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Tidal dissipation compared to seismic dissipation: in small bodies, in earths, and in superearths

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Abstract

While the seismic quality factor and phase lag are defined solely by the bulk properties of the mantle, their tidal counterparts are determined both by the bulk properties and the size effect (self-gravitation of a body as a whole). For a qualitative estimate, we model the body with a homogeneous sphere, and express the tidal phase lag through the lag in a sample of material. Although simplistic, our model is sufficient to understand that the lags are not identical. The difference emerges because self-gravitation pulls the tidal bulge down. At low frequencies, this reduces strain and the damping rate, and makes tidal damping less efficient in larger objects. At higher frequencies, competition between self-gravitation and rheology becomes more complex, though for sufficiently large superearths the same rule applies: the larger the planet, the weaker tidal dissipation in it. Being negligible for small terrestrial planets and moons, the difference between the seismic and tidal lagging (and likewise between the seismic and tidal damping) becomes very considerable for large exoplanets (superearths). In those, it is much lower than what one might expect from using a seismic quality factor.

The tidal damping rate deviates from the seismic damping rate especially in the zero-frequency limit, and this difference takes place for bodies of *any* size. So the equal in magnitude but opposite in sign tidal torques, exerted on one another by the primary and the secondary, have their orbital averages going smoothly through zero as the secondary crosses the synchronous orbit.

We describe the mantle rheology with the Andrade model, allowing it to lean toward the Maxwell model at the lowest frequencies. To implement this additional flexibility, we reformulate the Andrade model by endowing it with a free parameter ζ which is the ratio of the anelastic timescale to the viscoelastic Maxwell time of the mantle. Some uncertainty in this parameter's frequency dependence does not influence our principal conclusions.

1 The goal and the plan

As the research on exoplanetary systems is gaining momentum, more and more accurate theoretical tools of planetary dynamics come into demand. Among those tools are the methods of calculation of tidal evolution of both orbital and rotational motion of planets and their moons. Such calculations involve two kind of integral parameters of celestial bodies – the Love numbers and the tidal quality factors. The values of these parameters depend upon the rheology of a body, as well as its size, temperature, and the tidal frequency.

It has recently become almost conventional in the literature to assume that the tidal quality factor of superearths should be of the order of one hundred to several hundred (Carter et al. 2011, Léger et al. 2009). Although an acceptable estimate for the seismic Q, this range of numbers turns out to fall short, sometimes by orders of magnitude, of the tidal Q of superearths.

In our paper, the frequency dependence of tidal damping in a near-spherical homogeneous body is juxtaposed with the frequency dependence of damping in a sample of the material of which the body consists. For brevity, damping in a sample will be termed (somewhat broadly) as "seismic damping".

We shall demonstrate that, while the tidal Q of the solid Earth happens not to deviate much from the solid-Earth seismic Q, the situation with larger telluric bodies is considerably different. The difference stems from the presence of self-gravitation, which suppresses the tidal bulge and thereby acts as extra rigidity – a long-known circumstance often neglected in astronomical studies.[1] Due to self-gravitation ("size effect"), tidal damping in superearths is much less efficient than in earths, and the difference may come to orders of magnitude, as will be demonstrated below. Thus, while the *seismic* Q of a superearth may be comparable to the seismic Q of the solid Earth, the *tidal* Q of a superearth may exceed this superearth's seismic Q greatly. This is the reason why it is inappropriate to approximate superearths' tidal quality factors with that of the solid Earth.

We shall show that the difference between the frequency dependence of the tidal Q factor and the seismic Q may explain the "improper" frequency dependence of the tidal dissipation rate measured by the lunar laser ranging (LLR) method. We also shall point out that the correct frequency dependence of the tidal dissipation rate, especially at low frequencies, plays an important role in modeling the process of entrapment into spin-orbit resonances. In greater detail, the latter circumstance will be discussed in Efroimsky (2011).

The rate of the "seismic damping" (a term that we employ to denote also damping in a sample of the material) is defined, at each frequency, by the material's rheology only, i.e., by the constitutive equation linking the strain and stress at this frequency. The rate of the tidal damping however is determined both by the rheology and by the intensity of self-gravitation of the body. At a qualitative level, this can be illustrated by the presence of two terms, 1 and $19\mu(\infty)/(2\rho \mathrm{g} R)$, in the denominator of the expression for the static Love number k_2 of a homogeneous sphere. Here $\mu(\infty)$ denotes the relaxed shear modulus, g signifies the surface gravity, while ρ and R stand for the mean density and the radius of the body. The first of these terms, 1, is responsible for the size effect (self-gravitation), the second for the bulk properties of the medium. Within the applicability realm of an important theorem called *elastic-viscoelastic analogy* (also referred to as the *correspondence principle*), the same

[1] Including the size effect via k_l is common. Unfortunately, it is commonly assumed sufficient. This treatment however is inconsistent in that it ignores the inseparable connection between the Love number and the tidal quality factor (or the tidal phase lag). In reality, both the Love number and the sine of the tidal lag should be derived from the rheology and geometry of the celestial body, and cannot be adjusted separately from one another.

expression interconnects the Fourier component $\bar{k}_2(\chi)$ of the time derivative of the Love number with the Fourier component $\bar{\mu}(\chi)$ of the stress-relaxation function at frequency χ. This renders the frequency-dependence of the tangent of the tidal lag, which is the negative ratio of the imaginary and real parts of $\bar{k}_2(\chi)$.

This preliminary consideration illustrates the way rheology enters the picture. First, the constitutive equation defines the frequency dependence of the complex compliance, $\bar{J}(\chi)$, and of the complex rigidity $\bar{\mu}(\chi) = 1/\bar{J}(\chi)$. The functional form of this dependence determines the frequency dependence of the complex Love number, $\bar{k}_l(\chi)$. The latter furnishes the frequency dependence of the products $|\bar{k}_l(\chi)| \, \sin\epsilon_l(\chi)$ which enter the tidal theory.

In Section 2, we briefly recall the standard description of stress-strain relaxation and dissipation in linear media. In Section 3, we describe a rheological model, which has proven to be adequate to the experimental data on the mantle minerals and partial melts. The goal of the subsequent sections will be to build the rheology into the theory of bodily tides, and to compare a tidal response of a near-spherical body to a seismic response rendered by the medium. Finally, several examples will be provided. Among these, will be the case of the Moon, whose "improper" tidal-dissipation frequency-dependence finds an explanation as soon as the difference between the seismic and tidal friction is brought to light. In the closing section, we shall compare our results with those obtained by Goldreich (1963).

2 Formalism

Everywhere in this paper we shall take into consideration only the deviatoric stresses and strains, thus neglecting compressibility.

2.1 Compliance and rigidity. The standard linear formalism in the time domain

The value of strain in a material depends only on the present and past values taken by the stress and not on the current *rate* of change of the stress. Hence the compliance operator $\hat{J}$ mapping the stress $\sigma_{\gamma\nu}$ to the strain $u_{\gamma\nu}$ must be just an integral operator, linear at small deformations:

$$2\, u_{\gamma\nu}(t) \;=\; \hat{J}(t)\;\sigma_{\gamma\nu} \;=\; \int_{-\infty}^{t} J(t\,-\,t\,')\;\;\dot{\sigma}_{\gamma\nu}\;(t\,')\;dt\,' \quad , \tag{1}$$

where $t\,' < t$, while overdot denotes $d/dt\,'$. The kernel $J(t - t\,')$ is termed the *compliance function* or the *creep-response function*.

Integration by parts renders:

$$2\, u_{\gamma\nu}(t) \;=\; \hat{J}(t)\;\sigma_{\gamma\nu} \;=\; J(0)\;\sigma_{\gamma\nu}(t) \;-\; J(\infty)\;\sigma_{\gamma\nu}(-\infty) \;+\; \int_{-\infty}^{t} \dot{J}\;(t\,-\,t\,')\;\sigma_{\gamma\nu}(t\,')\;dt\,' \quad . \tag{2}$$

As the load in the infinite past may be set zero, the term containing the relaxed compliance $J(\infty)$ may be dropped. The unrelaxed compliance $J(0)$ can be absorbed into the integral if we agree that the elastic contribution enters the compliance function not as

$$J(t-t\,') \;=\; J(0) \;+\; \text{viscous and hereditary terms} \quad , \tag{3}$$

but as

$$J(t-t\,')\;=\;J(0)\,\Theta(t\,-\,t\,')\,+\,\mbox{viscous and hereditary terms}\quad . \tag{4}$$

The Heaviside step-function $\Theta(t\,-\,t\,')$ is set unity for $t-t\,'\,\geq\,0$, and zero for $t-t\,'\,<\,0$, so its derivative is the delta-function $\delta(t\,-\,t\,')$. Keeping this in mind, we reshape (2) into

$$2\,u_{\gamma\nu}(t)\;=\;\hat{J}(t)\;\sigma_{\gamma\nu}\;=\;\int_{-\infty}^{t}\dot{J}\;(t-t\,')\;\sigma_{\gamma\nu}(t\,')\,dt\,'\quad,\quad\mbox{with}\;\;J(t-t\,')\;\;\mbox{containing}\;\;J(0)\,\Theta(t-t\,')\quad . \tag{5}$$

Inverse to the compliance operator

$$2\,u_{\gamma\nu}\;=\;\hat{J}\;\sigma_{\gamma\nu}\quad . \tag{6}$$

is the rigidity operator

$$\sigma_{\gamma\nu}\;=\;2\,\hat{\mu}\;u_{\gamma\nu}\quad . \tag{7}$$

In the presence of viscosity, operator $\hat{\mu}$ is not integral but is integrodifferential, and thus cannot be expressed as $\sigma_{\gamma\nu}(t)\;=\;2\int_{-\infty}^{t}\dot{\mu}(t\,-\,t\,')\,u_{\gamma\nu}(t\,')\,dt\,'$. It can though be written as

$$\sigma_{\gamma\nu}(t)\;=\;2\int_{-\infty}^{t}\mu(t\,-\,t\,')\,\dot{u}_{\gamma\nu}(t\,')\,dt\,'\quad , \tag{8}$$

if its kernel, the stress-relaxation function $\mu(t\,-\,t\,')$, is imparted with a term $2\,\eta\,\delta(t-t\,')$, integration whereof renders the viscous portion of stress, $2\,\eta\,\dot{u}_{\gamma\nu}$. The kernel also incorporates an unrelaxed part $\mu(0)\,\Theta(t-t\,')$, whose integration furnishes the elastic portion of the stress. The unrelaxed rigidity $\mu(0)$ is inverse to the unrelaxed compliance $J(0)$.

Each term in $\mu(t-t\,')$, which is neither constant nor proportional to a delta function, is responsible for hereditary reaction.

2.2 In the frequency domain

To Fourier-expand a real function, nonnegative frequencies are sufficient. Thus we write:

$$\sigma_{\gamma\nu}(t)\;=\;\int_0^{\infty}\bar{\sigma}_{\gamma\nu}(\chi)\;e^{\textstyle{^{i\chi t}}}\;d\chi\qquad\mbox{and}\qquad u_{\gamma\nu}(t)\;=\;\int_0^{\infty}\bar{u}_{\gamma\nu}(\chi)\;e^{\textstyle{^{i\chi t}}}\;d\chi\quad , \tag{9}$$

where the complex amplitudes are

$$\bar{\sigma}_{\gamma\nu}(\chi)=\sigma_{\gamma\nu}(\chi)\;\,e^{i\varphi_{\sigma}(\chi)}\quad\;\;,\qquad\bar{u}_{\gamma\nu}(\chi)=u_{\gamma\nu}(\chi)\;\,e^{i\varphi_{u}(\chi)}\quad\;, \tag{10}$$

while the initial phases $\varphi_{\sigma}(\chi)$ and $\varphi_{u}(\chi)$ are set to render the real amplitudes $\sigma_{\gamma\nu}(\chi_n)$ and $u_{\gamma\nu}(\chi_n)$ non-negative. To ensure convergence, the frequency is, whenever necessary, assumed to approach the real axis from below: $\mathcal{I}m(\chi)\rightarrow 0-$.

With the same caveats, the complex compliance $\bar{J}(\chi)$ is introduced as the Fourier image of the time derivative of the creep-response function:

$$\int_0^{\infty}\bar{J}(\chi)\,e^{i\chi\tau}d\chi\;=\dot{J}\;(\tau)\quad . \tag{11}$$

The inverse expression,

$$\bar{J}(\chi) \;=\; \int_0^{\infty} \dot{J}\,(\tau)\, e^{-i\chi\tau}\, d\tau \quad , \tag{12}$$

is often written down as

$$\bar{J}(\chi) \;=\; J(0) \,+\, i\,\chi \int_0^{\infty} [\; J(\tau) \,-\, J(0)\,\Theta(\tau)\;]\; e^{-i\chi\tau}\, d\tau \quad . \tag{13}$$

For causality reasons, the integration over τ spans the interval $[\,0, \infty\,)$ only. Alternatively, we can accept the convention that *each* term in the creep-response function is accompanied with the Heaviside step function.

Insertion of the Fourier integrals (9 - 11) into (1) leads us to

$$2\int_0^{\infty} \bar{u}_{\gamma\nu}(\chi)\; e^{i\chi t}\; d\chi \;=\; \int_0^{\infty} \bar{\sigma}_{\mu\nu}(\chi)\; \bar{J}(\chi)\; e^{i\chi t}\; d\chi \quad , \tag{14}$$

whence we obtain:

$$2\;\bar{u}_{\gamma\nu}(\chi) \;=\; \bar{J}(\chi)\;\bar{\sigma}_{\gamma\nu}(\chi) \quad . \tag{15}$$

Expressing the complex compliance as

$$\bar{J}(\chi) \;=\; |\bar{J}(\chi)|\;\exp\left[-\,i\,\delta(\chi)\,\right] \quad , \tag{16}$$

where

$$\tan\delta(\chi) \;\equiv\; -\,\frac{\mathcal{I}m\left[\,\bar{J}\,(\chi)\,\right]}{\mathcal{R}e\left[\,\bar{J}\,(\chi)\,\right]} \quad , \tag{17}$$

we see that $\delta(\chi)$ is the phase lag of a strain harmonic mode relative to the appropriate harmonic mode of the stress:

$$\varphi_u(\chi) \;=\; \varphi_\sigma(\chi) \;-\; \delta(\chi) \quad . \tag{18}$$

2.3 The quality factor(s)

In the linear approximation, at each frequency χ the average (per period) energy dissipation rate $\langle \dot{E}\;(\chi)\rangle$ is defined by the deformation at that frequency only, and bears no dependence upon the other frequencies:

$$\langle\,\dot{E}(\chi)\,\rangle \;=\; -\,\frac{\chi E_{peak}(\chi)}{Q(\chi)} \tag{19}$$

or, the same:

$$\Delta E_{cycle}(\chi) \;=\; -\,\frac{2\,\pi\;E_{peak}(\chi)}{Q(\chi)} \quad , \tag{20}$$

$\Delta E_{cycle}(\chi)$ being the one-cycle energy loss, and $Q(\chi)$ being the quality factor related to the phase lag at the frequency χ. It should be clarified right away, to which of the lags we are linking the quality factor. When we are talking about a sample of material, this lag is simply $\delta(\chi)$ introduced above as the negative argument of the appropriate Fourier component of the complex compliance – see formulae (17 - 18). However, whenever we address tide, the quality factor becomes linked (via the same formulae) to the *tidal* phase lag $\epsilon(\chi)$. Within the same rheological model, the expression for $\epsilon(\chi)$ differs from that for $\delta(\chi)$, because the tidal lag depends not only upon the local properties of the material, but also upon self-gravitation of the body as a whole.

