MEAN AND APPARENT PLACE COMPUTATIONS IN THE NEW IAU SYSTEM. II. TRANSFORMATION OF MEAN STAR PLACES FROM FK4 B1950.0 TO FK5 J2000.0 USING MATRICES IN 6-SPACE

B. D. YALLOP AND C. Y. HOHENKERK
Royal Greenwich Observatory, Hailsham, East Sussex, England

C. A. SMITH, G. H. KAPLAN, J. A. HUGHES, AND P. K. SEIDELMANN
U. S. Naval Observatory, Washington, DC 20392

ABSTRACT

A 6×6 matrix method for transforming a catalog mean place from epoch and equinox B1950.0 on the FK4 system to epoch and equinox J2000.0 on the FK5 system is described. A step-by-step comparison is made between the matrix method and the classical spherical formulas.

I. INTRODUCTION

The main purposes of this paper are (1) to describe a 6×6 matrix method for transforming mean star places from the standard epoch and equinox of B1950.0 on the FK4 system to the standard epoch and equinox of J2000.0 on the FK5 system, and (2) to make a step-by-step comparison between this matrix method and the classical method described in Paper I.

The transformation from B1950.0 to J2000.0 is described in the next section. It is straightforward to express this transformation in the matrix notation adopted by Standish (1982) with the modifications communicated to Standish by Aoki and Soma (1983). There are two main advantages in the matrix method. First, if there are any changes in the transformation, such as a change in one of the parameters or a change in the order of operations, then it is easily incorporated into the matrix transformation. Second, it is easier to translate the matrix method into a high-level programming language such as FORTRAN or BASIC, and the program should run more efficiently. Third, there are no problems with singularities at the poles. Moreover, it is much easier to modify the algorithm if it is decided to change the transformation at a later stage (e.g., the alternative procedure suggested by C. A. Murray. See note added in proof to Paper I).

The various stages of the matrix transformation are discussed in more detail in Secs. IV, V, VII, IX, and X. In Sec. XI the complete matrix transformation is compared with the classical method. In most cases, the comparison is obvious. The exception is at the step in the transformation at which the proper motions are changed from the FK4 system to the FK5 system. In the matrix method, this transformation is implicit, while in the classical method it is explicit. It requires considerable algebraic manipulation to obtain explicit expressions from the matrix method which may then be compared with the classical expressions given in Paper I. When numerical comparisons are made between the two methods, they will agree to a precision of $\pm 0\rlap{.}''001$. The largest difference comes from the spherical equations for the E terms of aberration, partly from the corrections to position, but mainly from the corrections to the proper motions in right ascension and declination. Ignoring the effect of the E terms of aberration, the agreement is better than $\pm 2'' \times 10^{-10}$, provided the terms of order $\pm 1'' \times 10^{-7}$ involving l in the proper-motion equations are included in the classical case and all variables are calculated to the same precision in both methods. This includes the equinox correction, the precession angles, and m and n, the rates of change of the precession angles. In fact, the classical method calculates m and n at 1984 January 1.0 directly from their polynomial expressions, while in the 6×6 matrix method they are calculated at B1950.0 and J2000.0, and their values at 1984 January 1.0 are implicit. This produces differences between the two methods of about $\pm 3'' \times 10^{-6}$ in position and $\pm 2'' \times 10^{-5}$ per century in proper motion.

II. THE TRANSFORMATION FROM B1950.0 TO J2000.0

There has been much controversy in the literature over the correct procedure for the transformation (Aoki *et al.* 1983). The recommended transformation is given in Paper I. The transformation described here follows that recommendation. It should be noted that when transferring individual observations, as opposed to a catalog mean place, the safest method is to transform the observation back to the epoch of the observation, on the FK4 system (or in the system that was used to produce the observed mean place), convert to the FK5 system, and transform to the epoch and equinox J2000.0.

The transformation for a fundamental catalog position is as follows:

Step 1. Form the position and velocity vector from the FK4 catalog position, i.e., form the position and velocity vector from the right ascension, declination, proper motions, parallax, and radial velocity.

Step 2. Remove the E terms of aberration from the B1950.0 catalog mean place. There is also a question whether or not the E terms should be removed from the proper motions. The problem is discussed more fully in Paper I. If the FK4 catalog is used, they certainly do not have to be removed from stars within 10° of the poles because they have not been included (Lederle 1984).

Step 3. Apply space motion to the position vector to the epoch 1984 January 1.0, which is the epoch at which the sidereal time expression in terms of UT is changed (IAU 1977). This is an example of where one of the parameters of the transformation could be changed.

Step 4. Precess, using the FK4 precession constants, the position and velocity from B1950.0 to 1984 January 1.0.

Step 5. Apply the equinox correction FK4 to FK5 at 1984 January 1.0.

Step 6. Convert the proper motions from seconds of arc per tropical century to seconds of arc per Julian century.

Step 7. Precess, using the FK5 precession constants, the position and velocity from 1984 January 1.0 to J2000.0.

Step 8. Apply space motion to the position vector from

 0004-6256/89/010274-06$00.90

1984 January 1.0 to J2000.0.

Step 9. Convert the position and velocity vector back to right ascension and declination and proper motions in right ascension and declination, and extract the parallax and radial velocity in the FK5 system.

III. NOTATION AND DEFINITIONS

α—right ascension in degrees.

δ—declination in degrees.

μ—proper motion in right ascension in arcseconds per century. On the FK4 system, tropical centuries are used, while Julian centuries are used on the FK5 system.

