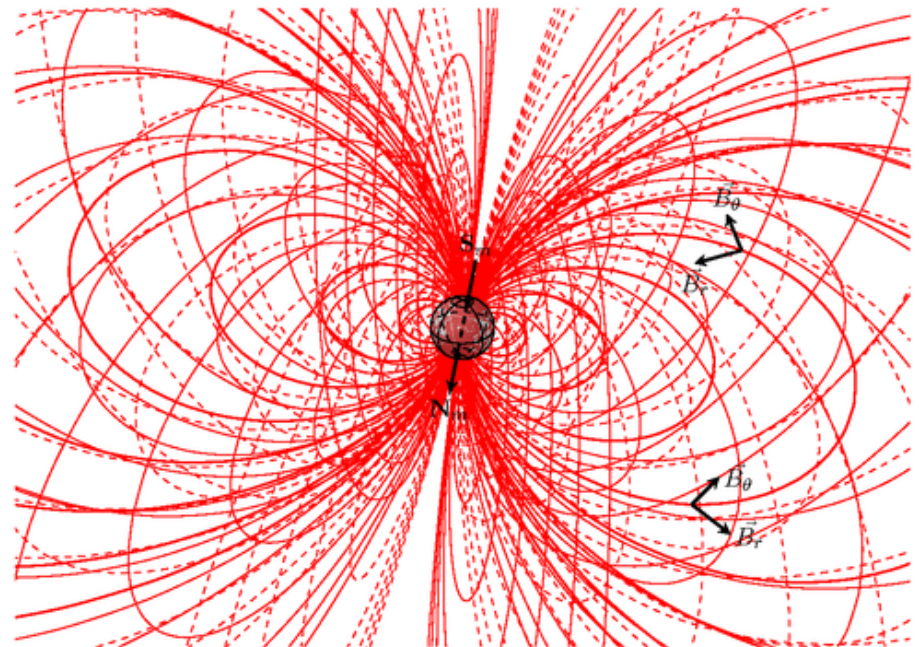


Properties of Magnetic Dipoles

To visualize the complex geometry of cosmic magnetic fields it is useful to represent the field by means of magnetic field lines, that is, lines that are everywhere parallel to the magnetic field vector. Since dipolar geometry underlies many examples of cosmic magnetic fields, we examine the dipole case in this module and its modifications in the next module.



[This figure will be replaced by one using a CCMC program after Lutz finished it.]

The equations of a dipole field in spherical polar coordinates (r, θ, ϕ) are

$$B_r = 2M \cos \theta / r^3 \quad (1)$$

$$B_\theta = M \sin \theta / r^3 \quad (2)$$

$$B_\phi = 0 \quad (3)$$

where M is the dipole moment, which can be positive or negative. In the case of Earth, $M = -8 \times 10^{15} \text{ T m}^3$ or $-31,000 \text{ nT R}_e^3$. It is often useful to use latitude (λ) instead of colatitude (θ); then the equations are

$$B_r = 2M \sin \lambda / r^3 \quad (4)$$

$$B_\lambda = -M \cos \lambda / r^3 \quad (5)$$

$$B_\phi = 0 \quad (6)$$

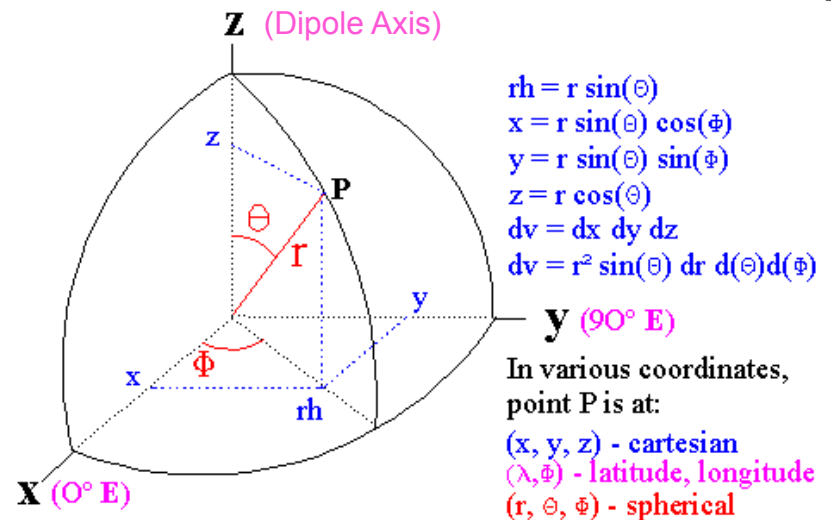
where λ is negative in the southern hemisphere so that $d\lambda = -d\theta$ at all latitudes. We will also find it useful to employ the cartesian (x, y, z) coordinate system, where x points sunward, z points northward and is coaxial with the dipole, and y completes the set. Then we have