The aforementioned "seismic-or-tidal" ambiguity in definition of Q becomes curable as soon as one points out to which kind of deformation the quality factor pertains. More serious is the ambiguity stemming from the freedom in defining $E_{peak}(\chi)$.

If $E_{peak}(\chi)$ in (19 - 20) signifies the peak *energy* stored at frequency χ, the resulting quality factor is related to the lag via

$$Q^{-1}_{energy} \;=\; \sin|\delta| \tag{21}$$

(not $\tan|\delta|$ as commonly believed – see the calculation in the Appendix to Efroimsky 2011).

If however $E_{peak}(\chi)$ is introduced as the absolute maximum of *work* carried out on the sample at frequency χ over a finite time interval, then the appropriate Q factor is connected to the lag via

$$Q^{-1}_{work} \;=\; \frac{\tan|\delta|}{1\,-\,\left(\frac{\pi}{2}\,-\,|\delta|\right)\,\tan|\delta|} \quad , \tag{22}$$

as was shown in *Ibid.*[2]

The third definition of the quality factor (offered by Goldreich 1963) is

$$Q^{-1}_{Goldreich} = \tan|\delta| \quad . \tag{23}$$

This definition, though, corresponds neither to the peak work nor to the peak energy.

In the limit of weak lagging, all three definitions entail

$$Q^{-1} \;=\; |\delta| \;+\; O(\delta^2) \quad . \tag{24}$$

For the lag approaching $\pi/2$, the quality factor defined as (21) assumes its minimal value, $Q_{energy} = 1$, while definition (22) renders $Q_{work} = 0$. The latter is natural, since in the considered limit the work performed on the system is negative, its absolute maximum being zero.[3]

In seismic studies or in exploration of attenuation in small samples, one's choice among the three definitions of Q is a matter of personal taste, for the quality factor is large and the definitions virtually coincide.

[2]In Efroimsky & Williams (2009), $E_{peak}(\chi)$ was miscalled "peak energy". However the calculation of Q was performed there for $E_{peak}(\chi)$ introduced as the peak *work*.

[3] As $Q < 2\pi$ implies $E_{peak} < \Delta E$, such small values of Q are unattainable in the case of damped free oscillations. Still, Q can assume such values under excitation, tides being the case.

In the theory of tides, the situation is different, because at times one has to deal with situations where the definitions of Q disagree noticeably – this happens when dissipation is intensive and Q is of order unity. To make a choice, recall that the actual quantities entering the Fourier expansion of tides over the modes ω_{lmpq} are the products[4]

$$k_l\,\sin\epsilon_l \;=\; k_l(\omega_{lmpq})\,\sin\epsilon_l(\omega_{lmpq}) \quad , \tag{25}$$

where $k_l(\omega_{lmpq})$ are the dynamical analogues to the Love numbers. It is these products that show up in the $lmpq$ terms of the expansion for the tidal potential (force, torque). From this point of view, a definition like (21) would be preferable, though this time with the tidal lag ϵ instead of the seismic lag δ :

$$Q_l^{-1} \;=\; \sin|\,\epsilon_l\,| \tag{26a}$$

or, in a more detailed manner:

$$Q_l^{-1}(\omega_{lmpq}) \;=\; \sin|\,\epsilon_l(\omega_{lmpq})\,| \quad . \tag{26b}$$

Under this definition, one is free to substitute $k_l\,\sin\epsilon_l$ with k_l/Q_l . The subscript l accompanying the tidal quality factor will then serve as a reminder of the distinction between the tidal quality factor and its seismic counterpart.

While the notion of the tidal quality factor has some illustrative power and may be employed for rough estimates, calculations involving bodily tides should be based not on the knowledge of the quality factor but on the knowledge of the overall frequency dependence of products $k_l\,\sin\epsilon_l \;=\; k_l(\omega_{lmpq})\,\sin\epsilon_l(\omega_{lmpq})$. Relying on these functions would spare one of the ambiguity in definition of Q and would also enable one to take into account the frequency dependence of the dynamical Love numbers.

3 The Andrade model and its reparameterisation

In the low-frequency limit, the mantle's behaviour is unlikely to differ much from that of the Maxwell body, because over timescales much longer than 1 yr viscosity dominates (Karato & Spetzler 1990). At the same time, the accumulated geophysical, seismological, and geodetic observations suggest that at shorter timescales anelasticity takes over and the mantle is described by the Andrade model. However, the near-Maxwell behavior expected at low frequencies can be fit into the Andrade formalism, as we shall explain below.

3.1 Experimental data: the power scaling law

Dissipation in solids may be effectively modeled using the empirical scaling law

$$\sin\delta \;=\; \left(\,\mathcal{E}\,\chi\,\right)^{-p} \quad , \tag{27}$$

[4] A historical tradition (originating from Kaula 1964) prescribes to denote the tidal phase lags with ϵ_{lmpq} , while keeping for the dynamical Love numbers the same notation as for their static predecessors: k_l . These conventions are in conflict because the product $k_l\,\sin\epsilon_{lmpq}$ is the negative imaginary part of the complex Love number $\bar{k}_l$. More logical is to use the unified notation as in (25). At the same time, it should not be forgotten that for triaxial bodies the functional form of the dependence of $\bar{k}_l$ on frequency is defined not only by l but also by $m,\,p,\,q$. In those situations, one has to deal with $k_{lmpq}\,\sin\epsilon_{lmpq}$, see Section 4.

$\mathcal{E}$ being a constant having the dimensions of time. This "constant" may itself bear a (typically, much slower) dependence upon the frequency χ. The dependence of $\mathcal{E}$ on the temperature is given by the Arrhenius law (Karato 2008).

Experiments demonstrate that the power dependence (27) is surprisingly universal, with the exponential p robustly taking values within the interval from 0.14 to 0.4 (more often, from 0.14 to 0.3).

For the first time, dependence (27) was measured on metals in a lab. This was done by Andrade (1910), who also tried to pick up an expression for the compliance compatible with this scaling law. Later studies have demonstrated that this law works equally well, and with similar values of p, both for silicate rocks (Weertman & Weertman 1975, Tan et al. 1997) and ices (Castillo-Rogez 2009, McCarthy et al 2007).

Independently from the studies of samples in the lab, the scaling behaviour (27) was obtained via measurements of dissipation of seismic waves in the Earth (Mitchell 1995, Stachnik et al. 2004, Shito et al. 2004).

The third source of confirmation of the power scaling law came from geodetic experiments that included: (a) satellite laser ranging (SLR) of tidal variations in the J_2 component of the gravity field of the Earth; (b) space-based observations of tidal variations in the Earth's rotation rate; and (c) space-based measurements of the Chandler wobble period and damping (Benjamin et al. 2006, Eanes & Bettadpur 1996, Eanes 1995).[5]

While samples of most minerals furnish the values of α lying within the interval $0.15 - 0.4$, the geodetic measurements give $0.14 - 0.2$. At least a fraction of this difference may be attributed to the presence of partial melt, which is known to have lower values of p (Fontaine et al. 2005).

On all these grounds, it is believed that mantles of terrestrial planets are adequately described by the Andrade model, at least in the higher frequency band where anelasticity dominates (Gribb & Cooper 1998, Birger 2007, Efroimsky & Lainey 2007, Zharkov & Gudkova 2009). Some of the other models were considered by Henning et al. (2009).

The Andrade model is equally well applicable to celestial bodies with ice mantles (for application to Iapetus see Castillo-Rogez et al. 2011) and to bodies with considerable hydration in a silicate mantle.[6] The model can also be employed for modeling of the tidal response *of the solid parts* of objects with significant liquid-water layers.[7]

3.2 The Andrade model in the time domain

The compliance function of the Andrade body (Cottrell & Aytekin 1947, Duval 1978),

$$J(t-t') \;=\; \left[\; J \;+\; (t-t')^{\alpha}\;\beta + \;(t-t')\;\eta^{-1}\,\right]\,\Theta(t-t') \quad , \tag{28}$$

[5] It should be noted that in reality the geodetic measurements were confirming the power law (27) not for the seismic lag δ but for the tidal lag ϵ, an important detail to be addressed shortly.

[6] Damping mechanisms in a wet planet will be the same as in a dry one, except that their efficiency will be increased. So the dissipation rate will have a similar frequency dependence but higher magnitude.

[7] In the absence of internal oceans, a rough estimate of the tidal response can be obtained through modeling the body with a homogeneous sphere. However the presence of such oceans makes it absolutely necessary to calculate the overall response through integration over the solid and liquid layers. As demonstrated by Tyler (2009), tidal dissipation in internal ocean layers can play a big role in rotational dynamics of the body.

contains empirical parameters α and β, the steady-state viscosity η, and the unrelaxed compliance $J \equiv J(0) = 1/\mu(0) = 1/\mu$. We endow the right-hand side of (28) with the Heaviside step-function $\Theta(t-t')$, to ensure that insertion of (28) into (5), with the subsequent differentiation, yields the elastic term $J\,\delta(t-t')$ under the integral. The model allows for modeling dissipation mechanisms over a continuum of frequencies, which is useful for complex inhomogeneous materials with a range of grain sizes.

The Andrade model can be thought of as the Maxwell model equipped with an extra term $(t-t')^{\alpha}\,\beta$ describing hereditary reaction of strain to stress. The Maxwell model

$$J^{(Maxwell)}(t-t') \;=\; \left[\,J \;+\; (t-t')\;\eta^{-1}\,\right]\,\Theta(t-t') \tag{29}$$

is a simple rheology, which has a long history of application to planetary problems, but generally has too strong a frequency dependence at frequencies higher than about $1\,/\mathrm{yr}$ (Karato 2008). Insertion of (29) into (5) renders strain consisting of two separate inputs. The one proportional to J implements the instantaneous (elastic) reaction, while the one containing η^{-1} is responsible for the viscous part of the reaction.

Just as the viscous term $(t-t')\,\eta^{-1}$ showing up in (28 - 29), so the anelastic term $(t-t')^{\alpha}\,\beta$ emerging in the Andrade model (28) is delayed – both terms reflect how the past stressing is influencing the present deformation. At the same time, the anelastic reaction differs from viscosity both mathematically and physically, because it is produced by different physical mechanisms.

A disadvantage of the formulation (28) of the Andrade model is that it contains a parameter of fractional dimensions, β. To avoid fractional dimensions, we shall express this parameter, following Efroimsky (2011), as

$$\beta \;=\; J\,\tau_{_A}^{-\alpha} \;=\; \mu^{-1}\,\tau_{_A}^{-\alpha}\quad, \tag{30a}$$

the new parameter $\tau_{_A}$ having dimensions of time. This is the timescale associated with the Andrade creep, wherefore it may be named as the “Andrade time” or the “anelastic time”.

Another option is to express β as

$$\beta \;=\; \zeta^{-\alpha}\,J\,\tau_{_M}^{-\alpha} \;=\; \zeta^{-\alpha}\,\mu^{-1}\,\tau_{_M}^{-\alpha}\quad, \tag{30b}$$

where the dimensionless parameter ζ is related through

$$\zeta \;=\; \frac{\tau_{_A}}{\tau_{_M}} \tag{31}$$

to the anelastic timescale $\tau_{_A}$ and to the Maxwell time

$$\tau_{_M} \;\equiv\; \frac{\eta}{\mu} \;=\; \eta\,J\quad. \tag{32}$$

In terms of the so-introduced parameters, the compliance assumes the form of

$$J(t-t') \;\;=\;\; J\,\left[\,1 \;+\; \left(\frac{t-t'}{\tau_{_A}}\right)^{\alpha} \;+\; \frac{t-t'}{\tau_{_M}}\,\right]\,\Theta(t-t') \tag{33a}$$

$$\;\;=\;\; J\,\left[\,1 \;+\; \left(\frac{t-t'}{\zeta\;\tau_{_M}}\right)^{\alpha} \;+\; \frac{t-t'}{\tau_{_M}}\,\right]\,\Theta(t-t')\quad. \tag{33b}$$

For $\tau_{_A} \ll \tau_{_M}$ (or, equivalently, for $\zeta \ll 1$), anelasticity plays a more important role than viscosity. On the other hand, a large $\tau_{_A}$ (or large ζ) would imply suppression of anelasticity, compared to viscosity.

It has been demonstrated by Castillo-Rogez that under low stressing (i.e., when the grain-boundary diffusion is the dominant damping mechanism – like in Iapetus) β obeys the relation

$$\beta \approx J\,\tau_{_M}^{-\alpha} = J^{1-\alpha}\,\eta^{-\alpha} = \mu^{\alpha-1}\,\eta^{-\alpha} \quad , \tag{34a}$$

(see, e.g., Castillo-Rogez et al. 2011). Comparing this to (30), we can say that the anelastic and viscoelastic timescales are close to one another:

$$\tau_{_A} \approx \tau_{_M} \tag{34b}$$

or, equivalently, that the dimensionless parameter ζ is close to unity:

$$\zeta \approx 1 \quad . \tag{34c}$$

Generally, we have no reason to expect the anelastic and viscoelastic timescales to coincide, nor even to be comparable under all possible circumstances. While (34) may work when the grain-boundary diffusion dominates anelastic friction, we are also aware of a case when the timescales $\tau_{_A}$ and $\tau_{_M}$ differ considerably. This is a situation when stressing is stronger, and the anelastic part of dissipation is dominated by dislocations unpinning. This is what happens in mantles of earths and superearths.

On theoretical grounds, Karato & Spetzler (1990) point out that the dislocation-unpinning mechanism remains effective in the Earth's mantle down to the frequency threshold $\chi_0 \sim 1\ \mathrm{yr}^{-1}$. At lower frequencies, this mechanism becomes less efficient, giving way to viscosity. Thus at low frequencies the mantle's behaviour becomes closer to that of the Maxwell body.[8] This important example tells us that the anelastic time $\tau_{_A}$ and the dimensionless parameter ζ may, at times, be more sensitive to the frequency than the Maxwell time would be. Whether $\tau_{_A}$ and ζ demonstrate this sensitivity or not – may, in its turn, depend upon the intensity of loading, i.e., upon the damping mechanisms involved.

3.3 The Andrade model in the frequency domain

Through (12), it can be demonstrated (Findley et al. 1976) that in the frequency domain the compliance of an Andrade material reads as

$$\bar{J}(\chi) = J + \beta\,(i\chi)^{-\alpha}\,\Gamma(1+\alpha) - \frac{i}{\eta\chi} \tag{35a}$$

$$= J\left[1 + (i\,\chi\,\tau_{_A})^{-\alpha}\,\Gamma(1+\alpha) - i\,(\chi\,\tau_{_M})^{-1}\right] \quad , \tag{35b}$$

$$= J\left[1 + (i\,\chi\,\zeta\,\tau_{_M})^{-\alpha}\,\Gamma(1+\alpha) - i\,(\chi\,\tau_{_M})^{-1}\right] \quad , \tag{35c}$$

[8] Using the Andrade model as a fit to the experimentally observed scaling law (27), we see that the exponential p coincides with the Andrade parameter $\alpha < 1$ at frequencies above the said threshold, and that p becomes closer to unity below the threshold – see subsection 3.4 below.

