A Navigation Solution Involving Changes to Course and Speed

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Abstract In a previous paper [1], an approach to celestial navigation was presented that allows a vessel's latitude, longitude, course, and speed to be simultaneously estimated. The development assumed that the vessel was sailing a rhumb-line track at a constant speed. In this paper, the approach is extended to cover the case where observations are taken over several legs of a voyage. A single least-squares solution for all sailing parameters is developed. The algorithm presented in this paper is not limited to celestial navigation, but is applicable to any situation in which observations over several legs of a voyage are to be combined in a single solution.

Key Words celestial navigation, celestial fix, motion of observer

Introduction

A development of celestial navigation was presented in [1] in which the motion of the observer was included as an essential part of the basic mathematics. This allows celestial observations to be used for the determination of not just latitude and longitude at a specific instant (the traditional fix), but also course and speed—providing, of course, that a sufficient number of observations are available, suitably distributed in azimuth and time. The development in [1] assumes that the observer's vessel is sailing a rhumb-line track at constant speed during the observations. For altitude sights made with a hand-held sextant (typical accuracy ± 1 arcmin), at least eight observations, spread over several hours, are generally needed for reliable course and speed estimates. However, the time span over which the observations need to be made would be less if their accuracy were better. For example, an automated shipboard star tracker might make the necessary observations and obtain a good solution in a matter of minutes.

The algorithm described in [1] has been implemented in software specifically designed for U.S. Navy operational use. Navy ships are subject to changes in orders en route, and are often involved in maneuvers. The software was therefore required to handle an arbitrary number of changes to course and speed. Even on an uncomplicated port-to-port voyage, the great-circle route would usually be approximated by a series of rhumb lines. The possibility that a celestial fix might involve observations that span two or more voyage legs had to be considered.

In chart-based navigation, lines of position (whether celestial or terrestrial) can be easily advanced or retired over changes to course and speed (see Bowditch [2], pp. 130–132). The assumption is simply made that the difference between a vessel's true position and its estimated position—the error in position—remains constant, in azimuth and distance, as the vessel moves, despite course and speed changes. The procedure for advancing a line of position is thus no more complicated when several voyage legs are involved than when there is only a single leg to consider. A popular mathematical approach to celestial sight reduction [3,4] is a direct mathematical translation of chart-based navigation, and it adopts the same assumption for combining observations spanning multiple legs.

However, as pointed out in [5], even when only one leg is involved, this assumption cannot be rigorously true, and it is often not even approximately true. A vessel's estimated course and speed over bottom may be considerably in error due to components of current and wind not taken into account. In practice, then, the error in position changes with time. The situation is complicated further by changes to direction or speed, which modify the net force of the wind on the vessel. So not only is the basic assumption (constant positional error) dubious, but so is its first derivative (constant rate of change of positional error) when multiple legs are involved.

The development presented in [1] assumed at the outset that the observer's estimated course and speed are probably not accurate, so that the error in position is not constant. That algorithm allows for a simultaneous solution for latitude, longitude, course, and speed. However, the algorithm is based on the geometry of a single rhumb-line leg—that is, the course and speed are assumed constant. This paper explores how the approach in [1] can be generalized for the multiple-leg case. The multiple-leg algorithm presented here, although developed for rhumb-line tracks and celestial navigation, is applicable to any kind of track and any set of observations in which a single observable is a known function of the observer's instantaneous latitude and longitude.

Single-Leg Solution

The algorithm presented in [1] is a least-squares solution for a vessel's latitude and longitude at a specific time, along with its course and speed (assumed constant). We assume that initially we know the vessel's track reasonably well. Errors of several degrees in position or course or several knots in speed are routinely handled, and larger errors usually only require additional iterations for convergence. What we are solving for are corrections to our *a priori* estimates of these four quantities. The equation of condition for the least-squares solution, which is computed for each observation, is:

$$\begin{aligned} a = &\left(\frac{\partial \mathrm{Hc}}{\partial \phi}\frac{\partial f}{\partial \phi_0} + \frac{\partial \mathrm{Hc}}{\partial \lambda}\frac{\partial g}{\partial \phi_0}\right)\Delta\phi_0 + \left(\frac{\partial \mathrm{Hc}}{\partial \phi}\frac{\partial f}{\partial \lambda_0} + \frac{\partial \mathrm{Hc}}{\partial \lambda}\frac{\partial g}{\partial \lambda_0}\right)\Delta\lambda_0 \\ &+ \left(\frac{\partial \mathrm{Hc}}{\partial \phi}\frac{\partial f}{\partial C} + \frac{\partial \mathrm{Hc}}{\partial \lambda}\frac{\partial g}{\partial C}\right)\Delta C \; + \left(\frac{\partial \mathrm{Hc}}{\partial \phi}\frac{\partial f}{\partial S} + \frac{\partial \mathrm{Hc}}{\partial \lambda}\frac{\partial g}{\partial S}\right)\Delta S \end{aligned} \tag{1}$$

where
- a is the altitude intercept ($a = \mathrm{Ho} - \mathrm{Hc}$),
- Ho is the observed altitude,
- Hc is the computed altitude,
- ϕ is the estimated latitude for the time of observation,
- λ is the estimated longitude for the time of observation,
- ϕ_0 is the estimated latitude for the time of the fix,
- λ_0 is the estimated longitude for the time of the fix,
- C is the course of the observer's vessel,
- S is the speed of the observer's vessel,
- f is the rhumb-line sailing formula for latitude, and
- g is the rhumb-line sailing formula for longitude.

This equation expresses the altitude intercept a in terms of small corrections to the estimated latitude, longitude, course, and speed, respectively denoted $\Delta\phi_0$, $\Delta\lambda_0$, ΔC, and ΔS. These latter four quantities are unknowns to be determined, and once their values are known, we can correct our estimate of the vessel's track. The corrections to latitude and longitude have 0 subscripts to denote that they apply to the specific time, t_0, for which the fix is desired. All angles, including latitude and longitude, are expressed in radians.

The quantities within parentheses in equation (1) are the coefficients of the unknowns. These coefficients, along with the altitude intercept a, must be numerically evaluated for each observation. To do that, we need algebraic expressions for the partial derivatives. There are ten partial derivatives that appear:

$$\frac{\partial \mathrm{Hc}}{\partial \phi}, \frac{\partial \mathrm{Hc}}{\partial \lambda}, \frac{\partial f}{\partial \phi_0}, \frac{\partial f}{\partial \lambda_0}, \frac{\partial f}{\partial C}, \frac{\partial f}{\partial S}, \frac{\partial g}{\partial \phi_0}, \frac{\partial g}{\partial \lambda_0}, \frac{\partial g}{\partial C}, \frac{\partial g}{\partial S}$$

These are obtained from three formulas: the equation for Hc as a function of ϕ and λ; the sailing formula for latitude, represented by the function f; and the sailing formula for longitude, represented by the function g. The sailing formulas are functions that together provide the position of the vessel as a function of time t; they involve the parameters C, S, ϕ_0, and λ_0 (the sailing formulas are derived in [6]). Algebraic expressions for all the partials needed above are given in the Appendix to [1].

For a single-leg solution, the mathematical procedure can be summarized as follows. First, the time t_0 for the fix is chosen. Initial estimates of the values of ϕ_0, λ_0, C, and S are made for time t_0. The sailing formulas f and g then allow the instantaneous latitude, ϕ, and longitude, λ, to be computed for any other time t. For each observation, therefore, an estimate of ϕ and λ is made. The sextant altitude, hs, is reduced to an observed altitude, Ho, in the usual way. An almanac or ephemeris routine is then used to provide the celestial coordinates (GHA and Dec) of the observed body, and Hc and a are computed for the observation. Using the expressions for the partials given in [1], the coefficients of the unknowns in equation (1) can be computed. To include the observation in the least-squares solution, the coefficients are added as a new row in the design matrix, and a is added to the column vector of measurements. After all observations have been processed in this way, the least-squares solution is computed, yielding values for $\Delta\phi_0$, $\Delta\lambda_0$, ΔC, and ΔS. These are then used to correct the original estimates of ϕ_0, λ_0, C, and S for time t_0. The corrected values of ϕ_0 and λ_0 define the fix. The whole process can be reiterated if necessary, using the new values of ϕ_0, λ_0, C, and S as the basis for the computations. Details, including a numerical example, are given in [1]. For a description of least-squares techniques, see any text on scientific data analysis, for example, [7,8].

