

Derivatives

Notation Notes:

$\frac{d}{dx}$ in front of a function means "take the derivative."

$f'(x) =$ means that whatever is after the equals sign, is the derivative of $f(x)$.

Constant Rule

$$\frac{d}{dx}c = 0$$

The constant rule says that when you take the derivative of a constant, it is 0. We will see why this is in a second, while we talk about power rule.

Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The power rule says that if you have x to some power n , the derivative is the power times x raised to the power minus 1.

Examples:

1. $\frac{d}{dx}5$

You might be confused on how to use the power rule on 5 since it doesn't have an x raised to a power.

However, it actually does. 5 has an x^0 next to it, since anything raised to the zero power is 1, therefore $5 = 5x^0$. So, using the power rule gives us $5 \cdot 0x^{0-1} = 0x^{-1} = 0$. This is why the constant rule is true for any constant.

2. $\frac{d}{dx}(7x^3)$

Using the power rule, we have $7 \cdot (3x^{3-1}) = 21x^2$.

$$\text{So, } \frac{d}{dx}(7x^3) = 21x^2.$$

3. Find the derivative of $f(x) = 6x^3 - 9x + 4$.

We can use the power rule on each individual term since we are adding and subtracting. Using the power rule on each term, we get

$$\begin{aligned} &6(3x^{3-1}) - 9(1x^{1-1}) + 4(0x^{0-1}) \\ &= 18x^2 - 9x^0 + 0x^{-1} \\ &= 18x^2 - 9 \end{aligned}$$

Practice:

Find the derivative.

1. $5x^4$
2. $2x^7 + 4x^4 - 7x + 140$
3. $\sqrt{x} + 5x$

Log and Exponential Function Rules

General Logarithm Rule:

$$\frac{d}{dx}[\log_a(f(x))] = \frac{1}{f(x)\ln(a)} * f'(x)$$

Special case:

$$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$$

Super special case:

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

General Exponential Rule:

$$\frac{d}{dx}[a^{f(x)}] = a^{f(x)} * \ln(a) * f'(x)$$

Special case:

$$\frac{d}{dx}[e^{f(x)}] = e^{f(x)} * f'(x)$$

Super special case:

$$\frac{d}{dx}[e^x] = e^x$$

These are the typical exponential and logarithm derivatives used in calculus. The general exponential and logarithm derivatives come from using the limit definition of the derivative to find the derivative. The special cases are actually the general rules in disguise. If we apply the general log rule to natural log of $f(x)$, we get

$$\frac{d}{dx}[\ln(f(x))] = \frac{1}{f(x) * \ln(e)} * f'(x) = \frac{f'(x)}{f(x)}$$

which is our special case for logs. Then, if we apply the general rule for log to natural log of x , we get

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x * \ln(e)} * 1 = \frac{1}{x}$$

which is our super special case for logs. Similarly, if the general rule for exponentials is applied to e raised to the $f(x)$ we get

$$\frac{d}{dx}[e^{f(x)}] = e^{f(x)} * \ln(e) * f'(x) = e^{f(x)} * f'(x)$$

which is our special case for exponentials. If we apply the general rule for exponentials to e raised to the x

we get

$$\frac{d}{dx} [e^x] = e^x * \ln(e) * 1 = e^x$$

which is our super special case for exponentials.

Product Rule

$$\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Examples:

1. Find the derivative of $h(x) = (6x^2 - 7x + 9)(x^3 + 4x^2 - 10x - 3)$

Now, if we just had the power rule, we would have to multiply everything out and then take the derivative of each individual term, which seems like a lot of work. Luckily, we have the product rule to make things a little easier on us. So let's go ahead and label what is going to be $f(x)$ and what is going to be $g(x)$.

To match our product rule, let's have $6x^2 - 7x + 9 = f(x)$ and $x^3 + 4x^2 - 10x - 3 = g(x)$. The product rule says that we also need the derivatives of $f(x)$ and $g(x)$. So let's find those derivatives using the power rule.

$f(x) = 6x^2 - 7x + 9$	$f'(x) = 12x - 7$
$g(x) = x^3 + 4x^2 - 10x - 3$	$g'(x) = 3x^2 + 8x - 10$

Now that we have all the pieces, let's put them together.

$$h'(x) = (12x - 7)(x^3 + 4x^2 - 10x - 3) + (6x^2 - 7x + 9)(3x^2 + 8x - 10)$$

You could multiply everything out, but usually it is acceptable to just leave it as shown above. However, it will depend on what your specific professor wants.

2. Find the derivative of $j(x) = e^{2x^2}(4x^3 + 7)$

For this example, even multiplying out wouldn't help since you would still end up having to do the product rule, except you would have to do it twice and not just once. Like before, let's label our different functions.

$f(x) = e^{2x^2}$	$f'(x) =$
$g(x) = 4x^3 + 7$	$g'(x) =$

Now, let's find the derivatives.

