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ORIGINAL ARTICLE

Long-term evolution of orbits about a precessing oblate planet. 2. The case of variable precession*

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Abstract We continue the study undertaken in Efroimsky [Celest. Mech. Dyn. Astron. **91**, 75–108 (2005a)] where we explored the influence of spin-axis variations of an oblate planet on satellite orbits. Near-equatorial satellites had long been believed to keep up with the oblate primary's equator in the cause of its spin-axis variations. As demonstrated by Efroimsky and Goldreich [Astron. Astrophys. **415**, 1187–1199 (2004)], this opinion had stemmed from an inexact interpretation of a correct result by Goldreich [Astron. J. **70**, 5–9 (1965)]. Although Goldreich [Astron. J. **70**, 5–9 (1965)] mentioned that his result (preservation of the initial inclination, up to small oscillations about the moving equatorial plane) was obtained for *non-osculating* inclination, his admonition had been persistently ignored for forty years. It was explained in Efroimsky and Goldreich [Astron. Astrophys. **415**, 1187–1199 (2004)] that the equator precession influences the osculating inclination of a satellite orbit already in the first order over the perturbation caused by a transition from an inertial to an equatorial coordinate system. It was later shown in Efroimsky [Celest. Mech. Dyn. Astron. **91**, 75–108 (2005a)] that the *secular part* of the inclination is affected only in the second order. This fact, anticipated by Goldreich [Astron. J. **70**, 5–9 (1965)], remains valid for a constant rate of the precession. It turns out that non-uniform variations of the planetary spin state generate changes in the osculating elements, that are linear in $\mid \dot{\vec{\mu}} \mid$, where $\vec{\mu}$ is the planetary equator's total precession rate that includes the equinoctial precession, nutation, the Chandler wobble, and the polar wander. We work out a formalism which will help us to determine if these factors cause a drift of a satellite orbit away from the evolving planetary equator.

*By "precession," in its most general sense, we mean any change of the direction of the spin axis of the planet—from its long-term variations down to nutations down to the Chandler wobble and polar wander.

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1 The scope of the project

1.1 The motivation

Calculation of the obliquity of a planet (Ward 1973; Laskar and Robutel 1993; Touma and Wisdom 1994) are always obtained within a simplified model based on representation of the planet by a symmetrical rigid rotator, with no internal structure or dissipative phenomena taken into consideration. This model yields, via the Colombo (1966) equation, the history of the planet axis' inclination in an inertial frame. Thence the evolution of the obliquity can be found. The Colombo (1966) equation was derived for a rigid planet in the principal rotation state. These assumptions raise questions when it comes to real physics. First, a planet is deformable and, thereby, is subject to solar tides. It also tends to yield its shape to the instantaneous axis of rotation. (This phenomenon is always acknowledged in regard to the Chandler wobble, but never in regard to the equinoctial precession.) Second, a forced rotator is never in a principal spin state, and its angular–velocity vector is always slightly off its angular–momentum vector. These three phenomena influence the equinoctial precession and, through it, the obliquity variations. On the one hand, these phenomena are feeble; on the other hand, we are interested in their accumulation over the longest time scales, and therefore we are unsure of the outcome. Last, and by no means least, the Colombo description of the equinoctial precession ignores the possibility of planetary catastrophes that might have altered the planet's spin mode.

It would be good to develop a model-independent check of whether the planet could have maintained, through its entire past, the same equinoctial precession as it has today. Such a check is offered by Mars' two satellites. The present proximity of both moons to the Martian equatorial plane is hardly a mere coincidence. Hence, the question becomes: could Mars have maintained equinoctial precession, predicted by the Colombo model, through its entire history without pushing an initially near-equatorial satellite too far away from the equatorial plane?

1.2 The objective

If it turns out that the equinoctial precession, predicted by the Colombo (1966) model, does not drive the satellites away from the equator, or drives them away at a very slow rate, then this will become an independent confirmation of this model's applicability. If, however, it turns out that the predicted precession of the spin axis leads to considerable variations in the satellite inclination relative to the equator of date, this will mean that the Colombo model should be further improved or/and that a planetary catastrophe may have altered Mars' spin state.

According to Goldreich (1965) and Kinoshita (1993) the inclination of a near-equatorial satellite only oscillates about its initial value, provided the equinoctial precession is uniform. However, even within the simple Colombo model, the equinoctial precession is variable. Besides, in these works *non-osculating* elements were used, circumstance noticed by Goldreich (1965) but missed by many authors who employed and furthered his result.

Whenever the disturbance depends upon velocities (like a transition from inertial axes to ones co-precessing with the planet), a mere amendment of the disturbing function makes the planetary equations render not the osculating but the so-called contact orbital elements whose physical interpretation is non-trivial (Efroimsky and Goldreich 2003, 2004). To furnish osculating elements, the equations should be enriched with extra terms, some of which will not be additions to the disturbing function.[1] Some of them will be of the *first order* in the velocity-dependent perturbation, others of the second. For *uniform* precession, the first-order extra terms average out, except for a term showing up in the equation for $\mathrm{d}M_o/\mathrm{d}t$ (Efroimsky 2005a), as predicted by Goldreich 40 years ago. Thus, if we address not the elements per se but their *secular parts*, Goldreich's result obtained for the contact elements stays also for the osculating ones: the orbit will oscillate about the uniformly moving equator but will not shift away from it.

Under *variable* precession of the spin axis, the secular parts of the precession-caused first-order terms are of the first order in $|\,\dot{\vec{\mu}}\,|$ where $\vec{\mu}$ is the total precession rate of the equator (Efroimsky 2005a). Accordingly, the secular parts of the osculating elements may differ from those of their contact counterparts already in this order.

To understand if Mars could have kept through its entire past the same equinoctial precession, we need to determine if the satellite orbits might have shifted away from the equator in the cause of nonuniform precession. To see how the secular parts of the osculating elements evolve, we shall build the required mathematical formalism based on the averaged equations.

1.3 The means

The motion of a satellite about a precessing oblate planet is most naturally described with orbital elements defined in a coprecessing equatorial frame. It is also convenient to choose the elements to be osculating. The physical interpretation of such orbital variables will be most straightforward.

1.3.1 Exact planetary equations

The above defined setting is the two-body problem disturbed by two perturbations—the gravitational pull of the equatorial bulge and the transition to a non-inertial frame of reference associated with the precessing planetary equator. Together, they generate the following variation of the Hamiltonian (Efroimsky and Goldreich 2003; Efroimsky 2005a, b):

$$\Delta\mathcal{H}^{(\mathrm{osc})} \;=\; -\left[\,R_{\mathrm{oblate}}(\nu) \,+\, \vec{\mu}\cdot(\vec{f}\times\vec{\mathbf{g}}) \,+\, (\vec{\mu}\times\vec{f})\cdot(\vec{\mu}\times\vec{f})\,\right], \tag{1}$$

where the oblateness-caused disturbing potential is

$$R_{\mathrm{oblate}}(\nu) \;=\; \frac{G\,m\,J_2}{2}\,\frac{\rho_e^2}{r^3}\,\left[\,1 \,-\, 3\,\sin^2 i\,\sin^2(\omega \,+\, \nu)\,\right], \tag{2}$$

ρ_e being the mean equatorial radius of the planet, and ν denoting the true anomaly. The vector

[1] These terms will complicate both the Lagrange- and Delaunay-type equations. The Delaunay equations will no longer be Hamiltonian. This parallels a predicament with the Andoyer elements used in the theory of rigid-body rotation with angular–velocity-dependent perturbations (Efroimsky 2007; Gurfil et al. 2007).

$$\vec{\boldsymbol{\mu}} \;=\; \mu_1 \, \hat{\mathbf{x}} \;+\; \mu_2 \, \hat{\mathbf{y}} \;+\; \mu_3 \, \hat{\mathbf{z}} \;=\; \hat{\mathbf{x}} \, \frac{\mathrm{d}I_p}{\mathrm{d}t} \;+\; \hat{\mathbf{y}} \, \frac{\mathrm{d}h_p}{\mathrm{d}t} \, \sin I_p \;+\; \hat{\mathbf{z}} \, \frac{\mathrm{d}h_p}{\mathrm{d}t} \, \cos I_p \tag{3}$$

is the precession rate of the planetary spin axis (Sometimes $\vec{\boldsymbol{\mu}}$ is referred to as the rotational vector of the equator.) Angles I_p and h_p are the inclination and the longitude of the node of the planetary equator of date relative to that of epoch; unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ denote a coordinate system fixed on the moving equatorial plane of date, $\hat{\mathbf{z}}$ being orthogonal to the equator-of-date plane, and $\hat{\mathbf{x}}$ pointing towards the ascending node of the equator of date relative to the one of epoch. For details of calculation of I_p and h_p see Sect. 2.2.2 and Appendix A.

Notations $\vec{\boldsymbol{f}}$ and $\vec{\mathbf{g}}$ stand for two auxiliary vector functions which play an important role in the theory. These are the implicit functional dependencies of the unperturbed (two-body) position and velocity upon time and six orbital elements. These dependencies emerge as a solution

$$\begin{aligned} \vec{\mathbf{r}} \;&=\; \vec{\boldsymbol{f}} \, (C_1, \ldots, C_6, \, t) \, , \\ \vec{\mathbf{v}} \;&=\; \vec{\mathbf{g}} \, (C_1, \ldots, C_6, \, t) \, , \quad \vec{\mathbf{g}} \;\equiv\; \frac{\partial \vec{\boldsymbol{f}}}{\partial t} \end{aligned} \tag{4}$$

to the reduced two-body problem

$$\ddot{\vec{\mathbf{r}}} \;+\; \frac{G\,m}{r^2} \, \frac{\vec{\mathbf{r}}}{r} \;=\; 0 \tag{5}$$

and define, geometrically, a Keplerian ellipse or hyperbola parameterised with some set of six independent orbital elements C_i which are constants in the absence of disturbances. Under perturbation, these elements are endowed with time dependence.

This way, our Hamiltonian variation $\Delta\mathcal{H}^{(\mathrm{osc})}$, too, becomes, through composition, a function of time and the six elements used in (4)—these could be the Keplerian or Delaunay or Poincare or Jacobi elements. The Hamiltonian variation is equipped with superscript "(osc)" in order to emphasise that this is the form taken by the Hamiltonian expressed as a function of osculating orbital elements. This clause, seemingly trivial and therefore redundant, turns out to be crucial. As pointed out in Efroimsky and Goldreich (2004) and explained in detail in Efroimsky (2005a), a naive development of the planetary equations in precessing frames leads to a Hamiltonian variation different from (1); but that Hamiltonian variation tacitly turns out to be a function of non-osculating orbital elements. This tacit loss of osculation in problems with velocity-dependent perturbations is an old pitfall in orbit calculations. Although some 40 years ago Goldreich (1965) warned of these difficulties, the issue has until recently been ignored in the literature.

For some general-type parameterisation of the instantaneous conics through six orbital variables $C_1, \ldots, C_6$, the variation-of-parameters equations will read

$$\begin{aligned} [C_n \; C_i] \, \frac{\mathrm{d}C_i}{\mathrm{d}t} \;=\; & -\, \frac{\partial \, \Delta\mathcal{H}^{(\mathrm{osc})}}{\partial C_n} \\ & + \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\boldsymbol{f}}}{\partial C_n} \times \vec{\mathbf{g}} \;-\; \vec{\boldsymbol{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial C_n} \right) \;-\; \dot{\vec{\boldsymbol{\mu}}} \cdot \left(\vec{\boldsymbol{f}} \times \frac{\partial \vec{\boldsymbol{f}}}{\partial C_n} \right) \\ & - \left(\vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{f}} \right) \, \frac{\partial}{\partial C_n} \left(\vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{f}} \right) , \end{aligned} \tag{6}$$

provided these conics are chosen to be always tangent to the trajectory, i.e., provided the parameters are chosen to be osculating[2] (Efroimsky and Goldreich 2004; Efroimsky 2005a).

