Technical Note

Celestial Pole Offsets: Conversion from (dX,dY) to ($d\psi$,$d\epsilon$)

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Version 3 — 2005 May

1 Introduction

Resolutions B1.6–B1.8 of the 2000 IAU General Assembly established the IAU 2000A precession-nutation model, defined more precisely what is meant by the celestial pole, and provided a new definition of UT1 in terms of an angle, θ, that directly measures the rotation of Earth in the celestial reference system. These resolutions represent major changes in the way in which the instantaneous orientation of the Earth is to be computed. In particular, the equinox is no longer a fundamental reference point and sidereal time is no longer needed for this computation. However, Resolution B1.8 contains the following provision: "...the IERS will continue to provide users with data and algorithms for the conventional transformations." Thus, the IERS (International Earth Rotation and Reference Systems Service) has published a new expression for sidereal time consistent with the new definition of UT1.

Among the data that the IERS has regularly published for the conventional transformations are celestial pole offsets $d\psi$ and $d\epsilon$, representing the observed angular offset of the celestial pole from its modelled position, expressed in terms of changes in ecliptic longitude and obliquity. These parameters are simply the differential forms of the angles $\Delta\psi$ and $\Delta\epsilon$ used to represent the position of the celestial pole in the nutation theory. The IERS has interpreted the above provision of Resolution B1.8 as requiring only that it supply $d\psi$ and $d\epsilon$ values for the pre-2000 precession and nutation models, i.e., Lieske et al. (1977) precession and Wahr (1979) nutation; furthermore, it has stated its intention of discontinuing these data after 2005. The IERS publishes observed celestial pole offsets with respect to the IAU 2000A precession-nutation model in terms of the parameters dX and dY, which are small changes in the unit vector components of the celestial pole position. This reflects the new IERS computational procedure whereby the pole position is expressed in rectangular coordinates (X,Y) with respect to the axes of the International Celestial Reference System (ICRS)[1] (with $Z = \sqrt{1 - X^2 - Y^2} \approx 1$). Thus, in the long term, only dX and dY will be available from the IERS.

Because the IAU 2000A precession-nutation model matches modern VLBI observations quite well, the magnitudes of the pole offsets are now of order 1 milliarcsecond (mas) or less, compared

[1] More correctly, with respect to the axes of the Geocentric Celestial Reference System (GCRS).

to several tens of mas for the offsets from the previous models. Thus, some users may no longer need the pole offsets for their applications. Nevertheless, it seems likely that most of those who have been using the offsets will continue to do so.

Most existing software systems for positional astronomy applications use the "conventional transformations", that is, they express the series of rotations between the terrestrial and celestial systems in terms of the familiar angular quantities based on the equinox and sidereal time. It is possible to implement the IAU 2000A precession-nutation model and the new definition of UT1 without adopting the (X,Y) coordinate scheme for the pole coordinates now used by the IERS. Indeed, the new precession and nutation models still express the position of the pole in terms of conventional angles, and the X and Y components must be obtained from the angular quantities. Thus, there will likely be a continuing need to convert dX and dY values to the equivalent $d\psi$ and $d\epsilon$ values, even among those who implement the new IAU models.

2 Procedures

The conversion is trivial for epochs near J2000.0, when the celestial pole remains close to the ICRS (GCRS) Z-axis. Then Z=1 to high precision and dX and dY are effectively angles ($dZ \approx 0$). We have for this case

$$\begin{aligned} d\psi &= dX / \sin\epsilon \\ d\epsilon &= dY \end{aligned} \tag{1}$$

where ϵ is the mean obliquity of the ecliptic of date ($\approx$ J2000.0).

As the celestial pole precesses farther from the ICRS Z-axis, two effects must be accounted for. First, dX and dY can no longer be considered angles as dZ becomes non-zero; second, the directions corresponding to $d\psi$ and $d\epsilon$ change with respect to the ICRS axes, according to the position of the pole in the ICRS. The IERS Conventions (2003), Chapter 5, gives the following relationship in eq. (23):

$$\begin{aligned} dX &= d\psi \sin\epsilon_A + (\psi_A \cos\epsilon_0 - \chi_A)\, d\epsilon \\ dY &= d\epsilon - (\psi_A \cos\epsilon_0 - \chi_A)\, d\psi \sin\epsilon_A \end{aligned} \tag{2}$$

where ϵ_A is the mean obliquity, and ψ_A and χ_A are accumulated precession angles, all evaluated for the date of interest. The IERS Conventions gives expansions for ϵ_A, ψ_A and χ_A in eq. (32). The above relation is said to be accurate to 1 microarcsecond (μas) over one century. Solving for $d\psi$ and $d\epsilon$ we obtain

$$\begin{aligned} d\psi &= \frac{dX - f_p\, dY}{(1 + f_p^2)\sin\epsilon_A} \\ d\epsilon &= \frac{f_p\, dX + dY}{1 + f_p^2} \end{aligned} \tag{3}$$

where $f_p = \psi_A \cos\epsilon_0 - \chi_A \approx 0.02236\, T$ is the precession factor, where T is the number of centuries from J2000.0. Note that we can still formally use milliarcsecond units for dX and dY — we simply convert to radians and use the results as small changes to unit vector component values.