$$B_x = 3M xz / r^5 \quad (7)$$

$$B_y = 3M yz / r^5 \quad (8)$$

$$B_z = M (3z^2 - r^2) / r^5 \quad (9)$$

Dipole Field Equations



Digression on the distinction between dipole moment, M , used in planetology and magnetic moment, μ , used in plasma physics, notably the magnetic moment of a charged particle in a magnetic field. $\mu = 4\pi M / \mu_0 = M \times 10^7$. Whereas M has the dimensions of T m^3 , μ has the dimensions of J/T . Thus, M is more convenient when discussing the spatial properties of the dipole field, but μ is more convenient when discussing magnetic energy.

Potential Form of the Dipole Field Equations

The field of a dipole is both curl-free (no currents) and divergence-free (like all magnetic fields). Therefore, there exist a scalar potential, Φ , and a vector potential, A , from which the field equations may be generated by differentiation.

$$\mathbf{B} = -\nabla\Phi \quad (10)$$

where

$$\Phi = M \cos \theta / r^2 \quad (11)$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (12)$$

where

$$\mathbf{A} = M \sin \theta / r^2 \boldsymbol{\varphi} \quad (13)$$

and $\boldsymbol{\varphi}$ is the unit vector in the φ direction.

Integrating the field-line equation

$$dr/B_r = r d\theta/B_\theta \quad (14)$$

gives

$$r = L \sin^2\theta \quad (15)$$

where the integration constant, written as L by convention, denotes the distance at which the field line crosses the equator.

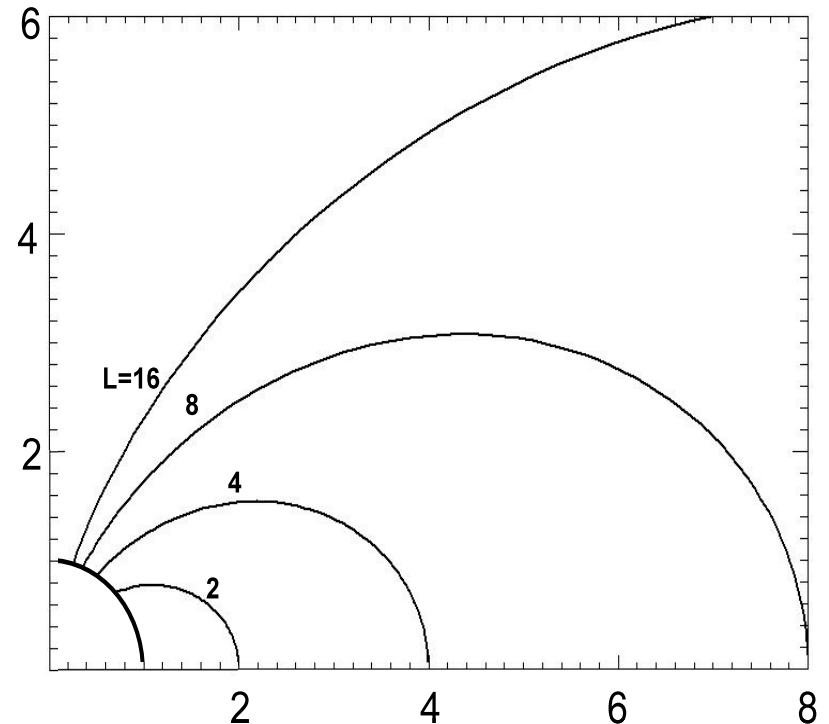
This equation was used to plot the field lines in adjacent figure. The field lines are self-similar in the sense that normalized to their equatorial crossing distance, L , they have the same shape:

$$\eta = \sin^2\theta \quad (16)$$

where η is normalized radial distance, r/L .