A more convenient representation of the above equation will be achieved if one includes the $-\left(\vec{\mu}\times\vec{f}\right)\,\partial\left(\vec{\mu}\times\vec{f}\right)/\partial C_n$ term in the Hamiltonian:

$$[C_n\;C_i]\,\frac{\mathrm{d}C_i}{\mathrm{d}t} = -\frac{\partial\;\text{“}\Delta\mathcal{H}\text{”}}{\partial C_n} + \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial C_n}\times\vec{\mathbf{g}} - \vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial C_n}\right) - \dot{\vec{\mu}}\cdot\left(\vec{f}\times\frac{\partial\vec{f}}{\partial C_n}\right), \tag{7}$$

the amended “Hamiltonian” being defined through

$$\text{“}\Delta\mathcal{H}\text{”} = -\left[R_{\text{oblate}}(\nu) + \vec{\mu}\cdot(\vec{f}\times\vec{\mathbf{g}}) + \frac{1}{2}\,(\vec{\mu}\times\vec{f})\cdot(\vec{\mu}\times\vec{f})\right]. \tag{8}$$

Here the quotation marks are necessary to emphasise that “$\Delta\mathcal{H}$" is not the real Hamiltonian variation but merely a convenient mathematical entity. Under this convention, and under the assumption that the parameterisation is implemented through the Kepler elements, (7) yields the following system of Lagrange-type planetary equations:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\left[\frac{\partial\,(-\text{“}\Delta\mathcal{H}\text{”})}{\partial M_o} - \dot{\vec{\mu}}\cdot\left(\vec{f}\times\frac{\partial\vec{f}}{\partial M_o}\right)\right], \tag{9}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1-e^2}{n\,a^2\,e}\left[\frac{\partial\,(-\text{“}\Delta\mathcal{H}\text{”})}{\partial M_o} - \dot{\vec{\mu}}\cdot\left(\vec{f}\times\frac{\partial\vec{f}}{\partial M_o}\right)\right] - \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\times\left[\frac{\partial\,(-\text{“}\Delta\mathcal{H}\text{”})}{\partial\omega} + \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial\omega}\times\vec{\mathbf{g}} - \vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial\omega}\right) - \dot{\vec{\mu}}\cdot\left(\vec{f}\times\frac{\partial\vec{f}}{\partial\omega}\right)\right], \tag{10}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{-\cos i}{na^2(1-e^2)^{1/2}\,\sin i}\times\left[\frac{\partial\,(-\text{“}\Delta\mathcal{H}\text{”})}{\partial i} + \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial i}\times\vec{\mathbf{g}} - \vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial i}\right) - \dot{\vec{\mu}}\cdot\left(\vec{f}\times\frac{\partial\vec{f}}{\partial i}\right)\right] + \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\left[\frac{\partial\,(-\text{“}\Delta\mathcal{H}\text{”})}{\partial e} + \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial e}\times\vec{\mathbf{g}} - \vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial e}\right) - \dot{\vec{\mu}}\cdot\left(\vec{f}\times\frac{\partial\vec{f}}{\partial e}\right)\right], \tag{11}$$

[2] Had we simply amended the Hamiltonian by the above variation $\Delta\mathcal{H}^{(\text{osc})}$, without inserting the extra $\vec{\mu}$-dependent terms into (6), such equations would yield non-osculating elements, ones parametrising a family of non-tangent conics. This would happen because the Hamiltonian perturbation depends not only upon positions but also upon the canonical momenta. Another way of getting into this hidden trap is to start with the Cartesian or spherical coordinates and momenta, and to perform the Hamilton–Jacobi operation. The resulting variables C_j will then come out canonical and will be the well-known Delaunay elements. In case the Hamiltonian perturbation depends upon the momenta, these Delaunay elements will be non-osculating, i.e., will parameterise a sequence of instantaneous conics non-tangent to the physical orbit (Efroimsky and Goldreich 2003; Efroimsky 2005a).

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{\cos i}{na^2\,(1-e^2)^{1/2}\sin i}\left[\frac{\partial\,(-\text{“}\Delta\mathcal{H}\text{”})}{\partial\omega} + \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial\omega}\times\vec{\mathbf{g}} - \vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial\omega}\right)\right.$$
$$\left. - \dot{\vec{\mu}}\cdot\left(\vec{f}\times\frac{\partial\vec{f}}{\partial\omega}\right)\right] - \frac{1}{na^2\,(1-e^2)^{1/2}\,\sin i}\left[\frac{\partial\,(-\text{“}\Delta\mathcal{H}\text{”})}{\partial\Omega}\right.$$
$$\left. + \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial\Omega}\times\vec{\mathbf{g}} - \vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial\Omega}\right) - \dot{\vec{\mu}}\cdot\left(\vec{f}\times\frac{\partial\vec{f}}{\partial\Omega}\right)\right], \tag{12}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2\,(1-e^2)^{1/2}\,\sin i}\left[\frac{\partial\,(-\text{“}\Delta\mathcal{H}\text{”})}{\partial i} + \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial i}\times\vec{\mathbf{g}} - \vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial i}\right)\right.$$
$$\left. - \dot{\vec{\mu}}\cdot\left(\vec{f}\times\frac{\partial\vec{f}}{\partial i}\right)\right], \tag{13}$$

$$\frac{\mathrm{d}M_o}{\mathrm{d}t} = -\,\frac{1-e^2}{n\,a^2\,e}\left[\frac{\partial\,(-\text{“}\Delta\mathcal{H}\text{”})}{\partial e} + \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial e}\times\vec{\mathbf{g}} - \vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial e}\right) - \dot{\vec{\mu}}\cdot\left(\vec{f}\times\frac{\partial\vec{f}}{\partial e}\right)\right]$$
$$-\,\frac{2}{n\,a}\left[\frac{\partial\,(-\text{“}\Delta\mathcal{H}\text{”})}{\partial a} + \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial a}\times\vec{\mathbf{g}} - \vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial a}\right) - \dot{\vec{\mu}}\cdot\left(\vec{f}\times\frac{\partial\vec{f}}{\partial a}\right)\right], \tag{14}$$

where terms $\vec{\mu}\cdot\left((\partial\vec{f}/\partial M_o)\times\vec{\mathbf{g}} - (\partial\vec{\mathbf{g}}/\partial M_o)\times\vec{f}\right)$ have been omitted in (9) and (10), because these terms vanish identically (see the appendix to Efroimsky 2005a).

1.3.2 The approximation

To obtain the first-order (over $\vec{\mu}$) secular parts of the osculating elements, we shall carry out two operations:

(1) First, we shall drop the $O(\vec{\mu}^2)$ contribution to “$\Delta\mathcal{H}$” and shall assume that preservation of the first-order terms and neglect of the second-order ones in the equations makes them render solutions valid in the first-order. This assumption should remain valid for some interval of time, an interval whose actual duration can be determined only through accurate numerical simulation. In our analytical developments we shall hope that this interval is sufficiently long.[3]
(2) Second, we shall substitute both the disturbing function $(-\text{“}\Delta\mathcal{H}\text{”})$ and the other precession-generated (i.e., $\vec{\mu}$-dependent) terms with their orbital averages. To be more exact, the rate $\vec{\mu}$ and each of the elements will be considered as a function of the true anomaly ν and expanded into a Fourier integral which will

[3] In (9)–(14), the $\vec{\mu}$-terms on the right-hand side are of order $|\vec{\mu}|^2/n$. According to Ward (1973), the range of values of $|\vec{\mu}|$ for Mars hardly ever exceeded 10^{-3} year^{-1}. The value of n for the Martian satellites is of order one day^{-1}. If we now look for example at (1.3.1), we shall see that the quadratic in $\vec{\mu}$ terms surely cannot contribute to $\mathrm{d}i/\mathrm{d}t$ more than an angular degree over a million of years, and are quite likely to remain insignificant over dozens of millions years. Whether these terms may be omitted at timescales of 100 millions of years and longer—should be checked by numerical computation. As demonstrated in Lainey et al. (2007), the model remains surprisingly exact for, at least, 20 Myr.

then be split into two pieces—an integral over the band of frequencies less than the orbital frequency and an integral over the higher frequencies:

$$C_j \;=\; \langle\, C_j \,\rangle \;+\; C_j^{\heartsuit}, \qquad \vec{\boldsymbol{\mu}} \;=\; \langle\, \vec{\boldsymbol{\mu}} \,\rangle \;+\; \vec{\boldsymbol{\mu}}^{\,\heartsuit} \tag{15}$$

The first term, $\langle\vec{\boldsymbol{\mu}}\rangle$ or $\langle C_j\rangle$, will be regarded as the secular part, while the second one, $\vec{\boldsymbol{\mu}}^{\heartsuit}$ or $C_j^{\heartsuit}$, will be averaged out. The left-hand sides of the averaged planetary equations will now contain the time derivatives not of the elements but of their secular parts. To understand the structure of the averaged right-hand sides, consider some product $A(\nu)B(\nu)$, where A and B denote some of the elements or the projection of $\vec{\boldsymbol{\mu}}$ onto the instantaneous normal to the satellite orbit:

$$A\;B \;=\; \Big(\langle\, A \,\rangle \;+\; A^{\heartsuit}\Big)\;\Big(\langle\, B \,\rangle \;+\; B^{\heartsuit}\Big) \;=\; \langle A\;B\rangle \;+\; (A\;B)^{\heartsuit}. \tag{16}$$

The secular and high-frequency components of this product will read, correspondingly, as

$$\langle A\;B\rangle \;\equiv\; \langle\, A \,\rangle\;\;\langle\, B \,\rangle \;+\; \big\langle A^{\heartsuit}\,B^{\heartsuit}\big\rangle \tag{17}$$

and

$$(A\;B)^{\heartsuit} \;\equiv\; \langle\, A \,\rangle\;B^{\heartsuit} \;+\; \langle\, B \,\rangle\,A^{\heartsuit} \;+\; \Big(A^{\heartsuit}\,B^{\heartsuit} \;-\; \big\langle A^{\heartsuit}\,B^{\heartsuit}\big\rangle\Big)\,. \tag{18}$$

An obvious circumstance is that the secular part of the product consists not only of the product of the secular parts of the multipliers but also of the term $\big\langle A^{\heartsuit}\,B^{\heartsuit}\big\rangle$ containing resonances. A less evident but crucially important circumstance is that, technically, the above separation of timescales is never implemented exactly (unless one deals from the very beginning with the Fourier expansions of all the functions involved). Therefore, the (imperfectly calculated) high-frequency parts $A^{\heartsuit}$, $B^{\heartsuit}$ and $(A\;B)^{\heartsuit}$ are unavoidably contaminated with the lower-frequency modes, modes whose effect may considerably accumulate at large times and exert "back-reaction" upon the secular part of the product (Laskar 1990).

1.3.3 The planetary equations for the first-order secular parts of the osculating elements

Naively, the afore proposed approximation will lead us to a new system of planetary equations. It will be identical to the systems (9)–(14), except that now the letters $a,\ e,\ \omega,\ \Omega,\ i,\ M_o$ will denote not the osculating elements but their secular parts. Similarly, $\vec{\boldsymbol{\mu}}$ will now stand for the secular part of the precession rate. The Hamiltonian will now be substituted with

$$\begin{aligned} \Delta\mathcal{H}^{(\mathrm{eff})} &= -\,\Big[\;\langle R_{\mathrm{oblate}}\rangle \;+\; \langle\;\vec{\boldsymbol{\mu}}\cdot(\vec{\boldsymbol{f}}\times\vec{\mathbf{g}})\;\rangle\;\Big] \\ &= -\,\frac{G\,m\,J_2}{4}\;\frac{\rho_e^2}{a^3}\;\frac{3\,\cos^2 i\,-\,1}{\left(1\,-\,e^2\right)^{3/2}} \;-\; \sqrt{G\,m\,a\,\left(1\,-\,e^2\right)} \\ &\qquad \times\,(\,\mu_1\;\sin i\;\sin\Omega \;-\; \mu_2\;\sin i\;\cos\Omega \;+\; \mu_3\;\cos i\,)\,, \end{aligned} \tag{19}$$

where, once again, all letters denote not the appropriate variables but their averages.