χ being the frequency, and Γ denoting the Gamma function. The imaginary and real parts of the complex compliance are:

$$\mathcal{I}m[\bar{J}(\chi)] \;=\; -\,\frac{1}{\eta\,\chi}\;-\;\chi^{-\alpha}\,\beta\,\sin\left(\frac{\alpha\,\pi}{2}\right)\,\Gamma(\alpha\,+\,1) \tag{36a}$$

$$=\; -\,J\,(\chi\,\tau_{_M})^{-1}\;-\;J\,(\chi\,\tau_{_A})^{-\alpha}\,\sin\left(\frac{\alpha\,\pi}{2}\right)\,\Gamma(\alpha\,+\,1) \tag{36b}$$

$$=\; -\,J\,(\chi\,\tau_{_M})^{-1}\;-\;J\,(\chi\,\zeta\,\tau_{_M})^{-\alpha}\,\sin\left(\frac{\alpha\,\pi}{2}\right)\,\Gamma(\alpha\,+\,1) \tag{36c}$$

and

$$\mathcal{R}e[\bar{J}(\chi)] \;=\; J\;+\;\chi^{-\alpha}\,\beta\,\cos\left(\frac{\alpha\,\pi}{2}\right)\,\Gamma(\alpha\,+\,1) \tag{37a}$$

$$=\; J\;+\;J\,(\chi\tau_{_A})^{-\alpha}\,\cos\left(\frac{\alpha\,\pi}{2}\right)\,\Gamma(\alpha\,+\,1) \tag{37b}$$

$$=\; J\;+\;J\,(\chi\,\zeta\,\tau_{_M})^{-\alpha}\,\cos\left(\frac{\alpha\,\pi}{2}\right)\,\Gamma(\alpha\,+\,1)\quad . \tag{37c}$$

The ensuing frequency dependence of the phase lag will look as[9]

$$\tan\delta(\chi) \;=\; -\,\frac{\mathcal{I}m\left[\bar{J}(\chi)\right]}{\mathcal{R}e\left[\bar{J}(\chi)\right]} \;=\; \frac{(\eta\,\chi)^{-1}\;+\;\chi^{-\alpha}\,\beta\,\sin\left(\frac{\alpha\,\pi}{2}\right)\Gamma\,(\alpha\,+\,1)}{\mu^{-1}\;+\;\chi^{-\alpha}\,\beta\,\cos\left(\frac{\alpha\,\pi}{2}\right)\,\Gamma\,(\alpha\,+\,1)} \tag{38a}$$

$$=\; \frac{\frac{1}{\chi\,\tau_{_M}}\;+\;\left(\frac{1}{\chi\,\tau_{_A}}\right)^{\alpha}\;\Gamma\,(\alpha\,+\,1)\;\;\sin\left(\frac{\alpha\,\pi}{2}\right)}{1\,+\;\left(\frac{1}{\chi\,\tau_{_A}}\right)^{\alpha}\;\Gamma\,(\alpha\,+\,1)\;\;\cos\left(\frac{\alpha\,\pi}{2}\right)} \;=\; \frac{z^{-1}\,\zeta\;+\;z^{-\alpha}\,\sin\left(\frac{\alpha\,\pi}{2}\right)\Gamma\,(\alpha\,+\,1)}{1\,+\;z^{-\alpha}\,\cos\left(\frac{\alpha\,\pi}{2}\right)\;\Gamma\,(\alpha\,+\,1)}\quad , \tag{38b}$$

with z being the dimensionless frequency defined through

$$z\;\equiv\;\chi\,\tau_{_A}\;=\;\chi\,\tau_{_M}\,\zeta\quad . \tag{39}$$

Evidently, for $\beta\rightarrow 0$ (that is, for $\zeta\rightarrow\infty$ or $\tau_{_A}\rightarrow\infty$), expression (38) approaches

$$\tan\delta(\chi)\;=\;(\tau_{_M}\,\chi)^{-1}\quad , \tag{40}$$

which is the frequency dependence appropriate to the Maxwell body.

3.4 Low frequencies: from Andrade toward Maxwell

The Andrade body demonstrates the so-called "elbow dependence" of the dissipation rate upon frequency. At high frequencies, the lag δ satisfies the power law

$$\tan\delta\;\sim\;\chi^{-p}\quad , \tag{41}$$

[9] In some publications (e.g., Nimmo 2008), formula (38a) is given as an expression for the inverse quality factor. This is legitimate when the latter is defined through (23).

the exponential being expressed via an empirical parameter α, where $0 < \alpha < 1$ for most materials. It follows from (38) that at higher frequencies $p = \alpha$, while at low frequencies $p = 1 - \alpha$.

The statement by Karato & Spetzler (1990), that the mantle's behaviour at low frequencies should lean toward that of the Maxwell body, can be fit into the Andrade formalism, if we agree that at low frequencies either α approaches zero (so p approaches unity) or ζ becomes large (so τ_A becomes much larger than τ_M). The latter option is more physical, because the increase of τ_A would reflect the slowing-down of the unpinning mechanism studied in *Ibid.*

One way or another, the so-parameterised Andrade model is fit to embrace the result from *Ibid.* Hence our treatment will permit us to describe both the high-frequency range where the traditional Andrade model is applicable, and the low-frequency band where the behaviour of the mantle deviates from the Andrade model toward the Maxwell body. Comparison of the two models in the frequency domain is presented in Figure 1.

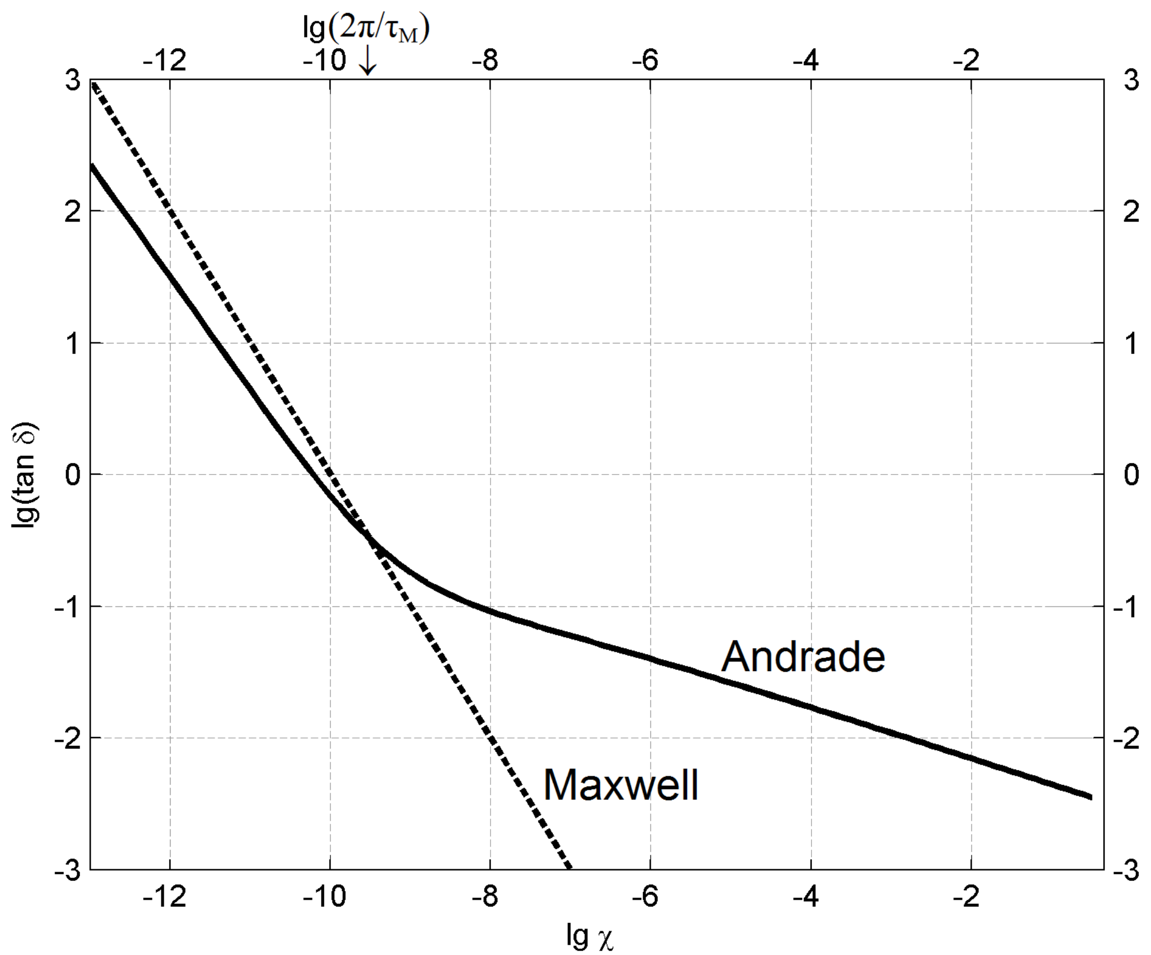

Figure 1: Andrade and Maxwell models in the frequency domain. The plot shows the decadic logarithm of $\tan\delta$, as a function of the decadic logarithm of the forcing frequency χ (in cycles s^{-1}). For the Andrade body, the tangent of phase lag is given by (38b), with $\alpha = 0.2$ and $\tau_A = \tau_M = 10^{10}$ s. For the Maxwell body, the tangent is rendered by (40), with $\tau_M = 10^{10}$ s.

4 Expanding a tidal potential or torque – over the tidal modes or over the forcing frequencies?

Consider a binary system with mean motion n, and suppose that tidal dissipation in one of the bodies much exceeds that in its companion. Then the former body may be treated as the tidally perturbed primary, the latter being its tide-raising secondary. The sidereal angle and the spin rate of the primary will be expressed with θ and $\dot{\theta}$, while the node, pericentre, and mean anomaly of the secondary, as seen from the primary,[10] will be denoted by Ω, ω, and $\mathcal{M}$.

In the Darwin-Kaula theory, bodily tides are expanded over the modes

$$\omega_{lmpq} \equiv (l-2p)\,\dot{\omega} + (l-2p+q)\,\dot{\mathcal{M}} + m\,(\dot{\Omega} - \dot{\theta}) \approx (l-2p+q)\,n - m\,\dot{\theta} \quad , \tag{42}$$

with l, m, p, q being integers. Dependent upon the values of the mean motion, spin rate, and the indices, the tidal modes ω_{lmpq} may be positive or negative or zero.

In the expansion of the tidal potential or torque or force, summation over the integer indices goes as $\sum_{l=2}^{\infty}\sum_{m=0}^{l}\sum_{p=0}^{\infty}\sum_{q=-\infty}^{\infty}$. For example, the secular polar component of the tidal torque will read as:

$$\mathcal{T} = \sum_{l=2}^{\infty}\sum_{m=0}^{l}\sum_{p=0}^{\infty}\sum_{q=-\infty}^{\infty} \ldots\; k_l(\omega_{lmpq})\; \sin\epsilon_l(\omega_{lmpq}) \quad , \tag{43}$$

where the ellipsis denotes a function of the primary's radius and the secondary's orbital elements. The functions $k_l(\omega_{lmpq})$ are the dynamical analogues of the static Love numbers, while the phase lags corresponding to the tidal modes ω_{lmpq} are given by

$$\epsilon_l(\omega_{lmpq}) = \omega_{lmpq}\;\Delta t_l(|\omega_{lmpq}|) = |\,\omega_{lmpq}\,|\;\Delta t_l(|\omega_{lmpq}|)\;\mathrm{sgn}\,\omega_{lmpq} = \chi_{lmpq}\;\Delta t_l(\chi_{lmpq})\;\mathrm{sgn}\,\omega_{lmpq} \quad . \tag{44}$$

Here the positively defined quantities

$$\chi_{lmpq} \equiv |\,\omega_{lmpq}\,| \tag{45}$$

are the forcing frequencies in the material, while the positively defined time lags Δt_{lmpq} are their functions.

Following Kaula (1964), the phase and time lags are often denoted with ϵ_{lmpq} and Δt_{lmpq}. For near-spherical bodies, though, the notations $\epsilon_l(\chi_{lmpq})$ and $\Delta t_l(\chi_{lmpq})$ would be preferable, because for such bodies the functional form of the dependency $\epsilon_{lmpq}(\chi)$ is defined by

[10] When the role of the primary is played by a planet and the role of the perturbing secondary is played by the host star, the argument of the pericentre of the star as seen from the planet, ω, differs by π from the argument of the pericentre of the planet as seen from the star.

Also mind that in equation (42) the letter ω with the subscript $lmpq$ denotes, as ever, a tidal mode, while the same letter without a subscript stands for the periapse. The latter use of this letter in equation (42) is exceptional, in that elsewhere in the paper the letter ω, with or without a subscript, always denotes a tidal mode.

ls, but is ignorant of the values of the other three indices.[11] The same therefore applies to the time lag.

The forcing frequencies in the material of the primary, χ_{lmpq}, are positively defined. While the general formula for a Fourier expansion of a field includes integration (or summation) over both positive and negative frequencies, it is easy to demonstrate that in the case of real fields it is sufficient to expand over positive frequencies only. The condition of the field being real requires that the real part of a Fourier term at a negative frequency is equal to the real part of the term at an opposite, positive frequency. Hence one can get rid of the terms with negative frequencies, at the cost of doubling the appropriate terms with positive frequencies. (The convention is that the field is the real part of a complex expression.)

The tidal theory is a rare exception to this rule: here, a contribution of a Fourier mode into the potential is not completely equivalent to the contribution of the mode of an opposite sign. The reason for this is that the tidal theory is developed to render expressions for tidal forces and torques, and the sign of the tidal mode ω_{lmpq} shows up explicitly in those expression. This happens because the phase lag in (43) is the product (44) of the tidal mode ω_{lmpq} and the positively defined time lag Δt_{lmpq}.

This way, if we choose to expand tide over the positively defined frequencies χ only, we shall have to insert "by hand" the multipliers

$$\mathrm{sgn}\,\omega_{lmpq} \;=\; \mathrm{sgn}\left[\;(l-2p+q)\,n \;-\; m\,\dot{\theta}\;\right] \tag{46}$$

into the expressions for the tidal torque and force, a result to be employed below in formula (65). The topic is explained in greater detail in Efroimsky (2011).

5 Complex Love numbers and the elastic-viscoelastic analogy

Let us recall briefly the switch from the stationary Love numbers to their dynamical counterparts, the Love operators. The method was pioneered, probably, by Zahn (1966) who applied it to a purely viscous planet. The method works likewise for an arbitrary linear rheological model, insofar as the elastic-viscoelastic analogy (also referred to as the correspondence principle) remains in force.