Equation (1) can be applied beyond the case of celestial navigation if we simply generalize the meanings of several variables. Ho and Hc can be regarded as the observed and computed values, respectively, of any scalar quantity that can be measured while underway and that is a function of the observer's instantaneous latitude and longitude. For example, range to a known point would be such a quantity. Then $a = \mathrm{Ho} - \mathrm{Hc}$ becomes simply the "observed minus computed" residual of this quantity. As long as we can compute a, $\partial\mathrm{Hc}/\partial\phi$, and $\partial\mathrm{Hc}/\partial\lambda$ for each observation (and the geometry of the observations is not degenerate), equation (1) applies. The development presented below does not depend on particular meanings of Ho, Hc, a, or the partial derivatives $\partial\mathrm{Hc}/\partial\phi$ and $\partial\mathrm{Hc}/\partial\lambda$.

General Considerations for Multiple-Leg Tracks

When we have observations spread over more than one leg, the conditional equation gains additional terms corresponding to the added degrees of freedom in the problem. That is, the geometry of the problem becomes more complex and can be described only with additional parameters, some of which we must solve for. We will consider a new voyage leg to begin with either a course change or a speed change (or both) at a known time. We model these changes as instantaneous, and we can ignore the finite turning radius of the vessel if (1) no observations are made during the turn, and (2) the turning radius is short compared with the length of the straight portions of the vessel's track. In the most general case, each leg has both course and speed independent of the course and speed on any other leg. (This allows for the case where a change of either course or speed alters the interaction of wind and current on the vessel and affects the other parameter.) Just as in the single-leg case, we assume that, initially, we know the vessel's track reasonably well (errors not exceeding a few degrees in position or course and a few knots in speed), and that what we need to solve for are corrections to our starting estimates of the relevant parameters.

To cover the multiple-leg case, some changes to our mathematical notation are necessary. We will use subscripts to designate specific legs. The first leg will be denoted by subscript 1. This leg begins at time t_1, when the vessel is at position (ϕ_1,λ_1), sailing course C_1 at speed S_1. The second leg begins at time t_2 when the vessel is at position (ϕ_2,λ_2); the new course and speed are C_2 and S_2. The third begins at time t_3, and so on. We assume that we know the times t_j exactly, and that we start with reasonable estimates for the other quantities.

We have used the symbols f and g to represent the sailing formulas for rhumb-line tracks, which respectively provide the latitude and longitude of the vessel as a function of time. For the multiple-leg case, we must use the symbols f_1 and g_1 to represent the sailing formulas to be used for leg 1, f_2 and g_2 to represent the sailing formulas to be used for leg 2, etc. The algebraic form of the sailing formulas is, of course, the same for all legs; it is just the parameter values used in the formulas that are different for each leg. We will continue to use the symbols f and g (without subscripts) to represent the generic form of the sailing formulas. The values of five parameters in these formulas effectively define the functions f_j and g_j for each leg j. The parameters are the time t_j at which the leg begins, the latitude ϕ_j and longitude λ_j of the vessel at time t_j, the course C_j, and the speed S_j. For each leg j we have

$$\text{for}\quad t_j \le t \le t_{j+1} \qquad \begin{cases} \phi(t) = f_j = f\left(t - t_j, \phi_j, \lambda_j, C_j, S_j\right) \\ \lambda(t) = g_j = g\left(t - t_j, \phi_j, \lambda_j, C_j, S_j\right) \end{cases} \tag{2}$$

The initial point of the leg (position (ϕ_j,λ_j) at time t_j) thus serves as a *reference point* for the sailing formulas for the leg. We have made the reference point of each leg to be the initial point, but that is a requirement only for the second and succeeding legs in a multiple-leg case; for the first (or only) leg, the reference point can be anywhere along the leg.

It should be noted that the functional dependence of the two sailing formulas on the parameters shown in equation (2) is implicit; the set of parameters used explicitly in the two formulas is somewhat different. As presented in [1] and [6], both formulas contain auxiliary variables that need to be expanded before the dependence of these formulas on the parameter set shown above becomes evident.

The naive approach to the multiple-leg problem would be to simply take equation (1) and generalize it to cover N legs, using the notation just introduced:

$$a_{ik} = \sum_{j=1}^{\mathrm{N}} \Bigg[\left(\frac{\partial \mathrm{Hc}}{\partial \phi} \frac{\partial f_k}{\partial \phi_j} + \frac{\partial \mathrm{Hc}}{\partial \lambda} \frac{\partial g_k}{\partial \phi_j} \right) \Delta\phi_j \; + \left(\frac{\partial \mathrm{Hc}}{\partial \phi} \frac{\partial f_k}{\partial \lambda_j} + \frac{\partial \mathrm{Hc}}{\partial \lambda} \frac{\partial g_k}{\partial \lambda_j} \right) \Delta\lambda_j + \left(\frac{\partial \mathrm{Hc}}{\partial \phi} \frac{\partial f_k}{\partial C_j} + \frac{\partial \mathrm{Hc}}{\partial \lambda} \frac{\partial g_k}{\partial C_j} \right) \Delta C_j \; + \left(\frac{\partial \mathrm{Hc}}{\partial \phi} \frac{\partial f_k}{\partial S_j} + \frac{\partial \mathrm{Hc}}{\partial \lambda} \frac{\partial g_k}{\partial S_j} \right) \Delta S_j \Bigg] \tag{3}$$

where the subscript k denotes the leg on which the observation was taken, a_{ik} represents the altitude intercept for the ith observation on leg k, and the summation is over all N legs in the problem. There are 4N free parameters to be determined: $\Delta\phi_j$, $\Delta\lambda_j$, ΔC_j, and ΔS_j for $j = 1$ through N. The partial derivatives of the sailing formulas that appear in equation (3) contain subscripts. The subscript in the "numerator", k, represents the leg number of the observation being processed and the subscript in the "denominator", j, represents the leg number of a particular unknown.

In this construction of the problem, each leg is independent of the others, and there is no requirement that the legs be continuous. With independent legs, only observations taken on a particular leg contribute to the parameters for that leg. That is because the sailing formulas for a particular leg depend only on parameters relating to that leg, and therefore the partial derivatives of the sailing formulas are zero if $j \neq k$. Therefore, in equation (3), as j cycles from 1 to N, the coefficients of all the unknowns are zero except when $j = k$, so that an observation has no effect except on parameters relating to its own leg. When $j = k$, the partials are evaluated from the single-leg formulas given in [1]. What we have done, therefore, is set up a single least-squares solution for a problem that could be handled as N individual least-squares solutions, one for each leg. The values of the unknowns that we solve for will be the same regardless of which approach is taken.

Of course, the idea that a multiple-leg voyage consists of a series of independent legs does not correspond to the real problem. The courses and speeds may be independent, but each leg must be continuous with its neighbors. Because we are modeling each leg change as instantaneous, continuity is required only in position, not in motion. The requirement that adjacent legs be continuous defines a set of constraints on the solution, and constraints always reduce the number of free parameters in a problem. Formal methods have been developed to deal with constrained least-squares problems, but the multiple-leg problem can be attacked more easily. In the next section, we will abandon equation (3) and take another approach to the problem.