By doing so, we would, however, ignore that the $\vec{\boldsymbol{\mu}}$-dependent terms in (9)–(14) contain products of high-frequency quantities (such as, e.g. the product of the true-anomaly-dependent expression $(\partial\vec{\boldsymbol{f}}/\partial\omega)\times\vec{\mathbf{g}}\;-\;(\partial\vec{\mathbf{g}}/\partial\omega)\times\vec{\boldsymbol{f}}$ by the high-frequency part of $\vec{\mu}$ in formula (10)). Averages of such products will contribute to the secular parts of the right-hand sides of the approximate planetary equations, as in the example (17). (As we shall see below, these inputs will be due to the commensurabilities between the orbital motion of the satellite and the short-term nutations of the primary.) Keeping this in mind, we should approximate the exact planetary equations rather with the following system:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\left[-\left\langle \dot{\vec{\mu}}\left(\vec{f}\times\frac{\partial\vec{f}}{\partial M_o}\right)\right\rangle\right], \tag{20}$$

$$\begin{aligned}\frac{\mathrm{d}e}{\mathrm{d}t} = {} & \frac{1-e^2}{n\,a^2\,e}\left[-\left\langle \dot{\vec{\mu}}\left(\vec{f}\times\frac{\partial\vec{f}}{\partial M_o}\right)\right\rangle\right] \\ & -\frac{(1-e^2)^{1/2}}{n\,a^2\,e}\left[\left\langle \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial\omega}\times\vec{\mathbf{g}}-\vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial\omega}\right)\right\rangle - \left\langle \dot{\vec{\mu}}\left(\vec{f}\times\frac{\partial\vec{f}}{\partial\omega}\right)\right\rangle\right],\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\mathrm{d}\omega}{\mathrm{d}t} = {} & \frac{-\cos i}{na^2(1-e^2)^{1/2}\,\sin i} \\ & \times\left[\frac{\partial\left(-\Delta\mathcal{H}^{(\mathrm{eff})}\right)}{\partial i} + \left\langle \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial i}\times\vec{\mathbf{g}}-\vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial i}\right)\right\rangle - \left\langle \dot{\vec{\mu}}\left(\vec{f}\times\frac{\partial\vec{f}}{\partial i}\right)\right\rangle\right] \\ & + \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\left[\frac{\partial\left(-\Delta\mathcal{H}^{(\mathrm{eff})}\right)}{\partial e} + \left\langle \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial e}\times\vec{\mathbf{g}}-\vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial e}\right)\right\rangle\right. \\ & \left. - \left\langle \dot{\vec{\mu}}\left(\vec{f}\times\frac{\partial\vec{f}}{\partial e}\right)\right\rangle\right],\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\mathrm{d}i}{\mathrm{d}t} = {} & \frac{\cos i}{na^2\,(1-e^2)^{1/2}\sin i}\left[\left\langle \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial\omega}\times\vec{\mathbf{g}}-\vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial\omega}\right)\right\rangle - \left\langle \dot{\vec{\mu}}\left(\vec{f}\times\frac{\partial\vec{f}}{\partial\omega}\right)\right\rangle\right] \\ & - \frac{1}{na^2\,(1-e^2)^{1/2}\,\sin i}\left[\frac{\partial\left(-\Delta\mathcal{H}^{(\mathrm{eff})}\right)}{\partial\Omega} + \left\langle \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial\Omega}\times\vec{\mathbf{g}}-\vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial\Omega}\right)\right\rangle\right. \\ & \left. - \left\langle \dot{\vec{\mu}}\left(\vec{f}\times\frac{\partial\vec{f}}{\partial\Omega}\right)\right\rangle\right],\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\mathrm{d}\Omega}{\mathrm{d}t} = {} & \frac{1}{na^2\,(1-e^2)^{1/2}\,\sin i} \\ & \times\left[\frac{\partial\left(-\Delta\mathcal{H}^{(\mathrm{eff})}\right)}{\partial i} + \left\langle \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial i}\times\vec{\mathbf{g}}-\vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial i}\right)\right\rangle - \left\langle \dot{\vec{\mu}}\left(\vec{f}\times\frac{\partial\vec{f}}{\partial i}\right)\right\rangle\right],\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{\mathrm{d}M_o}{\mathrm{d}t} = {} & -\frac{1-e^2}{n\,a^2\,e}\left[\frac{\partial\left(-\Delta\mathcal{H}^{(\mathrm{eff})}\right)}{\partial e} + \left\langle \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial e}\times\vec{\mathbf{g}}-\vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial e}\right)\right\rangle - \left\langle \dot{\vec{\mu}}\left(\vec{f}\times\frac{\partial\vec{f}}{\partial e}\right)\right\rangle\right] \\ & -\frac{2}{n\,a}\left[\frac{\partial\left(-\Delta\mathcal{H}^{(\mathrm{eff})}\right)}{\partial a} + \left\langle \vec{\mu}\cdot\left(\frac{\partial\vec{f}}{\partial a}\times\vec{\mathbf{g}}-\vec{f}\times\frac{\partial\vec{\mathbf{g}}}{\partial a}\right)\right\rangle - \left\langle \dot{\vec{\mu}}\left(\vec{f}\times\frac{\partial\vec{f}}{\partial a}\right)\right\rangle\right]\end{aligned} \tag{25}$$

the angular brackets denoting the secular parts. In (16), (17) and (19) we took into account the fact that the averaged and truncated Hamiltonian (15) depends neither on M_o nor on ω .

It should be emphasised that in this section and hereafter *the symbols* a, e, ω, Ω, i, M_o, $\vec{\mu}$ *stand not for the exact values but for the mean values of the appropriate variables*. A mean value of an element is understood to include the secular and long-period parts, the short-period components being averaged out.

The case of uniform planetary precession ($\vec{\mu} = \text{const}$) was studied in Efroimsky (2005a). In that case, the terms containing $\dot{\vec{\mu}}$ evidently vanish. Besides, it turns out that, for constant $\vec{\mu}$, the mean values of the other $\vec{\mu}$-dependent terms, except one, vanish too:

$$\vec{\mu} \cdot \left\langle \left(\frac{\partial \vec{f}}{\partial C_j} \times \vec{\mathbf{g}} - \vec{f} \times \frac{\partial \vec{\mathbf{g}}}{\partial C_j} \right) \right\rangle = 0, \quad C_j = e,\, \Omega,\, \omega,\, i,\, M_o, \tag{26}$$

$$\begin{aligned} \vec{\mu} \cdot \left\langle \left(\frac{\partial \vec{f}}{\partial a} \times \vec{\mathbf{g}} - \vec{f} \times \frac{\partial \vec{\mathbf{g}}}{\partial a} \right) \right\rangle &= \vec{\mu} \cdot \left(\frac{\partial \vec{f}}{\partial a} \times \vec{\mathbf{g}} - \vec{f} \times \frac{\partial \vec{\mathbf{g}}}{\partial a} \right) \\ &= \frac{3}{2}\, \mu_\perp \sqrt{\frac{G\, m\, \left(1 - e^2\right)}{a}}, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \mu_\perp &\equiv \mu_1 \sin i \, \sin \Omega - \mu_2 \sin i \, \cos \Omega + \mu_3 \cos i \\ &= \dot{I}_p \sin i \, \sin \Omega - \dot{h}_p \sin I_p \sin i \, \cos \Omega + \dot{h}_p \cos I_p \cos i \end{aligned} \tag{28}$$

is the projection of the planets' precession rate $\vec{\mu}$ onto the instantaneous normal to the satellite's orbit.[4]

Hence, in this approximation and under the assumption of constant $\vec{\mu}$, in order to compute the secular parts of the orbital elements, it is sufficient to amend the Hamiltonian with the $\vec{\mu}$-dependent addition and to ignore all the other $\vec{\mu}$-dependent terms except the one given by (27). This will no longer be the case for variable precession, i.e., for time-dependent $\vec{\mu}$. Section 2 of our article will address itself to calculation of the secular parts (26)–(27) in the case of time-dependent $\vec{\mu}$.

2 Equations for the first-order secular parts of the osculating elements

2.1 Two Fourier expansions of the precession spectrum

Precession of the planetary spin axis has a continuous spectrum that spans from the polar wander and the fastest nutations to the Chandler wobble to the long-term variations whose time scales go all way to billions of years. When the planet has a sufficiently massive moon capable of influencing the planetary precession, the rate of this precession, $\vec{\mu}$, should be regarded as a function not only of time but also of

[4] Here $\mu_\perp$ is expressed in the basis $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ associated with the planet's equator of date. Unit vector $\hat{\mathbf{z}}$ is perpendicular to the equator of date, while $\hat{\mathbf{x}}$ is pointing along the line of the ascending node of the equator of date on the equator of epoch; therefore, the components μ_j are given by (3).

the position of the satellite. We shall be interested, however, in the situation where the satellites are small and do not considerably influence rotation of their primary (while rotation variations of the primary still may affect the satellite orbits). This is, for example, the case of Mars whose tiny satellites affect its precession only in a very high order (Laskar 2004). Under these circumstances, it is fair to treat the precession rate as a function of time solely:

$$\vec{\mu}(t) \;=\; \int_0^\infty \left[\;\vec{\mu}^{(s)}(u)\;\sin(ut)\;+\;\vec{\mu}^{(c)}(u)\;\cos(ut)\;\right]\,\mathrm{d}u, \tag{29}$$

u standing for the angular frequency. In what follows, it will be convenient to describe the evolution not in terms of time but via the true anomaly ν of the satellite. For our present purposes, it will be advantageous to express the precession rate as function of the satellite's true anomaly:

$$\vec{\mu}(\nu) \;=\; \int_0^\infty \left[\;\vec{\mu}^{(s)}(W)\;\sin(W\nu)\;+\;\vec{\mu}^{(c)}(W)\;\cos(W\nu)\;\right]\,\mathrm{d}W, \tag{30}$$

W being the circular "frequency" related to the true anomaly ν . Evidently, $\vec{\mu}(t)$ $\vec{\mu}(\nu)$, $\vec{\mu}(u)$, and $\vec{\mu}(W)$ are four different functions. We nevertheless denote them with the same notation $\vec{\mu}(\ldots)$ because the argument will always single out which particular function we mean. The interconnection between functions $\vec{\mu}(\nu)$ and $\vec{\mu}(t)$ is given by

$$\vec{\mu}(t) \;=\; \vec{\mu}(\nu)\,|_{\;\nu=\int n\,\mathrm{d}t}.$$

The interconnection between the Fourier components is less obvious. However, it simplifies under the assumption of vanishing eccentricity and slowly changing semimajor axis:

$$\vec{\mu}(W) \;\approx\; n\,\vec{\mu}(u)\,|_{\;W=u/n}, \quad n \;\equiv\; (G\,m)^{1/2}\,a^{-3/2}. \tag{31}$$

A rigorous relation to be used below is:[5]

$$\frac{\mathrm{d}\vec{\mu}(\nu)}{\mathrm{d}\nu} \;=\; \frac{\mathrm{d}\vec{\mu}(t)}{\mathrm{d}t}\left(\frac{\partial t}{\partial \nu}\right)_{a,e,i,\omega,\Omega,M_o} \;=\; \dot{\vec{\mu}}\,\left(\frac{\partial t}{\partial M}\right)_{a,\ldots}\left(\frac{\partial M}{\partial \nu}\right)_{a,\ldots} \;=\; \frac{\dot{\vec{\mu}}\,\left(1-e^2\right)^{3/2}}{n\,(1+e\,\cos\nu)^2}. \tag{32}$$

2.2 The role of the $\langle\vec{\mu}\cdot(\ldots)\rangle$ and $\langle\dot{\vec{\mu}}\cdot(\ldots)\rangle$ terms

These terms, ignored in the literature hitherto, implement the subtle influence of the planet's orbit precession upon its satellites' motion. The physical content of this effect is as follows: first, the precession of the planetary orbit slowly alters the solar torque acting on the planet; second, the variations of this torque entail changes in the planetary spin axis' precession; and, finally, third: these changes influence the satellites' orbits. This three-step interaction is extremely weak; still, its effect may accumulate over very long periods of time.

[5] This relation immediately follows from the well known equality $\mathrm{d}M\;\;(1+e\cos\nu)^2\;\;=\;\;\mathrm{d}\nu\;\left(1-e^2\right)^{3/2}$, one upon which also the averaging rule (75) is based.

