{\phi}} = T\left(V_{\overline{\phi}}\right)$ for every $x \in X$. It is clear that the ϵ-optimal non-randomized stationary Markov policy is the universally measurable selector from Theorem 4.7. If universally measurable selection is assumed to be possible (i.e. the set I defined in Theorem 4.7 is the entire set D), then the supremum will be achieved. This assumption is weaker than requiring that Borel-measurable selection is possible, as in Stokey and Lucas [49]. The assumption is also weaker than the regularity conditions needed to solve semicontinuous models in [10], which have Borel-measurable optimal plans. The following lemma, which follows from Proposition 9.17 of [10] supplies a sufficient condition for the supremum to be achieved.

Lemma 5.16 *Under Criterion 5.14, if there exists a nonnegative integer $\overline{k}$ such that for each $x \in X$, $\lambda \in \mathbb{R}$, and $k \geqslant \overline{k}$, the set*

$$S_k(x, \lambda) = \left\{ y \in S(x) \mid g(y) + \beta \int V_k(x')\ t(dx' \mid x, y) \geqslant \lambda \right\} \tag{27}$$

is compact in Y then there exists a non-randomized optimal stationary policy for the infinite horizon problem.

This is a weaker condition than assuming that the constraint sets are compact or that Γ is upper hemi-continuous. If the supremum in (26) is achieved for the initial state $x_t \in X$, the boundedness assumption implies that a unique stationary non-random Markov optimal plan exists.

6 Stochastic Euler Equations

In this section, Euler equations for the stochastic decision are derived. The usefulness of stochastic Euler equations is discussed in [49, pg. 280-283]. Although necessary and sufficient conditions for the optimum to exist have been established, the stronger characterization given by Euler equations is often needed and is always useful. Since the principal of optimality has been shown to hold, Bellman's equation can be used to derive stochastic Euler equations. Of course, the optimal value function needs to satisfy additional regularity conditions. In particular, it needs to be differentiable. In addition, it must be possible to interchange the order of integration. The interchange is possible, for example, if each partial derivative of V is absolutely integrable ([38] and [14, 49, Theorem 9.10, pg. 266-257]). In the present case, it is sufficient to show that the value function is differentiable on an open subset of x_t, because of Criterion 5.14. Then the value function meets the conditions in Mattner [44], particularly the locally bounded assumption, and the interchange is valid. [17] The following two results are immediate implications of the envelope theorem. The first proves that the optimal solution inherits differentiability. The second formally derives the stochastic Euler equations proposed in Barnett et al. [3] and, more generally, demonstrates that stochastic Euler equations can be validly derived for the class of models.

Theorem 6.1 *If $U\left(\ \right)$ is concave and differentiable, then the value function is differentiable.*

PROOF. Let $\overline{\phi}$ denote the optimal stationary non-random Markov policy. Note that at time s, $\overline{\phi}$ is a function of x_s. To simplify notation, let $\overline{V}\left(x\right) = V_{\overline{\phi}}\left(x\right)$. Bellman's equation implies

$$\overline{V}\left(x\right) = g\left(\overline{\phi}\left(x\right)\right) + \beta \int_Z \overline{V}\left[f\left(\overline{\phi}\left(x\right), z\right)\right] p(dz \mid x, y) \tag{28}$$

holds for any x_t. Note $x_{t+1} = f\left(y_t, z_t\right)$, so the value function within the integral is being evaluated one period into the future. Let x^0 denote the actual initial state. For $x \in N\left(x^0\right)$, where $N\left(x^0\right)$ is a neighborhood of x^0, define

$$J\left(x\right) = g\left(x, \overline{\phi}\left(x^0\right)\right) + \beta \int_Z \overline{V}\left[f\left(\overline{\phi}\left(x^0\right), z\right)\right] p(dz \mid x, y). \tag{29}$$

In words, $J\left(x\right)$ is the value function with the policy constrained to be the optimal policy for x^0. Clearly, $J\left(x^0\right) = V_{\overline{\phi}}\left(x^0\right)$ and, $\forall x \in N\left(x^0\right)$, $J\left(x\right) \leq V_{\overline{\phi}}\left(x\right)$ because $\phi\left(x^0\right)$ is not the optimal policy for $x \neq x^0$. If the original utility function $U\left(\ \right)$ is concave and differentiable then so is $u\left(\ \right)$ and therefore $g\left(\ \right)$. This assumption implies that $J\left(x\right)$ is also concave and differentiable. The envelope theorem from [8] combined with the fact that the policy $\overline{\phi}$ is optimal uniformly in x then implies that $V_{\overline{\phi}}\left(x\right)$ is differentiable for all $x \in int\left(X\right)$, As prices and rates of return are assumed to be larger than zero, the only