Algorithm for Multiple-Leg Tracks

A viable solution to the real multiple-leg problem must explicitly or implicitly deal with the requirement of continuity at the leg intersections. The mathematical condition is that, for all legs beyond the first, the point at the start of leg j must be identical to that at the end of leg $j - 1$. At the start of leg $j > 1$, the vessel's coordinates (ϕ_j,λ_j) are therefore not free parameters. Only on the first leg are the coordinates of the reference point free parameters. Given the location of that reference point, the time of each course or speed change, and the course and speed values on each leg, the vessel's track is completely determined. Since all the times are known, for N legs, there are 2N+2 free parameters in the problem (rather than 4N, as in equation (3) above). We start with estimates of their values and seek to correct those estimates. The two quantities ϕ_1 and λ_1, which define the position of the vessel at the reference point of the first leg, represent the fix. The other free parameters of the problem are the course and speed along each leg. The conditional equation for this case must therefore take the form

$$
\begin{aligned}
a_{ik} = \quad & \left(\frac{\partial \text{Hc}}{\partial \phi} \frac{\partial f_k}{\partial \phi_1} + \frac{\partial \text{Hc}}{\partial \lambda} \frac{\partial g_k}{\partial \phi_1} \right) \Delta\phi_1 \; + \left(\frac{\partial \text{Hc}}{\partial \phi} \frac{\partial f_k}{\partial \lambda_1} + \frac{\partial \text{Hc}}{\partial \lambda} \frac{\partial g_k}{\partial \lambda_1} \right) \Delta\lambda_1 \\
& + \sum_{j=1}^{\text{N}} \left[\left(\frac{\partial \text{Hc}}{\partial \phi} \frac{\partial f_k}{\partial C_j} + \frac{\partial \text{Hc}}{\partial \lambda} \frac{\partial g_k}{\partial C_j} \right) \Delta C_j \; + \left(\frac{\partial \text{Hc}}{\partial \phi} \frac{\partial f_k}{\partial S_j} + \frac{\partial \text{Hc}}{\partial \lambda} \frac{\partial g_k}{\partial S_j} \right) \Delta S_j \right]
\end{aligned}
\tag{4}
$$

where the notation is the same as for equation (3). This equation must be evaluated numerically for each observation, and the observations are taken in chronological order so that all the observations from a given voyage leg k are processed before any observations from the next leg.

In equation (4), as in equations (1) and (3), the coefficients of the unknowns are composed of linear combinations of partial derivatives. The issue is how to evaluate these partial derivatives as the observations from each leg are processed. These partial derivatives are of two types. The partials of the observable Hc with respect to the instantaneous position of the vessel ($\partial\text{Hc}/\partial\phi$ and $\partial\text{Hc}/\partial\lambda$) are straightforward to evaluate; if Hc is celestial altitude, the expressions can be taken from [1]. The partials of the sailing formulas f_k and g_k are more difficult to deal with, because they describe the relationship between the ship's instantaneous position and all the free parameters in the problem (ϕ_1 and λ_1 as well as C_j and S_j for all j).

The sailing formula partials contain subscripts: k in the "numerator" represents the leg number of the current observation and j in the "denominator" represents the leg number of a particular unknown. (In the top line of equation (4), $j = 1$.) Within each evaluation of equation (4) (each observation), k remains fixed but j runs over all the legs in the problem. In evaluating the sailing-formula partials, we therefore have three cases to consider: $j < k$, $j = k$, and $j > k$. Keeping in mind the meanings of j and k, we recognize that for $j > k$, the sailing formula partials must be zero, reflecting the fact that the ship's current computed position cannot depend on the parameters of future legs. For $j = k$, the partials must be the same as in the single-leg case, and the expressions from [1] can be directly used. The $j < k$ case is the most complicated, because these partials must reflect the fact that the ship's computed position does depend on the parameters of past legs, propagated through the constraints at the leg intersections. Most of Appendix A deals with this case.

Thus, the net effect of the partials of the sailing formulas on equation (4) is that each observation contributes to determining not just the course and speed on its own leg k but also those of *all preceding legs*. However, the future is indeterminate, and observations cannot provide any information on the unknowns relating to legs that have not yet occurred (they may not occur). All observations contribute to determining the coordinates of the reference point on the first leg, which serves as the fix for the multiple-leg problem.

In Appendix A, an algorithm for the evaluation of the sailing formulas partials that appear in equation (4) is presented. The algorithm is quite straightforward in that all that is done for the $j < k$ case is to implicitly fold the continuity constraints into the sailing formulas, the functions f_k and g_k for leg k. Then, new expressions for the partial derivatives of f_k and g_k are immediately obtained using the chain rule. These expressions amount to recurrence relations. Partials computed for each leg are built on partials computed for previous legs (a process that follows logically from the nature of the problem). Some care with mathematical bookkeeping is necessary. However, the advantage of this scheme is that, if the observations are processed in chronological order, no calculations are required beyond those needed for the single-leg case, except trivial multiplications and additions. No changes are required in the least-squares procedure itself. Appendix A describes the procedure in detail. Because it represents a modification to the single-leg case, the algorithm is not difficult to implement in computer code.

In the development of the multiple-leg case presented here, the fix is always on the first leg. In practice, what is of interest is not really the vessel's position on the first leg, but the last, which is the leg that the vessel is currently sailing. It may seem as if the problem should be reversed. In fact, we can do just that—the problem is symmetric in time and the vessel can, mathematically, be sailed backwards as well as forwards. Therefore, what we have been calling the "first leg" can actually be the last and what we called the "initial point" of each leg can actually be the final point. The problem has been described in a manner that keeps the development and terminology as straightforward as possible. This allows us to visualize the vessel as moving forward in time, to discuss causes and effects in a natural order, and to refer to leg $j+1$ as the "subsequent" or "next" leg. However, when the algorithm presented here is used for real problems, the procedure can be implemented in a time-reversed fashion to provide information of practical use.

Sample Calculation

A numerical example using this algorithm is presented below. A set of 12 artificial sextant observations (hs values), spanning 3 legs of a hypothetical voyage (4 observations per leg), was reduced using the algorithm described in this paper. The solution yielded a position fix for a single specified time in the final leg and course and speed estimates for all legs. The artificial observations were generated by a computer program that numerically integrated a hypothetical ship's course, consisting of 3 rhumb-line tracks, as shown in Table 1 (north latitudes and east longitudes are positive).

Table 1—Ship's Track Used in Sample Calculation

Start of Leg A		Start of Leg B		Start of Leg C	
Date:	30 Aug 1996	Date:	31 Aug 1996	Date:	31 Aug 1996
Time:	14:00:00 UTC	Time:	00:18:30 UTC	Time:	14:06:26 UTC
Latitude:	+40.000°	Latitude:	+43.275°	Latitude:	+46.446°
Longitude:	−50.000°	Longitude:	−47.479°	Longitude:	−43.736°
Course:	30.00°	Course:	40.00°	Course:	70.00°
Speed:	22.00 kn	Speed:	18.00 kn	Speed:	15.00 kn

The data in Table 1, and any positions derived from them, constitute "truth" for this case. The latitude and longitude listed for the beginning of legs B and C were computed from the data for the previous leg; that is, the legs meet at these points and the changes to course and speed were assumed to be instantaneous. Astronomical positions of the celestial bodies chosen were computed that are consistent with those in the 1996 *Nautical Almanac* and transformed to local altitudes. Atmospheric refraction was included but the height of eye was considered to be zero, and the time scale UT1 was assumed to be identical to UTC (that is, DUT1=UT1–UTC=0). Random errors with a 0.7 arcmin standard deviation were added to the calculated hs values. The artificial observations, for 30-31 August 1996 (UTC), are given in Table 2.