2.2.1 The $\langle \vec{\boldsymbol{\mu}} \cdot (\ldots) \rangle$ terms

To illustrate the role of commensurabilities between the satellite orbital motion and the planetary nutations, let us consider the average $\left\langle \vec{\boldsymbol{\mu}} \cdot \left((\partial \vec{\boldsymbol{f}}/\partial e) \times \vec{\mathbf{g}} - (\partial \vec{\mathbf{g}}/\partial e) \times \vec{\boldsymbol{f}} \right) \right\rangle$ emerging in Eq. (22) for $\mathrm{d}\omega/\mathrm{d}t$ and in Eq. (25) for $\mathrm{d}M_o/\mathrm{d}t$: as shown in Sect. A.4 of the appendix to Efroimsky (2004),

$$\vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\boldsymbol{f}}}{\partial e} \times \vec{\mathbf{g}} - \vec{\boldsymbol{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial e} \right) = -\, \mu_\perp \, \frac{n\,a^2 \, \left(3\,e + 2\cos\nu + e^2 \cos\nu\right)}{(1 + e\cos\nu)\, \sqrt{1 - e^2}}, \tag{33}$$

$\mu_\perp$ being given by (29). By virtue of (75) and (30), its secular part at some ν will be:

$$\begin{aligned}
&\left\langle \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\boldsymbol{f}}}{\partial e} \times \vec{\mathbf{g}} - \vec{\boldsymbol{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial e} \right) \right\rangle \\
&= -\,\frac{1 - e^2}{2\pi}\, n a^2 \int_{\nu'=-\pi}^{\nu'=\pi} \mu_\perp(\nu + \nu')\, \frac{3e + 2\cos(\nu + \nu') + e^2 \cos(\nu + \nu')}{(1 + e\cos(\nu + \nu'))^3}\, \mathrm{d}\nu' \\
&= -\,\frac{1 - e^2}{2\pi}\, n\,a^2 \int_0^\infty dW \int_{-\pi}^{\pi} \mathrm{d}\nu' \left[\mu_\perp^{(s)}(W) \sin(W(\nu + \nu')) + \mu_\perp^{(c)}(W) \cos(W(\nu + \nu')) \right] \\
&\quad \times \left[\left(2 + e^2\right) \cos(\nu + \nu') + \left(-\,3\,e - \frac{5}{2}\,e^3 \right) \cos 2(\nu + \nu') \right. \\
&\quad\quad \left. +\; 3\,e^2 \cos 3(\nu + \nu') - \frac{5}{2}\,e^3 \cos 4(\nu + \nu') + O(e^4) \right] \\
&\quad -\,\frac{1 - e^2}{2\,\pi}\, n a^2 \left[\left(2 + e^2\right) \mu_\perp^{(c)}(1) + \left(-3e - \frac{5}{2}\,e^3 \right) \mu_\perp^{(c)}(2) \right. \\
&\quad\quad \left. +\, 3e^2\, \mu_\perp^{(c)}(3) - \frac{5}{2}\,e^3\, \mu_\perp^{(c)}(4) + O(e^4) \right],
\end{aligned} \tag{34}$$

W being the angular "frequency" related to the true anomaly ν, as in Eq. (30). Not surprisingly, the integral over W has been reduced to an infinite sum over the discrete values $W = 1, 2, 3, 4, \ldots$ corresponding to commensurabilities between the orbital frequency of the satellite and the nutational frequencies of the oblate planet.[6] The main resonant input comes from the principal commensurability $W = 1$, i.e., from the nutation mode resonant with the orbit. The higher-order resonant inputs originate from the faster nutations characterised by $W = 2,\ 3,\ 4, \ldots$ In Eqs. (20)–(25), almost all terms proportional to $\vec{\boldsymbol{\mu}}$ produce such resonances. At the time when we are writing this paper, our knowledge about the fast nutations and polar wander of Mars is yet very limited, and we shall not venture to offer quantitative estimates of the time scale over which the effect of these resonances upon the satellite orbit becomes considerable.

Slower than $W = 1$ variations of $\vec{\boldsymbol{\mu}}$ bring no non-resonant contributions into the average of the right-hand side of (33). It can be shown that none of the $\langle \vec{\boldsymbol{\mu}} \cdot (\ldots) \rangle$

[6] It should be emphasised that (34) was obtained by a certain approximation: the averaging ignored the back-reaction of the short-period motions upon the long-period ones (i.e., it ignored the fact that, after each orbital period, the satellite does not return to exactly the same point it started); for example, it was assumed that the elements e and a remained constant during the integration over ν' from 0 to 2π.

term emerging in (20)–(24) yield a non-resonant input (hence (26)). For these reasons, in the rest of this paper, the terms $\langle\, \vec{\boldsymbol{\mu}}\ (\ldots)\,\rangle$ will be omitted.

2.2.2 The $\langle\, \dot{\vec{\boldsymbol{\mu}}} \cdot (\ldots)\,\rangle$ *terms*

Let us consider, as an example, the term $\langle\, \dot{\vec{\boldsymbol{\mu}}} \cdot \left(-\vec{\boldsymbol{f}} \times (\partial\vec{\boldsymbol{f}}/\partial\omega) \right) \rangle$ showing up on the right-hand sides of Eqs. (21) and (23). We have from Appendix A11 of Efroimsky (2004):

$$\dot{\vec{\boldsymbol{\mu}}} \cdot \left(-\vec{\boldsymbol{f}} \times \frac{\partial\vec{\boldsymbol{f}}}{\partial\omega} \right) = -\, \dot{\mu}_{\perp}\ a^2\ \frac{(1\ -\ e^2)^2}{(1\ +\ e\ \cos\nu)^2}. \tag{35}$$

Just as in the preceding example (34), orbital averaging of this expression would yield resonant terms entailed by commensurabilities between the orbital frequency of the satellite and the fast variations of $\dot{\vec{\boldsymbol{\mu}}}$. For the reasons explained above, here we omit these contributions. However, in distinction from the $\langle\, \vec{\boldsymbol{\mu}} \cdot (\ldots)\,\rangle$ terms, some of the $\langle\, \dot{\vec{\boldsymbol{\mu}}} \cdot (\ldots)\,\rangle$ do have nonresonant components. For example, the mean part of (35) will be finite even for a constant $\dot{\vec{\boldsymbol{\mu}}}$:

$$\begin{aligned} \left\langle\, \dot{\vec{\boldsymbol{\mu}}} \cdot \left(-\vec{\boldsymbol{f}} \times \frac{\partial\vec{\boldsymbol{f}}}{\partial\omega} \right) \right\rangle &= -\, \dot{\mu}_{\perp} a^2 \left(1 - e^2\right)^2 \frac{\left(1 - e^2\right)^{3/2}}{2\,\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\nu}{(1 + e\ \cos\nu)^4} \\ &= -\, \dot{\mu}_{\perp} \frac{a^2}{2} \left(2 + 3\,e^2\right), \end{aligned} \tag{36}$$

the superscript dot denoting a time derivative taken in the frame co-precessing with the equator of date. In other words, $\dot{\mu}_{\perp}$ is, by definition, not a full time derivative but its projection onto the instantaneous normal to the satellite's orbit. So defined $\dot{\mu}_{\perp}$ contains only derivatives of μ_j but not of the angles:

$$\dot{\mu}_{\perp}\ =\ \dot{\mu}_1\ \sin i\ \sin\Omega\ -\ \dot{\mu}_2\ \sin i\ \cos\Omega\ +\ \dot{\mu}_3\ \cos i. \tag{37}$$

As shown in the Appendix below, $\dot{\mu}_{\perp}$ can be expressed via the longitude of the node, h_p, and the inclination, I_p, of the equator of date relative to the one of epoch:

$$\begin{aligned} \dot{\mu}_{\perp} &= \ddot{I}_p\ \sin i\ \sin\Omega\ -\ \left(\ddot{h}_p\ \sin I_p\ +\ \dot{h}_p\, \dot{I}_p\ \cos I_p \right) \sin i\ \cos\Omega \\ &\quad +\ \left(\ddot{h}_p\ \cos I_p\ -\ \dot{h}_p\, \dot{I}_p\ \sin I_p \right)\ \cos i \\ &\approx \ddot{h}_p\ \left(-\ \sin I_p\ \sin i\ \cos\Omega\ +\ \cos I_p\ \cos i \right). \end{aligned} \tag{38}$$

The quantities h_p, I_p and their time derivatives can be calculated from integration of the Colombo equation of spin precession in inertial space,

$$\frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t} = \alpha\ \left(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \right)\ \left(\hat{\mathbf{k}} \times \hat{\mathbf{n}} \right), \tag{39}$$

$\hat{\mathbf{k}}\ =\ \left(\ \sin I_p\ \sin h_p,\ \ -\ \sin I_p\ \cos h_p \text{ and } \cos I_p\ \right)^T$ being a unit vector pointing along the major-inertia axis of the planet, and $\hat{\mathbf{n}} = (\sin I_{\mathrm{orb}}\ \sin\Omega_{\mathrm{orb}}, -\sin I_{\mathrm{orb}} \cos\Omega_{\mathrm{orb}}$ and $\cos I_{\mathrm{orb}})^T$ being a unit normal to the planetary orbit plane defined (relative to some fiducial plane) through the inclination $I_{\rm orb}$ and longitude of the node $\Omega_{\rm orb}$. The constant (or, better to say, the slowly varying factor) α is given by

$$\alpha \;\equiv\; \frac{3\,n_p^2}{2\,s\,\left(1\,-\,e_p^2\right)^{3/2}}\;\frac{C\,-(A\,+\,B)/2}{C}, \tag{40}$$

where n_p, e_p, s and A, B, C are the mean motion, the eccentricity, the spin angular velocity, and the moments of inertia of the planet (as ever, $C\,\geq\,B\,\geq\,A$). Even in the relatively simple case of $C\,>\,B\,=\,A$, the planet's axis of rotation does not describe a circular cone because the unit normal to the planet's orbit, $\vec{\mathbf{n}}$, is subject to variations caused by the precession of the planet's orbit about the Sun. While integration of the Colombo equation is explained below in Appendix A, here we would emphasise that this equation describes the evolution of planetary spin only under a very strong assumption of this spin being principal, i.e., in neglect of the Chandler wobble and polar wander.

3 Evolution of the elements in the leading order of e

3.1 The semimajor axis and the eccentricity

As explained in subsection 2.2.1, the $\langle\,\vec{\boldsymbol{\mu}}\cdot(\,\dots)\,\rangle$ terms may be omitted. The expressions for the orbital averages of the $\dot{\vec{\boldsymbol{\mu}}}$-dependent terms, derived in the Appendix to Efroimsky (2006), have the form:

$$\left\langle\;\dot{\vec{\boldsymbol{\mu}}}\;\cdot\left(-\vec{\boldsymbol{f}}\;\times\;\frac{\partial\vec{\boldsymbol{f}}}{\partial M_o}\right)\right\rangle\;=\;-\;\dot{\mu}_\perp\;a^2\,\sqrt{1\;-\;e^2}, \tag{41}$$

$$\left\langle\;\dot{\vec{\boldsymbol{\mu}}}\;\cdot\left(-\vec{\boldsymbol{f}}\;\times\;\frac{\partial\vec{\boldsymbol{f}}}{\partial\omega}\right)\right\rangle\;=\;-\;\dot{\mu}_\perp\;a^2\,\left(1\;+\;\frac{3}{2}\,e^2\right). \tag{42}$$

Here

$$\mu_\perp\;\equiv\;\vec{\boldsymbol{\mu}}\,\cdot\,\vec{\boldsymbol{w}}\;=\;\mu_1\;\sin i\;\sin\Omega\;-\;\mu_2\;\sin i\;\cos\Omega\;+\;\mu_3\;\cos i, \tag{43}$$

where the unit vector

$$\vec{\boldsymbol{w}}\;=\;\hat{\mathbf{x}}\;\sin i\;\sin\Omega\;-\;\hat{\mathbf{y}}\;\sin i\;\cos\Omega\;+\;\hat{\mathbf{z}}\;\cos i$$

is the unit normal to the instantaneous plane of orbit, while the unit vectors $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ denote the basis of the co-precessing coordinate system $x,\,y,\,z$. (The axes x and y belong to the planet's equatorial plane, and the longitude of the node, Ω, is measured from x.)

The quantity $\dot{\mu}_\perp$ is defined as

$$\dot{\mu}_\perp\;\equiv\;\dot{\vec{\boldsymbol{\mu}}}\cdot\vec{\boldsymbol{w}}, \tag{44}$$

but not as $\mathrm{d}(\vec{\boldsymbol{\mu}}\cdot\vec{\boldsymbol{w}})/\mathrm{d}t$—a subtlety important to our further developments.

Insertion of these expressions into Eqs. (20)–(21) will give:

$$\frac{\mathrm{d}a}{\mathrm{d}t}\;=\;-\,2\,a\,\frac{\dot{\mu}_\perp}{n}\,\sqrt{1\,-\,e^2}\;=\;-\,\frac{2\,a^{5/2}}{\sqrt{G\,m}}\,\dot{\mu}_\perp\,\sqrt{1\,-\,e^2}, \tag{45}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{5}{2}\,\frac{\dot{\mu}_\perp}{n}\,e\,\sqrt{1\,-\,e^2} \;=\; \frac{5}{2}\,\frac{a^{3/2}\,e}{\sqrt{G\,m}}\;\dot{\mu}_\perp\,\sqrt{1\,-\,e^2}. \tag{46}$$

For small eccentricities, the approximate solution is:

$$a = a_o \exp\left[-\frac{2}{n_o}\left(\mu_\perp - \mu_{\perp_o}\right)\right]^{-2/3} + O(e^2) \approx a_o\left[1 + \frac{4}{3n_o}\left(\mu_\perp - \mu_{\perp_o}\right)\right], \tag{47}$$

$$e = e_o \exp\left[-\frac{2}{n_o}\left(\mu_\perp - \mu_{\perp_o}\right)\right]^{-5/4} + O(e^2) \approx e_o\left[1 + \frac{5}{2n_o}\left(\mu_\perp - \mu_{\perp_o}\right)\right], \tag{48}$$

where $n_o \;\equiv\; (G\,m)^{1/2}\,a_o^{-3/2}$. We see that variations in the primary's precession exert almost no influence upon the satellite's semimajor axis and eccentricity.