[17] The theorem actually applies to holomorphic functions, but the proof can be readily adapted for first-order (real) differentiable function.

initial conditions for which $V_{\overline{\phi}}(x)$ is not differentiable are infinite (positive or negative) initial asset endowments, which can be excluded. □

Theorem 6.2 (Stochastic Euler Equations) *If $U(\ )$ is concave and differentiable. Then the stochastic Euler equations for* (3) *are,*

$$\frac{\partial u(a_t^*, c_t^*)}{\partial a_i} = \frac{\partial u(a_t^*, c_t^*)}{\partial c'} - \frac{1}{1+\xi} E_t \left[\left(1+\rho_{i,t}\right) \frac{p_t^*}{p_{t+1}^*} \frac{\partial u\left(a_{t+1}^*, c_{t+1}^*\right)}{\partial c'} \right] \tag{30}$$

and

$$\frac{\partial u(a_t^*, c_t^*)}{\partial c'} = \frac{1}{1+\xi} E_t \left[(1+R_t) \frac{p_t^*}{p_{t+1}^*} \frac{\partial u\left(a_{t+1}^*, c_{t+1}^*\right)}{\partial c'} \right] \tag{31}$$

where a_t^* and c_t^* are the controls specified by the non-randomized optimal stationary policy and c' is an arbitrary numéraire.

PROOF. Using the notation from the previous proof, the envelope theorem implies that

$$\overline{V}_x\left(x^0\right) = J_x\left(x^0\right) = \left.\frac{\partial g(x, \overline{\phi}(x))}{\partial x}\right|_{x=x^0} = \left.\frac{\partial g(\overline{\phi}(x))}{\partial x}\right|_{x=x^0} \tag{32}$$

where $\overline{V}_x(x)$ is a vector-valued function whose i^{th} element is given by

$$\partial g(x, \overline{\phi}(x))/\partial x_i. \tag{33}$$

The differentiability of the value function combined with the ability to interchange differentiation with integration for the stochastic integral, imply that the necessary conditions for $\overline{\phi}(x^0)$ to be optimal are

$$\left.\frac{\partial \overline{V}_y(x)}{\partial y}\right|_{y=\overline{\phi}(x^0)} = 0 \tag{34}$$

where $\partial\overline{V}/\partial y$ is a vector-valued function whose i^{th} element is given by $\partial\overline{V}/\partial y_i$. It follows that the stochastic Euler equation is

$$\left.\frac{\partial g(y)}{\partial y} + \beta \int_Z \overline{V}'_y [f(y,z)] \frac{\partial f(y,z)}{\partial y} p(dz \mid x, y)\right|_{y=\overline{\phi}(x)} = 0 \tag{35}$$

where $\partial g/\partial y$ is a $k+n+1$ vector-valued function whose i^{th} element $\partial g/\partial y_i$, $\partial f/\partial y$ is a $k+n+1$ by $2k+n+2$ matrix with i,j element $\partial f_j/\partial y_i$. Equation

(32) can be used to replace the unknown value function, so that (35) becomes

$$\left.\frac{\partial g(y)}{\partial y} + \beta \int_Z \frac{\partial g(y)}{\partial x} \frac{\partial f(y,z)}{\partial y} p(dz \mid x, y)\right|_{y=\overline{\phi}(x)} = 0. \tag{36}$$

The simple form of the system equation implies that

$$\partial f/\partial y = \left[I_{(k+n+1)\times(k+n+1)} \;\; 0_{(k+n+1)\times(k+1)} \right], \tag{37}$$

so that (36) becomes,

$$\left.\frac{\partial g(y)}{\partial y} + \beta \int_Z \frac{\partial g(y)}{\partial x} p(dz \mid x, y)\right|_{y=\overline{\phi}(x)} = 0. \tag{38}$$