5.1 From the Love numbers to the Love operators

A homogeneous near-spherical incompressible primary alters its shape and potential, when influenced by a static point-like secondary. At a point $\vec{\boldsymbol{R}} = (R, \lambda, \phi)$, the potential due to a

[11] Within the applicability realm of the elastic-viscoelastic analogy employed in subsection 5.2 below, the functional form of the complex Love number $\bar{k}_l(\chi)$ of a near-spherical object is determined by index l solely, while the integers $m,\,p,\,q$ show up through the value of the frequency: $\bar{k}_l(\chi) \,=\, \bar{k}_l(\chi_{lmpq})$. This applies to the lag too, since the latter is related to $\bar{k}_l$ via (62).

For triaxial bodies, the functional forms of the frequency dependencies of the Love numbers and phase lags do depend upon $m,\,p,\,q$, because of coupling between spherical harmonics. In those situations, notations $\bar{k}_{lmpq}$ and ϵ_{lmpq} become necessary (Dehant 1987a,b; Smith 1974). The Love numbers of a slightly non-spherical primary differ from the Love numbers of the spherical reference body by a term of the order of the flattening, so a small non-sphericity can usually be ignored.

tide-generating secondary of mass M^*_{sec} , located at $\vec{\boldsymbol{r}}^{\,*} = (r^*,\,\lambda^*,\,\phi^*)$ with $r^* \geq R$, can be expressed through the Legendre polynomials $P_l(\cos\gamma)$ or the Legendre functions $P_{lm}(\sin\phi)$:

$$W(\vec{\boldsymbol{R}}\,,\,\vec{\boldsymbol{r}}^{\,*}) \;=\; \sum_{l=2}^{\infty} W_l(\vec{\boldsymbol{R}}\,,\;\vec{\boldsymbol{r}}^{\,*}) \;=\; -\;\frac{G\;M^*_{sec}}{r^*}\;\sum_{l=2}^{\infty}\left(\frac{R}{r^{\,*}}\right)^{l}\;P_l(\cos\gamma)$$

$$=\;-\;\frac{G\;M^*_{sec}}{r^*}\sum_{l=2}^{\infty}\left(\frac{R}{r^{\,*}}\right)^{l}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}(2-\delta_{0m})P_{lm}(\sin\phi)P_{lm}(\sin\phi^*)\;\cos m(\lambda-\lambda^*)\quad, \qquad (47)$$

G being Newton's gravitational constant, and γ being the angle between the vectors $\vec{\boldsymbol{r}}^{\,*}$ and $\vec{\boldsymbol{R}}$ originating from the primary's centre. The latitudes $\phi,\,\phi^*$ are reckoned from the primary's equator, while the longitudes $\lambda,\,\lambda^*$ are reckoned from a fixed meridian on the primary.

The l^{th} spherical harmonic $U_l(\vec{\boldsymbol{r}})$ of the resulting change of the primary's potential at an exterior point $\vec{\boldsymbol{r}}$ is connected to the l^{th} spherical harmonic $W_l(\vec{\boldsymbol{R}}\,,\,\vec{\boldsymbol{r}}^{\,*})$ of the perturbing exterior potential via $U_l(\vec{\boldsymbol{r}}) = (R/r)^{l+1}\,k_l\,W_l(\vec{\boldsymbol{R}}\,,\,\vec{\boldsymbol{r}}^{\,*})$, so the total change in the exterior potential of the primary becomes:

$$U(\vec{\boldsymbol{r}}) \;=\; \sum_{l=2}^{\infty} U_l(\vec{\boldsymbol{r}}) \;=\; \sum_{l=2}^{\infty}\left(\frac{R}{r}\right)^{l+1} k_l\;\,W_l(\vec{\boldsymbol{R}},\;\vec{\boldsymbol{r}}^{\,*})\quad. \qquad (48)$$

While in (47) $\vec{\boldsymbol{R}}$ could lie either outside or on the surface of the primary, in (48) it would be both convenient and conventional to make $\vec{\boldsymbol{R}}$ a surface point. Both in (47) and (48), the vector $\vec{\boldsymbol{r}}$ denotes an exterior point located above the surface point $\vec{\boldsymbol{R}}$ at a radius $r \;\geq\; R$ (with the same latitude and longitude), while $\vec{\boldsymbol{r}}^{\,*}$ signifies the position of the tide-raising secondary. The quantities k_l are the static Love numbers.

Under dynamical stressing, the Love numbers turn into operators:

$$U_l(\vec{\boldsymbol{r}},\,t) \;=\; \left(\frac{R}{r}\right)^{l+1}\hat{k}_l(t)\;W_l(\vec{\boldsymbol{R}},\;\vec{\boldsymbol{r}}^{\,*},\,t\,')\quad, \qquad (49)$$

where integration over the semi-interval $t\,' \in (\,-\,\infty,\,t\;]$ is implied:

$$U_l(\vec{\boldsymbol{r}},\,t) \;=\; \left(\frac{R}{r}\right)^{l+1}\int_{t\,'=-\infty}^{t\,'=t} k_l(t-t\,')\;\dot{W}_l\;(\vec{\boldsymbol{R}}\,,\,\vec{\boldsymbol{r}}^{\,*},\,t\,')\,dt\,' \qquad (50a)$$

$$=\;\left(\frac{R}{r}\right)^{l+1}\,[k_l(0)W(t)\;-\;k_l(\infty)W(-\infty)]\;+\;\left(\frac{R}{r}\right)^{l+1}\int_{-\infty}^{t}\dot{k}_l(t-t\,')\;\;W_l(\vec{\boldsymbol{R}}\,,\,\vec{\boldsymbol{r}}^{\,*},\,t\,')\,dt\,'\quad. \qquad (50b)$$

Like in the compliance operator (1 - 2), here we too obtain the boundary terms: one corresponding to the instantaneous elastic reaction, $k_l(0)W(t)$, another caused by the perturbation in the infinite past, $-\,k_l(\infty)W(-\infty)$. The latter term can be dropped by setting $W(-\infty)$ zero, while the former term may be included into the kernel in the same manner as in (3 - 5):

$$\left(\frac{R}{r}\right)^{l+1}\; k_l(0)W(t) \;+\; \left(\frac{R}{r}\right)^{l+1}\int_{-\infty}^{t}\stackrel{\bullet}{k}_l(t-t\,')\;\; W_l(\vec{\boldsymbol{R}}\,,\,\vec{\boldsymbol{r}}^{\,*},\,t\,')\,dt\,'$$

$$=\left(\frac{R}{r}\right)^{l+1}\int_{-\infty}^{t}\frac{d}{dt}\left[\;k_l(t\,-\,t\,')\;-\;k_l(0)\;+\;k_l(0)\Theta(t\,-\,t\,')\;\right]\;W_l(\vec{\boldsymbol{R}},\vec{\boldsymbol{r}}^{\,*},t\,')\,dt\,'\quad. \tag{50c}$$

All in all, neglecting the unphysical term with $W(-\infty)$, and inserting the elastic term into the Love number not as $k_l(0)$ but as $k_l(0)\,\Theta(t-t\,')$, we arrive at

$$U_l(\vec{\boldsymbol{r}},\,t)\;=\;\left(\frac{R}{r}\right)^{l+1}\int_{-\infty}^{t}\stackrel{\bullet}{k}_l(t-t\,')\;W_l(\vec{\boldsymbol{R}}\,,\;\vec{\boldsymbol{r}}^{\,*},\;t\,')\,dt\,'\;, \tag{51}$$

with $k_l(t-t\,')$ now incorporating the elastic reaction as $k_l(0)\,\Theta(t-t\,')$ instead of $k_l(0)$. For a perfectly elastic primary, the elastic reaction would be the only term present in the expression for $k_l(t-t\,')$. Then the time derivative of k_l would be $\stackrel{\bullet}{k}_l(t-t\,')\;=\;k_l\,\delta(t-t\,')$, with $k_l\;\equiv\;k_l(0)$ being the static Love number, and (51) would reduce to $U_l(\vec{\boldsymbol{r}},\,t)\;=\;\left(\frac{R}{r}\right)^{l+1}k_l\,W_l(\vec{\boldsymbol{R}}\,,\;\vec{\boldsymbol{r}}^{\,*},\;t)$, like in the static case.

Similarly to (11), the complex Love numbers are defined as the Fourier images of $\stackrel{\bullet}{k}_l\;(\tau)$:

$$\int_{0}^{\infty}\bar{k}_l(\chi)e^{i\chi\tau}d\chi\;=\stackrel{\bullet}{k}_l\;(\tau)\quad, \tag{52}$$

overdot denoting $d/d\tau$. Following Churkin (1998), the time derivatives $\stackrel{\bullet}{k}_l\;(t)$ can be named *Love functions*.[12] Inversion of (52) renders:

$$\bar{k}_l(\chi)\;=\;\int_{0}^{\infty}\stackrel{\bullet}{k}_l(\tau)\;e^{-i\chi\tau}\,d\tau\;=\;k_l(0)\;+\;i\;\chi\;\int_{0}^{\infty}\left[\,k_l(\tau)\;-\;k_l(0)\,\Theta(\tau)\,\right]\;e^{-i\chi\tau}\,d\tau\quad, \tag{53}$$

where integration from 0 is sufficient, as the future disturbance contributes nothing to the present distortion, wherefore $k_l(\tau)$ vanishes at $\tau<0$. Recall that the time τ denotes the difference $t-t\,'$ and thus is reckoned from the present moment t backward into the past

In the frequency domain, (50) will take the shape of

$$\bar{U}_l(\chi)\;=\;\left(\frac{R}{r}\right)^{l+1}\bar{k}_l(\chi)\;\bar{W}_l(\chi)\quad, \tag{54}$$

χ being the frequency, while $\bar{U}_l(\chi)$ and $\bar{W}_l(\chi)$ being the Fourier or Laplace components of the potentials $U_l(t)$ and $W_l(t)$. The frequency-dependencies $\bar{k}_l(\chi)$ should be derived from the expression for $\bar{J}(\chi)$ or $\bar{\mu}(\chi)=1/\bar{J}(\chi)$. These expressions follow from the rheological model of the medium.

Rigorously speaking, we ought to assume in expressions (52 - 54) that the spectral components are functions of the tidal mode ω and not of the forcing frequency χ . However, as

[12] Churkin (1998) used functions, which he denoted $k_l(t)$ and which were, due to a difference in notations, the same as our $\stackrel{\bullet}{k}_l\;(\tau)$.

explained in the end of Section 4, employment of the positively defined forcing frequencies is legitimate, insofar as we do not forget to attach the sign multipliers (46) to the terms of the Darwin-Kaula expansion for the tidal torque. Therefore here and hereafter we shall expand over χ, with the said caveat kept in mind.

5.2 Complex Love numbers as functions of the complex compliance. The elastic-viscoelastic analogy

The dependence of the static Love numbers on the static rigidity modulus $\mu(\infty)$ looks as

$$k_l^{(static)} = \frac{3}{2\,(l\,-\,1)}\;\frac{1}{1\,+\,A_l^{(static)}} \quad , \tag{55}$$

where

$$A_l^{(static)} \equiv \frac{(2\,l^2\,+\,4\,l\,+\,3)}{l\,\mathrm{g}\,\rho\,R}\;\mu(\infty) = \frac{3\,(2\,l^2\,+\,4\,l\,+\,3)}{4\,l\,\pi\,G\,\rho^2\,R^2}\;\mu(\infty) = \frac{3\,(2\,l^2\,+\,4\,l\,+\,3)}{4\,l\,\pi\,G\,\rho^2\,R^2\;\;J(\infty)} \quad , \tag{56}$$

with ρ, g, and R being the density, surface gravity, and radius of the body, and G being the Newton gravitational constant. The static rigidity modulus and its inverse, the static compliance, are denoted here with $\mu(\infty)$ and $J(\infty)$, correspondingly. These notations imply that we identify *static* with *relaxed*.

Specifically, the static quadrupole Love number will read:

$$k_2^{(static)} = \frac{3}{2}\;\frac{1}{1\,+\,A_2^{(static)}} \quad , \tag{57}$$

where the quantity

$$A_2^{(static)} = \frac{57}{8\,\pi}\;\frac{\mu(\infty)}{G\,\rho^2\,R^2} \tag{58}$$

is sometimes referred to as $\tilde{\mu}$. Clearly, $A_2^{(static)}$ in (57), as well as $A_l^{(static)}$ in (55), is a dimensionless measure of strength-dominated versus gravity-dominated behaviour.

It is not immediately clear whether the same expression interconnects also $\bar{k}_l(\chi)$ with $\bar{\mu}(\chi)$. Fortunately, though, a wonderful theorem called *elastic-viscoelastic analogy*, also known as the *correspondence principle*, ensures that the viscoelastic operational moduli $\bar{\mu}(\chi)$ or $\bar{J}(\chi)$ obey the same algebraic relations as the elastic parameters μ or J (see, e.g., Efroimsky 2011 and references therein). For this reason, the Fourier or Laplace images of the viscoelastic equation of motion [13] and of the constitutive equation look as their static counterparts, except that the stress, strain, and potentials get replaced with their Fourier or Laplace images, while k_l, μ, and J get replaced with the Fourier or Laplace images of $\dot{k}_l\,(t-t\,')$, $\dot{\mu}\,(t-t\,')$, and $\dot{J}\,(t-t\,')$. For example, the constitutive equation will look like: $\bar{\sigma}_{\gamma\nu} \;=\; 2\;\bar{\mu}\;\bar{u}_{\gamma\nu}$. Therefore the solution to the problem will retain the mathematical form

[13] In the equation of motion, we should neglect the acceleration term, which is justified at realistic frequencies.

of $\bar{U}_l = \bar{k}_l \bar{W}_l$, with $\bar{k}_l$ keeping the same functional dependence on ρ , R , and $\bar{\mu}$ (or $\bar{J}$) as in (56), except that now μ and J are equipped with overbars:

$$\bar{k}_l(\chi) \;=\; \frac{3}{2\,(l\,-\,1)}\;\frac{1}{1\,+\,A_l\,\bar{\mu}(\chi)/\mu} \tag{59a}$$

$$=\; \frac{3}{2\,(l\,-\,1)}\;\frac{1}{1\,+\,A_l\,J/\bar{J}(\chi)} \;=\; \frac{3}{2\,(l\,-\,1)}\;\frac{\bar{J}(\chi)}{\bar{J}(\chi)\,+\,A_l\,J}\quad, \tag{59b}$$

where

$$A_l\;\equiv\;\frac{(2\,l^2\,+\,4\,l\,+\,3)\,\mu}{l\,\mathrm{g}\,\rho\,R}\;=\;\frac{3\,(2\,l^2\,+\,4\,l\,+\,3)\,\mu}{4\,l\,\pi\,G\,\rho^2\,R^2}\;=\;\frac{3\,(2\,l^2\,+\,4\,l\,+\,3)\;\;J^{-1}}{4\,l\,\pi\,G\,\rho^2\,R^2}\quad, \tag{60}$$

Although expression (60) for factors A_l is very similar to expression (56) for their static counterparts, an important difference between (56) and (60) should be pointed out. While in (56) we had the static (relaxed) rigidity and compliance, $\mu(\infty)$ and $J(\infty) = 1/\mu(\infty)$, in (60) the letters μ and J may stand for any benchmark values satisfying $J = 1/\mu$. This freedom stems from the fact that the products $A_l\,J$ entering (59b) bear no dependence upon J or μ . The second term in the denominator of (59b) contains $\bar{\mu}$. For convenience, we multiply and then divide it by some μ , and make the multiplier μ a part of A_l as in (60). This makes it easier for us to compare (59b) with its static predecessor (56). However the constant μ here is essentially arbitrary, and is not obliged to coincide with, say, unrelaxed or relaxed rigidity. Accordingly, $J = 1/\mu$ is not obliged to be the unrelaxed or relaxed compliance.