μ'—proper motion in declination in arcseconds per century.

π—parallax in arcseconds.

V—radial velocity in km s^{-1}.

C—the length of the century in days. Julian centuries C_J consist of 36 525 days, while tropical centuries $C_B = 36524.21987817305$ days.

$\mathscr{E}$—Epoch, e.g., 1984 January 1.0.

JD(date)—The Julian date, which is a function of the date or epoch. Julian dates for some relevant epochs are JD(B1950.0) = 2433282.42345905, JD(1984 January 1.0) = 2445700.5, and JD(J2000.0) = 2451545.0.

k—Conversion from km s^{-1} to AU per century. $k = 86400C/1.49597870\times 10^8$.

P—the precession matrix. $\mathbf{P}_n$, $\mathbf{P}_o$ are the precession matrices on the new FK5 and the old FK4 systems, respectively. $\mathbf{P}^{-1}$ and $\dot{\mathbf{P}}$ represent the inverse and the differential precession matrix. Whenever it occurs in this paper, $\dot{\mathbf{P}}^{-1}$ means $(\dot{\mathbf{P}})^{-1}$.

r—a column vector of position, where the transpose $\mathbf{r}' = (x, y, z)$.

$\dot{\mathbf{r}}$—a column vector of velocity, where the transpose $\dot{\mathbf{r}}' = (\dot{x}, \dot{y}, \dot{z})$.

$\mathbf{R}_i(\phi)$—the standard orthonormal rotation matrices $\mathbf{R}_i(\phi)$, $i = 1,2,3$, which rotate a right-handed set of axes x, y, z through an angle ϕ anticlockwise about the ith axis.

v—a vector in 6-space, where the transpose $\mathbf{v}' = (\mathbf{r}', \dot{\mathbf{r}}') = (x, y, z, \dot{x}, \dot{y}, \dot{z})$.

The rotations about the x, y, and z axes are represented by the following matrices

$$\mathbf{R}_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix},$$

$$\mathbf{R}_2(\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix},$$

$$\mathbf{R}_3(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These matrices are orthonormal and therefore have the following properties:

$$\mathbf{R}_i^{\mathrm{T}}(\phi)\mathbf{R}_i(\phi) = \mathbf{I} = \mathbf{R}_i(\phi)\mathbf{R}_i^{\mathrm{T}}(\phi),$$

$$\mathbf{R}_i^{\mathrm{T}}(\phi)\mathbf{R}_j(\phi) = \mathbf{O}, \quad i \neq j,$$

where $\mathbf{R}_i^{\mathrm{T}}$ is the transpose of $\mathbf{R}_i$ and equals the inverse $\mathbf{R}_i^{-1}$. **I** is the unit matrix and **O** is the zero (null) matrix. Matrix inversion is very efficient for orthonormal matrices as it is just a matter of exchanging rows for columns. In all applications in this paper, ϕ is a function of time; thus the derivatives of the rotation matrices are given by

$$\dot{\mathbf{R}}_i(\phi) = \dot{\phi}\frac{\partial}{\partial\phi}\mathbf{R}_i(\phi), \quad i = 1,2,3,$$

and

$$\frac{\partial}{\partial\phi}\mathbf{R}_1(\phi) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & \cos\phi \\ 0 & -\cos\phi & -\sin\phi \end{bmatrix},$$

$$\frac{\partial}{\partial\phi}\mathbf{R}_2(\phi) = \begin{bmatrix} -\sin\phi & 0 & -\cos\phi \\ 0 & 0 & 0 \\ \cos\phi & 0 & -\sin\phi \end{bmatrix},$$

$$\frac{\partial}{\partial\phi}\mathbf{R}_3(\phi) = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using the above definitions for the two 3×3 matrices $\mathbf{R}_i(\phi)$ and $\dot{\mathbf{R}}_i(\phi)$, we now need to work in 6-space and define a 6×6 matrix

$$\mathbf{Q}_i(\phi,\dot{\phi}) = \begin{bmatrix} \mathbf{R}_i(\phi) & \mathbf{O} \\ \dot{\phi}\dfrac{\partial}{\partial\phi}\mathbf{R}_i(\phi) & \mathbf{R}_i(\phi) \end{bmatrix}.$$

The reason for introducing **Q** and working in 6-space is as follows. Consider a position vector $\mathbf{r}_0$ which is rotated about the ith axis to produce a new vector $\mathbf{r}_1$, then

$$\mathbf{r}_1 = \mathbf{R}_i(\phi)\mathbf{r}_0.$$

Differentiating this equation with respect to time gives the relation between the two velocity vectors $\dot{\mathbf{r}}_0$, $\dot{\mathbf{r}}_1$ as follows:

$$\dot{\mathbf{r}}_1 = \dot{\phi}\frac{\partial}{\partial\phi}\mathbf{R}_i(\phi)\mathbf{r}_0 + \mathbf{R}_i(\phi)\dot{\mathbf{r}}_0.$$

Hence

$$\mathbf{v}_1 = \begin{bmatrix} \mathbf{R}_i(\phi) & \mathbf{O} \\ \dot{\phi}\dfrac{\partial}{\partial\phi}\mathbf{R}_i(\phi) & \mathbf{R}_i(\phi) \end{bmatrix} \begin{bmatrix} \mathbf{r}_0 \\ \dot{\mathbf{r}}_0 \end{bmatrix} = \mathbf{Q}_i(\phi,\dot{\phi})\mathbf{v}_0.$$

In astronomy, the higher-order derivatives $\ddot{\mathbf{r}}$, etc., are negligible and therefore these two sets of equations in three dimensions are all that are required, and it is found to be more efficient to use one set of equations in 6-space.