Table 2—Artificial Observations for Sample Calculation

Leg A — 30 Aug			Leg B — 31 Aug			Leg C — 31 Aug		
UTC	**Object**	**hs** °	**UTC**	**Object**	**hs** °	**UTC**	**Object**	**hs** °
14:58:54.3	Sun	57.872	00:36:29.7	Moon	17.659	22:01:35.5	Alpheratz	19.411
18:25:04.2	Sun	36.890	05:42:28.2	Moon	48.486	22:23:11.4	Rasalhague	54.089
22:50:08.4	Arcturus	34.624	06:54:27.9	Markab	37.710	22:44:47.3	Kochab	53.487
23:04:51.9	Jupiter	23.795	11:42:26.5	Sun	33.730	23:06:23.2	Alpheratz	30.225

All of the Sun and Moon observations apply to the lower limb. The star and planet observations represent twilight observations; the two Moon observations represent night observations. The Moon's phase was 0.92 (age 17 days).

Once computed and stored in a file, these artificial observations were used as input to a separate computer program that implemented the algorithm described in this paper. This second program was given the incorrect data on the ship's position and motion listed in Table 3.

Table 3—Incorrect Data on Ship's Track
Supplied to Multiple-Leg Algorithm

Start of Leg A		Start of Leg B		Start of Leg C	
Date:	30 Aug 1996	Date:	31 Aug 1996	Date:	31 Aug 1996
Time:	14:00:00 UTC	Time:	00:18:30 UTC	Time:	14:06:26 UTC
Latitude:	+41.700°	Latitude:	+44.469°	Latitude:	+47.159°
Longitude:	−48.000°	Longitude:	−45.639°	Longitude:	−42.050°
Course:	32.00°	Course:	43.00°	Course:	67.50°
Speed:	19.00 kn	Speed:	16.00 kn	Speed:	12.00 kn

The dates and times shown in Table 3 are the same as in Table 1, but the course and speed values, as well as the position of the first point of leg A, are considerably different. Just as in Table 1, the latitude and longitude listed in Table 3 for the beginning of legs B and C are computed from the data listed for the previous leg. Clearly, this track (Table 3) is a rather poor approximation to the true track (Table 1). The test of the algorithm is whether, given only the 12 observations (Table 2), it can determine the true track of the vessel, limited only by the error in the observations. The date and time for the desired fix were chosen to be 1 Sep 1996 at 07:30:00 UTC, which is on leg C, 8.4 hours after the last observation. In addition to the latitude and longitude for that time, course and speed for all 3 legs were to be determined. Based on the incorrect data on the ship's track it was given, the second program computed the estimated latitude and longitude for the time of the fix to be (+48.490°, −37.280°). It also computed the data in Table 4 for the 12 observations.

Table 4—Estimates of Sight-Reduction Quantities for Observations

Leg	**UTC**	**Est ϕ** °	**Est λ** °	**Object**	**Ho** °	**Hc** °	**a** ′
A	14:58:54.3	+41.964	−47.780	Sun	57.862	56.407	+87.3
A	18:25:04.2	+42.887	−47.001	Sun	36.868	34.944	+115.5
A	22:50:08.4	+44.073	−45.983	Arcturus	34.600	33.126	+88.4
A	23:04:51.9	+44.139	−45.926	Jupiter	23.758	22.465	+77.6
B	00:36:29.7	+44.527	−45.563	Moon	17.607	18.608	−60.0
B	05:42:28.2	+45.522	−44.255	Moon	48.472	47.192	+76.8
B	06:54:27.9	+45.756	−43.944	Markab	37.688	36.198	+89.4
B	11:42:26.5	+46.692	−42.687	Sun	33.706	34.379	−40.4
C	22:01:35.5	+47.765	−39.893	Alpheratz	19.364	20.332	−58.1
C	22:23:11.4	+47.793	−39.795	Rasalhague	54.077	53.217	+51.5
C	22:44:47.3	+47.820	−39.696	Kochab	53.474	53.770	−17.8
C	23:06:23.2	+47.848	−39.597	Alpheratz	30.197	31.023	−49.6

In Table 4, the altitude intercept, a, is Ho − Hc and positive values indicate "toward." The Ho and Hc values are topocentric and, for the Sun and Moon, they refer to the lower limb. The large a values are a result of the poor data on the ship's track that the program was given, that we hope to correct.

As described previously, the algorithm described in this paper will usually be implemented in a time-reversed sense. The program followed this strategy, working backwards through the observations and effectively treating leg C, containing the fix, as the "first" leg. In recognition of this strategy, and in order to maintain consistency with equation (4), the following notation will therefore be used:

Leg A → Leg 3: Course C_3, Speed S_3

Leg B → Leg 2: Course C_2, Speed S_2

Leg C → Leg 1: Course C_1, Speed S_1, Fix Coordinates ϕ_1, λ_1

As it processed each observation, the program computed the values of the coefficients of all the unknowns in equation (4). It sent these coefficient values, along with the value of a, to a standard least-squares routine.

All the observations were given equal weight. After all 12 observations had been processed in this way, the least-squares solution was formed. The results are given in Table 5, which lists the corrections to the parameters describing the ship's track.

Table 5—Least-Squares Solution

Parameter	No. Obs.	Value	
$\Delta\phi_1$	12	-0.579 ± 0.041	°
$\Delta\lambda_1$	12	-0.517 ± 0.045	°
ΔC_1	12	$+2.76 \pm 0.85$	°
ΔS_1	12	$+2.77 \pm 0.20$	kn
ΔC_2	8	-3.01 ± 0.38	°
ΔS_2	8	$+2.25 \pm 0.21$	kn
ΔC_3	4	-2.30 ± 0.48	°
ΔS_3	4	$+2.87 \pm 0.20$	kn

The second column in Table 5 indicates the number of observations contributing to each parameter. Table 5 shows rather large corrections to the parameter values, as we expect from the crudeness of the track data that the program used. The mean error of unit weight of the solution was computed to be 0.8 arcmin. If we add the parameter values from Table 5 to the estimated (incorrect) data on the ship's track, we obtain the final results of the entire process, summarized in Table 6.

Table 6—Results of Solution for Ship's Track

	Truth	Solution	Error
ϕ_1	+47.932	+47.911	−0.021
λ_1	−37.745	−37.797	−0.052
C_1	70.00	70.26	+0.26
S_1	15.00	14.77	−0.23
C_2	40.00	39.99	−0.01
S_2	18.00	18.25	+0.25
C_3	30.00	29.70	−0.30
S_3	22.00	21.87	−0.13

The units used in Table 6 are degrees and knots, as in the previous tables. "Error" simply refers to the difference solution–truth. Obviously a great improvement in the data on the ship's track was made. Of course, in a real situation, "Truth" would not be known—the question at this point would be whether another iteration of entire procedure would be advisable, using the updated values of the parameters describing the ship's track given in Table 6. In fact, the program did perform two iterations of the algorithm, and the results given in Tables 5 and 6 reflect the sum of the corrections obtained from the two iterations. If the solution is reliable, the iterations will converge rapidly; that is, the parameter corrections from a second or third iteration will all be much smaller than those from the previous iteration. That was the case here.

The algorithm did recover the ship's true track, although the fix coordinates are perhaps not as accurate as we might desire. The error in the fix position is 2.5 nmi, even though the observations had a scatter of 0.7 arcmin. This example is adversely affected by the small number of observations involved—there are only $1\frac{1}{2}$ times as many observations as there are parameters, leaving only 4 degrees of freedom. The parameter correlation matrix from the solution showed a 0.96 correlation between $\Delta\phi_1$ and ΔC_1, indicating that the geometry of the observations does not allow a clean separation of the effects of these errors. Such problems are not uncommon in cases with so few observations. Generally, at least a half-dozen observations on each leg, well spread out in time, should be used. In one scenario tested, adding just 4 more observations to this example reduced the error in the fix to less than 1 nmi. The extra observations not only doubled the number of degrees of freedom in the problem but also improved its geometry.

In any event, the multiple-leg algorithm described in this paper may provide the best answers attainable for the 12-observation case we have been considering. For example, if we try something more conventional, and process only the observations in leg 1 (leg C), we obtain the results in Table 7.