The above caveat is important, because in certain rheological models some of the unrelaxed or relaxed moduli may be zero or infinite. This will happen, for example, if we start with the Maxwell or Kelvin-Voigt body and perform a transition to a purely viscous medium. Fortunately, in realistic rheologies such things do not happen. Hence it will be convenient (and possible) to identify the J from (60) with the *unrelaxed* compliance $J = J(0)$ emerging in the rheological model (33). Accordingly, the rigidity $\mu = 1/J$ from (60) will be identified with the *unrelaxed* rigidity $\mu(0) = 1/J(0)$. This convention will play a crucial role down the road, when we derive formula (63).

Writing the l th complex Love number as

$$\bar{k}_l(\chi)\;=\;\mathcal{R}e\left[\bar{k}_l(\chi)\right]\;+\;i\;\mathcal{I}m\left[\bar{k}_l(\chi)\right]\;=\;|\bar{k}_l(\chi)|\;e^{-i\epsilon_l(\chi)} \tag{61}$$

we express the phase lag $\epsilon_l(\chi)$ as:

$$|\bar{k}_l(\chi)|\;\sin\epsilon_l(\chi)\;=\;-\,\mathcal{I}m\left[\bar{k}_l(\chi)\right]\quad. \tag{62}$$

The importance of the products $|\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi)$ *lies in the fact that they show up in the terms of the Darwin-Kaula expansion of the tidal potential. As a result, it is these products, and not* k_l/Q *as some think, which emerge in the expansions for tidal forces and torques, and for the dissipation rate.*

In an attempt to preserve the popular notation k_l/Q , one may *define* the inverse quality factor as the sine of the lag – see the discussion in subsection 2.3. In this case, though, one would have to employ the tidal lag ϵ_l, and not the lag δ in the material (which we call the "seismic" lag). Accordingly, one will have to write not k_l/Q but k_l/Q_l, where $1/Q_l \equiv \sin\epsilon_l$.

Importantly, the functional form of the frequency-dependence $\sin\epsilon_l(\chi)$ is different for different l. Thus an attempt of naming $\sin\epsilon_l$ as $1/Q$ would give birth to a whole array of different functions $Q_l(\chi)$. For a triaxial body, things will become even more complicated – see footnote 11. To conclude, it is not advisable to denote $\sin\epsilon_l$ with $1/Q$.

It should be mentioned that the Darwin-Kaula theory of tides is equally applicable to tides in despinning and librating bodies. In all cases, the phase angle $\epsilon_l = \epsilon_l(\chi_{lmpq})$ parameterises the lag of the appropriate component of the bulge, while the absolute value of the complex Love number $|\bar{k}_l| = |\,\bar{k}_l(\chi_{lmpq})\,|$ determines the magnitude of this component. The overall bulge being a superposition of these components, its height may vary in time.

5.3 The tangent of the tidal lag

In the denominator of (59a) the term 1 emerges due to self-gravitation, while $A_l\, J/\bar{J}(\chi) = A_l\,|\bar{\mu}(\chi)|/\mu$ describes how the bulk properties of the medium contribute to deformation and damping. So for a vanishing $A_l\, J/|\bar{J}(\chi)|$ we end up with the hydrostatic Love numbers $k_l = \dfrac{3}{2\,(l\,-\,1)}$, while the lag becomes nil, as will be seen shortly from (64). On the contrary, for very large $A_l\, J/\bar{J}(\chi)$, we expect to obtain the Love numbers and lags ignorant of the shape of the body.

To see how this works out, combine formulae (35) and (59b), to arrive at

$$\tan\epsilon_l \;=\; -\;\frac{\mathcal{I}m\left[\bar{k}_l(\chi)\right]}{\mathcal{R}e\left[\bar{k}_l(\chi)\right]} \;=\; -\;\frac{A_l\;J\;\mathcal{I}m\left[\bar{J}(\chi)\right]}{\left\{\mathcal{R}e\left[\bar{J}(\chi)\right]\right\}^2 \,+\, \left\{\mathcal{I}m\left[\bar{J}(\chi)\right]\right\}^2 \,+\, A_l\;J\;\mathcal{R}e\left[\bar{J}(\chi)\right]} \;= \tag{63a}$$

$$\frac{A_l\;\left[\zeta\;z^{-1}\,+\,z^{-\alpha}\;\sin\left(\frac{\alpha\,\pi}{2}\right)\;\Gamma(1+\alpha)\right]}{\left[1+z^{-\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\Gamma(1+\alpha)\right]^2+\left[\zeta z^{-1}+z^{-\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\Gamma(1+\alpha)\right]^2+A_l\left[1+z^{-\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\Gamma(1+\alpha)\right]}\;, \tag{63b}$$

z being the dimensionless frequency defined by (39).

Comparing this expression with expression (38), over different frequency bands, we shall be able to explore how the tidal lag ϵ_l relates to the lag in the material δ (the "seismic lag").

5.4 The negative imaginary part of the complex Love number

As we already mentioned above, rheology influences the tidal behaviour of a planet through the following sequence of steps. A rheological model postulates the form of $\bar{J}(\chi)$. This function, in its turn, determines the form of $\bar{k}_l(\chi)$, while the latter defines the frequency dependence of the products $|\bar{k}_l(\chi)|\;\sin\epsilon(\chi)$ which enter the tidal expansions.

To implement this concatenation, one has to express $|\bar{k}_l(\chi)|\;\sin\epsilon(\chi)$ via $\bar{J}(\chi)$. This can be done by combining (59) with (62). It renders:

$$|\bar{k}_l(\chi)|\;\sin\epsilon_l(\chi) \;=\; -\,\mathcal{I}m\left[\bar{k}_l(\chi)\right] \;=\; \frac{3}{2\,(l\,-\,1)}\;\frac{-\,A_l\;J\,\mathcal{I}m\left[\bar{J}(\chi)\right]}{\left(\,\mathcal{R}e\left[\bar{J}(\chi)\right]\,+\,A_l\;J\,\right)^2\,+\,\left(\,\mathcal{I}m\left[\bar{J}(\chi)\right]\,\right)^2}\quad, \tag{64}$$

a quantity often mis-denoted[14] as k_l/Q . Together, formulae (35) and (64), give us the frequency dependencies for the factors $|\bar{k}_l(\chi)|\ \sin\epsilon_l(\chi)$ entering the theory of bodily tides. For detailed derivation of those dependencies, see Efroimsky (2011).

As explained in Section 4, employment of expressions (62 - 64) needs some care. Since both $\bar{U}$ and $\bar{k}_l$ are in fact functions not of the forcing frequency χ but of the tidal mode ω , formulae (62 - 64) should be equipped with multipliers $\mathrm{sgn}\,\omega_{lmpq}$, when plugged into the expression for the *lmpq* component of the tidal torque. With this important caveat in mind, and with the subscripts *lmpq* reinstalled, the complete expression will read:

$$|\bar{k}_l(\chi_{lmpq})|\ \sin\epsilon_l(\chi_{lmpq})\ =\ \frac{3}{2\,(l-1)}\ \frac{-\ A_l\ J\ \mathcal{I}m\left[\bar{J}(\chi_{lmpq})\right]}{\left(\mathcal{R}e\left[\bar{J}(\chi_{lmpq})\right]\ +\ A_l\ J\right)^2+\ \left(\mathcal{I}m\left[\bar{J}(\chi_{lmpq})\right]\right)^2}\ \mathrm{sgn}\,\omega_{lmpq}\quad . \qquad (65)$$

To make use of this and other formulae, it would be instructive to estimate the values of A_l for terrestrial objects of different size. In Table 1, we present estimates of A_2 for Iapetus, Mars, solid Earth, a hypothetical solid superearth having a density and rigidity of the solid Earth and a radius equal to 2 terrestrial radii ($R=2R_{\bigoplus}$), and also a twice larger hypothetical superearth ($R=4R_{\bigoplus}$) of the same rheology.

Taken the uncertainty of structure and the roughness of our estimate, all quantities in the table have been rounded to the first decimal. The values of Iapetus' and Mars' rigidity were borrowed from Castillo-Rogez et al. (2011) and Johnson et al. (2000), correspondingly.

In Figure 2, we compare the behaviour of $k_2\ \sin\epsilon_2=|\bar{k}_2(\chi)|\ \sin\epsilon_2(\chi)$ for the values of A_2 appropriate to Iapetus, Mars, solid Earth, and hypothetical superearths with $R\ =\ 2\,R_{\bigoplus}$ and $R\ =\ 4\,R_{\bigoplus}$, as given in Table 1. Self-gravitation pulls the tides down, mitigating their magnitude and the value of the tidal torque. Hence, the heavier the body the lower the appropriate curve. This rule is observed well at low frequencies (the viscosity-dominated range). In the intermediate zone and in the high-frequency band (where anelasticity dominates friction), this rule starts working only for bodies larger than about twice the Earth size. If we fix the tidal frequency at a sufficiently high value, we shall see that the increase of the size from that of Iapetus to that of Mars and further to that of the Earth results in an *increase* of the intensity of the tidal interaction. For a $R\ =\ 2\,R_{\bigoplus}$ superearth, the tidal factor $k_2\ \sin\epsilon_2$ is about the same as that for the solid Earth, and begins to decrease for larger radii (so the green curve for the larger superearth is located fully below the cyan curve for a smaller superearth).

6 Tidal dissipation versus seismic dissipation, in the anelasticity-dominated band

In this section, we shall address only the higher-frequency band of the spectrum, i.e., the range where anelasticity dominates viscoelasticity and the Andrade model is applicable safely. Mind though that the Andrade model can embrace also the near-Maxwell behaviour, and thus can be applied to the low frequencies, provided we "tune" the dimensionless parameter ζ appropriately – see subsection 3.4 above.

[14] One can denote (64) with k_l/Q only if the quality factor is defined through (26) and endowed with the subscript l .

Table 1: Estimates of $A_2^{(static)}$ for rigid celestial bodies. The values of $A_2^{(static)}$ are calculated using equation (58) and are rounded to the second figure.

	radius R	mean density ρ	mean relaxed shear rigidity $\mu(\infty)$	the resulting estimate for A_2
Iapetus	7.4×10^5 m	1.1×10^3 kg/m^3	4.0×10^9 Pa	200
Mars	3.4×10^6 m	3.9×10^3 kg/m^3	1.0×10^{11} Pa	19
The Earth	6.4×10^6 m	5.5×10^3 kg/m^3	0.8×10^{11} Pa	2.2
A hypothetical superearth with $R = 2\,R_{\oplus}$ and the same rheology as the Earth	4.5×10^8 m	5.5×10^3 kg/m^3	0.8×10^{11} Pa	0.55
A hypothetical superearth with $R = 4\,R_{\oplus}$ and the same rheology as the Earth	9.0×10^8 m	5.5×10^3 kg/m^3	0.8×10^{11} Pa	0.14

6.1 Response of a sample of material

At frequencies higher than some threshold value χ_0 , dissipation in minerals is mainly due to anelasticity rather than to viscosity.[15] Hence at these frequencies ζ should be of order unity or smaller, as can be seen from (33b). This entails two consequences. First, the condition $\chi \gg \frac{1}{\zeta\,\tau_{_M}}$, i.e., $z \gg 1$ is obeyed reliably, for which reason the first term dominates the denominator in (38). Second, either the condition $z \gg 1$ is stronger than $z \gg \zeta^{\frac{1}{1-\alpha}}$ or the two conditions are about equivalent. Hence the anelastic term dominates the numerator in (38): $z^{-\alpha} \gg z^{-1}\,\zeta$.

Altogether, over the said frequency range, (38) simplifies to:

$$\tan\delta \;\approx\; (\chi\,\tau_{_A})^{-\alpha}\,\sin\left(\frac{\alpha\,\pi}{2}\right)\;\Gamma(\alpha+1)\;=\;(\chi\,\zeta\,\tau_{_M})^{-\alpha}\,\sin\left(\frac{\alpha\,\pi}{2}\right)\;\Gamma(\alpha+1)\quad . \tag{66}$$

Clearly, $\tan\delta \ll 1$, wherefore $\tan\delta \approx \sin\delta \approx \delta$. For the seismic quality factor, we then have:

$$^{(seismic)}Q^{-1} \;\approx\; (\chi\,\tau_{_A})^{-\alpha}\,\sin\left(\frac{\alpha\,\pi}{2}\right)\;\Gamma(\alpha+1)\quad , \tag{67}$$

no matter which of the three definitions (21 - 23) we accept. Be mindful, that here we use the term *seismic* broadly, applying it also to a sample in a lab.

6.2 Tidal response of a homogeneous near-spherical body

Recall that defect unpinning stays effective at frequencies above some threshold χ_0 , which is likely to be above or, at least, not much lower than the inverse Maxwell time.[16] E.g., for the solid Earth, $\chi_0 \sim 1\,\mathrm{yr}^{-1}$ while $\tau_{_M} \sim 500\,\mathrm{yr}$. Over this frequency band, the free parameter ζ may be of order unity or slightly less than that. (This parameter grows as the frequencies become short of χ_0 .) Under these circumstances, in equation (63) we have: $\zeta z^{-1} \ll z^{-\alpha} \ll 1$, whence equation (63) becomes:

$$\tan\epsilon_l \;\approx\; \frac{A_l}{1\,+\,A_l}\;z^{-\alpha}\;\sin\left(\frac{\alpha\pi}{2}\right)\;\Gamma(1+\alpha)\quad . \tag{68}$$

In combination with (66), this renders:

$$\tan\epsilon_l \;=\; \frac{A_l}{1\,+\,A_l}\;\tan\delta\quad . \tag{69}$$

Had we defined the quality factors as cotangents, like in (23), then we would have to conclude from (69) that the tidal and seismic quality factors coincide for small objects (with $A_l \gg 1$)

[15] For the solid Earth, this threshold is about 1 yr^{-1} (Karato & Spetzler 1990). Being temperature sensitive, the threshold may assume different values for other terrestrial planets.

Also mind that the transition is not sharp and can extend over a decade or more.

[16] Dislocations may break away from the pinning agents (impurities, nodes, or jogs), or the pinning agents themselves may move along with dislocations. These two processes are called "unpinning", and they go easier at low frequencies, as the energy barriers become lower (Karato & Spetzler 1990, section 5.2.3).

and differ very considerably for superearths (i.e., for $A_l \ll 1$). Specifically, the so-defined quality factor Q_l of a superearth would be larger than its seismic counterpart Q by a factor of about A_l^{-1}.