It should, nevertheless, be kept in mind that the satellite orbital elements evolve not only under the influence of the primary's precession but also under the action of tides. Within the truncated model developed in this paper, we shall neglect the tides, but shall introduce them on a subsequent stage of the project.

3.2 The periapse, the inclination, and the node—in the leading order of e

Under the assumption of a and e remaining virtually unchanged, Eqs. (22)–(24) will make a closed system, provided we omit the $\langle\,\vec{\boldsymbol{\mu}}\cdot(\ldots)\,\rangle$ (for the reasons explained above) and also substitute the orbital averages of the $\dot{\vec{\boldsymbol{\mu}}}$-dependent terms with their approximations in the leading order of the eccentricity. This level of approximation would be consistent with the approximation used in (47) and (48). As shown in the Appendix to Efroimsky (2006),

$$\left\langle\,\dot{\vec{\boldsymbol{\mu}}}\,\cdot\left(-\vec{\boldsymbol{f}}\,\times\,\frac{\partial\vec{\boldsymbol{f}}}{\partial e}\right)\right\rangle \;=\; 0, \tag{49}$$

$$\left\langle\,\dot{\vec{\boldsymbol{\mu}}}\,\cdot\left(-\vec{\boldsymbol{f}}\,\times\,\frac{\partial\vec{\boldsymbol{f}}}{\partial\omega}\right)\right\rangle = -\,a^2\;(\,\dot{\mu}_1\;\sin i\;\sin\Omega\,-\,\dot{\mu}_2\;\sin i\;\cos\Omega \\ +\,\dot{\mu}_3\;\cos i\,)\,+\,O(e^2), \tag{50}$$

$$\left\langle\dot{\vec{\boldsymbol{\mu}}}\cdot\left(-\vec{\boldsymbol{f}}\times\frac{\partial\vec{\boldsymbol{f}}}{\partial\Omega}\right)\right\rangle = \frac{a^2}{2}\left[-\,\dot{\mu}_1\;\sin i\;\sin\Omega\;\cos i + \dot{\mu}_2\;\sin i\;\cos\Omega\;\cos i \right.\\ \left. -\dot{\mu}_3\left(2-\,\sin^2 i\right)\right] + O(e^2), \tag{51}$$

$$\left\langle\,\dot{\vec{\boldsymbol{\mu}}}\cdot\left(-\vec{\boldsymbol{f}}\,\times\,\frac{\partial\vec{\boldsymbol{f}}}{\partial i}\right)\right\rangle \;=\; -\,\frac{a^2}{2}\;(\,\dot{\mu}_1\;\cos\Omega\;+\;\dot{\mu}_2\;\sin\Omega\,)\;+\;O(e^2). \tag{52}$$

Substitution of (19) and of the above expressions for the $\dot{\vec{\boldsymbol{\mu}}}$-terms into (22)–(24) will give us:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{3\,n\,J_2}{4}\;\left(\frac{\rho_e}{a}\right)^2\;\left(5\;\cos^2 i\,-\,1\right)\,+\,\mu_n\;\cot i\,-\,\mu_\perp \\ +\,\frac{1}{2}\left(\frac{\dot{\mu}_1}{n}\;\cos\Omega\;+\;\frac{\dot{\mu}_2}{n}\;\sin\Omega\right)\,+\,O(e^2), \tag{53}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = -\mu_1 \cos\Omega - \mu_2 \sin\Omega - \frac{\dot{\mu}_\perp}{n} \cot i - \frac{\dot{\mu}_n}{2n} + \frac{1}{\sin i} \frac{\dot{\mu}_3}{n} + O(e^2), \tag{54}$$

$$\begin{aligned}\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= -\frac{3}{2} n J_2 \left(\frac{\rho_e}{a}\right)^2 \cos i - \frac{\mu_n}{\sin i}\\ &+ \frac{1}{2 \sin i} \left[- \left(\frac{\dot{\mu}_1}{n} \cos\Omega + \frac{\dot{\mu}_2}{n} \sin\Omega \right) \right] + O(e^2),\end{aligned} \tag{55}$$

where $\mu_\perp$ and $\dot{\mu}_\perp$ are given by (43) and (44). The quantity

$$\mu_n \equiv -\mu_1 \sin\Omega \cos i + \mu_2 \cos\Omega \cos i + \mu_3 \sin i \tag{56}$$

is the component of $\vec{\mu}$, pointing from the gravitating centre towards the ascending node of the orbit, while

$$\dot{\mu}_n = -\dot{\mu}_1 \sin\Omega \cos i + \dot{\mu}_2 \cos\Omega \cos i + \dot{\mu}_3 \sin i \tag{57}$$

is its time derivative taken in the frame co-precessing with the satellite orbit plane. (Taking the derivative in this frame, we differentiate only the components of $\vec{\mu}$, but not the angles.)

Under the assumption of constant a and small e, Eqs. (54) and (55) make a closed system.

3.3 Goldreich's approximation

It would now be tempting to introduce an even stronger assumption that both $|\dot{\vec{\mu}}|/\left(n^2 J_2 \sin i\right)$ and $|\vec{\mu}|/(n J_2 \sin i)$ are much less than unity, and to derive therefrom the system

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} \approx -\frac{3}{2} n J_2 \left(\frac{\rho_e}{a}\right)^2 \frac{\cos i}{\left(1-e^2\right)^2}, \tag{58}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} \approx -\mu_1 \cos\Omega - \mu_2 \sin\Omega, \tag{59}$$

whose solution,

$$\begin{aligned} i &= -\frac{\mu_1}{\chi} \cos[-\chi\,(t-t_o)+\Omega_o] + \frac{\mu_2}{\chi} \sin[-\chi\,(t-t_o)+\Omega_o] + i_o,\\ \Omega &= -\chi\,(t-t_o)+\Omega_o \quad \text{where} \quad \chi \equiv \frac{3}{2} n J_2 \left(\frac{\rho_e}{a}\right)^2 \frac{\cos i}{\left(1-e^2\right)^2}, \end{aligned} \tag{60}$$

seems to indicate that, in the course of planet precession (the term "precession" including, as agreed above, also nutations and the Chandler wobble and polar wander), the satellite inclination oscillates about i_o. Approximation (60) has already appeared in the literature. Goldreich (1965) derived such equations for the orbital averages of some nonosculating elements (which later were termed, by Brumberg (1992), "contact elements"). In our case, however, the approximation (60) was derived for the secular parts of osculating elements. We see that, in neglect of $\vec{\mu}^2$-terms and under the assumption of constant $\vec{\mu}$, the equations for the secular parts of osculating elements coincide with those for the secular parts of the contact ones (for a detailed explanation of this fact see Efroimsky 2005a).

The evident flaw of approximation (58)–(60) is its invalidity in the closest vicinity of the equator. In this vicinity, the parameters $|\dot{\vec{\mu}}|/\left(n^2 J_2 \sin i\right)$ and $|\vec{\mu}|/(n J_2 \sin i)$ are no longer small; so the entire approximation falls apart and gives no immediate indication on whether the inclination will go through zero and alter its sign or will "bounce off" the equator. At the first glance, this technical subtlety does not affect this approximation's main physical outcome, one that the inclination remains limited and shows no secular increase. In reality, though, the matter needs further exploration. For example, if the orbit keeps bouncing off the equator and the sign of i stays unaltered for long, then the term $\dot{\mu}_n/(2n)$ in (54) may, potentially, keep accumulating through aeons, creating a drift of the inclination. Whether this is so or not can be learned numerically through a more accurate approximation based on Eqs. (53)–(55). A more definite thing is that the Goldreich approximation is intended only for low inclinations: as can be seen from Eq. (55), at high inclinations it will fail, because the term $\mu_\perp/\sin i$ will dominate over the $J_2 \cos i$ term. All these issues will be addressed in our subsequent paper (Lainey et al. 2007, submitted).

3.4 Can precession cause secular changes of the inclination?

Above we saw that the Goldreich approximation reveals no secular terms in the expression for the inclination relative to the moving equator. While a reliable quest into this matter will demand numerical integration of the entire system of the planetary equations, we shall try to work out a qualitative estimate based on the approximation less crude than that of Goldreich. To this end we shall plug (44) and (56) into (54), and shall omit all the long-period terms. Thus we shall be left with the following estimate for the secular part of i:

$$\frac{\mathrm{d}i^{(\mathrm{sec})}}{\mathrm{d}t} = \frac{\dot{\mu}_3}{2n} \sin i + O(e^2) + \text{long-period terms}. \tag{61}$$

3.4.1 Small initial inclinations

For small inclinations, the above equation will look:

$$\frac{\mathrm{d}i^{(\mathrm{sec})}}{\mathrm{d}t} \approx A\, i^{(\mathrm{sec})}, \tag{62}$$

where

$$A \equiv \frac{\dot{\mu}_3}{2n} \approx \frac{\ddot{h}_p \cos I_p}{2n} \approx \frac{\ddot{h}_p}{2n}, \tag{63}$$

h_p and I_p being the longitude of the node and the inclination of the equator of date on that of epoch (see Appendix A8). From here we see that the osculating component of i will, approximately, obey

$$i^{(\mathrm{sec})} \approx i_o\, e^{At}. \tag{64}$$

The exponential dependence evidences of the presence of chaos in the system. It should be mentioned, though that the chaos will be weak, because A is extremely small. Besides, the second derivative of the precessing equator's node, $\ddot{h}_p$, which enters the expression for A, does not keep the same sign through aeons. The rate, at which node h_p and the inclination I_p evolve, can be computed, via the Colombo equation, from the rate of precession of the planet's orbit about the Sun (for details on the Colombo model see Appendix A). Qualitatively, one may expect the spectrum of h_p and I_p to resemble the frequency content of the planet's orbital precession (see the table in Appendix A). On all these grounds, the time dependence of i is constituted by the high-frequency oscillations (60) superimposed on a much slower evolution (64). We expect this slow evolution to look as a saw-tooth plot, because the sign of $\ddot{h}_p$ (and therefore of A) alters from time to time for the reason explained above. Due to this saw-tooth nature of the long-term evolution of $i^{(\mathrm{sec})}$, no considerable secular increase of the satellite's inclination should be expected, at least in the case of a small initial i_o. Numerical calculations performed in the e^3 order confirm this conclusion. Moreover, it turns out that even at not too small initial inclinations no secular changes in i accumulate over time scales of order billion years.

The said numerical results and plots are presented in our subsequent paper Lainey et al. (2007), devoted to a numerical implementation of our semianalytical model in the e^3 order.

3.4.2 Large initial inclinations

For near-polar orbits, Eq. (61) will read:

$$\frac{\mathrm{d}i^{(\mathrm{sec})}}{\mathrm{d}t} \approx A, \tag{65}$$

whence

$$i^{(\mathrm{sec})} \approx A\,t. \tag{66}$$

Once again, due to the undulatory sign alterations of A, we shall get a "saw-tooth" behaviour, though this time the teeth will be less steep than in the small-inclination case governed by the exponent (64). The teeth will be expected to cross the polar orbit once in a while. This kind of time dependence (so-called "crankshaft") is indeed what results from the numerical computations (Ibid.).

All in all, unless we begin very close to the pole, the variable equinoctial precession is not expected to entail secular changes in the satellite inclination relative to the equator of date. Exceptional is the case of near-polar orbits: in that case, leaps across the pole are possible (see Ibid. for details and plots).

4 Preparation for computation in the e^3 order

Insertion of (19) into Eqs.[7] (20)–(24) will lead us to the following system:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\,2\,\frac{\dot{\mu}_\perp}{n}\;a\;\left(1-e^2\right)^{1/2}, \tag{67}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{5}{2}\,\frac{\dot{\mu}_\perp}{n}\;e\;\left(1-e^2\right)^{1/2}, \tag{68}$$

[7] As within this model both the Hamiltonian perturbation and the $\dot{\boldsymbol{\mu}}$-dependent terms are substituted with their orbital averages, M_o becomes a nuisance parameter, so the planetary equation for $\mathrm{d}M_o/\mathrm{d}t$ is omitted.