The following properties for the **Q** matrix will be required later and are easily verified by substituting the above expressions for $\mathbf{R}_i(\phi)$ and $\partial\mathbf{R}_i(\phi)/\partial\phi$:

$$\mathbf{Q}_i^{-1}(\phi,\dot{\phi})\mathbf{Q}_i(\phi,\dot{\phi}) = \mathbf{I} = \mathbf{Q}_i(\phi,\dot{\phi})\mathbf{Q}_i^{-1}(\phi,\dot{\phi}),$$

$$\mathbf{Q}_i^{-1}(\phi,\dot{\phi}) = \mathbf{Q}_i(-\phi,-\dot{\phi}),$$

$$\mathbf{Q}_i^{-1}(\phi,\dot{\phi}) = \begin{bmatrix} R_i^{\mathrm{T}}(\phi) & \mathbf{O} \\ \dot{\phi}\dfrac{\partial}{\partial\phi}\mathbf{R}_i^{\mathrm{T}}(\phi) & \mathbf{R}_i^{\mathrm{T}}(\phi) \end{bmatrix}.$$

IV. CONVERSION FROM SPHERICAL TO VECTOR COORDINATES

In either method, matrix or classical, it is necessary at some stage of the calculation to convert from spherical to rectangular coordinates of position and velocity. Given that a star has position (α,δ) in degrees, proper motions (μ,μ') in seconds of arc per century, parallax (π) in seconds of arc, and radial velocity V in km s^{-1}, then the direction cosines of the position vector **r** and velocity vector $\dot{\mathbf{r}}$ in arcseconds per century are

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{bmatrix}$$

and

$$\dot{\mathbf{r}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\mu\sin\alpha\cos\delta - \mu'\cos\alpha\sin\delta \\ +\mu\cos\alpha\cos\delta - \mu'\sin\alpha\sin\delta \\ \mu'\cos\delta \end{bmatrix} + kV\pi\mathbf{r},$$

where $k = 86400C/(1.49597870\times 10)^8$ and C is the length of the century in days. The units of $\dot{\mathbf{r}}$ are arcseconds per century, Julian or tropical as appropriate, and $\mathbf{r}$ is automatically a unit vector.

Conversely, it is also necessary at some stage of the calculation to convert back from rectangular coordinates of position and velocity to spherical coordinates. It may no longer be the case that $\mathbf{r}$ is a unit vector. Hence given the vectors $\mathbf{r}' = (x, y, z)$ and $\dot{\mathbf{r}}' = (\dot{x}, \dot{y}, \dot{z})$, where $\dot{\mathbf{r}}'$ is in arcseconds per century, then the right ascension and declination (α,δ) and proper motions (μ,μ') in arcseconds per century are obtained from

$$\cos\alpha\cos\delta = x/r, \quad \sin\alpha\cos\delta = y/r, \quad \sin\delta = z/r,$$

$$\mu = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}, \quad \mu' = \frac{\dot{z}(x^2 + y^2) - z(x\dot{x} + y\dot{y})}{r^2\sqrt{x^2 + y^2}},$$

where

$$r = \sqrt{x^2 + y^2 + z^2}.$$

In the conversion from B1950.0 to J2000.0, the parallax and velocity at J2000.0 (π_1, V_1) can be obtained from $\mathbf{r}$ and $\dot{\mathbf{r}}$ together with the parallax (π) and velocity (V) at B1950.0; thus

$$\pi_1 = \pi/r, \quad V_1 = (x\dot{x} + y\dot{y} + z\dot{z})/k\pi r.$$

However, if $\pi = 0$, then $V_1 = V$.

V. THE REMOVAL OF THE ELLIPTIC TERMS OF ABERRATION

The equatorial velocity components of the E terms of aberration (Emerson 1973) referred to the equinox of date are given by

$$\mathbf{B} = \begin{bmatrix} -\Delta D \\ +\Delta C \\ +\Delta C\tan\epsilon \end{bmatrix}.$$

In a fixed frame at B1950.0, the components at date are given by

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}, \qquad (1)$$

where $\mathbf{P}$ is the precession matrix from B1950.0 to date. Provided $\mathbf{A}$ is in the appropriate units, then the direction to the star $\mathbf{r}_0$, corrected for the E terms of aberration, is given by

$$\mathbf{r}_0 = \mathbf{r}_{\text{cat}} - [\mathbf{A} - (\mathbf{r}'_{\text{cat}}\,\mathbf{A})\mathbf{r}_{\text{cat}}], \qquad (2)$$

where $\mathbf{r}_{\text{cat}}$ is the catalog position vector, $[\mathbf{A} - (\mathbf{r}'_{\text{cat}}\,\mathbf{A})\mathbf{r}_{\text{cat}}]$ is the component of $\mathbf{A}$ in the direction perpendicular to $\mathbf{r}_{\text{cat}}$, since $\mathbf{r}'_{\text{cat}}$ is the transpose of $\mathbf{r}_{\text{cat}}$, and $(\mathbf{r}'_{\text{cat}}\,\mathbf{A})$ is the scalar product. The numerical values of the elements of $\mathbf{A}$ at B1950.0 when $\mathbf{P} = \mathbf{I}$ are

$$\mathbf{A} = \begin{bmatrix} -1.62557 \\ -0.31919 \\ -0.13843 \end{bmatrix} \times 10^{-6} \text{ radians}$$

using the expressions for ΔC and ΔD given in Paper I, Sec. II *d 1*, Eq. (7). The classical formula (Paper I, Sec. II *d 1*, Eqs. (5) and (6)) may be derived from this expression using a first-order approximation.