In reality, the quality factors should be used for illustrative purposes only, because practical calculations involve the factor $|\bar{k}_l(\chi_{lmpq})|\,\sin\epsilon_l(\chi_{lmpq})$ rendered by (65). It is this factor which enters the $lmpq$ term of the Fourier expansion of tides. Insertion of (36 - 37) into (65) furnishes the following expression valid in the anelasticity-dominated band:

$$|\bar{k}_l(\chi_{lmpq})|\,\sin\epsilon_l(\chi_{lmpq}) \approx$$

$$\frac{3}{2\,(l-1)}\;\frac{A_l}{(A_l+1)^2}\;\sin\left(\frac{\alpha\pi}{2}\right)\;\Gamma(\alpha+1)\;\zeta^{-\alpha}\;\left(\tau_{_M}\;\chi_{lmpq}\right)^{-\alpha}\;\mathrm{sgn}\,\omega_{lmpq}\quad,\quad\mathrm{for}\quad\chi_{lmpq}\gg\chi_{_{HI}}\quad, \tag{70}$$

$\chi_{_{HI}}$ being the boundary between the high and intermediate frequencies, i.e., between the anelasticity-dominated band and the transitional zone. Expression (70) resembles the frequency dependency (36) for $|\bar{J}(\chi)|\,\sin\delta(\chi) = -\mathcal{I}m[\bar{J}(\chi)]$ at high frequencies. In Figure 2, dependency (70) corresponds the slowly descending slope on the right.

A detailed derivation of (70) from formulae (36 - 37) and (65) is presented in the Appendix to Efroimsky (2011). For terrestrial objects several times smaller than the Earth (so $A_l \gg 1$), the threshold turns out to be

$$\chi_{_{HI}} = \tau_{_M}^{-1}\,\zeta^{\frac{\alpha}{1-\alpha}} \quad . \tag{71}$$

For superearths (i.e., for $A_l \ll 1$), the threshold becomes

$$\chi_{_{HI}} = \tau_{_A}^{-1} = \tau_{_M}^{-1}\,\zeta^{-1} \quad . \tag{72}$$

Near the borderline between the anelasticity-dominated band and the transitional zone, the parameter ζ could be of order unity. It may as well be lower than unity, though not much (hardly by an order of magnitude), because too low a value of ζ would exclude viscosity from the play completely. We however expect viscosity to be noticeable near the transitional zone.

Finally, it should be reiterated that at frequencies lower than some χ_0 the defect-unpinning process becomes less effective, so anelasticity becomes less effective than viscosity, and the free parameter ζ begins to grow. Hence, if the above estimates for $\chi_{_{HI}}$ turn out to be lower than χ_0, we should set $\chi_{_{HI}} = \chi_0$ "by hand".

7 Tidal dissipation versus seismic dissipation, in the viscosity-dominated band

When frequency χ becomes short of some χ_0, the rate of defect-unpinning-caused anelastic dissipation decreases and viscosity begins to take over anelasticity.

If we simply assume the free parameter ζ to be of order unity everywhere, i.e., assume that the Maxwell and Andrade timescales are everywhere comparable, then application of the Andrade model will set χ_0 to be of order $\tau_{_M}^{-1}$. Anelasticity will dominate at frequencies above that threshold, while below it the role of viscosity will be higher. This approach however would be simplistic, because the actual location of the threshold should be derived from microphysics and may turn out to differ from $\tau_{_M}^{-1}$ noticeably. For example, in the terrestrial mantle the transition takes place at frequencies as high as 1 yr^{-1} (Karato & Spetzler 1990) and may be spread over a decade or more into lower frequencies, as we shall see from equation (73).

Another somewhat simplistic option would be to assume that $\zeta \sim 1$ at frequencies above χ_0, and to set $\zeta = \infty$ at the frequencies below χ_0. The latter would be equivalent to claiming that below this threshold the mantle is described by the Maxwell model. In reality, here we are just entering a transition zone, where ζ increases with the decrease of the frequency. While it is clear that in the denominator of (38) the first term dominates, the situation with the numerator is less certain. Only after the condition

$$\zeta \gg (\chi\,\tau_{_M})^{\frac{1-\alpha}{\alpha}} \approx (\chi\,\tau_{_M})^4 \tag{73}$$

is obeyed, the viscous term $1/(\chi\,\tau_{_M})$ becomes leading. This way, although ζ begins to grow as the frequency decreases below χ_0, the frequency may need to decrease by another decade or more before threshold (73) is reached.

7.1 Response of a sample of material

Accepting the approximation that the transition zone is narrow [17] and that the predominantly viscous regime is reached already at χ_0 or shortly below, we approximate the tangent of the lag with

$$\tan\delta \approx (\chi\,\tau_{_M})^{-1} \quad , \tag{74}$$

whence

$$\sin\delta \approx \begin{cases} (\chi\,\tau_{_M})^{-1} & \text{for} \quad \tau_{_M}^{-1} \ll \chi \ll \chi_0 \quad , \\ 1 & \text{for} \quad 0 \leq \chi \ll \tau_{_M}^{-1} \quad . \end{cases} \tag{75}$$

7.2 Tidal response of a homogeneous near-spherical body

When viscosity dominates anelasticity, expression (63) gets reduced to the following form:

$$\tan\epsilon_l \approx \frac{A_l}{1 + A_l + \left(\zeta\, z^{-1}\right)^2}\;\zeta\, z^{-1} = \frac{A_l}{1 + A_l + \left(\chi\,\tau_{_M}\right)^{-2}}\;\frac{1}{\chi\,\tau_{_M}} \quad , \tag{76}$$

comparison whereof with (74) renders:

$$\tan\epsilon_l \approx \frac{A_l}{1 + A_l + \left(\chi\,\tau_{_M}\right)^{-2}}\;\tan\delta = \frac{A_l}{1 + A_l + \tan^2\delta}\;\tan\delta \quad . \tag{77}$$

Now two special cases should be considered separately.

[17] For a broader transition zone, the rheology will approach that of Maxwell at lower frequencies. This though will not influence our main conclusions.

7.2.1 Small bodies and small terrestrial planets

As illustrated by Table 1, small bodies and small terrestrial planets have $A_l \gg 1$. So formulae (76) and (77) take the form of

$$\tan\epsilon_l \;\approx\; \begin{cases} \dfrac{1}{\chi\,\tau_{_M}} & \text{for} \quad \dfrac{1}{\tau_{_M}\,\sqrt{A_l+1}} \ll \chi \ll \chi_0 \quad , \\ \\ A_l\,\chi\,\tau_{_M} & \text{for} \quad 0 \leq \chi \ll \dfrac{1}{\tau_{_M}\,\sqrt{A_l+1}} \;\approx\; \dfrac{1}{\tau_{_M}\,\sqrt{A_l}} \quad , \end{cases} \tag{78}$$

and

$$\tan\epsilon_l \;\approx\; \begin{cases} \dfrac{A_l}{1\,+\,A_l}\,\tan\delta \;\approx\; \tan\delta & \text{for} \quad \dfrac{1}{\tau_{_M}\,\sqrt{A_l+1}} \ll \chi \ll \chi_0 \quad , \\ \\ \dfrac{A_l}{\tan\delta} & \text{for} \quad 0 \leq \chi \ll \dfrac{1}{\tau_{_M}\,\sqrt{A_l+1}} \;\approx\; \dfrac{1}{\tau_{_M}\,\sqrt{A_l}} \quad . \end{cases} \tag{79}$$

Had we defined the quality factors as cotangents of ϵ_l and δ , we would be faced with a situation that may at first glance appear embarrassing: in the zero-frequency limit, the so-defined tidal Q_l would becomes *inversely* proportional to the so-defined seismic Q factor. This would however correspond well to an obvious physical fact: when the satellite crosses the *lmpq* commensurability, the *lmpq* term of the tidal torque acting on a satellite must smoothly pass through nil, together with the *lmpq* tidal mode. (For example, the principal tidal torque $lmpq\,=\,2200$ must vanish when the satellite crosses the synchronous orbit.) For a more accurate explanation in terms of the $|\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi)$ factors see subsection 7.3 below.

7.2.2 Superearths

For superearths, we have $A_l \ll 1$, so (77) becomes

$$\tan\epsilon_l \;\approx\; A_l\,\chi\,\tau_{_M} \;=\; \frac{A_l}{\tan\delta} \qquad \text{for} \qquad 0 \leq \chi \ll \frac{1}{\tau_{_M}\,\sqrt{A_l+1}} \;\approx\; \frac{1}{\tau_{_M}} \quad . \tag{80}$$

Here we encounter the same apparent paradox: had we defined the quality factors as cotangents of ϵ_l and δ , we would end up with a tidal Q_l inversely proportional to its seismic counterpart Q . A qualitative explanation to this "paradox" is the same as in the subsection above, a more accurate elucidation to be given shortly in subsection 7.3 .

Another seemingly strange feature is that in this case (i.e., for $A_l \ll 1$) the tangent of the tidal lag skips the range of inverse-frequency behaviour and becomes linear in the frequency right below the inverse Maxwell time. This however should not surprise us, because the physically meaningful products $k_l\,\sin\epsilon_l$ still retain a short range over which they demonstrate the inverse-frequency behaviour. This can be understood from Figure 2. There, on the yellow line corresponding to a superearth, a short segment to the right of the maximum corresponds to the situation when $k_l\,\sin\epsilon_l$ scales as inverse frequency – see formula (81) below.

Thus we once again see that the illustrative capacity of the quality factor is limited. To spare ourselves of surprises and "paradoxes", we should always keep in mind that the actual calculations are based on the frequency dependence of $|\bar{k}_l(\chi_{lmpq})|\,\sin\epsilon_l(\chi_{lmpq})$.

7.3 Tidal response in terms of $|\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi)$

Combining (36 - 37) with (65), one can demonstrate that in the intermediate-frequency zone the tidal factors scale as

$$|\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi)\,\approx\,\frac{3}{2\,(l-1)}\,\frac{A_l}{(A_l+1)^2}\,\,(\,\tau_{_M}\,\chi\,)^{-1}\quad,\qquad\text{for}\qquad\tau_{_M}^{-1}\gg\chi\gg\tau_{_M}^{-1}\,(A_l+1)^{-1}\quad,\tag{81}$$

which corresponds to the short segment on the right of the maximum on Figure 2.

From the same formulae (36 - 37) and (65), it ensues that the low-frequency behaviour looks as

$$|\bar{k}_l(\chi)|\,\,\sin\epsilon_l(\chi)\,\approx\,\frac{3}{2\,(l-1)}\,A_l\,\,\tau_{_M}\,\chi\qquad,\qquad\qquad\text{for}\qquad\tau_{_M}^{-1}\,(A_l+1)^{-1}\,\gg\,\chi\quad,\tag{82}$$

a regime illustrated by the slope located on the left of the maximum on Figure 2.

Details of derivation of (81) and (82) can be found in the Appendix to Efroimsky (2011).

Just as expression (70) resembled the frequency dependency (36) for $|\bar{J}(\chi)|\,\sin\delta(\chi)$ at high frequencies, so (81) resembles the behaviour of $|\bar{J}(\chi)|\,\sin\delta(\chi)$ at low frequencies. At the same time, (82) demonstrates a feature inherent only in tides, and not in the behaviour of a sample of material: at $\chi<\tau_{_M}^{-1}(A_l+1)^{-1}\,=\,\frac{\mu}{\eta}\,(A_l+1)^{-1}$, the factor $|\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi)$ becomes linear in χ. This is not surprising, as the *lmpq* component of the average tidal torque or force must pass smoothly through zero and change its sign when the *lmpq* commensurability is crossed (and the *lmpq* tidal mode goes through zero and changes sign).

8 Why the *lmpq* component of the tidal torque does not scale as R^{2l+1}

A Fourier component $\mathcal{T}_{lmpq}$ of the tidal torque acting on a perturbed primary is proportional to $R^{2l+1}\,k_l\,\sin\epsilon_l$, where R is the primary's mean equatorial radius. Neglect of the $R-$dependence of the tidal factors $k_l\,\sin\epsilon_l$ has long been source of misunderstanding on how the torque scales with the radius.

From formulae (70) and (81), we see that everywhere except in the closest vicinity of the resonance the tidal factors are proportional to $A_l/(1+A_l)^2$ where $A_l\sim R^{-2}$ according to (60). Thence the overall dependence of the tidal torque upon the radius becomes:

Over the frequency band $\chi\,\gg\,\tau_{_M}^{-1}\,(1\,+\,A_l)^{-1}$,

$$\mathcal{T}_{lmpq}\sim R^{2l+1}k_l\,\sin\epsilon_l\sim\frac{R^{2l+1}\,A_l}{(1+A_l)^2}\sim\left\{\begin{array}{l}R^{2l-1}\;,\quad\text{for}\;\;A_l\ll 1\;\;\text{(superearths)}\;,\\ \\ R^{2l+3}\;,\quad\text{for}\;\;A_l\gg 1\;\;\text{(small bodies,}\;\;\text{small terrestrial planets)}\;.\end{array}\right.\tag{83}$$

In the closest vicinity of the *lmpq* commensurability, i.e., when the tidal frequency χ_{lmpq} approaches zero, the tidal factors's behaviour is described by (82). This furnishes a different scaling law for the torque, and the form of this law is the same for telluric bodies of all sizes:

Over the frequency band $\chi \ll \tau_{_M}^{-1}\,(1\,+\,A_l)^{-1}$,

$$\mathcal{T}_{lmpq} \sim R^{2l+1} k_l\,\sin\epsilon_l \sim R^{2l+1}\,A_l \sim\,R^{2l-1}\quad . \tag{84}$$

9 Conclusions and examples

Within the anelasticity-dominated band, the phase lags in a homogeneous near-spherical body and in a sample of material interrelate as

$$\tan\epsilon_l\;=\;\frac{A_l}{1\,+\,A_l}\;\tan\delta\;\approx\;\begin{cases} A_l\;\tan\delta & \text{for}\quad A_l\ll 1\;\;\text{(superearths)}\;\;,\\[2ex] \tan\delta & \text{for}\quad A_l\gg 1\;\;\text{(small bodies,\;\;small terrestrial planets)}\;\;.\end{cases} \tag{85}$$

However within the transitional zone, the link between the seismic and tidal dissipation rates becomes more complicated.

The interrelation between the tidal and seismic damping becomes apparently paradoxical at low frequencies, where viscosity dominates. As can be seen from (78 - 80), in the zero-frequency limit the tidal and seismic $Q\,s$ (if defined as cotangents of the appropriate lags) become *inversely* proportional to one another:

$$\tan\epsilon_l\;\;\approx\;A_l\,\chi\,\tau_{_M}\;=\;\frac{A_l}{\tan\delta}\qquad\qquad\text{for}\qquad 0\leq\chi\ll\frac{1}{\tau_{_M}\,\sqrt{A_l+1}}\quad . \tag{86}$$

This behaviour however has a good qualitative explanation – the average tidal torque $lmpq$ should vanish on crossing of the $lmpq$ resonance.