It is useful to consider the more general case, where the celestial pole is many degrees away from the ICRS Z-axis. The computed direction of the celestial pole in the ICRS, at time t, is given by the unit vector $\mathbf{n}$:

$$\mathbf{n}(t) = \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \mathbf{B}^{\mathrm{T}}\ \mathbf{P}^{\mathrm{T}}(t)\ \mathbf{N}^{\mathrm{T}}(t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where $\mathbf{B}^{\mathrm{T}}$ is the dynamical-to-ICRS frame bias matrix, $\mathbf{P}^{\mathrm{T}}$(t) is the precession matrix from date t to J2000.0, $\mathbf{N}^{\mathrm{T}}$(t) is the inverse nutation matrix (true-to-mean) for date t, and (0,0,1) is a vector that here represents the position of the true (dynamical) celestial pole with respect to the true (dynamical) equator of date. Dropping the explicit time dependence, the above equation becomes

$$\mathbf{n} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{B}^{\mathrm{T}}\ \mathbf{P}^{\mathrm{T}}\ \mathbf{N}^{\mathrm{T}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

If we have the values of dX and dY for time t, we can add them respectively to X and Y to get the position of the observed pole, $\mathbf{n_o}$, in the ICRS:

$$\mathbf{n_o} = \begin{pmatrix} X_o \\ Y_o \\ Z_o \end{pmatrix} = \begin{pmatrix} X + dX \\ Y + dY \\ Z + dZ \end{pmatrix} \tag{5}$$

where

$$dZ = -\left(\frac{X}{Z}\right) dX - \left(\frac{Y}{Z}\right) dY \qquad \text{and} \qquad Z_o = \sqrt{1 - X_o^2 - Y_o^2} \tag{6}$$

To obtain $d\psi$ and $d\epsilon$, we can use eq. (1), providing that all quantities are expressed with respect to the *mean equator and equinox of date*. So we have to precess $\mathbf{n_o}$ and $\mathbf{n}$ from the ICRS to the mean equator and equinox of date, difference them to form dX and dY in this system, then apply eq. (1). That is, if we use primes ($'$) to indicate quantities in the equator-of-date system, we form the vector

$$\mathbf{dn}' = \begin{pmatrix} dX' \\ dY' \\ dZ' \end{pmatrix} = \mathbf{PB} \begin{pmatrix} X + dX \\ Y + dY \\ Z + dZ \end{pmatrix} - \mathbf{PB} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{PB} \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix} \approx \mathbf{P} \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix} \tag{7}$$

where $\mathbf{P}$ is the precession matrix from J2000.0 to date t and $\mathbf{B}$ is the ICRS-to-dynamical frame bias matrix. In the rightmost expression, $\mathbf{B}$ is ignored; it works quite well because $\mathbf{B}$ is nearly the identity matrix (total rotation $\approx 1{\times}10^{-7}$ radian) and dX, dY, and dZ are also small and known to only a few significant digits. With dX' and dY' in hand we compute

$$\begin{aligned} d\psi &= dX'/\sin\epsilon \\ d\epsilon &= dY' \end{aligned} \tag{8}$$

where ϵ is the mean obliquity of the ecliptic of date t.

Equations (7) and (8) look simple enough, but remember that dZ is not provided by the IERS and must be obtained from eq. (6), which in turn requires the values of X, Y, and Z obtained from eq. (4). Two shortcuts to obtaining dZ are possible, both of which are restricted to centuries near J2000.0. In the first, we approximate the components of the pole vector $\mathbf{n}$ as follows: $X = 2004''.19\,T$, $Y = 0$, $Z = \sqrt{1 - X^2}$; this follows from the fact that, near J2000.0, the pole moves

mostly in X at an angular rate of $2004''.19$/century (the rate of precession in declination). In effect, this converts the expression for dZ in eq. (6) to

$$dZ = -\left(X + \frac{1}{2}\,X^3\right)\,dX \qquad \text{where} \qquad X = 2004''.19\,T \tag{9}$$

The other shortcut is to simply set $dZ = 0$.

3 Results

We end up with four possible procedures for converting dX and dY to $d\psi$ and $d\epsilon$:

- Apply eqs. (7) & (8), obtaining dZ from eq. (6), which in turn requires eq. (4);
- Apply eqs. (7) & (8), obtaining dZ from eq. (9);
- Apply eqs. (7) & (8), setting dZ=0; or
- Apply eq. (3).

The precision of the procedures decreases in the order listed. The first is rigorous, although it seems somewhat awkward, requiring both a "forward" and "backward" application of precession. The second (dZ from eq. (9)) has errors well below 1 μas for years 1700 through 2300. The third (dZ=0) has errors reaching a few μas over the same time period. The fourth, derived from the expression in the IERS Standards, can have errors of about 10 μas over that time. For the two centuries 1900 through 2100, the errors exceed 1 μas only for the last procedure, and in that case not by much. These are errors in $d\psi$ (without a $\sin\epsilon$ factor) and $d\epsilon$ and were computed using random input values of dX and dY of order 1 mas.

Obviously these errors are quite small for any of the procedures, considering that the observational uncertainties in the celestial pole offsets are currently at about the 300 μas level for IERS data. Also, of course, celestial pole offsets in any form have been available only for the past few decades, so a development designed to deal with values for, say, year 2200 is clearly an academic exercise. However, outlining a rigorous procedure for the (dX,dY) to ($d\psi$,$d\epsilon$) transformation is useful for the record and provides a basis for evaluating more convenient, although approximate, transformation schemes. Such a procedure might also become more appropriate in the future as the observational precision increases.