Table 7—Results of Solution for Leg 1 Only

	Truth	Solution	Error
ϕ_1	+47.932	+47.847	−0.085
λ_1	−37.745	−38.411	−0.666
ϕ_1	+47.932	+47.720	−0.212
λ_1	−37.745	−37.500	+0.245
C_1	70.00	76.45	+6.45
S_1	15.00	15.76	+0.76

Two solutions for leg 1 (leg C) are given in Table 7: above the line, only the latitude and longitude of the fix have been solved for; below the line, course and speed values have also been determined (the second solution has no degrees of freedom). Clearly, neither is satisfactory—the errors in the fix position are 27.6 and 16.1 nmi, respectively. The difficulty in treating leg 1 alone is that the observations represent a group of sights taken within a 1-hour period during evening twilight. Since the course and speed are not well known, and cannot be accurately determined using these observations, the extrapolation to the time of the fix fails. In this case, the use of observations from the previous legs can add substantial information to the process, providing a reasonable determination of the course and speed values for leg 1 (the final leg), and allowing a relatively accurate extrapolation of the ship's position forward.

What would happen to our 3-leg, 12-observation case if the observations were substantially better (or worse) than those listed in Table 2? We can easily change the random scatter in the artificial observations and run through the entire procedure again. The results of this experiment are summarized in Table 8, the body of which shows the error of each parameter (solution–truth), in the units we have been using, as a function of observational scatter.

Table 8—Parameter Accuracy vs. Observational Accuracy

	— Observational Uncertainty —				
	0	$\pm0.05'$	$\pm0.2'$	$\pm0.7'$	$\pm1.5'$
ϕ_1	+0.001	−0.001	−0.006	−0.021	−0.047
λ_1	+0.001	−0.003	−0.015	−0.052	−0.113
C_1	−0.02	0.00	+0.06	+0.26	+0.60
S_1	0.00	−0.02	−0.06	−0.23	−0.49
C_2	0.00	0.00	0.00	−0.01	−0.02
S_2	0.00	+0.01	+0.07	+0.25	+0.54
C_3	0.00	−0.02	−0.08	−0.30	−0.64
S_3	0.00	−0.01	−0.04	−0.13	−0.27

There are no surprises in Table 8; it simply indicates that the errors in the parameters solved for are directly proportional to the scatter in the observations.

Use of the Multiple-Leg Algorithm

The algorithm outlined above and detailed in Appendix A has been implemented in software for celestial navigation. It was produced by the U.S. Naval Observatory for Navy shipboard use [9]. This software is currently operational in the fleet. The algorithm has been extensively tested using artificial observations generated by independent programs, where "truth" can be known absolutely and controlled. The algorithm is accurate and remarkably robust across a wide variety of voyage track scenarios.

The algorithm cannot, however, be applied blindly, and the distribution of the observations in time and azimuth should be considered before proceeding. As the sample calculation above shows, it is advisable for there to be at least a half-dozen observations on each leg. However, valid solutions can sometimes be formed with fewer observations, or even when one or more legs are "empty." The example above also shows the utility of the algorithm in establishing a mathematical linkage between the legs that allows observations from subsequent (previous) legs to contribute to the solution on the first (last) leg—which includes the adjustment to the fix coordinates. It is difficult to provide hard and fast *a priori* rules for the algorithm's application that apply in all cases. As in any least-squares procedure, the parameter correlation matrix (obtained from the variance-covariance matrix) from the solution should be examined to determine whether the solution is reliable. When that is not the case, it may be inadvisable to use the algorithm in its fullest generality as presented above; course and speed corrections for certain legs may be excluded. In such cases, in the summation term of equation (4), the j for any leg can simply be skipped over and the corresponding ΔC_j and ΔS_j not included in the solution. If the distribution of observations is very sparse, the entire summation term of equation (4) can be omitted, and the solution confined to $\Delta\phi_1$ and $\Delta\lambda_1$, that is, only corrections to the fix coordinates are computed. Clearly, however, the algorithm is most useful, and reliable, when there is a good distribution of observations on each leg. In the case of celestial navigation, an automated observing system might be required to provide the number and accuracy of the observations needed.

The algorithm is quite general. As previously noted, equation (4) is not limited to celestial observations—Hc can represent any observable which is a known function of latitude and longitude. The only requirement is that the functions f and g provide the coordinates of the vessel as a function of time and the latitude, longitude, course, and speed at the beginning of the leg.

The algorithm might be useful even when a ship is nominally holding a fixed course and speed. If significant changes in current or wind occur, a single rhumb line may not adequately model the ship's track over bottom. In such cases, the ship's motion might be approximated by a series of rhumb-line legs, and the algorithm given in this paper would then be applicable.

Finally, in [1], the navigation problem for vessels on the surface of the Earth was described as being analogous to the orbit determination problem for bodies in the solar system. That paper derived the conditional equation for the single-leg navigation problem, reproduced here as equation (1), from considerations similar to those used by astronomers for the orbit problem. It may seem as if the multiple-leg navigation problem addressed in this paper has no analog in orbit theory, but that is not so. Spacecraft trajectories are adjusted by rocket firings, and the orbits of asteroids, comets, and spacecraft are perturbed by close approaches to massive bodies. In most of these cases the agent of change (a rocket burn or close gravitational encounter) is a transient phenomenon. An approximate treatment of such a problem, adequate for many purposes, would be to consider the "before" and "after" trajectories to meet at a point in space where the motion of the body instantaneously changes. Thus, even when orbits in the solar system are considered, we have situations that are quite similar to the multiple-leg navigation case. The strategy for dealing with the multiple-leg navigation problem described in this paper could be generalized to the orbit-determination problem as well.

Conclusion

The running fix—the determination of position at sea using observations taken over extended time spans—is a common navigational problem. Traditional chart-based techniques advance or retire lines of position to a common time. The technique works best when the vessel's motion is constant, but it is also applied across changes to course or speed, that is, to multiple-leg tracks. A commonly used mathematical development of celestial navigation mimics the chart-based scheme. The weakness of this approach to the running-fix problem is that it is based on the assumption that the difference between a vessel's estimated and true positions remains constant in two dimensions as the ship moves. The notion that a ship's positional error is constant is unlikely to be true, even approximately, except over very short periods, and any change to course or speed only complicates the situation.

An algorithm has been presented in this paper that allows celestial or other observations made over several voyage legs to be combined into a single solution for all relevant sailing parameters: the latitude and longitude of one point on the track and the course and speed along each leg. In this algorithm, the error in position is assumed to change, and its rate of change is assumed to be different along each leg. The algorithm is conceptually simple, uses ordinary least-squares procedures, makes efficient use of the available observations, and is relatively easy to implement in computer code.

References

1. Kaplan, G. H., "Determining the Position and Motion of a Vessel from Celestial Observations," *Navigation* Vol. 42, No. 4, Winter 1995, pp. 631–648.

2. *The American Practical Navigator*, Pub. No. 9, Defense Mapping Agency Hydrographic/Topographic Center, Bethesda, Md., 1995.

3. De Wit, C., "Optimal Estimation of a Multi-Star Fix," *Navigation* Vol. 21, No. 4, Winter 1974–75, pp. 320–325.

4. Yallop, B. D., and Hohenkerk, C. Y., *Compact Data for Navigation and Astronomy 1996–2000*, Her Majesty's Stationery Office, London, 1995.

5. Kaplan, G. H., "The Motion of the Observer in Celestial Navigation," *Navigator's Newsletter*, Issue 51, Spring 1996, pp. 10-13.

6. Kaplan, G. H., "Practical Sailing Formulas for Rhumb-Line Tracks on an Oblate Earth," *Navigation* Vol. 42, No. 2, Summer 1995, pp. 313–326.