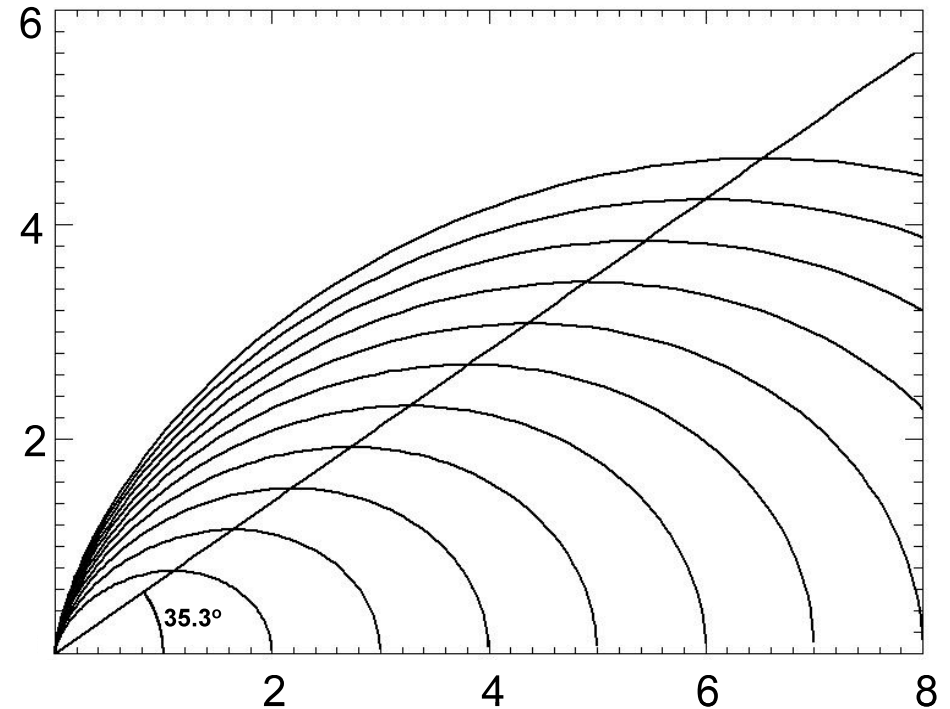
This means that that if you expand or contract a dipole field line keeping its shape the same but moving its equatorial crossing distance from L to L' , then it will lie on top of the dipole field line with L' as its equatorial crossing distance.

Dipole Field Line Geometry



Self Similarity of Dipole Field Lines

An interesting property of self-similarity of dipole field lines is that at fixed latitude they all make the same angle with respect to the radial. For example, at latitude 35.3° every dipole field line reaches its maximum distance from the equatorial plane and at that point is parallel to the equatorial plane. As a corollary, this means that the north-south (z) component of every dipole field changes sign at 35.3° latitude.

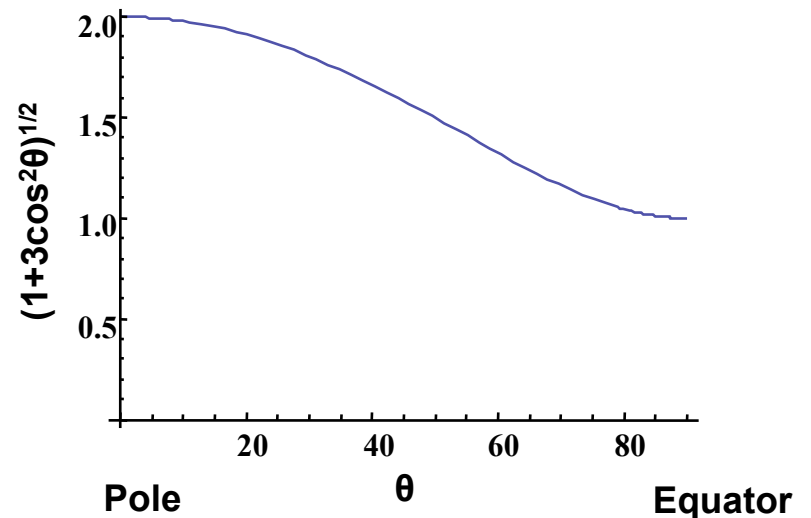


Dipole Field Strength

The dipole field equations (1) to (3) say that a dipole field is parallel to the radial direction over the poles and perpendicular to the radial direction on the equator. It is twice as strong at the pole as at the equator at a fixed radial distance. At any latitude, the field strength decreases with radial distance as $1/r^3$. The following equation makes explicit the dependence of field strength on distance and latitude.

$$B = |M| (1+3\cos^2\theta)^{1/2}/r^3 \quad (17)$$

The latitude dependence of field strength is plotted in the figure.