While in qualitative discussions it is easier to deal with the quality factors Q_l , in practical calculations we should rely on the factors $k_l\,\sin\epsilon_l$, which show up in the Darwin-Kaula expansion of tides. Just as $\tan\epsilon_l$, so the quantity $k_l\,\sin\epsilon_l$ too becomes linear in χ for low values of χ . As we saw in subsection 7.3, this happens over the frequencies below $\chi\ll\dfrac{1}{\tau_{_M}\,(A_l+1)}$. The slight difference between this threshold and the one shown in (86) stems from the fact that not only the lag but also the Love number is frequency dependent.

The factors $k_l\,\sin\epsilon_l$ bear dependence upon the radius R of a tidally disturbed primary, and the form of this dependence is not always trivial. At low frequencies, this dependence follows the intuitively obvious rule that the heavier the body the stronger it mitigates tides (and thence the smaller the value of $k_l\,\sin\epsilon_l$). However at high frequencies the calculated frequency dependence obeys this rule only beginning from sizes about or larger than the double size of the Earth, i.e., when self-gravitation clearly plays a larger role in tidal friction than the rheology does – see the discussion at the end of subsection 5.4.

The dependence of $k_l\,\sin\epsilon_l$ upon R helps one to write down the overall R-dependence of the tidal torque. Contrary to the common belief, the $lmpq$ component of the torque does *not* scale as R^{2l+1} , see formulae (83) and (84).

Here follow some examples illustrating how our machinery applies to various celestial bodies.

1. For small bodies and small terrestrial planets, the effect of self-gravitation is negligible, except in the closest vicinity of the zero frequency. Accordingly, for these bodies there is no difference between the tidal and seismic dissipations. [18]

Things change in the closest vicinity of the zero frequency. As can be observed from the second line of (78), for small bodies and small planets the tangent of the tidal lag becomes linear in the tidal frequency χ when the frequency χ becomes short of a certain threshold:[19] $\chi \ll \frac{1}{\tau_{_M}\sqrt{A_l+1}} \approx \frac{1}{\tau_{_M}\sqrt{A_l}}$. As can be seen from (82), the tidal factor $k_l\,\sin\epsilon_l \equiv |\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi)$ becomes linear in χ for $\chi \ll \tau_{_M}^{-1}\,(A_l+1)^{-1} \approx \tau_{_M}^{-1}\,A_l^{-1}$.

2. Tidal dissipation in superearths is much less efficient than in smaller terrestrial planets or moons – a circumstance that should reduce considerably the rates of orbit circularisation. This cautionary point has ramifications also upon the other tidal-dynamic timescales (e.g., despinning, migration).

In simple words, self-gravity reduces tidal dissipation because gravitational attraction pulls the tidal bulge back down, and thus reduces strain in a way similar to material strength.

As can be seen from (80), at tidal frequencies χ lower than the inverse Maxwell time,[20] tangent of the tidal lag changes its behaviour considerably, thereby avoiding divergence at the zero frequency. According to (82), the same pertains to the factor $k_l\,\sin\epsilon_l$.

3. While the role of self-gravity is negligible for small planets and is dominant for superearths, the case of the Earth is intermediate. For our mother planet, the contribution of self-gravitation into the Love numbers and phase lags is noticeable, though probably not leading. Indeed, for $\mu \approx 0.8\times 10^{11}$ Pa, one arrives at:

$$A_2 \;\approx\; 2.2 \quad , \qquad (87)$$

so formula (69) tells us that the Earth's tidal quality factor is a bit larger than its

[18] This can be understood also through the following line of reasoning. For small objects, we have $A_l \gg 1$; so the complex Love numbers (59b) may be approximated with

$$\bar{k}_l(\chi) \;=\; -\,\frac{3}{2}\,\frac{\bar{J}\,(\chi)}{\bar{J}\,(\chi)\;+\;A_l\,J} \;=\; -\,\frac{3}{2}\,\frac{\bar{J}\,(\chi)}{A_l\,J}\;+\;O\left(\,|\,\bar{J}/(A_l\,J)\,|^2\,\right) \quad .$$

The latter entails:

$$\tan\epsilon_l(\chi) \;\equiv\; -\,\frac{\mathcal{I}m\left[\bar{k}_l(\chi)\right]}{\mathcal{R}e\left[\bar{k}_l(\chi)\right]} \;\approx\; -\,\frac{\mathcal{I}m\left[\bar{J}(\chi)\right]}{\mathcal{R}e\left[\bar{J}(\chi)\right]} \;=\; \tan\delta(\chi) \quad ,$$

which is, in fact, correct *up to a sign* – see the closing paragraph of subsection 5.1.

[19] Recall that for small objects $A_l \gg 1$.

[20] For superearths, $A_l \ll 1$.

seismic counterpart, *taken at the same frequency:*[21]

$$ {}^{(tidal)}Q_2^{(solid\ Earth)} \approx 1.5 \times {}^{(seismic)}Q^{(solid\ Earth)} \quad . \tag{88}$$

The geodetic measurements of semidiurnal tides, carried out by Ray et al. (2001), yield ${}^{(tidal)}Q_2^{(solid\ Earth)} \approx 280$. The seismic quality factor ${}^{(seismic)}Q^{(solid\ Earth)}$ varies over the mantle, assuming values from 100 through 300. Accepting 200 for an arguable average, we see that (88) furnishes a satisfactory qualitative estimate.

This close hit should not of course be accepted too literally, taken the Earth's complex structure and the uncertainty in our knowledge of the Earth's rigidity. Still, on a qualitative level, we may enjoy this proximity with cautious optimism.

4. The case of the Moon deserves a special attention. Fitting of the LLR data to the power scaling law $Q \sim \chi^p$ has rendered a small *negative* value of the exponential: $p = -0.19$ (Williams et al. 2001). Further attempts by the JPL team to reprocess the data have led to $p = -0.07$. According to Williams & Boggs (2008),

"*There is a weak dependence of tidal specific dissipation Q on period. The Q increases from ~ 30 at a month to ~ 35 at one year. Q for rock is expected to have a weak dependence on tidal period, but it is expected to decrease with period rather than increase. The frequency dependence of Q deserves further attention and should be improved.*"

To understand the origin of the small negative value of the power, recall that it emerged through fitting of the tidal Q_2 and not of the seismic Q. If the future laser ranging confirms these data, this will mean that the principal tide in the Moon is located close to the maximum of the inverse tidal quality factor, i.e., close to the maximum taken by $\tan\epsilon_2$ in (78) at the frequency inverse to $\tau_{_M}\sqrt{A_l}$. Rigorously speaking, it was of course the factor $k_2 \sin\epsilon_2$ which was actually observed. The maximum of this factor is attained at the frequency $\tau_{_M}^{-1}(A_l + 1)^{-1}$, as can be seen from (81 - 82). It then follows from the LLR data that the corresponding timescale $\tau_{_M}(A_l + 1)$ should be of order 0.1 year. As explained in Efroimsky (2011), this would set the mean viscosity of the Moon as low as

$$\eta_{Moon} = 3 \times 10^{16} \text{ Pa s} \quad , \tag{89}$$

which in its turn would imply a very high concentration of the partial melt in the low mantle – quite in accordance with the existing models (Nakamura et al. 1974, Weber et al. 2011).

The future LLR programs may be instrumental in resolving this difficult issue. The value of the exponential p will have ramifications for the current models of the lunar mantle.

[21] When Benjamin et al. (2006) say that, according to their data, the tidal quality factor is slightly lower than the seismic one, these authors compare the two Q factors measured at different frequencies. Hence their statement is in no contradiction to our conclusions.

10 Comparison of our result with that by Goldreich (1963)

A formula coinciding with our (69) was obtained, through remarkably economic and elegant semi-qualitative reasoning, by Peter Goldreich (1963).

The starting point in *Ibid.* was the observation that the peak work performed by the second-harmonic disturbing potential should be proportional to this potential taken at the primary's surface, multiplied by the maximal surface inequality:

$$E_{peak} \sim R^3 \frac{R^3}{\frac{19\,\mu}{2\,\mathrm{g}\,\rho\,R} + 1} \sim \frac{R^7}{19\,\mu + 2\,\mathrm{g}\,\rho\,R} \quad , \tag{90}$$

R being the primary's radius.

In the static theory of Love, the surface strain is proportional to $R^2/(\,19\,\mu + 2\,\mathrm{g}\,\rho\,R\,)$. The energy loss over a cycle must be proportional to the square of the surface strain. Integration over the volume will give an extra multiplier of R^3, up to a numerical factor:

$$\Delta E_{cycle} \sim -\frac{R^7}{(\,19\,\mu + 2\,\mathrm{g}\,\rho\,R\,)^2} \quad . \tag{91}$$

Comparison of (90) and (91) rendered

$$Q = -\frac{2\pi\;E_{peak}}{\Delta E_{cycle}} \sim (\,19\,\mu + 2\,\mathrm{g}\,\rho\,R\,) \quad ,$$

wherefrom Goldreich (1963) deduced that

$$\frac{Q}{Q_0} = 1 + \frac{2\,\mathrm{g}\,\rho\,R}{19\,\mu} \quad ,$$

Q_0 being the value of Q for a body where self-gravitation is negligible. This coincides with our formula (69).

In reality, the coincidence of our results is only partial, for two reasons:

- First, our derivation of the right-hand side of (63) was based on the prior convention that the quantity J entering expression (60) be the *unrelaxed* compliance $J(0)$ of the mantle. Accordingly, the quantity $\mu = 1/J$ entering the expression for A_l should be the *unrelaxed* rigidity $\mu(0) = 1/J(0)$. In Goldreich (1963) however, the static, i.e., *relaxed* moduli were implied.

  In *Ibid.*, this mismatch was tolerable, because the paper was devoted to small bodies. For these objects, A_l is large, no matter whether we plug the relaxed or unrelaxed μ into (56). Thence the difference between the tidal and seismic Q factors is small, as can be seen from the second line of (85).

  For earths and superearths, however, the distinction between the unrelaxed and relaxed (static) moduli is critical. As can be seen from the first line of (85), the tidal Q factor is inversely proportional to A_l and, thereby, is inversely proportional to the mantle rigidity μ. As well known (e.g., Ricard et al. 2009, Figure 3), the unrelaxed μ of the mantle exceeds the relaxed μ by about two orders of magnitude.

- Second, as our calculation demonstrates, the simple interrelation given by (69) and (85) works *only in the anelasticity-dominated band*. In the transition zone (which begins, in the solid Earth, at timescales longer than ~ 1 yr) and in the viscosity-dominated band of lower frequencies, the interrelation between the tidal and seismic lagging is more complicated, and it deviates from Goldreich's formula in a fundamental way. Specifically, in the zero-frequency limit the cleavage between the tidal and seismic dissipation laws gets even larger: the tidal and seismic Q s become not proportional but *inversely* proportional to one another. Description of tidal lagging in all these, low-frequency bands requires a rheological model and the subsequent mathematics, and cannot be obtained through the simple arguments used by Goldreich (1963).

Despite these differences, the estimate by Goldreich (1963) provided as close a hit as was possible without resorting to heavy mathematics. The elegance of Peter Goldreich's arguments and the depth of his insight are especially impressive, taken the complexity of the problem and the volume of calculations required to obtain the exact answer.

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I also wish to pay tribute to the late Vladimir Churkin, whose tragic death prevented him from publishing his preprint cited in this paper. Written with a great pedagogical mastership, the preprint helped me to understand how the Love-number formalism should be combined with rheology.

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Appendix

Symbol Key

A_l	Dimensionless product emerging in the denominator of the expression for the Love number k_l
E	Energy
$\mathcal{E}$	Empirical constant having the dimensions of time, in the generic rheological law (27)
g	Surface gravity
G	Newton's gravitational constant
l	Degree (spherical harmonics, Legendre polynomials)
m	Order (spherical harmonics, associated Legendre polynomials)
$J,\ J(0)$	Unrelaxed compliance
$J(\infty)$	Relaxed compliance
$J(t\,-\,t\,')$	Creep-response function (compliance function, kernel of the compliance operator)
$\hat{J}$	Compliance operator
k_l	Tidal Love number of degree l
$k_l(t-t\,')$	Kernel of the Love operator of degree l
$\hat{k}_l$	The Love operator of degree l
$\bar{k}_l(\chi)$	Fourier component, at frequency χ, of the time derivative of the kernel $k_l(t-t\,')$
$\mathcal{M}$	Mean anomaly
n	Mean motion
p	Exponential in the generic rheological law (27)
P_l	Legendre polynomials of degree l
P_{lm}	Legendre associated functions (associated Legendre polynomials) of degree l and order m
Q	Dissipation quality Factor
r	Distance
$\vec{\boldsymbol{r}}$	Vector connecting the centre of the tidally-perturbed body (interpreted as the primary) with a point exterior to this body
$\vec{\boldsymbol{r}}^{\,*}$	Vector connecting the centre of the tidally-perturbed body (the primary) with a point-like tide-raising secondary
R	Primary's mean radius
t	Time
$u_{\gamma\nu}$	Shear strain tensor
$\bar{u}_{\gamma\nu}(\chi)$	Fourier component, at frequency χ, of the shear strain tensor
U	Change in the potential of the tidally-perturbed body (interpreted as the primary)
W	Disturbing potential generated by the tide-raising body (interpreted as the secondary)
α, β	Parameters of the Andrade model
γ, ν	Tensor indices
Γ	the Gamma function
δ	Material phase lag
Δt	Time lag
ϵ	Tidal phase lag
ϵ_{lmpq}	Tidal phase lag of the mode $lmpq$ in the Darwin-Kaula expansion
λ	Longitude
ζ	Parameter of the reformulated Andrade model (ratio of the anelastic timescale $\tau_{_A}$ to the Maxwell time $\tau_{_M}$)
η	Viscosity
$\mu,\ \mu(0)$	Unrelaxed shear modulus (unrelaxed rigidity)
$\mu(\infty)$	Relaxed shear modulus (relaxed rigidity)
$\mu(t-t\,')$	Stress-relaxation function (kernel of the rigidity operator)
$\hat{\mu}$	Rigidity operator
ϕ	Latitude
ρ	Mass density
$\sigma_{\gamma\nu}$	Shear stress tensor
$\bar{\sigma}_{\gamma\nu}(\chi)$	Fourier component, at frequency χ, of the shear stress tensor
τ	Time
$\tau_{_M}$	Maxwell time (viscoelastic timescale)
$\tau_{_A}$	Andrade time (anelastic timescale)
$\mathcal{T}$	Tidal torque
$\Theta(t-t\,')$	the Heaviside function
θ	Sidereal angle of the tidally-disturbed body (interpreted as the the primary)
$\dot{\theta}$	Spin rate of the primary
$\varphi_\sigma\ ,\ \varphi_u$	Initial phases of the stress and strain
χ	Frequency
χ_0	Frequency threshold marking the boundary between the anelasticity- and viscosity-dominated frequency bands
χ_{lmpq}	Physical frequencies of deformation emerging in the tidal theory (absolute values of the tidal modes ω_{lmpq})
ω_{lmpq}	Tidal modes in the Darwin-Kaula expansion of tides
ω	Argument of the pericentre
Ω	Longitude of the node

References

[1] Andrade, E. N. da C. 1910. "On the Viscous Flow in Metals, and Allied Phenomena." *Proceedings of the Royal Society of London. Series A.* Vol. **84**, pp. 1 - 12

[2] Benjamin, D.; Wahr, J. ; Ray, R. D.; Egbert, G. D.; and Desai, S. D. 2006. "Constraints on mantle anelasticity from geodetic observations, and implications for the J_2 anomaly." *Geophysical Journal International*, Vol. **165**, pp. 3 - 16

[3] Birger, B. I. 2007. "Attenuation of Seismic Waves and the Universal Rheological Model of the Earth's Mantle." *Izvestiya. Physics of the Solid Earth.* Vol. **49**, pp. 635 - 641

[4] Carter, J. A.; Winn, J. N.; Holman, M. J.; Fabrycky, D.; Berta, Z. K.; Burke, C. J.; and Nutzman, P. 2011. "The Transit Light Curve Project. XIII. Sixteen Transits of the Super-Earth GJ 1214b." *The Astrophysical Journal*, Vol. **730**, p. 82

[5] Castillo-Rogez, J. 2009. "New Approach to Icy Satellite Tidal Response Modeling." American Astronomical Society, DPS meeting 41, Abstract 61.07.