Aoki (1983) has pointed out that the E terms of aberration may have affected the proper motions as well. In vector notation, the effect is derived by differentiating Eq. (2) as follows:

$$\dot{\mathbf{r}}_0 = \dot{\mathbf{r}}_{\text{cat}} - \dot{\mathbf{A}} + (\mathbf{r}'_{\text{cat}}\,\dot{\mathbf{A}})\mathbf{r}_{\text{cat}} + (\dot{\mathbf{r}}'_{\text{cat}}\,\mathbf{A})\mathbf{r}_{\text{cat}} + (\mathbf{r}'_{\text{cat}}\,\mathbf{A})\dot{\mathbf{r}}_{\text{cat}}, \qquad (3)$$

where the terms $(\dot{\mathbf{r}}'_{\text{cat}}\,\mathbf{A})\mathbf{r}_{\text{cat}}$ and $(\mathbf{r}'_{\text{cat}}\,\mathbf{A})\dot{\mathbf{r}}_{\text{cat}}$ are very small and may be neglected unless a precision better than $\pm 1'' \times 10^{-3}$ is required.

In the fixed frame at B1950.0, the expression for $\dot{\mathbf{A}}$ at date is given by differentiating Eq. (1); thus

$$\dot{\mathbf{A}} = \dot{\mathbf{P}}^{-1}\mathbf{B} + \mathbf{P}^{-1}\dot{\mathbf{B}},$$

where

$$\dot{\mathbf{P}}^{-1} = \begin{bmatrix} 0 & \dot{\zeta}_A + \dot{z}_A & \dot{\theta}_A \\ -\dot{\zeta}_A - \dot{z}_A & 0 & 0 \\ -\dot{\theta}_A & 0 & 0 \end{bmatrix}$$

and ζ_A, z_A, and θ_A are the precession angles. At B1950.0, differentiating Andoyer's expressions for the precession angles which are given in Sec. VIII on the FK4 system, with $T = 1$ and $t = 0$, gives

$$\dot{\zeta}_A + \dot{z}_A = 4609''\!.90 \text{ per tropical century},$$

$$\dot{\theta}_A = 2004''\!.26 \text{ per tropical century}.$$

Also, $\mathbf{P}^{-1} = \mathbf{I}$ and

$$\dot{\mathbf{B}} = \begin{bmatrix} -\Delta\dot{D} \\ +\Delta\dot{C} \\ \Delta\dot{C}\tan\epsilon + \Delta C\dot{\epsilon}_r/\cos^2\epsilon \end{bmatrix} = \begin{bmatrix} +2''\!.9941 \\ -9''\!.0738 \\ -3''\!.9174 \end{bmatrix} \times 10^{-3} \text{ per tropical century}$$

using the expressions for $\Delta\dot{C}$ and $\Delta\dot{D}$ given in Paper I, Sec. II *d2*, Eqs. (10). Hence

$$\dot{\mathbf{A}} = \begin{bmatrix} +1''\!.245 \\ -1''\!.580 \\ -0''\!.659 \end{bmatrix} \times 10^{-3} \text{ per tropical century}.$$

The classical formula for the correction to the proper motions for the E terms is equivalent to Eq. (3) to first order. In the case where the E terms have been allowed for when deriving the catalog proper motions, the procedure in the vector method has to follow the classical method more closely. After the position vector $\mathbf{r}_{\text{cat}}$ has been corrected for the E terms using Eq. (2), the corrected right ascension and declination of the position vector $\mathbf{r}_0$ have to be determined and they are used to form the velocity vector.

VI. THE PRECESSION MATRIX IN 3-SPACE

The precession matrix $\mathbf{P}$, which precesses equatorial rectangular coordinates from a fixed equinox and equator $\mathscr{E}_F$ to one of date $\mathscr{E}_D$ in 3-space, is given by

$$\mathbf{P}[\mathscr{E}_F,\mathscr{E}_D] = \mathbf{R}_3(-z_A)\mathbf{R}_2(+\theta_A)\mathbf{R}_3(-\zeta_A),$$

where ζ_A, z_A, θ_A are the precession angles, which are evaluated using the appropriate time arguments. The expressions for the precession angles in seconds of arc are given in Sec. VIII. The precession matrix is made up of three rotations, applied in an appropriate order. The first rotation is through the angle $-\zeta_A$ about the z axis, the second rotation is through the angle $+\theta_A$ about the y axis, and the final rotation is through the angle $-z_A$ about the z axis.