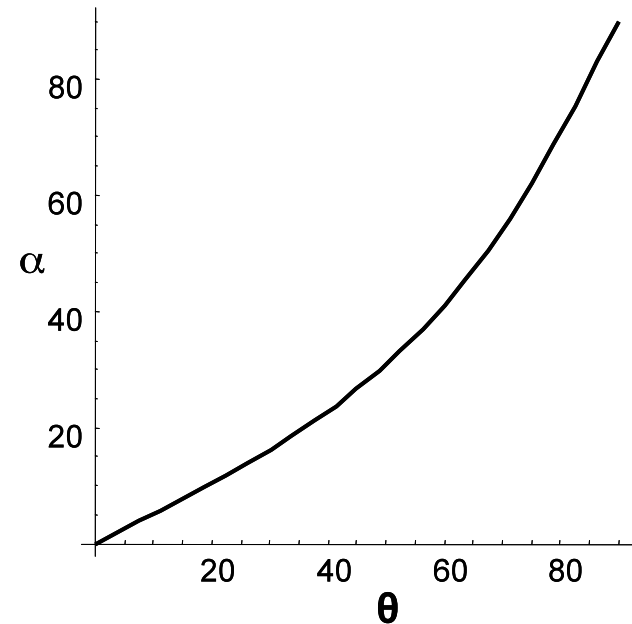


Field Line Inclination

Right at the pole a dipole field is parallel (or antiparallel) to the radial direction and right on the equator a dipole field is perpendicular to the radial direction. At other latitudes the angle between the radial direction and the magnetic field of a dipole (the complement of the inclination of the field) is given by

$$\alpha(\theta) = \cos^{-1}(2 \cos \theta / (1 + 3 \cos^2 \theta)^{1/2}) \quad (18)$$

This angle is plotted in the adjacent figure, from which we see that $\alpha(0)=0^\circ$ and $\alpha(90)=90^\circ$ as claimed.

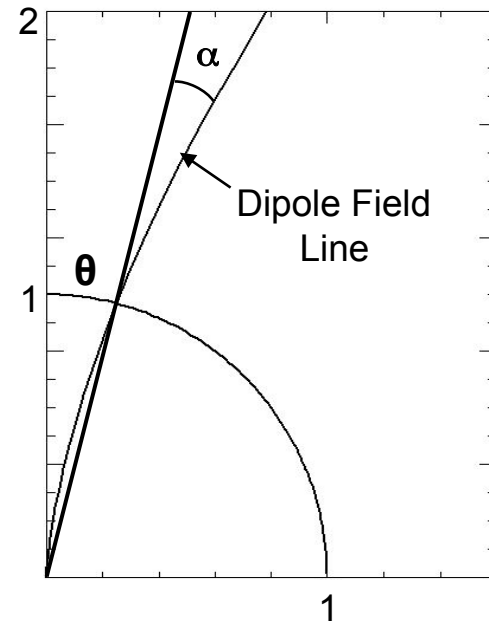


The Polar Region

Around the pole ($\theta = 0$) the behavior of $\alpha(\theta)$ is nearly linear. To determine the linear coefficient at the pole, we differentiate

$$d\alpha/d\theta = 2/(1+3 \cos^2\theta) \quad (19)$$

which when evaluated at $\theta = 0$ gives $d\alpha/d\theta = 1/2$. That is, around the pole the angle, α , between a dipole field line and the vertical direction is about half the polar angular, θ .



Exercises

1. Verify that potentials (11) and (13) yield the dipole equations (1) to (3).
2. Verify the field-line equation (15).
3. Verify that at latitude 35.3° every dipole field line reaches its maximum distance from the equatorial plane.
4. Verify the field-strength equation (17)
5. Verify equation (18) for the angle between the vertical direction and the magnetic field at colatitude θ . (Hint: The dot product between a unit radial vector and a unit vector in the direction of the magnetic field is the cosine of the desired angle.)
6. Show that for a dipole field line with equatorial crossing distance L , the radius of curvature* of at the equator is $L/3$. (Note that this is $1/3$ the radius of curvature of a geocentric circle of radius L).

*For a function of the form $r(\theta) = f(\theta)$, the radius of curvature, R_c , is given by

$$R_c = (r^2 + r'^2)^{3/2} / (r^2 + 2r'r'' - rr''')$$

where primes denote differentiation with respect to θ . Evaluate all terms at the equator, $\theta=90^\circ$.