7. Meyer, S. L., *Data Analysis for Scientists and Engineers*, John Wiley & Sons, Inc., New York, 1975.

8. Mikhail, E. M., *Observations and Least Squares*, Harper & Row Publishers, Inc., New York, 1976.

9. Janiczek, P. M., "STELLA: Toward Automated Celestial Navigation," *Surface Warfare* Vol. 21, No. 2, March/April 1996, pp. 34-37.

10. IMSL STAT/LIBRARY, FORTRAN Subroutines for Statistical Analysis, IMSL, Inc., Houston, 1987.

Appendix A

Evaluating the Partial Derivatives of the Sailing Formulas in the Multiple-Leg Case

Equation (4) in the main part of the paper is the conditional equation for a least-squares solution for the unknowns of the multiple-leg navigation problem. To use it, we need to be able to numerically evaluate the quantities within the parentheses in the equation, which are the coefficients of the unknowns. Equation (4) is evaluated for each observation. It is advantageous to take the observations in chronological order, so that all observations on a given leg are processed before any observations on the next leg.

The coefficients of the unknowns in equation (4) consist of linear combinations of various partial derivatives. As mentioned in the main text, the partials that appear are either partials of the observable Hc or partials of the sailing formulas f and g. For celestial observations, expressions for the partial derivatives $\partial\mathrm{Hc}/\partial\phi$ and $\partial\mathrm{Hc}/\partial\lambda$ are given in [1] and will not be discussed further here. For the multiple-leg case, the quantities that need special attention are the partial derivatives of the sailing formulas, for example, $\partial f_k/\partial\phi_1$ or $\partial g_k/\partial C_j$. These appear in equation (4) in the generic form $\partial y_k/\partial x_j$, where y represents either the function f or g, and x represents any of the four parameters ϕ, λ, C, or S. We will refer to these partial derivatives simply as the "sailing partials." The subscript k represents the leg number of the current observation, while j represents the leg number of a particular unknown. In each evaluation of equation (4), k remains fixed but j runs over all legs in the problem.

There are three fundamental points to be noted in evaluating equation (4):

(1) Each partial derivative that appears in equation (4) is numerically evaluated for the time and estimated position of the observation being processed, using the current best estimates of the values of the parameters of the problem.

(2) When the subscripts in the "numerator" and "denominator" of the sailing partials are the same, that is, when $j = k$, the partials correspond to those of the single-leg case. The expressions from the appendix to [1] can be used directly, although the notation is somewhat different. For example, $\partial g_2/\partial\phi_2$ is evaluated using the equation from [1] for $\partial g/\partial\phi_0$, applied to leg 2.

(3) The summation in the second line of equation (4) effectively ends at $j = k$, since all the sailing partials are zero when $j > k$, for reasons stated in the main body of the paper.

The major issue addressed in this appendix is the evaluation of the sailing partials in equation (4) when $j < k$. These occur in the computation of the coefficients of unknowns relating to legs preceding the one on which the observation was taken. To facilitate the computation of these partials, we will add an artificial observation at the end of each leg; that is, for leg j the added computation point is at $t = t_{j+1}$. We compute all the sailing partials that appear in equation (4) for this point just as if it were an actual observation at the very end of the leg. The values of these partials are stored for future reference. This point is not included in the solution since it is not an actual observation (we have no value for a_{ik}). The artificial observation is processed this way for each leg except the final one, where it is superfluous. The utility of these artificial observations will soon become evident.

First Leg. The sailing formulas for the first leg, which apply to times t where $t_1 \le t \le t_2$, can be represented as

$$\begin{aligned} \phi(t) &= f_1 = f\,(t - t_1, \phi_1, \lambda_1, C_1, S_1) \\ \lambda(t) &= g_1 = g\,(t - t_1, \phi_1, \lambda_1, C_1, S_1) \end{aligned} \tag{A1}$$

For observations on the first leg, where $k = 1$, we only have to deal with $j = 1$ since the sailing partials are all zero for $j > k$. Since the only subscript we encounter in either "numerators" or "denominators" is 1, all the observations on the first leg are processed just as in the single-leg problem.

Second Leg. The sailing formulas for the second leg, which apply to times t where $t_2 \le t \le t_3$, are:

$$\begin{aligned} \phi(t) &= f_2 = f\,(t - t_2, \phi_2, \lambda_2, C_2, S_2) \\ \lambda(t) &= g_2 = g\,(t - t_2, \phi_2, \lambda_2, C_2, S_2) \end{aligned} \tag{A2}$$

However, there is a constraint on the point (ϕ_2, λ_2), which is the start of leg 2: it is also the end of leg 1. That is, at $t = t_2$, we have $f_2 = f_1$ and $g_2 = g_1$. More explicitly,

$$\begin{aligned}\phi_2 = \phi(t_2) = f_2\Big|_{t_2} = f_1\Big|_{t_2} = f\left(t_2 - t_1, \phi_1, \lambda_1, C_1, S_1\right) \\ \lambda_2 = \lambda(t_2) = g_2\Big|_{t_2} = g_1\Big|_{t_2} = g\left(t_2 - t_1, \phi_1, \lambda_1, C_1, S_1\right)\end{aligned} \tag{A3}$$

where the vertical bar means "evaluated at."

For observations on the second leg, where $k = 2$, we have to deal with $j = 1$ and 2; the sailing partials are all zero for $j > k$.

Partials for $\Delta\phi_1$ and $\Delta\lambda_1$. To obtain expressions for the sailing partials needed in the first line of equation (4), we substitute the expressions for ϕ_2 and λ_2 from the right side of equations (A3) into equations (A2), then differentiate using the generalized chain rule. The results are:

$$\begin{aligned}\frac{\partial f_2}{\partial \phi_1} &= \frac{\partial f_2}{\partial \phi_2}\frac{\partial f_1}{\partial \phi_1} + \frac{\partial f_2}{\partial \lambda_2}\frac{\partial g_1}{\partial \phi_1} \\ \frac{\partial f_2}{\partial \lambda_1} &= \frac{\partial f_2}{\partial \phi_2}\frac{\partial f_1}{\partial \lambda_1} + \frac{\partial f_2}{\partial \lambda_2}\frac{\partial g_1}{\partial \lambda_1} \\ \frac{\partial g_2}{\partial \phi_1} &= \frac{\partial g_2}{\partial \phi_2}\frac{\partial f_1}{\partial \phi_1} + \frac{\partial g_2}{\partial \lambda_2}\frac{\partial g_1}{\partial \phi_1} \\ \frac{\partial g_2}{\partial \lambda_1} &= \frac{\partial g_2}{\partial \phi_2}\frac{\partial f_1}{\partial \lambda_1} + \frac{\partial g_2}{\partial \lambda_2}\frac{\partial g_1}{\partial \lambda_1}\end{aligned} \tag{A4}$$

On the right sides of equations (A4), the partials of f_2 and g_2 ($\partial f_2/\partial\phi_2$, etc., the first factors in each term) are evaluated for the particular time of the observation being processed, and are computed from the single-leg formulas just as if leg 2 were the only leg in the problem. However, the partials of f_1 and g_1 ($\partial f_1/\partial\phi_1$, etc., the second factors in each term) are evaluated for time $t = t_2$ (as required by equation (A3)), so their values have already been computed—they are the partials computed for the artificial observation at the end of leg 1.