Having calculated the precession angles for the matrix **P**, the inverse matrix $\mathbf{P}^{-1}$ can be calculated in various ways, for example,

$$\mathbf{P}^{-1} = \mathbf{R}_3^{-1}(-\zeta_A)\mathbf{R}_2^{-1}(+\theta_A)\mathbf{R}_3^{-1}(-z_A)$$
$$= \mathbf{R}_3^{T}(-\zeta_A)\mathbf{R}_2^{T}(+\theta_A)\mathbf{R}_3^{T}(-z_A)$$
$$= \mathbf{R}_3(+\zeta_A)\mathbf{R}_2(-\theta_A)\mathbf{R}_3(+z_A).$$

Alternatively, the precession angles may be recalculated since $\mathbf{P}^{-1}[\mathscr{E}_F,\mathscr{E}_D] = \mathbf{P}[\mathscr{E}_D,\mathscr{E}_F]$. Differentiating **P**, we find that the differential precession matrix is given by

$$\dot{\mathbf{P}} = \dot{\mathbf{R}}_3(-z_A)\mathbf{R}_2(+\theta_A)\mathbf{R}_3(-\zeta_A)$$
$$+ \mathbf{R}_3(-z_A)\dot{\mathbf{R}}_2(+\theta_A)\mathbf{R}_3(-\zeta_A)$$
$$+ \mathbf{R}_3(-z_A)\mathbf{R}_2(+\theta_A)\dot{\mathbf{R}}_3(-\zeta_A),$$

where $\dot{\mathbf{R}}$ is defined in Sec. III. $\dot{\mathbf{P}}$ is required when comparing the two methods (Sec. XI).

VII. THE PRECESSION MATRIX IN 6-SPACE

In 6-space, the notation for the precession matrix has to be modified to include a further parameter s, where $s = 0$ when precessing from one inertial frame to another inertial frame and $s = 1$ when precessing from an inertial frame to a non-inertial frame (i.e., rotating frame of date). The expression for **P** becomes

$$\mathbf{P}[\mathscr{E}_F,\mathscr{E}_D,s] = \begin{bmatrix} \mathbf{P} & \mathbf{O} \\ s\dot{\mathbf{P}} & \mathbf{P} \end{bmatrix}$$
$$= \mathbf{Q}_3(-z_A, -s\dot{z}_A)\mathbf{Q}_2(+\theta_A, +s\dot{\theta}_A)$$
$$\times \mathbf{Q}_3(-\zeta_A, -s\dot{\zeta}_A).$$

For the conversion from FK4 B1950.0 to FK5 J2000.0 in Sec.X, the case $s = 1$ is required. The inverse is also required with $s = 1$, and is given by

$$\mathbf{P}^{-1}[\mathscr{E}_F,\mathscr{E}_D,1] = \begin{bmatrix} \mathbf{P} & \mathbf{O} \\ \dot{\mathbf{P}} & \mathbf{P} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{O} \\ \dot{\mathbf{P}}^{-1} & \mathbf{P}^{-1} \end{bmatrix}$$
$$= [\mathbf{Q}_3(-z_A, -\dot{z}_A)\mathbf{Q}_2(+\theta, +\dot{\theta}_A)$$
$$\times \mathbf{Q}_3(-\zeta, -\dot{\zeta}_A)]^{-1}.$$

VIII. NUMERICAL EXPRESSIONS FOR THE PRECESSION ANGLES AND THEIR RATES OF CHANGE

The equatorial precession angles ζ_A, z_A, θ_A are given as polynomial functions of T and t. T transforms the equations from the basic epoch $\mathscr{E}_0$ (e.g., B1850.0 or J2000.0) to the required fixed (initial) epoch $\mathscr{E}_F$ (e.g., B1950.0), while t transforms the equations from the fixed (initial) epoch $\mathscr{E}_F$ (e.g., B1950.0 or J2000.0) to the epoch of date $\mathscr{E}_D$.

$$\zeta_A = \zeta_A(T,t), \quad z_A = z_A(T,t), \quad \theta_A = \theta_A(T,t),$$

where

$$T = [\mathrm{JD}(\mathscr{E}_F) - \mathrm{JD}(\mathscr{E}_0)]/C$$

and

$$t = [\mathrm{JD}(\mathscr{E}_D) - \mathrm{JD}(\mathscr{E}_F)]/C,$$

with C the number of days in the century, Julian or tropical as appropriate. The conversion of mean star places from B1950.0 on the FK4 system to J2000.0 on the FK5 system requires not only the precession angles but also their rates of change with respect to time to be defined on both systems. The precession rates are given by

$$\dot{\zeta}_A = \frac{d}{dt}\zeta_A(T,t), \quad \dot{z}_A = \frac{d}{dt}z_A(T,t), \quad \dot{\theta}_A = \frac{d}{dt}\theta_A(T,t).$$

There is still some argument as to whose (i.e., Newcomb, or Andoyer, or Kinoshita) definitions of precession to use with the FK4 system. In this paper, we have used Andoyer's, and in these equations the basic epoch is $\mathscr{E}_0 = $ B1850.0, and the time arguments are fractions of a tropical century.

$$\zeta_A = (2303''.5545 + 1''.39720T + 0''.000060T^2)t + (0''.30240 - 0''.000270T)t^2 + 0''.017995t^3,$$
$$z_A = (2303''.5545 + 1''.39720T + 0''.000060T^2)t + (1''.09480 + 0''.000390T)t^2 + 0''.018325t^3,$$
$$\theta_A = (2005''.112 - 0''.8529T - 0''.00037T^2)t + (-0''.4265 - 0''.00037T)t^2 - 0''.04180t^3.$$

In this application, $T = 1$ as the fixed epoch is $\mathscr{E}_F = $ B1950.0, and $t = [\mathrm{JD}(\mathscr{E}_D) - \mathrm{JD}(\mathscr{E}_F)]/C_B$. The precession angles for the FK5 system have been defined by Lieske (1979) and adopted by the IAU. In these equations, the basic epoch of the equations is $\mathscr{E}_0 = $ J2000.0 or $\mathrm{JD}(\mathscr{E}_0) = 2451545.0$, and the time arguments are fractions of a Julian century.