Using the fact that there is a redundant control at the optimum, an element can be eliminated from $\overline{\phi}(x)$. In particular, choose an arbitrary element of c. Denote this numéraire element by c', and the remaining $n-1$ elements of c by c_-. Assume without loss of generality that $p' = p^*$ so that $\psi' = 1$ where ψ' is the element of $\boldsymbol{\psi}$ that coincides with c'. To further simplify notation, let y_s^* denote y evaluated at the optimum at time s. Then using the obvious notation, $g(y_s^*) = g\left(\theta_s^*, c_-^*, c'^*\right)$ and (38) implies that, for $i \in \{1, \ldots, k\}$,

$$\frac{\partial g\left(\theta_t^*, c_{-t}^*, c_t'^*\right)}{\partial \theta_i} - \frac{\partial g\left(\theta_t^*, c_{-t}^*, c_t'^*\right)}{\partial c'} + \beta \int_Z \left(\psi_{i,t+1}\right) \frac{\partial g(\theta_{t+1}^*, c_{-t+1}^*, c_{t+1}'^*)}{\partial c'} p(dz \mid x, y^*) = 0.$$

Also, taking the derivative with regards to y_{k+1} (the benchmark asset) implies

$$-\frac{\partial g\left(\theta_t^*, c_{-t}^*, c_t'^*\right)}{\partial c'} + \beta \int_Z \left(\psi_{k+1,t+1}\right) \frac{\partial g(\theta_{t+1}^*, c_{-t+1}^*, c_{t+1}'^*)}{\partial c'} p(dz \mid x, y^*) = 0. \tag{39}$$

Substituting the original notation proves the result. □

The stochastic Euler equations define an asset pricing rule that is a strict generalization of the consumption CAPM asset pricing rule.[18] Substituting from (31) into (30) and using the linearity of the conditional expectations operator implied by the linearity of the integral, produces

$$\partial u(a_t^*, c_t^*)/\partial a_i = \frac{1}{1+\xi} E_t \left[\left(R_t - \rho_{i,t}\right) \frac{p_t^*}{p_{t+1}^*} u_{c'}\left(a_{t+1}^*, c_{t+1}^*\right) \right] \tag{40}$$

[18] There are, of course, n other equations for differentiation with respect to elements of c. These equations are simpler in that they are non-stochastic.

where $u_{c'}\left(a_{t+1}^{*}, c_{t+1}^{*}\right) = \partial u\left(a_{t+1}^{*}, c_{t+1}^{*}\right)/\partial c'$. The first order condition for a simple utility maximization problem for consumption goods is $\frac{\partial u(c)/\partial c_i}{\partial u(c)/\partial c_j} = \frac{p_i}{p_j}$. Similarly, the right-hand side of (40) defines the relevant information for assets' relative prices.

7 Conclusion

Ljungqvist and Sargent [39, pg. xxi] state that there is an "art" to choosing the right state variables so that a problem can be solved through recursive techniques. They further argue that increasing the range of problems amenable to recursive techniques has been one of the key advances in macroeconomic theory. This paper has applied a different art: carefully defining the characteristics of the state and control spaces. But the motivation is similar. The choice of space and the subsequent measurability assumptions allow stochastic MIUF models to be solved through the DP recursion. The results mirror those that are available for deterministic dynamic problems: an unique solution exists that can be differentiated to derive (stochastic) Euler equations.

The method used in this paper requires regularity conditions that are less restrictive than other approaches. Even more importantly, the regularity conditions do not restrict the economics of the problem. Consequently, the results are broadly applicable to monetary and financial models, particularly the many models where the existence of a solution was just assumed. The approach to modeling uncertainty can be applied to other stochastic economic models, but introducing risk, as is done in this paper, is probably the clearest motivation for introducing contemporaneous uncertainty.

The stochastic MIUF problem integrates monetary and finance models, containing important examples from each as special cases. Further work on integrating aspects of finance into models of money could address both the fact that technological and theoretical advances have been steadily increasing the liquidity of risky assets and the fact there is little consensus on how to model risk. The expected utility framework, whose underpinnings are formally established in this paper, is the most commonly used approach, but it is not universally accepted. It is not yet clear whether stochastic MIUF models can incorporate alternative models of risk. Furthermore, the current model addresses the decision of an individual consumer. Embedding the decision problem in a market context would strengthen the connection between the two literatures.

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