$f(x) = e^{2x^2}$	$f'(x) = 4xe^{2x^2}$
$g(x) = 4x^3 + 7$	$g'(x) = 12x^2$

Putting it all together:

$$j'(x) = (4xe^{2x^2})(4x^3 + 7) + (e^{2x^2})(12x^2)$$

Practice:

Find the derivative.

4. $(x + 3)(x^2 - 2x + 5)$

5. $\sqrt{x}(x^3 + 6)$

6. $\ln(x)(5x^3 - 3x^2)$

Quotient Rule

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Memorization Tip: The top of the quotient rule is the same as the product rule, just with a minus sign instead of a plus sign.

Examples:

1. Find the derivative of $h(x) = \frac{\ln(x)}{2x}$

Similar to the product rule, we are going to label, find the derivatives of those labelled pieces, then plug it into our rule.

$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$
$g(x) = 2x$	$g'(x) = 2$

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h'(x) = \frac{\left(\frac{1}{x}\right)(2x) - (\ln(x))(2)}{(2x)^2}$$

$$h'(x) = \frac{2 - 2 \ln(x)}{4x^2}$$

Typically, you only have to simplify your answer if the question says to, or if it's not too much trouble to simplify it.

2. Find the derivative of $j(x) = \frac{4x^3 - 9x}{5x^2 + 2}$

$f(x) = 4x^3 - 9x$	$f'(x) = 12x^2 - 9$
$g(x) = 5x^2 + 2$	$g'(x) = 10x$

$$j'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$j'(x) = \frac{(12x^2 - 9)(5x^2 + 2) - (4x^3 - 9x)(10x)}{(5x^2 + 2)^2}$$

$$j'(x) = \frac{(12x^2 - 9)(5x^2 + 2) - 40x^4 + 90x^2}{(5x^2 + 2)^2}$$

Practice:

Find the derivative.

7. $\frac{3x + 9}{2 - x}$

8. $\frac{x^2 + 4x + 20}{3 - x}$

9. $\frac{x + 5}{x^3 - 7}$

Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Examples:

1. $\frac{d}{dx} (3x^3 - 6x^2)^8$

Similar to the product and quotient rules, we are going to label and take the individual derivatives then put it into the chain rule. However, it can be a little confusing at first to figure out what should be labelled as what, but after some practice it becomes more clear. Typically, when talking about the chain rule, there are two parts. The "inside" ($g(x)$) and the "outside" ($f(x)$). The "inside" usually refers to whatever is being raised to a power or inside the square root. The "outside" is usually the power or the square root.

In this example, the "inside" is $3x^3 - 6x^2$ since that is the function being raised to a power. The "outside" is the power of 8. To take the derivative, we leave what's on the inside the same and take the derivative of the outside function, then multiply by the derivative of the inside.

$$\begin{aligned} &= 8 \left[\underbrace{3x^3 - 6x^2}_{g(x)} \right]^7 * \frac{d}{dx} (3x^3 - 6x^2) \\ &= 8(3x^3 - 6x^2)^7 (9x^2 - 12x) \\ &= 8(9x^2 - 12x) (3x^3 - 6x^2)^7 \\ &= (72x^2 - 96x) (3x^3 - 6x^2)^7 \end{aligned}$$

2. Find the derivative of $y = \sqrt{3x^2 + 5}$

Rewrite so that it looks like something we are more familiar with: $(3x^2 + 5)^{\frac{1}{2}}$. Now it looks similar to the last example.

$$\begin{aligned} y' &= \frac{1}{2} \left[\underbrace{3x^2 + 5}_{g(x)} \right]^{\frac{1}{2}-1} * \frac{d}{dx} (3x^2 + 5) \\ y' &= \frac{1}{2} (3x^2 + 5)^{-\frac{1}{2}} (6x) \\ y' &= \frac{3x}{\sqrt{3x^2 + 5}} \end{aligned}$$

Practice:

Find the derivative.

10. $(2x + 5)^3$

11. $\sqrt{x^3 - 4x + 8}$

12. e^{3x-2}

Answers to Practice Problems:

1. $20x^3$

2. $14x^6 + 16x^3 - 7$

3. $\frac{1}{2\sqrt{x}} + 5$

4. $(x^2 - 2x + 5) + (x + 3)(2x - 2)$ which when simplified fully, gives us $3x^2 + x - 1$

5. $\frac{1}{2\sqrt{x}}(x^3 + 6) + 3x^2\sqrt{x}$

6. $(5x^2 - 3x) + \ln(x)(15x^2 - 6x)$

7. $\frac{-3}{(2-x)^2}$

8. $\frac{-x^2 + 6x + 32}{(3-x)^2}$

9. $\frac{-2x^3 - 15x^2 - 7}{(x^3 - 7)^2}$

10. $6(2x + 5)^2$

11. $\frac{3x^2 - 4}{2\sqrt{x^3 - 4x + 8}}$

12. $3e^{3x-2}$