Partials for ΔC_1 and ΔS_1. Similarly, we differentiate equations (A2), with ϕ_2 and λ_2 substituted from equations (A3), to obtain the sailing partials needed for the second line of equation (4) when $j = 1$:

$$\begin{aligned}\frac{\partial f_2}{\partial C_1} &= \frac{\partial f_2}{\partial \phi_2}\frac{\partial f_1}{\partial C_1} + \frac{\partial f_2}{\partial \lambda_2}\frac{\partial g_1}{\partial C_1} \\ \frac{\partial f_2}{\partial S_1} &= \frac{\partial f_2}{\partial \phi_2}\frac{\partial f_1}{\partial S_1} + \frac{\partial f_2}{\partial \lambda_2}\frac{\partial g_1}{\partial S_1} \\ \frac{\partial g_2}{\partial C_1} &= \frac{\partial g_2}{\partial \phi_2}\frac{\partial f_1}{\partial C_1} + \frac{\partial g_2}{\partial \lambda_2}\frac{\partial g_1}{\partial C_1} \\ \frac{\partial g_2}{\partial S_1} &= \frac{\partial g_2}{\partial \phi_2}\frac{\partial f_1}{\partial S_1} + \frac{\partial g_2}{\partial \lambda_2}\frac{\partial g_1}{\partial S_1}\end{aligned} \tag{A5}$$

The partials of f_2 and g_2 on the right side of equation (A5) are the same as those on the right side of equation (A4). The partials of f_1 and g_1 are evaluated for time $t = t_2$, the end point of leg 1, so their values have also been previously computed.

Partials for ΔC_2 and ΔS_2. For the second line of equation (4) when $j = 2$, we encounter only partials where the "numerator" and "denominator" have the same subscript (i.e., $j = k = 2$). These are evaluated for each observation from leg 2 just as if it were a single-leg case.

Third Leg. The sailing formulas in latitude and longitude for leg 3 can be represented as

$$\begin{aligned} \phi(t) &= f_3 = f\left(t - t_3, \phi_3, \lambda_3, C_3, S_3\right) \\ \lambda(t) &= g_3 = g\left(t - t_3, \phi_3, \lambda_3, C_3, S_3\right) \end{aligned} \tag{A6}$$

We have the usual constraint on the point (ϕ_3, λ_3), which is the start of leg 3, but also the end of leg 2:

$$\begin{aligned} \phi_3 = \phi(t_3) = f_3\Big|_{t_3} = f_2\Big|_{t_3} &= f\left(t_3 - t_2, \phi_2, \lambda_2, C_2, S_2\right) \\ \lambda_3 = \lambda(t_3) = g_3\Big|_{t_3} = g_2\Big|_{t_3} &= g\left(t_3 - t_2, \phi_2, \lambda_2, C_2, S_2\right) \end{aligned} \tag{A7}$$

For observations on the third leg, where $k = 3$, we have to deal with $j = 1, 2$, and 3.

Partials for $\Delta\phi_1$ *and* $\Delta\lambda_1$*.* The sailing partials needed for the first line of equation (4) are obtained by differentiating equations (A6), with ϕ_3 and λ_3 substituted from equations (A7). With the help of the chain rule, the following results are obtained:

$$\begin{aligned} \frac{\partial f_3}{\partial \phi_1} &= \frac{\partial f_3}{\partial \phi_3}\frac{\partial f_2}{\partial \phi_1} + \frac{\partial f_3}{\partial \lambda_3}\frac{\partial g_2}{\partial \phi_1} \\ \frac{\partial f_3}{\partial \lambda_1} &= \frac{\partial f_3}{\partial \phi_3}\frac{\partial f_2}{\partial \lambda_1} + \frac{\partial f_3}{\partial \lambda_3}\frac{\partial g_2}{\partial \lambda_1} \\ \frac{\partial g_3}{\partial \phi_1} &= \frac{\partial g_3}{\partial \phi_3}\frac{\partial f_2}{\partial \phi_1} + \frac{\partial g_3}{\partial \lambda_3}\frac{\partial g_2}{\partial \phi_1} \\ \frac{\partial g_3}{\partial \lambda_1} &= \frac{\partial g_3}{\partial \phi_3}\frac{\partial f_2}{\partial \lambda_1} + \frac{\partial g_3}{\partial \lambda_3}\frac{\partial g_2}{\partial \lambda_1} \end{aligned} \tag{A8}$$

Here, the partials of f_3 and g_3 on the right side are evaluated for the particular time of the observation being processed, from the single-leg formulas applied to leg 3. The partials of f_2 and g_2 are evaluated for time $t = t_3$ (a requirement of equation (A7)), so we already have their values—they were computed for the artificial observation at the end of leg 2 when equation (A4) was applied.

Partials for ΔC_1 *and* ΔS_1*.* The sailing partials needed for the second line of equation (4) when $j = 1$ are obtained by the same procedure as used for equations (A8):

$$\begin{aligned} \frac{\partial f_3}{\partial C_1} &= \frac{\partial f_3}{\partial \phi_3}\frac{\partial f_2}{\partial C_1} + \frac{\partial f_3}{\partial \lambda_3}\frac{\partial g_2}{\partial C_1} \\ \frac{\partial f_3}{\partial S_1} &= \frac{\partial f_3}{\partial \phi_3}\frac{\partial f_2}{\partial S_1} + \frac{\partial f_3}{\partial \lambda_3}\frac{\partial g_2}{\partial S_1} \\ \frac{\partial g_3}{\partial C_1} &= \frac{\partial g_3}{\partial \phi_3}\frac{\partial f_2}{\partial C_1} + \frac{\partial g_3}{\partial \lambda_3}\frac{\partial g_2}{\partial C_1} \\ \frac{\partial g_3}{\partial S_1} &= \frac{\partial g_3}{\partial \phi_3}\frac{\partial f_2}{\partial S_1} + \frac{\partial g_3}{\partial \lambda_3}\frac{\partial g_2}{\partial S_1} \end{aligned} \tag{A9}$$

The partials of f_3 and g_3 on the right are the same as those in equation (A8). The partials of f_2 and g_2 are evaluated for time $t = t_3$, the end point of leg 2, and their values were computed when equation (A5) was evaluated for the artificial observation at the end of leg 2.

Partials for ΔC_2 *and* ΔS_2. The sailing partials used in the second line of equation (4) when $j = 2$ are:

$$\begin{aligned}
\frac{\partial f_3}{\partial C_2} &= \frac{\partial f_3}{\partial \phi_3}\frac{\partial f_2}{\partial C_2} + \frac{\partial f_3}{\partial \lambda_3}\frac{\partial g_2}{\partial C_2} \\
\frac{\partial f_3}{\partial S_2} &= \frac{\partial f_3}{\partial \phi_3}\frac{\partial f_2}{\partial S_2} + \frac{\partial f_3}{\partial \lambda_3}\frac{\partial g_2}{\partial S_2} \\
\frac{\partial g_3}{\partial C_2} &= \frac{\partial g_3}{\partial \phi_3}\frac{\partial f_2}{\partial C_2} + \frac{\partial g_3}{\partial \lambda_3}\frac{\partial g_2}{\partial C_2} \\
\frac{\partial g_3}{\partial S_2} &= \frac{\partial g_3}{\partial \phi_3}\frac{\partial f_2}{\partial S_2} + \frac{\partial g_3}{\partial \lambda_3}\frac{\partial g_2}{\partial S_2}
\end{aligned} \tag{A10}$$

The partials of f_3 and g_3 on the right side of equation (A10) are the same as those in equations (A8) and (A9). The partials of f_2 and g_2 are evaluated for time $t = t_3$, the end point of leg 2, and their values are also available from previous computation.

Partials for ΔC_3 *and* ΔS_3. For the second line of equation (4) when $j = 3$, all the sailing partials have the same subscript, 3, in their "numerator" and "denominator." These are evaluated for the observation using the single-leg formulas applied to leg 3.