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{3}{2}\frac{n J_2}{\left(1-e^2\right)^2}\left(\frac{\rho_e}{a}\right)^2\left(\frac{5}{2}\cos^2 i - \frac{1}{2}\right)$$

$$-\mu_\perp + \mu_n \cot i - \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\left\langle \dot{\vec{\mu}}\left(-\vec{f}\times\frac{\partial\vec{f}}{\partial i}\right)\right\rangle, \tag{69}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = -\mu_1\cos\Omega - \mu_2\sin\Omega$$

$$+\frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\left\langle \dot{\vec{\mu}}\left(-\vec{f}\times\frac{\partial\vec{f}}{\partial\omega}\right)\right\rangle$$

$$-\frac{1}{na^2(1-e^2)^{1/2}\sin i}\left\langle \dot{\vec{\mu}}\left(-\vec{f}\times\frac{\partial\vec{f}}{\partial\Omega}\right)\right\rangle, \tag{70}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -\frac{3}{2}nJ_2\left(\frac{\rho_e}{a}\right)^2\frac{\cos i}{\left(1-e^2\right)^2} - \frac{\mu_n}{\sin i}$$

$$+\frac{1}{n\,a^2\,(1-e^2)^{1/2}\,\sin i}\left\langle \dot{\vec{\mu}}\left(-\vec{f}\times\frac{\partial\vec{f}}{\partial i}\right)\right\rangle, \tag{71}$$

where, according to Appendix A,

$$\left\langle \dot{\vec{\mu}}\cdot\left(-\vec{f}\times\frac{\partial\vec{f}}{\partial i}\right)\right\rangle$$

$$= \frac{a^2}{4}\left\{\dot{\mu}_1\left[-\left(2+3e^2\right)\cos\Omega + 5e^2\left(\cos\Omega\cos 2\omega - \sin\Omega\sin 2\omega\cos i\right)\right]\right.$$

$$+\dot{\mu}_2\left[-\left(2+3e^2\right)\sin\Omega + 5e^2\left(\sin\Omega\cos 2\omega + \cos\Omega\sin 2\omega\cos i\right)\right]$$

$$\left.+\dot{\mu}_3\left[5e^2\sin 2\omega\sin i\right]\right\} \tag{72}$$

$$\left\langle \dot{\vec{\mu}}\cdot\left(-\vec{f}\times\frac{\partial\vec{f}}{\partial\omega}\right)\right\rangle = -\frac{a^2}{2}\left(2+3e^2\right)$$

$$\times\left(\dot{\mu}_1\sin i\sin\Omega - \dot{\mu}_2\sin i\cos\Omega + \dot{\mu}_3\cos i\right), \tag{73}$$

$$\left\langle \dot{\vec{\mu}}\cdot\left(-\vec{f}\times\frac{\partial\vec{f}}{\partial\Omega}\right)\right\rangle$$

$$= \frac{a^2}{4}\left\{\dot{\mu}_1\sin i\left[-\left(2+3e^2\right)\sin\Omega\cos i + 5e^2\left(\cos\Omega\sin 2\omega\right.\right.\right.$$

$$\left.\left.+\sin\Omega\cos 2\omega\cos i\right)\right]$$

$$+\dot{\mu}_2\sin i\left[\left(2+3e^2\right)\cos\Omega\cos i\right.$$

$$\left.+5e^2\left(\sin\Omega\sin 2\omega - \cos\Omega\cos 2\omega\cos i\right)\right]$$

$$\left.-\dot{\mu}_3\left[\left(2+3e^2\right)\left(2-\sin^2 i\right) + 5e^2\sin^2 i\cos 2\omega\right]\right\} \tag{74}$$

5 Conclusions

In this article, we continued our analytical investigation of the behaviour of orbits about a precessing oblate planet. We built up a reasonably simplified model that takes into account both the long-term variability of the planetary precession (variability caused by the planet's orbit precession) and the short-term variability (polar wonder, etc.).

We have written down equations (67)–(71) that describe evolution of the satellite orbit at long time scales. The equations include known functions of time, μ_1, μ_2, μ_3, $\mu_\perp$, μ_n, $\dot{\mu}_1$, $\dot{\mu}_2$, $\dot{\mu}_3$, $\dot{\mu}_\perp$, $\dot{\mu}_n$, which are various projections of the planet axis' precession rate and of this rate's time derivative. An algorithm for computation of these functions of time is presented in Appendix A. These functions vary in time as a result of precession of the primary's orbit about the Sun. This way, we have analytically established connection between the precession of the planet's orbit and the evolution of its satellites. Physically, this connection comes into being through the following concatenation of circumstances: precession of the planetary orbit leads to variations in the Solar torque acting on the planet; the torque variations cause changes in the planet axis' precession; these changes, in their turn, entail variations of orbits of the planet's satellite. This effect is extremely weak and accumulates over very long time scales. Our preliminary analytical estimates have shown that no considerable secular alterations of the inclination should be expected, except in the case of near-polar orbits.

All in all, we have fully prepared a launching pad for computation of the evolution of near-equatorial circummartian orbits at long time scales. The methods and results of this integration, and their physical interpretation will be presented in our next publication (Lainey et al. 2007). Briefly speaking, those results are to be threefold. First, it turns out that our semianalytical model is robust beyond expectations. Despite the averaging and the neglect of $\vec{\mu}^{\,2}$-terms, it works very well over timescales up to, at least, 20 Myr. Second, it turns out that precession by itself (i.e., in the absence of the other physical factors like the tides, the pull of the Sun, etc.) cannot cause accumulating secular changes in the satellite inclination, *provided the initial inclination is not too large*. This means that, for orbits not too close to the polar one, the main prediction of the Goldreich model stays valid, even though the model cannot adequately describe the entire dynamics (which becomes weakly chaotic). Third, it turns out that in the vicinity of the polar orbit precession of the primary can cause major alteration of the satellite orbits, including unusual features in the behaviour of the inclination. See *Ibid.* for more details. Further work along this line of research will be aimed at including more factors into the model—the tidal forces, the pull of the Sun, the triaxiality of the planet, etc.

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Appendix A

Here we calculate, in neglect of nutation-caused resonances, the secular parts of the $\vec{\mu}$-dependent terms emerging in the planetary equations (19)–(25). We also explain how to compute the time dependence of various projections of the planetary precession rate $\vec{\mu}$ and of its time derivatives.

A.1 The averaging rule

The mean values are to be calculated via the averaging rule:

$$\langle \cdots \rangle \equiv \frac{\left(1 - e^2\right)^{3/2}}{2\,\pi} \int_{-\pi}^{\pi} \cdots \frac{\mathrm{d}\nu}{\left(1 + e\,\cos\nu\right)^2}. \qquad (75)$$

Since the averaging is carried out over the true anomaly, it will be convenient to express the precession rate not as a function of time, $\vec{\mu}(t)$, but as a function of the true anomaly:

$$\vec{\mu}(\nu) = \int_0^\infty \left[\, \vec{\mu}^{(s)}(W)\ \sin(W\nu) + \vec{\mu}^{(c)}(W)\ \cos(W\nu) \right] \mathrm{d}W, \qquad (76)$$

where W being the circular "frequency" related to the true anomaly ν .

In what follows, we shall need the average of the projection of $\vec{\mu}(t)$ onto the instantaneous normal to the orbit. This projection, $\mu_\perp$, will be expressed by

$$\mu_\perp \equiv \vec{\mu} \cdot \vec{w} = \mu_1\ \sin i\ \sin\Omega - \mu_2\ \sin i\ \cos\Omega + \mu_3\ \cos i, \qquad (77)$$

where the unit vector

$$\vec{w} = \hat{\mathbf{x}}\ \sin i\ \sin\Omega - \hat{\mathbf{y}}\ \sin i\ \cos\Omega + \hat{\mathbf{z}}\ \cos i, \qquad (78)$$

stands for the normal to the instantaneous osculating ellipse, and the unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are the basis of the co-precessing coordinate system x, y, z. (The axes x and y belong to the planet's equatorial plane, and the longitude of the node, Ω, is measured from x.) Expressions of μ_j via the longitude of the node and inclination of the equator of date relative to that of epoch are given in Sect. A.3.

A.2 Calculation of the secular and long-period parts of $\dot{\vec{\mu}} \cdot \left(-\vec{f} \times \frac{\partial \vec{f}}{\partial \Omega} \right)$

As an example, here we shall present calculation of the secular and long-term parts of the expression $\dot{\vec{\mu}} \cdot \left(-(\vec{f} \times \partial\vec{f}/\partial\Omega)\right)$. Calculation of the secular and long-term components of $\dot{\vec{\mu}} \cdot \left(-(\vec{f} \times \partial\vec{f}/\partial C_i)\right)$, with $C_i = a, e, i, \omega, M_o$, are performed in a similar manner. They can be found in the extended version of this paper, which is available on-line (Efroimsky 2006).

For the purpose of this calculation we shall need the following auxiliary integrals:

$$\Upsilon_0 \equiv \int_{-\pi}^{\pi} \frac{1}{(1 + e\,\cos\nu)^4}\,\mathrm{d}\nu = \pi\,\frac{2 + 3\,e^2}{\left(1 - e^2\right)^{7/2}}, \qquad (79)$$

$$\Upsilon_1 \equiv \int_{-\pi}^{\pi} \frac{\sin(\omega+\nu)\ \cos(\omega+\nu)}{(1\,+\,e\,\cos\nu)^4}\,d\nu \;=\; \frac{5}{2}\,\pi\,e^2\,\frac{\sin(2\omega)}{\left(1\,-\,e^2\right)^{7/2}}, \tag{80}$$

$$\Upsilon_2 \equiv \int_{-\pi}^{\pi} \frac{\sin^2(\omega+\nu)}{(1\,+\,e\,\cos\nu)^4}\,d\nu \;=\; \frac{1}{2}\,\pi\,\frac{2\,+\,3\,e^2\,-\,5\,e^2\,\cos(2\omega)}{\left(1\,-\,e^2\right)^{7/2}}. \tag{81}$$

The first component:

$$\begin{aligned}
&\left\langle\,\dot{\mu}_1\left(\frac{\partial\vec{f}}{\partial\Omega}\times\vec{f}\right)_1\right\rangle\\
&=\left\langle\,\dot{\mu}_1\,a^2\,\frac{\left(1\,-\,e^2\right)^2}{(1\,+\,e\,\cos\nu)^2}\right.\\
&\qquad\left.\times\left[\,\cos\Omega\,\cos(\omega+\nu)\,-\,\sin\Omega\,\sin(\omega+\nu)\,\cos i\,\right]\,\sin(\omega+\nu)\,\sin i\right\rangle\\
&=\,\dot{\mu}_1\,a^2\left(1\,-\,e^2\right)^2\,\frac{\left(1\,-\,e^2\right)^{3/2}}{2\,\pi}\,\left\{\,\Upsilon_1\,\cos\Omega\,\sin i\,-\,\Upsilon_2\,\sin\Omega\,\cos i\,\sin i\,\right\}\\
&=\dot{\mu}_1\frac{a^2}{4}\sin i\left\{-\left(2+3e^2\right)\sin\Omega\cos i+5e^2\left[\cos\Omega\sin(2\omega)+\sin\Omega\cos(2\omega)\cos i\right]\right\}.
\end{aligned} \tag{82}$$

The second component:

$$\begin{aligned}
&\left\langle\,\dot{\mu}_2\left(\frac{\partial\vec{f}}{\partial\Omega}\times\vec{f}\right)_2\right\rangle\\
&=\left\langle\,\dot{\mu}_2\,a^2\,\frac{\left(1\,-\,e^2\right)^2}{(1\,+\,e\,\cos\nu)^2}\right.\\
&\qquad\left.\times\left[\,\sin\Omega\,\cos(\omega+\nu)\,+\,\cos\Omega\,\sin(\omega+\nu)\,\cos i\,\right]\,\sin(\omega+\nu)\,\sin i\right\rangle\\
&=\,\dot{\mu}_2\,a^2\left(1\,-\,e^2\right)^2\,\frac{\left(1\,-\,e^2\right)^{3/2}}{2\,\pi}\,\left\{\,\Upsilon_1\,\sin\Omega\,\sin i\,+\,\Upsilon_2\,\cos\Omega\,\sin i\,\cos i\,\right\}\\
&=\dot{\mu}_2\,a^2\,\frac{\left(1\,-\,e^2\right)^{7/2}}{2\,\pi}\,\left\{\left[\frac{5}{2}\,\pi\,e^2\,\frac{\sin(2\omega)}{\left(1\,-\,e^2\right)^{7/2}}\right]\sin\Omega\right.\\
&\qquad\left.+\left[\frac{1}{2}\,\pi\,e^2\,\frac{2\,+\,3\,e^2\,-\,5\,e^2\,\cos(2\omega)}{\left(1\,-\,e^2\right)^{7/2}}\right]\cos\Omega\right\}\,\sin i\\
&=\,\dot{\mu}_2\,\frac{a^2}{4}\,\sin i\,\left\{\left(2\,+\,3\,e^2\right)\,\cos\Omega\,\cos i\right.\\
&\qquad\qquad\left.+\,5e^2\,\left[\sin\Omega\,\sin(2\omega)\,-\,\cos\Omega\,\cos(2\omega)\,\cos i\right]\right\}.
\end{aligned} \tag{83}$$