$$\zeta_A = (2306''.2181 + 1''.39656T - 0''.000139T^2)t + (0''.30188 - 0''.000344T)t^2 + 0''.017998t^3,$$
$$z_A = (2306''.2181 + 1''.39656T - 0''.000139T^2)t + (1''.09468 + 0''.000066T)t^2 + 0''.018203t^3,$$
$$\theta_A = (2004''.3109 - 0''.85330T - 0''.000217T^2)t + (-0''.42665 - 0''.000217T)t^2 - 0''.041833t^3.$$

In this application, $T = [\mathrm{JD}(\mathscr{E}_F) - \mathrm{JD}(\mathscr{E}_0)]/C_J$ and $t = [\mathrm{JD}(\mathscr{E}_D) - \mathrm{JD}(\mathscr{E}_F)]/C_J$. In both systems, the precession angles are subscripted by the symbol A, which indicates that the precession angles are accumulated, and the rates of change are expressed in seconds of arc per century.

IX. EQUINOX CORRECTION

Fricke (1982) has determined that the FK4 right ascension system requires a correction of $+0''.525$ at B1950.0, and in general at epoch $\mathscr{E}$ the correction to right ascension should be

$$E_{\mathscr{E}} = E_{50} + \dot{E}[\mathrm{JD}(\mathscr{E}) - \mathrm{JD(B1950.0)}]/C_B,$$

where

$$E_{50} = 0''.525 \text{ and } \dot{E} = 1''.275.$$

In the classical method, this correction is applied separately to the right ascensions and proper motions in right ascension. In the vector method, the correction is applied by means of a rotation about the z axis. The total effect on both the position and velocity vectors is given by $\mathbf{Q}_3(-E_{\mathscr{E}}, -\dot{E})$, where $E_{\mathscr{E}}$ is expressed in degrees and $\dot{E}$ in arcseconds per tropical century.

X. THE COMPLETE TRANSFORMATION IN MATRIX NOTATION

The six steps 3, 4, 5, 6, 7, and 8 in Sec. II may be represented by successive multiplication by the six matrices M_1, M_2, M_3, M_4, M_5, and M_6 on the position and velocity vector $\mathbf{v}_0$ at B1950.0 producing the position and velocity vector $\mathbf{v}_1$ at J2000.0, where

$$\mathbf{v}_1 = \mathsf{M}_6\mathsf{M}_5\mathsf{M}_4\mathsf{M}_3\mathsf{M}_2\mathsf{M}_1\mathbf{v}_0 \tag{4}$$

and the six matrices are defined as follows:

M_1—Adds space motion between the standard epoch B1950.0 and $\mathscr{E}$ to the position vector at B1950.0.

$$\mathsf{M}_1 = \begin{bmatrix} \mathbf{I} & t_0\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{bmatrix},$$

where $t_0 = c[\mathrm{JD}(\mathscr{E}) - \mathrm{JD}(\mathrm{B1950.0})]/C_B$ and $c = \pi/(180\times 3600)$ is a factor that converts seconds of arc to radians.

M_2—Applies FK4 precession from B1950.0 to $\mathscr{E}$, to the position and velocity in 6-space.

$$\mathsf{M}_2 = \mathbf{P}_{\mathrm{o}}[\mathrm{B1950.0}, \mathscr{E}, 1]$$
$$= \mathbf{Q}_3(-z_{\mathrm{A}}, -\dot{z}_{\mathrm{A}})Q_2(\theta_{\mathrm{A}}, \dot{\theta}_{\mathrm{A}})Q_3(-\zeta_{\mathrm{A}}, -\dot{\zeta}_{\mathrm{A}}).$$

M_3—Adds the equinox correction to the right ascension at epoch $\mathscr{E}$.

$$\mathsf{M}_3 = \mathbf{Q}_3(-E_{\mathscr{E}}, -\dot{E}),$$

where $E_{\mathscr{E}} = E_{50} + \dot{E}[\mathrm{JD}(\mathscr{E}) - \mathrm{JD}(\mathrm{B1950.0})]/C_B$ and $E_{50} = 0\overset{''}{.}525$ and $\dot{E} = 1\overset{''}{.}275$.

M_4—Converts the proper motions from tropical centuries to Julian centuries.

$$\mathsf{M}_4 = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & F\mathbf{I} \end{bmatrix},$$

where $F = C_J/C_B$.

M_5—Applies FK5 precession from $\mathscr{E}$ to J2000.0, to the position and velocity in 6-space.

$$\mathsf{M}_5 = \mathbf{P}_{\mathrm{n}}^{-1}[\mathrm{J2000.0}, \mathscr{E}, 1]$$
$$= \mathbf{Q}_3(+\zeta_{\mathrm{A}}, +\dot{\zeta}_{\mathrm{A}})\mathbf{Q}_2(-\theta_{\mathrm{A}}, -\dot{\theta}_{\mathrm{A}})$$
$$\times \mathbf{Q}_3(+z_{\mathrm{A}}, +\dot{z}_{\mathrm{A}})$$

M_6—Adds space motion between $\mathscr{E}$ and J2000.0 to the position vector at $\mathscr{E}$.

$$\mathsf{M}_6 = \begin{bmatrix} \mathbf{I} & -t_1\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{bmatrix},$$

where $t_1 = c[\mathrm{JD}(\mathscr{E}) - \mathrm{JD}(\mathrm{J2000.0})]/C_J$ and $c = \pi/(180\times 3600)$.