[6] Castillo-Rogez, J. C.; Efroimsky, M., and Lainey, V. 2011. "The tidal history of Iapetus. Dissipative spin dynamics in the light of a refined geophysical model". *Journal of Geophysical Research – Planets*, Vol. **116**, p. E09008
doi:10.1029/2010JE003664

[7] Churkin, V. A. 1998. "The Love numbers for the models of inelastic Earth." Preprint No 121. Institute of Applied Astronomy. St.Petersburg, Russia. /in Russian/

[8] Cottrell, A. H., and Aytekin, V. 1947. "Andrade's creep law and the flow of zinc crystalls." *Nature*, Vol. **160**, pp. 328 - 329

[9] Dehant V. 1987a. "Tidal parameters for an inelastic Earth." *Physics of the Earth and Planetary Interiors*, Vol. **49**, pp. 97 - 116

[10] Dehant V. 1987b. "Integration of the gravitational motion equations for an elliptical uniformly rotating Earth with an inelastic mantle." *Physics of the Earth and Planetary Interiors*, Vol. **49**, pp. 242 - 258

[11] Duval, P. 1978. "Anelastic behaviour of polycrystalline ice." *Journal of Glaciology*, Vol. **21**, pp. 621 - 628

[12] Eanes, R. J. 1995. *A study of temporal variations in Earth's gravitational field using LAGEOS-1 laser ranging observations*. PhD thesis, University of Texas at Austin

[13] Eanes, R. J., and Bettadpur, S. V. 1996. "Temporal variability of Earth's gravitational field from laser ranging." In: Rapp, R. H., Cazenave, A. A., and Nerem, R. S. (Eds.) *Global gravity field and its variations. Proceedings of the International Association of Geodesy Symposium No 116 held in Boulder CO in July 1995.* IAG Symposium Series,

Vol. **116**, pp. 30 - 41. Springer, NY 1997
ISBN: 978-3-540-60882-0

[14] Efroimsky, M., and V. Lainey. 2007. "The Physics of Bodily Tides in Terrestrial Planets, and the Appropriate Scales of Dynamical Evolution." *Journal of Geophysical Research – Planets*, Vol. **112**, p. E12003. doi:10.1029/2007JE002908

[15] Efroimsky, M., and Williams, J. G. 2009. "Tidal torques. A critical review of some techniques." *Celestial mechanics and Dynamical Astronomy,* Vol. **104**, pp. 257 - 289
arXiv:0803.3299

[16] Efroimsky, M. 2011. "Bodily tides near spin-orbit resonances." *Celestial mechanics and Dynamical Astronomy,* in press. Extended version available at: arXiv:1105.6086

[17] Findley, W. N.; Lai, J. S.; & Onaran, K. 1976. *Creep and relaxation of nonlinear viscoelastic materials.* Dover Publications, 384 pp.

[18] Fontaine, F. R.; Ildefonse, B.; and Bagdassarov, N. 2005. "Temperature dependence of shear wave attenuation in partially molten gabbronorite at seismic frequencies." *Geophysical Journal International*, Vol. **163**, pp. 1025 - 1038

[19] Goldreich, P. 1963. "On the eccentricity of the satellite orbits in the Solar System." *Monthly Notices of the Royal Astronomical Society of London*, Vol. **126**, pp. 257 - 268

[20] Gribb, T.T., & Cooper, R.F. 1998. "Low-frequency shear attenuation in polycrystalline olivine: Grain boundary diffusion and the physical significance of the Andrade model for viscoelastic rheology." *Journal of Geophysical Research – Solid Earth*, Vol. **103**, pp. B27267 - B27279

[21] Henning, W. G.; O'Connell, R.; and Sasselov, D. 2009. "Tidally Heated Terrestrial Exoplanets: Viscoelastic Response Models." *The Astrophysical Journal*, Vol **707**, pp 1000-1015

[22] Johnson, C. L.; Solomon, S. C.; Head, J. W.; Phillips, R. J.; Smith, D. E.; and Zuber, M. T. 2000. "Lithospheric Loading by the Northern Polar Cap on Mars." *Icarus*, Vol. **144**, pp. 313 - 328

[23] Karato, S.-i. 2008. *Deformation of Earth Materials. An Introduction to the Rheology of Solid Earth*. Cambridge University Press, UK.

[24] Karato, S.-i., and Spetzler, H. A. 1990. "Defect Microdynamics in Minerals and Solid-State Mechanisms of Seismic Wave Attenuation and Velocity Dispersion in the Mantle." *Reviews of Geophysics*, Vol. **28**, pp. 399 - 423

[25] Kaula, W. M. 1964. "Tidal Dissipation by Solid Friction and the Resulting Orbital Evolution." *Reviews of Geophysics*, Vol. **2**, pp. 661 - 684

[26] Léger, A.; Rouan, D.; Schneider, J.; Barge, P.; Fridlund, M.; Samuel, B.; Ollivier, M.; Guenther, E.; Deleuil, M.; Deeg, H. J.; and 151 coauthors. 2009. "Transiting exoplanets from the CoRoT space mission. VIII. CoRoT-7b: the first super-Earth with measured radius." *Astronomy and Astrophysics*, Vol. **506** pp. 287 - 302

[27] McCarthy, C.; Goldsby, D. L.; and Cooper, R. F. 2007. "Transient and Steady-State Creep Responses of Ice-I/Magnesium Sulfate Hydrate Eutectic Aggregates." *38th Lunar and Planetary Science Conference XXXVIII*, held on 12 - 16 March 2007 in League City, TX. LPI Contribution No 1338, p. 2429

[28] Mitchell, B. 1995. "Anelastic structure and evolution of the continental crust and upper mantle from seismic surface wave attenuation." *Reviews of Geophysics*, Vol **33**, pp 441 - 462

[29] Nakamura, Y.; Latham, G.; Lammlein, D.; Ewing, M.; Duennebier, F.; and Dorman, J. 1974. "Deep lunar interior inferred from recent seismic data." *Geophysical Research Letters*, Vol. **1**, pp. 137 - 140

[30] Nimmo, F. 2008. "Tidal Dissipation and Faulting." Paper presented at the *Science of Solar System Ices Workshop*. Lunar and Planetary Institute, Oxnard CA

[31] Ray, R. D.; Eanes, R. J.; and Lemoine, F. G. 2001. "Constraints on energy dissipation in the Earth's body tide from satellite tracking and altimetry." *Geophysical Journal International,* Vol. **144**, pp. 471 - 480

[32] Ricard, Y.; Matas, J.; and Chambat, F. 2009. "Seismic attenuation in a phase change coexistence loop." *Physics of the Earth and Planetary Interiors.*, Vol. **176**, pp. 124 - 131

[33] Shito, A., Karato, S.-I.,, Park, J. 2004. "Frequency dependence of Q in Earth's upper mantle inferred from continuous spectra of body waves." *Geophysical Research Letters* 31, L12603, doi:10.1029/2004GL019582.

[34] Smith M. 1974. "The scalar equations of infinitesimal elastic-gravitational motion for a rotating, slightly elliptical Earth." *The Geophysical Journal of the Royal Astronomical Society*, Vol. **37**, pp. 491 - 526

[35] Stachnik, J. C.; Abers, G. A.; and Christensen, D. H. 2004. "Seismic attenuation and mantle wedge temperatures in the Alaska subduction zone." *Journal of Geophysical Research – Solid Earth*, Vol. **109**, No B10, p. B10304, doi:10.1029/2004JB003018

[36] Tan, B. H.; Jackson, I.; and Fitz Gerald J. D. 1997. "Shear wave dispersion and attenuation in fine-grained synthetic olivine aggregates: preliminary results." *Geophysical Research Letters*, Vol. **24**, No 9, pp. 1055 - 1058, doi:10.1029/97GL00860

[37] Tyler, R. H. 2009. "Ocean tides heat Enceladus." *Geophysical Research Letters*, Vol. **36**, p. L15205, doi:10.1029/2009GL038.300

[38] Weber, R. C.; Lin, Pei-Ying; Garnero, E.; Williams, Q.; and Lognonné, P. 2011. "Seismic Detection of the Lunar Core." *Science*, Vol. **331**, Issue 6015, pp. 309 - 312

[39] Weertman, J., and Weertman, J. R. 1975. "High Temperature Creep of Rock and Mantle Viscosity." *Annual Review of Earth and Planetary Sciences*, Vol. **3**, pp. 293 - 315

[40] Williams, J. G., Boggs, D. H., Yoder, C. F., Ratcliff, J. T., and Dickey, J. O. 2001. "Lunar rotational dissipation in solid-body and molten core." *The Journal of Geophysical Research – Planets*, Vol. **106**, No E11, pp. 27933 - 27968. doi:10.1029/2000JE001396

[41] Williams, J. G., and Boggs, D. H. 2008. "Lunar Core and Mantle. What Does LLR See?"
Proceedings of the 16th International Workshop on Laser Ranging held on 12-17 October 2008 in Poznan, Poland. Edited by S. Schilliak. Published by: Space Research Centre, Polish Academy of Sciences, Warsaw. Vol. **101**, pp. 101 - 120
http://cddis.gsfc.nasa.gov/lw16/docs/papers/sci_1_Williams_p.pdf
http://cddis.gsfc.nasa.gov/lw16/docs/papers/proceedings_vol2.pdf

[42] Zahn, J.-P. 1966. "Les marées dans une étoile double serrée." *Annales d'Astrophysique,* Vol. 29, pp. 313 - 330

[43] Zharkov, V.N., and Gudkova, T.V. 2009. "The period and Q of the Chandler wobble of Mars." *Planetary and Space Science*, Vol. **57**, pp. 288 - 295

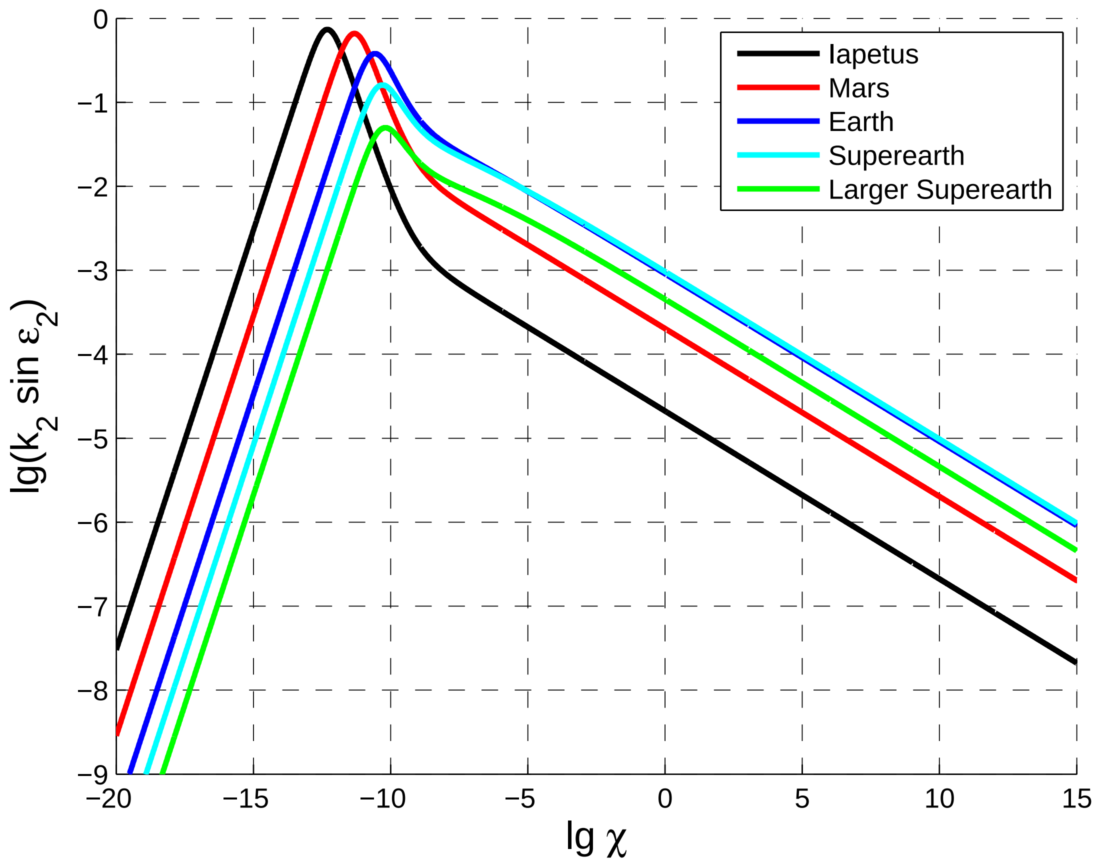

Figure 2: Negative imaginary part of the complex quadrupole Love number, $k_2 \sin\epsilon_2 = -\mathcal{I}m\left[\bar{k}_2(\chi)\right]$, as a function of the tidal frequency χ. The black, red, and blue curves refer, correspondingly, to Iapetus, Mars, and the solid Earth. The cyan and green curves refer to the two hypothetical superearths described in Table 1. These superearths have the same rheology as the solid Earth, but have sizes $R = 2\,R_{\bigoplus}$ and $R = 4\,R_{\bigoplus}$. Each of these five objects is modeled with a homogeneous near-spherical self-gravitating Andrade body with $\alpha = 0.2$ and $\tau_{_A} = \tau_{_M} = 10^{10}$ s. In the limit of vanishing tidal frequency χ, the factors $k_2 \sin\epsilon_2$ approach zero, which is natural from the physical point of view. Indeed, an $lmpq$ term in the expansion for tidal torque contains the factor $k_l(\chi_{lmpq})\sin\epsilon_l(\chi_{lmpq})$. On crossing the $lmpq$ resonance, where the frequency χ_{lmpq} goes through zero, the factor $k_l(\chi_{lmpq})\sin\epsilon_l(\chi_{lmpq})$ must vanish, so that the $lmpq$ term of the torque could change its sign.