The third component:

$$\left\langle \dot{\mu}_3 \left(\frac{\partial \vec{f}}{\partial \Omega} \times \vec{f} \right)_3 \right\rangle$$

$$= \left\langle -\,\dot{\mu}_3\, a^2\, \frac{\left(1-e^2\right)^2}{(1+e\,\cos\nu)^2} \left[\cos^2(\omega+\nu) + \sin^2(\omega+\nu)\,\cos^2 i\right] \right\rangle$$

$$= -\dot{\mu}_3\, a^2\, \frac{\left(1-e^2\right)^{7/2}}{2\,\pi} \left(\Upsilon_0 - \Upsilon_2 \sin^2 i\right)$$

$$= -\dot{\mu}_3\, a^2\, \frac{\left(1-e^2\right)^{7/2}}{2\,\pi} \left\{\pi\, \frac{2+3e^2}{\left(1-e^2\right)^{7/2}} - \pi\, \frac{2+3e^2-5e^2\,\cos(2\omega)}{2\,\left(1-e^2\right)^{7/2}}\, \sin^2 i\right\}$$

$$= -\,\dot{\mu}_3\, \frac{a^2}{4} \left\{ \left(2 + 3\,e^2\right)\left[2 - \sin^2 i\right] + 5\,e^2\,\sin^2 i\,\cos(2\omega) \right\} \tag{84}$$

Total:

$$\left\langle \dot{\vec{\mu}} \cdot \left(-\vec{f} \times \frac{\partial \vec{f}}{\partial \Omega} \right) \right\rangle$$

$$= \frac{a^2}{4} \left\{ \dot{\mu}_1 \sin i \left[-\left(2+3e^2\right) \sin\Omega \cos i + 5e^2\,(\cos\Omega \sin 2\omega + \sin\Omega \cos 2\omega \cos i)\right] \right.$$
$$+\,\dot{\mu}_2 \sin i \left[\left(2+3e^2\right) \cos\Omega \cos i + 5e^2\,(\sin\Omega \sin 2\omega - \cos\Omega \cos 2\omega \cos i)\right]$$
$$\left. -\,\dot{\mu}_3 \left[\left(2 + 3\,e^2\right)\left(2 - \sin^2 i\right) + 5\,e^2\,\sin^2 i\,\cos 2\omega \right] \right\}. \tag{85}$$

Interestingly, even in the limit of vanishing eccentricity this sum survives and becomes

$$\frac{a^2}{2} \left\{ -\,\dot{\mu}_1\,\sin i\,\sin\Omega\,\cos i + \dot{\mu}_2\,\sin i\,\cos\Omega\,\cos i - \dot{\mu}_3 \left(2 - \sin^2 i\right) \right\}$$
$$= \frac{a^2}{2}\,\dot{\mu}_n\,\sin i - a^2\,\dot{\mu}_3. \tag{86}$$

Moreover, even when both the eccentricity and inclination are nil, this sum still remains non-zero.

A.3 The planetary precession rate $\vec{\mu}$ and its projection $\mu_\perp$ onto the satellite's orbital momentum

Let the inertial axes $(X,\ Y,\ Z)$ and the corresponding unit vectors $(\hat{\mathbf{X}},\ \hat{\mathbf{Y}},\ \hat{\mathbf{Z}})$ be fixed in space so that X and Y belong to the equator of epoch. A rotation within the equator-of-epoch plane by longitude h_p, from axis X, will define the line of nodes, x. A rotation about this line by an inclination angle I_p will give us the planetary equator of date. The line of nodes x, along with axis y naturally chosen within the equator-of-date plane, and with axis z orthogonal to this plane, will constitute the precessing coordinate system, with the appropriate basis denoted by $(\hat{\mathbf{x}},\ \hat{\mathbf{y}},\ \hat{\mathbf{z}})$.

In the inertial basis $(\hat{\mathbf{X}},\ \hat{\mathbf{Y}},\ \hat{\mathbf{Z}})$, the direction to the North Pole of date is given by

$$\hat{\mathbf{z}} = \left(\sin I_p\, \sin h_p,\quad -\sin I_p\, \cos h_p \ \text{ and } \ \cos I_p \right)^T, \tag{87}$$

while the total angular velocity reads:

$$\vec{\boldsymbol{\omega}}^{(\text{inertial})}_{\text{total}} \;=\; \hat{\mathbf{z}}\, s \;+\; \vec{\boldsymbol{\mu}}^{(\text{inertial})} \tag{88}$$

the first term denoting the rotation about the precessing axis $\hat{\mathbf{z}}$, the second term being the precession rate of $\hat{\mathbf{z}}$ relative to the inertial frame $(\hat{\mathbf{X}},\, \hat{\mathbf{Y}},,\, \hat{\mathbf{Z}})$, and s standing for the angular velocity of rotation about the axis $\hat{\mathbf{z}}$. This precession rate is given by

$$\vec{\boldsymbol{\mu}}^{(\text{inertial})} \;=\; \left(\dot{I}_p \, \cos h_p, \dot{I}_p \, \sin h_p, \dot{h}_p \right)^T, \tag{89}$$

because this expression satisfies $\vec{\boldsymbol{\mu}}^{(\text{inertial})} \times \hat{\mathbf{z}} = \dot{\hat{\mathbf{z}}}$.

In a frame precessing with the equator, the precession rate will be represented by vector

$$\vec{\boldsymbol{\mu}} \;=\; \hat{\mathbf{R}}_{i\to p} \;\; \vec{\boldsymbol{\mu}}^{(\text{inertial})}, \tag{90}$$

where the matrix of rotation from the equator of epoch to that of date (i.e., from the inertial frame to the precessing one) is given by

$$\hat{\mathbf{R}}_{i\to p} \;=\; \begin{bmatrix} \cos h_p & \sin h_p & 0 \\ -\,\cos I_p \;\sin h_p & \cos I_p \;\cos h_p & \sin I_p \\ \sin I_p \;\sin h_p & -\sin I_p \;\cos h_p & \cos I_p \end{bmatrix}. \tag{91}$$

From here we get the components of the precession rate, as seen in the co-precessing coordinate frame $(x\,,\; y\,,\; z)$:

$$\vec{\boldsymbol{\mu}} \;=\; (\,\mu_1,\;\; \mu_2,\;\; \mu_3\,)^T \;=\; \left(\dot{I}_p, \quad \dot{h}_p\, \sin I_p, \quad \dot{h}_p\, \cos I_p \right)^T. \tag{92}$$

In our paper we also need the components of $\dot{\vec{\boldsymbol{\mu}}}$, dot standing for derivatives calculated in the frame co-precessing with the equator:

$$\dot{\vec{\boldsymbol{\mu}}} = (\dot{\mu}_1 \;\;,\;\; \dot{\mu}_2 \;\;,\;\; \dot{\mu}_3)^T = \left(\ddot{I}_p \;\;,\;\; \ddot{h}_p\, \sin I_p + \dot{h}_p \dot{I}_p \cos I_p \;\;,\;\; \ddot{h}_p \cos I_p - \dot{h}_p \dot{I}_p \sin I_p\right)^T. \tag{93}$$

The matrix of rotation from the precessing frame of the equator of date to the frame associated with the satellite's orbital plane will look:

$$\hat{\mathbf{R}}_{p\to o} \;=\; \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\,\cos i\;\sin\Omega & \cos i\;\cos\Omega & \sin i \\ \sin i\;\;\sin\Omega & -\sin i\;\;\cos\Omega & \cos i \end{bmatrix}. \tag{94}$$

This will give us the precession rate as seen in the instantaneous orbit frame:

$$\vec{\boldsymbol{\mu}}^{(\text{orb})} \;=\; \hat{\mathbf{R}}_{p\to o} \;\; \vec{\boldsymbol{\mu}}. \tag{95}$$

This vector's component pointing towards the ascending node of the satellite orbit relative to the equator of date) is what we need in our formulae (52) and (54):

$$\begin{aligned} \mu_n \;&=\; -\,\mu_1\;\sin\Omega\;\cos i \,+\, \mu_2\;\cos\Omega\;\cos i \,+\, \mu_3\;\sin i \\ &=\, -\,\dot{I}_p\;\sin\Omega\;\cos i \,+\, \dot{h}_p\;\sin I_p\;\cos\Omega\;\cos i \,+\, \dot{h}_p\;\cos I_p\;\sin i. \end{aligned} \tag{96}$$

Its time derivative (taken in the frame of reference precessing with the equator of date) is:

$$\dot{\mu}_n = -\dot{\mu}_1 \,\sin\Omega\, \cos i + \dot{\mu}_2 \,\cos\Omega\, \cos i + \dot{\mu}_3 \,\sin i$$

$$= -\ddot{I}_p \,\sin\Omega\, \cos i + \left(\ddot{h}_p \,\sin I_p + \dot{h}_p \dot{I}_p \,\cos I_p\right) \cos\Omega\, \cos i$$
$$+ \left(\ddot{h}_p \,\cos I_p - \dot{h}_p \dot{I}_p \,\sin I_p\right) \sin i. \quad (97)$$

The third component of $\vec{\boldsymbol{\mu}}^{(\text{orb})}$ (i.e., the component orthogonal to the instantaneous plane of the orbit) is exactly what we need in (27) and (33):

$$\mu_\perp = \mu_1 \,\sin i\, \sin\Omega - \mu_2 \,\sin i\, \cos\Omega + \mu_3 \,\cos i$$
$$= \dot{I}_p \,\sin i\, \sin\Omega - \dot{h}_p \,\sin I_p \,\sin i\, \cos\Omega + \dot{h}_p \,\cos I_p \,\cos i. \quad (98)$$

Its time derivative $\dot{\mu}_\perp$ defined in the axes co-precessing with the equator (and therefore equal to $\dot{\vec{\boldsymbol{\mu}}} \cdot \vec{\boldsymbol{w}}$, not to $\mathrm{d}(\vec{\boldsymbol{\mu}} \cdot \vec{\boldsymbol{w}})/\mathrm{d}t$) will now be expressed by

$$\dot{\mu}_\perp \equiv \dot{\mu}_1 \,\sin i\, \sin\Omega - \dot{\mu}_2 \,\sin i\, \cos\Omega + \dot{\mu}_3 \,\cos i$$
$$= \ddot{I}_p \,\sin i\, \sin\Omega - \left(\ddot{h}_p \,\sin I_p + \dot{h}_p \dot{I}_p \,\cos I_p\right) \sin i\, \cos\Omega$$
$$+ \left(\ddot{h}_p \,\cos I_p - \dot{h}_p \dot{I}_p \,\sin I_p\right) \cos i \quad (99)$$
$$\approx \ddot{h}_p \left(-\sin I_p \,\sin i\, \cos\Omega + \cos I_p \,\cos i\right), \quad (100)$$

I_p and h_p being the inclination and the longitude of the node of the equator of date relative to the one of epoch.

The expression for $\dot{\mu}_\perp$ permitted approximations shown above because, for Mars' equator, the speed of the nodes' motion, $|\dot{h}_p| \approx 360°/(1.75 \times 10^5\,\text{year}) \approx 2 \times 10^{-3}\,\text{year}^{-1}$, much exceeds the rate of its inclination change, $|\dot{I}_p| \approx 5°/(0.5 \times 10^6\,\text{year}) \approx 10^{-5}\,\text{year}^{-1}$ (Ward 1974).

A.4 Calculation of h_p and I_p

The question now becomes as to how to calculate the time dependence of h_p and I_p . As very well known, these two angles evolve in time because a non-spherical planet behaves itself as an unsupported top whose precession is instigated by the solar torque. The torques produced by the satellites are irrelevant (Laskar 2004), the cases of the Moon and Charon being exceptional. When the moments of inertia of the planet relate as $C > B = A$, the solar torque is

$$\vec{\mathbf{T}} = \frac{3\,G\,M}{R^3}\,(C - A)\,\left(\hat{\mathbf{r}} \cdot \hat{\mathbf{k}}\right)\,\left(\hat{\mathbf{r}} \times \hat{\mathbf{k}}\right), \quad (101)$$

while in the general case of $C \geq B \geq A$ it is equal to

$$\vec{\mathbf{T}} = \frac{3\,G\,M}{R^3}\,\left(C - \frac{A+B}{2}\right)\,\left(\hat{\mathbf{r}} \cdot \hat{\mathbf{k}}\right)\,\left(\hat{\mathbf{r}} \times \hat{\mathbf{k}}\right), \quad (102)$$

provided that the spin mode is not too deviant from the principal one, and that this spin is much faster than the planet's orbital revolution about the Sun.