The product of the six matrices may be represented by the single matrix

$$\mathsf{M} = \mathsf{M}_6\mathsf{M}_5\mathsf{M}_4\mathsf{M}_3\mathsf{M}_2\mathsf{M}_1.$$

In particular, when the epoch $\mathscr{E}$ = 1984 January 1.0, the numerical expression for M, printed to 15 decimal places, is

$$\begin{bmatrix}
0.999925678186902 & -0.011182059642247 & -0.004857946558960 & 0.000002423950176 & -0.000000027106627 & -0.000000011776558 \\
0.011182059571766 & 0.999937478448132 & -0.000027176441185 & 0.000000027106627 & 0.000002423978783 & -0.000000000065874 \\
0.004857946721186 & -0.000027147426498 & 0.999988199738770 & 0.000000011776559 & -0.000000000065816 & 0.000002424101735 \\
-0.000541652366951 & -0.237968129744288 & 0.436227555856097 & 0.999947035154614 & -0.011182506121805 & -0.004857669684959 \\
0.237917612131583 & -0.002660763319071 & -0.008537771074048 & 0.011182506007242 & 0.999958833818833 & -0.000027184471371 \\
-0.436111276039270 & 0.012259092261564 & 0.002119110818172 & 0.004857669948650 & -0.000027137309539 & 1.000009560363559
\end{bmatrix}$$

XI. THE COMPARISON

This section compares the method that uses matrices in six space with the classical method in three space. Equation (4) may be written with $\mathbf{P} = \mathbf{P}_{\mathrm{n}}$ and $E = E_{\mathscr{E}}$:

$$\begin{pmatrix} \mathbf{r}_1 \\ \dot{\mathbf{r}}_1 \end{pmatrix} = \begin{bmatrix} \mathbf{I} & -t_1\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{O} \\ \dot{\mathbf{P}}^{-1} & \mathbf{P}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & F\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{R}_3(-E) & \mathbf{O} \\ \dot{\mathbf{R}}_3(-E) & \mathbf{R}_3(-E) \end{bmatrix} \begin{bmatrix} \mathbf{P}_{\mathrm{o}} & \mathbf{O} \\ \dot{\mathbf{P}}_{\mathrm{o}} & \mathbf{P}_{\mathrm{o}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & t_0\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{r}_0 \\ \dot{\mathbf{r}}_0 \end{pmatrix}.$$

The transformation from $\mathscr{E}$ to J2000.0 on the FK5 system is

$$\begin{pmatrix} \mathbf{r}_1 \\ \dot{\mathbf{r}}_1 \end{pmatrix} = \mathsf{M}_6\mathsf{M}_5\mathbf{v}^5_{\mathscr{E}} = \begin{bmatrix} \mathbf{I} & -t_1\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{O} \\ \dot{\mathbf{P}}^{-1} & \mathbf{P}^{-1} \end{bmatrix} \begin{pmatrix} \mathbf{r}^5_{\mathscr{E}} \\ \dot{\mathbf{r}}^5_{\mathscr{E}} \end{pmatrix}. \tag{5}$$

Hence at $\mathscr{E}$ the middle part of the transformation from FK4 to FK5 is given by

$$\begin{pmatrix} \mathbf{r}^5_{\mathscr{E}} \\ \dot{\mathbf{r}}^5_{\mathscr{E}} \end{pmatrix} = \mathsf{M}_4\mathsf{M}_3\mathbf{v}^4_{\mathscr{E}} = \begin{pmatrix} \mathbf{R}_3(-E)\mathbf{r}^4_{\mathscr{E}} \\ F[\dot{\mathbf{R}}_3(-E)\mathbf{r}^4_{\mathscr{E}} + \mathbf{R}_3(-E)\dot{\mathbf{r}}^4_{\mathscr{E}}] \end{pmatrix}. \tag{6}$$

The transformation from B1950.0 to $\mathscr{E}$ on the FK4 system is

$$\begin{pmatrix} \mathbf{r}^4_{\mathscr{E}} \\ \dot{\mathbf{r}}^4_{\mathscr{E}} \end{pmatrix} = \mathsf{M}_2\mathsf{M}_1\mathbf{v}_0 = \begin{pmatrix} \mathbf{P}_{\mathrm{o}}(\mathbf{r}_0 + t_0\dot{\mathbf{r}}_0) \\ \dot{\mathbf{P}}_{\mathrm{o}}(\mathbf{r}_0 + t_0\dot{\mathbf{r}}_0) + \mathbf{P}_{\mathrm{o}}\dot{\mathbf{r}}_0 \end{pmatrix}. \tag{7}$$

Consider the space part of the transformation from Eq. (5),

$$\mathbf{r}_1 = \mathbf{P}^{-1}\mathbf{r}^5_{\mathscr{E}} - t_1(\mathbf{P}^{-1}\dot{\mathbf{r}}^5_{\mathscr{E}} + \dot{\mathbf{P}}^{-1}\mathbf{r}^5_{\mathscr{E}}).$$

From the velocity part of Eq. (5)

$$\dot{\mathbf{r}}_1 = \mathbf{P}^{-1}\dot{\mathbf{r}}^5_{\mathscr{E}} + \dot{\mathbf{P}}^{-1}\mathbf{r}^5_{\mathscr{E}}.$$

Hence

$$\mathbf{r}_1 = \mathbf{P}^{-1}(\mathbf{r}^5_{\mathscr{E}} - t_1\mathbf{P}\dot{\mathbf{r}}_1).$$