General Strategy. The pattern should be obvious by now. The partials of the sailing formulas that appear in equation (4) have the general form $\partial y_k/\partial x_j$, where y represents either the function f or g; x represents any of the four parameters ϕ, λ, C, or S; k is the number of the current observation's leg; and j is an index that represents any of the N legs in the problem. When $j > k$ the partial is zero. The non-zero partials are of two types. In the first type, $j = k$, that is, the partial has the form $\partial y_k/\partial x_k$. These partials are computed from the single-leg formulas given in [1], applied to leg k, and are evaluated for the time of the observation being processed. The second type of partial has $j < k$, and we have seen that these partials are expanded as follows:

$$\frac{\partial y_k}{\partial x_j} = \frac{\partial y_k}{\partial \phi_k}\frac{\partial f_{k-1}}{\partial x_j} + \frac{\partial y_k}{\partial \lambda_k}\frac{\partial g_{k-1}}{\partial x_j} \tag{A11}$$

Looking at the two terms on right side of equation (A11), we see that the first factor in each is a partial of the first type, which we know how to evaluate. For a given observation, these first factors are used repeatedly. The second factor is a partial that must be evaluated for time $t = t_k$, which is the instant at the end of leg $k-1$ (the previous leg). Since we have inserted an artificial observation at the end of each leg, these partials have already been computed. For each sailing-formula partial that appears in equation (4), expanded as above, these second factors remain the same for all observations within a leg. It should be noted that some of the partials used in equation (A11) are zero if the rhumb-line sailing formulas from [1] are used. (For example, latitude is not a function of longitude, so $\partial f_k/\partial \lambda_k = 0$.) The algorithm has been presented in its fullest generality so that it is not limited to a specific set of sailing formulas.

This algorithm can be considered an extension of the one used for the single-leg case. It requires no new mathematics beyond trivial multiplications and additions. Equation (A11) amounts to a recurrence relation that builds the sailing partials for the $j < k$ case from those already computed for the previous leg, together with partials computed using the single-leg formulas applied to the current leg. No changes in the least-squares procedure are needed beyond providing for the extra unknown parameters that the multiple-leg case requires. Using this algorithm, a computer implementation of the single-leg case can be modified to handle the multiple-leg case with relatively little difficulty—the additional code is mostly involved with bookkeeping. Some new arrays are required for storage of the sailing partials computed for the artificial observations at the end of each leg. However, the storage required is small since in practice the number of legs that would be included in a solution would probably never exceed about five.

As mentioned in the main text, for practical applications, the algorithm should in most cases be implemented in a time-reversed sense. That is, leg 1 should be the most recent (latest) leg and leg N should be the first (earliest) leg. That allows the fix (ϕ_1,λ_1), which can be anywhere on leg 1, to represent the current position of the observer's vessel if desired. In a time-reversed implementation, the observations are taken in reverse chronological order. Each of the artificial observations must then be added at the *beginning* of its leg, so that it is the last computational point encountered on the leg as the observations are processed.

Appendix B

Alternative Multiple-Leg Solution Algorithms

What alternative algorithms are available for the multiple-leg navigational solution? As mentioned in the main part of the paper, the multiple-leg case is an example of a constrained least-squares problem. Least-squares theory allows constraint equations—known relationships among the parameters being solved for—to be included as part of the solution (see, for example, [8], Chapter 9). In principle, this theory would allow us to use equation (3), with four unknown parameters per leg, in combination with linearized versions of constraint equations such as (A3) and (A7) in Appendix A. However, this approach is considerably more difficult computationally, involving, among other things, another matrix inversion. Generally, "canned" least-squares packages do not provide for this type of solution (although at least one, IMSL [10], does).

Another approach, discussed in [8], uses the constraint equations to eliminate the dependent (redundant) parameters in the problem. That is, the constraints are used to solve for a set of independent (free) parameters "up front," and all the relevant equations are rewritten in terms of this set of independent unknowns. The algorithm outlined above is actually a variant of this approach. In our case, the coordinates of the starting points of legs 2 through N are the dependent parameters. The algorithm does not explicitly solve for and eliminate these parameters—rather, the algorithm works entirely with the derivatives of the parameters, and uses the chain rule to effectively remove the starting point coordinates of legs 2 through N from the solution. The solution, therefore, involves only a set of independent parameters.

The direct elimination of the dependent parameters is difficult with the sailing formulas used above but the method can be illustrated if the vessel's track is approximated by piecewise polynomials. (Spline theory is not really applicable, since continuity in derivatives is not required at the junction points.) For short rhumb lines, latitude can be represented as a linear function of time, although longitude requires a quadratic polynomial (great circles would require quadratics in both coordinates). Suppose we look at the equations for longitude for a three-leg problem:

$$\begin{aligned}
&\text{Leg 1:} \qquad \lambda(t) = g_1 = \lambda_1 + \dot{\lambda}_1(t - t_1) + \tfrac{1}{2}\ddot{\lambda}_1(t - t_1)^2 \\
&\text{Leg 2:} \qquad \lambda(t) = g_2 = \lambda_2 + \dot{\lambda}_2(t - t_2) + \tfrac{1}{2}\ddot{\lambda}_2(t - t_2)^2 \\
&\text{Leg 3:} \qquad \lambda(t) = g_3 = \lambda_3 + \dot{\lambda}_3(t - t_3) + \tfrac{1}{2}\ddot{\lambda}_3(t - t_3)^2
\end{aligned} \tag{B1}$$

These equations contain nine parameters: λ_1, $\dot{\lambda}_1$, $\ddot{\lambda}_1$, λ_2, $\dot{\lambda}_2$, $\ddot{\lambda}_2$, λ_3, $\dot{\lambda}_3$, and $\ddot{\lambda}_3$. However, two of these, λ_2 and λ_3, the longitudes at the beginning of legs 2 and 3, are dependent parameters. They can be related to the other parameters by the usual constraints, which provide for continuity in position at the leg intersections. For this case, the constraints, analogous to equations (A3) and (A7) in Appendix A, are $g_2 = g_1$ at $t = t_2$ and $g_3 = g_2$ at $t = t_3$. These conditions allow us to eliminate λ_2 and λ_3 from equations (B1), yielding

$$\begin{aligned}
g_1 &= \lambda_1 + \dot{\lambda}_1(t - t_1) + \tfrac{1}{2}\ddot{\lambda}_1(t - t_1)^2 \\
g_2 &= \lambda_1 + \dot{\lambda}_1(t_2 - t_1) + \tfrac{1}{2}\ddot{\lambda}_1(t_2 - t_1)^2 + \dot{\lambda}_2(t - t_2) + \tfrac{1}{2}\ddot{\lambda}_2(t - t_2)^2 \\
g_3 &= \lambda_1 + \dot{\lambda}_1(t_2 - t_1) + \tfrac{1}{2}\ddot{\lambda}_1(t_2 - t_1)^2 + \dot{\lambda}_2(t_3 - t_2) + \tfrac{1}{2}\ddot{\lambda}_2(t_3 - t_2)^2 \\
&\quad + \dot{\lambda}_3(t - t_3) + \tfrac{1}{2}\ddot{\lambda}_3(t - t_3)^2
\end{aligned} \tag{B2}$$

Equations (B2) are the new sailing formulas for longitude to be used for the three legs, which involve no dependent parameters. The latitude equations (which have no quadratic terms) can be treated similarly. We are left with only one geographic point in the problem, defined by the parameters ϕ_1 and λ_1, plus, for each leg j, three motion parameters, $\dot{\phi}_j$, $\dot{\lambda}_j$, and $\ddot{\lambda}_j$. We can then form a conditional equation, similar to equation (4) in the main text, but with these eleven parameters as unknowns. A least-squares solution would then be completely straightforward.

This seems to be a simpler approach to the problem, but there are prices to be paid. First, the motion parameters for each leg j—that is, $\dot{\phi}_j$, $\dot{\lambda}_j$, and $\ddot{\lambda}_j$—are somewhat abstract and would have to be transformed to quantities such as course or speed that would be useful to the navigator. Second, there are now three motion parameters to solve for per leg instead of two, requiring more observations for a reliable solution. Third, the polynomial representation of each leg's track is simply a Taylor expansion of more exact formulas (here, for rhumb lines) and extra terms might be required for long legs, depending on the accuracy requirements. Every extra term adds another free parameter to the problem, and the requirement for more observations. The algorithm presented in the main text of this paper and Appendix A is thus more economical in its use of the available observations, a critical consideration.