Using Eqs. (6) and (7) to express $\mathbf{r}^5_{\mathscr{E}}$ in terms of $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$, the position at J2000.0 can be written in terms that are directly comparable with the classical method. Thus

$$\mathbf{r}_1 = \mathbf{P}^{-1}[\mathbf{R}_3(-E)\mathbf{P}_o(\mathbf{r}_0 + t_0\dot{\mathbf{r}}_0) - t_1\mathbf{P}\dot{\mathbf{r}}_1].$$

To complete the comparison, an expression connecting $\dot{\mathbf{r}}^5_{\mathscr{E}}$ with $\dot{\mathbf{r}}^4_{\mathscr{E}}$ is required. This is obtained by considering the velocity part of the transformation. From Eq. (5),

$$\mathbf{r}^5_{\mathscr{E}} = \mathbf{P}(\mathbf{r}_1 + t_1\dot{\mathbf{r}}_1),$$

$$\dot{\mathbf{r}}^5_{\mathscr{E}} = \dot{\mathbf{P}}(\mathbf{r}_1 + t_1\dot{\mathbf{r}}_1) + \mathbf{P}\dot{\mathbf{r}}_1.$$

Eliminating $\mathbf{r}_1$ from this pair of equations, and using Eq. (7) to express $\dot{\mathbf{r}}^4_{\mathscr{E}}$ in a similar manner, gives

$$\dot{\mathbf{r}}^5_{\mathscr{E}} = \dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{r}^5_{\mathscr{E}} + \mathbf{P}\dot{\mathbf{r}}_1,$$

$$\dot{\mathbf{r}}^4_{\mathscr{E}} = \dot{\mathbf{P}}_o\mathbf{P}_o^{-1}\mathbf{r}^4_{\mathscr{E}} + \mathbf{P}_o\dot{\mathbf{r}}_0.$$

Using the above equations and the relationship between $\dot{\mathbf{r}}^5_{\mathscr{E}}$ and $\dot{\mathbf{r}}^4_{\mathscr{E}}$ given in Eq. (6), we have

$$\mathbf{P}\dot{\mathbf{r}}_1 + \dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{r}^5_{\mathscr{E}} = F\left[-\dot{E}\frac{\partial}{\partial E}\mathbf{R}_3(-E)\mathbf{r}^4_{\mathscr{E}} + \mathbf{R}_3(-E)\dot{\mathbf{P}}_o\mathbf{P}_o^{-1}\mathbf{r}^4_{\mathscr{E}} + \mathbf{R}_3(-E)\mathbf{P}_o\dot{\mathbf{r}}_0\right]. \tag{8}$$

The matrix $\dot{\mathbf{P}}\mathbf{P}^{-1}$ is given by

$$\dot{\mathbf{P}}\mathbf{P}^{-1} = \begin{bmatrix} 0 & -m & -n \\ m & 0 & -l \\ n & l & 0 \end{bmatrix},$$

where $l = -\dot{\theta}_A \sin(-z_A) - \dot{\zeta}_A \cos(-z_A)\sin\theta_A$, $m = \dot{z}_A + \dot{\zeta}_A\cos\theta_A$, and $n = \dot{\theta}_A\cos(-z_A) - \dot{\zeta}_A \times\sin(-z_A)\sin\theta_A$. At epoch $\mathscr{E}$ = 1984 January 1.0 in the FK5 system $l = 1'' \times 10^{-9}$ per Julian century. On the right-hand side of the equation, in the FK4 system at epoch $\mathscr{E}$ = 1984 January 1.0, the numerical value of l_o in the matrix $\dot{\mathbf{P}}_o\mathbf{P}_o^{-1}$ is $l_o = 5'' \times 10^{-8}$ per tropical century.

The vector equation (8) represents three scalar equations at epoch $\mathscr{E}$. By multiplying these equations by $x^5_{\mathscr{E}}, y^5_{\mathscr{E}}, z^5_{\mathscr{E}}$ as appropriate and combining them to form μ and μ' (see Sec. IV), we obtain

$$\mu + m + n\sin\alpha\tan\delta - l\cos\alpha\tan\delta = (\mu_o + m_o + n_o\sin\alpha_o\tan\delta_o - l_o\cos\alpha_o\tan\delta_o)F$$

$$\mu' + n\cos\alpha + l\sin\alpha = (\mu'_o + n_o\cos\alpha_o + l_o\sin\alpha_o)F,$$

where $\alpha = \alpha_o + E$ and $\delta = \delta_o$ are the right ascension and declination at epoch $\mathscr{E}$ on the FK5 and FK4 systems. These equations are identical to the equations used in the classical method (Paper I) except for the terms in l and l_o, which are too small to be included in the classical formulas.

REFERENCES

Andoyer, H. (1911). Bull. Astron. **28**, 67.

Aoki, S., and Soma, M. (1983). Communication with Standish.

Aoki, S., Soma, M., Kinoshita, H., and Inoue, K. (1983). Astron. Astrophys. **128**, 263.

Emerson, B. (1973). R. Obs. Bull. No. 178, 229.

Fricke, W. (1982). Astron. Astrophys. **107**, L13.

IAU Sixteenth General Assembly (1976). Trans. IAU **XVIB**, 56, 58.

Lederle, T., and Schwan, H. (1984). Astron. Astrophys. **134**, 1.

Lieske, J. H. (1979). Astron. Astrophys. **73**, 282.

Standish, E. M. (1982). Astron. Astrophys. **115**, 20.