In the above two formulae, M is the solar mass, R denotes the distance between the centres of masses of the planet and the Sun, the unit vector $\hat{\mathbf{r}}$ points from the planet towards the Sun, and the unit vector $\hat{\mathbf{k}}$ points in the direction of the major-inertia axis of the planet.

The precession of the angular momentum $\vec{\mathbf{L}}$ of the planetary spin obeys

$$\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \vec{\mathbf{T}}. \tag{103}$$

Colombo (1966) averaged this equation over the planet's year, under the assumption that the perturbing torque causes only very small variations of spin. This averaging yields the Colombo equation valid at timescales much exceeding 1 year:

$$\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \frac{3\,n_p^2}{2\left(1 - e_p^2\right)^{3/2}} \left(C - \frac{A+B}{2}\right) \left(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\right) \left(\hat{\mathbf{k}}\times\hat{\mathbf{n}}\right), \tag{104}$$

n_p and e_p being the mean motion and the eccentricity of the planet's orbit about the Sun, and $\hat{\mathbf{n}}$ being the unit vector normal to the planetary orbit; while $\mathrm{d}\vec{\mathbf{L}}/\mathrm{d}t$ should be understood as a change of $\vec{\mathbf{L}}$ over a year, divided by the length of the year: $\Delta\vec{\mathbf{L}}/P$. The angular velocity of the planet about its axis being denoted with letter s, the Colombo equation may be rewritten as

$$\frac{1}{s\,C}\,\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \alpha\,\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\right)\,\left(\hat{\mathbf{k}}\times\hat{\mathbf{n}}\right) \tag{105}$$

the factor α being defined as

$$\begin{aligned}\alpha &\equiv \frac{3\,n_p^2}{2\,s\,\left(1 - e_p^2\right)^{3/2}}\;\frac{C - \frac{A+B}{2}}{C}\\ &= \frac{2\pi}{P}\,\frac{D}{P}\,\frac{1}{\left(1 - e_p^2\right)^{3/2}}\left[\frac{3}{2}\;\frac{C - \frac{A+B}{2}}{C}\right],\end{aligned} \tag{106}$$

where $P = 2\pi/n_p$ is the duration of the planet's year, and $D = 2\pi/s$ is that of its day. The relative difference between the moments of inertia may be expressed through the parameter J_2 emerging in the expression for potential via the planetocentric latitude ϕ

$$\begin{aligned}V = &-\frac{Gm}{r}\left[1 - \sum_{n=2}^{\infty} J_n\,\left(\frac{\rho_e}{r}\right)^n\,P_n(\sin\phi)\right.\\ &\left.+\sum_{n=2}^{\infty}\sum_{j=1}^{n} J_{nj}\,\left(\frac{\rho_e}{r}\right)^n\,P_{nj}\,(\sin\phi)\,\cos j\left(\lambda - \lambda_{nj}\right)\right]\end{aligned} \tag{107}$$

(where ρ_e stands for the mean *equatorial* radius of the planet):

$$J_2 = \frac{C - \frac{A+B}{2}}{M\,\rho_e^2} = \frac{C - \frac{A+B}{2}}{C}\;\frac{C}{M\,\rho_e^2}. \tag{108}$$

It is also interconnected with the nonsphericity parameter J:

$$J \equiv \frac{3}{2}\left(\frac{\rho_e}{\rho}\right)^2 J_2 = \frac{3}{2}\,\frac{C - \frac{A+B}{2}}{C}\;\frac{C}{M\,\rho^2} = \frac{3}{2}\,\frac{C - \frac{A+B}{2}}{C}\;K, \tag{109}$$

where ρ is simply the mean (not the mean equatorial) radius of the planet, while the quantity

$$K \equiv \frac{C}{m\,\rho^2} \tag{110}$$

is the squared ratio of the gyration radius $\sqrt{C/m}$ of the planet to its mean radius ρ. We thus see that

$$\frac{3}{2}\,\frac{C - \dfrac{A+B}{2}}{C} = \frac{J}{K} = \frac{3}{2}\,J_2\,\frac{m\,\rho_e^2}{C}, \tag{111}$$

whence

$$\alpha = \frac{2\pi}{P}\,\frac{D}{P}\,\frac{1}{\left(1-e_p^2\right)^{3/2}}\,\frac{J}{K} = \frac{2\pi}{P}\,\frac{D}{P}\,\frac{1}{\left(1-e_p^2\right)^{3/2}}\,\frac{3}{2}\,J_2\,\frac{m\,\rho_e^2}{C}. \tag{112}$$

Ward (1974) used, for Mars, $K = 0.359$ and $J = 2.95 \times 10^{-3}$, which gave him the value:[8] $\alpha_{\mathrm{Mars}} = 1.26 \times 10^{-12}$ rad/s $= 8.19$ arc sec/year.

To further simplify the above expression (105), Colombo (1966) assumed that the spin angular momentum $\vec{\mathbf{L}}$ is parallel to the spin angular velocity $\vec{\mathbf{s}}$:

$$\vec{\mathbf{L}} \approx C\,\vec{\mathbf{s}} \approx C\,s\,\hat{\mathbf{k}}. \tag{113}$$

While investigating the dynamics of the Moon at less than cosmic time scales, Colombo certainly could afford this approximation. Compare the latter with the exact expression for $\vec{\mathbf{L}}$ through $\vec{\mathbf{s}}$ and through the moments of inertia $C \geq B \geq A$:

$$\vec{\mathbf{L}} = \hat{\mathbf{i}}\,s_1\,A + \hat{\mathbf{j}}\,s_2\,B + \hat{\mathbf{k}}\,s_3\,C = \hat{\mathbf{i}}\,s_1\,(A - C) + \hat{\mathbf{j}}\,s_2\,(B - C) + C\,s\left(\hat{\mathbf{p}} - \hat{\mathbf{k}}\right) + C\,s\,\hat{\mathbf{k}}, \tag{114}$$

$\hat{\mathbf{s}} \equiv \left(\hat{\mathbf{i}}\,s_1 + \hat{\mathbf{j}}\,s_2 + \hat{\mathbf{k}}\,s_3\right)s^{-1}$ being the instantaneous direction of the angular velocity of the planet's spin. We see that Colombo's approximation stems from the frivolous assertion that the planet always remains in a principal spin state. Indeed, insofar as $\hat{\mathbf{s}}$ coincides with $\vec{\mathbf{k}}$ the components s_1 and s_2 are nil, and (113) becomes exact. Under such an assertion, the equation for unit vector aimed in the direction of the major-inertia axis, $\vec{\mathbf{k}}$, assumes the form[9]

$$\frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t} = \alpha\left(\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}\right)\left(\hat{\mathbf{k}}\times\hat{\mathbf{n}}\right) \tag{115}$$

[8] In his formulae, Ward (1973) missed or deliberately neglected the factor $\left(1-e_p^2\right)^{-3/2}$. In the case of Mars, this may look legitimate because nowadays this factor amounts to 1.013 . However, the Martian eccentricity is wont to have varied through aeons within the interval of $e = 0.01 - 0.14$ (Murray et al. 1973). This means that the said factor $\left(1-e_p^2\right)^{-3/2}$, might have varied from almost unity through 1.03 . This 3 % increase will look less than innocent if we recall that several authors (Ward 1974; Laskar and Robutel 1993; Touma and Wisdom 1994) insist on the stochastic nature of Mars' obliquity variations.

[9] By basing his research on the approximation (115), Ward (1974) implicitly made the strong assumption of Mars always remaining in the principal spin state, polar wander and nutations and the Chandler wobble being neglected. By employing a Hamiltonian, that generates equation (115), Laskar and Robutel (1993) and Touma and Wisdom (1994), too, rested their study on the same assumption.

the unit normal to the planet's orbit, $\vec{\mathbf{n}}$, being subject to variations described by the formulae and tables worked out by Brouwer and van Woerkom (1950).[10] The quantities h_p, I_p and their time derivatives can be calculated from integration of the above equation for $\hat{\mathbf{k}}$. To this end, let us recall that our unit vector $\hat{\mathbf{k}}$ coincides with the afore discussed unit vector $\hat{\mathbf{z}}$ (see formula (87) above). Therefore, in the frame of the equator of epoch (which we assume, for convenience, to coincide with the ecliptic of 1950), $\hat{\mathbf{k}}$ and $\mathrm{d}\hat{\mathbf{k}}/\mathrm{d}t$ will read:

$$\hat{\mathbf{k}} = \left(\sin I_p \sin h_p \;,\; -\sin I_p \cos h_p \;,\; \cos I_p\right)^T \;, \tag{116}$$

$$\frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t} = \Big(\dot{I}_p \cos I_p \;\sin h_p + \dot{h}_p \;\sin I_p \;\cos h_p \;,\; -\dot{I}_p \;\cos I_p \;\cos h_p + \dot{h}_p \;\sin I_p \;\sin h_p \;,\; -\dot{I}_p \;\sin I_p\Big)^T \;, \tag{117}$$

while the components of $\hat{\mathbf{n}}$,

$$\hat{\mathbf{n}} \;=\; (\sin I_{\mathrm{orb}} \;\sin \Omega_{\mathrm{orb}} \;,\; -\;\sin I_{\mathrm{orb}} \;\cos \Omega_{\mathrm{orb}} \;,\; \cos I_{\mathrm{orb}})^T \;, \tag{118}$$

may be expressed through the auxiliary variables

$$q = \sin I_{\mathrm{orb}} \;\sin \Omega_{\mathrm{orb}}, \quad p = \sin I_{\mathrm{orb}} \;\cos \Omega_{\mathrm{orb}}, \tag{119}$$

whose evolution will be found from

$$q \;=\; \sum_{j=1}^{\infty} N_j \;\sin \left(s'_j t \;+\; \delta_j\right) \;, \tag{120}$$

$$p = \sum_{j=1}^{\infty} N_j \;\cos \left(s'_j t \;+\; \delta_j\right) . \tag{121}$$

Under the assumption that the orbital elements are defined relative to the ecliptic plane of 1950, Brouwer and van Woerkom (1950) calculated the values of the amplitudes, frequencies, and phases used in the above formulae. Below follow the triples of numbers:

j	N_j	s'_j (arc sec/year)	$\delta'_j(°)$
1	0.0084889	−5.201537	19.43255
2	0.0080958	−6.570802	318.05685
3	0.0244823	−18.743586	255.03057
4	0.0045254	−17.633305	296.54103
5	0.0275703	+0.000004	107.10201
6	0.0028112	−25.733549	127.36669
7	−0.0017308	−2.902663	315.06348
8	−0.0012969	− 0.677522	202.29272

[10] Brouwer and van Woerkom (1950) chose the ecliptic of 1950 as the reference plane. Since our eventual goal is to simply estimate the range of variations of i and Ω over large time scales, we can accept, without loss of generality, that at some distant epoch the Martian equator coincided with that plane.

Technically, the computation of time evolution of I_p and h_p can be implemented through a set of differential equations obtained by substitution of

$$\hat{\mathbf{n}} = (\,q\,,\,-p\,,\,u\,)^T \quad , \quad q \equiv \sin I_{\mathrm{orb}}\,\sin\Omega_{\mathrm{orb}} \;,$$
$$p \equiv \sin I_{\mathrm{orb}}\,\cos\Omega_{\mathrm{orb}}\;,\; u \equiv \cos I_{\mathrm{orb}}\;, \tag{122}$$

$$\hat{\mathbf{k}} = (\,Q\,,\,-P\,,\,U\,)^T, \quad Q = \sin I_p\,\sin h_p,\;\; P = \sin I_p\,\cos h_p,\;\; U \equiv \cos I_p, \tag{123}$$

into the Colombo equation (115). Here follow these equations:

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = -\,\alpha\,\left[\,q(t)\,Q(t) + p(t)\,P(t) + u(t)\,U(t)\,\right]\,\left[\,-\,p(t)\,U(t) + u(t)\,P(t)\,\right], \tag{124}$$

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = \alpha\,\left[\,q(t)\,Q(t) + p(t)\,P(t) + u(t)\,U(t)\,\right]\,\left[\,u(t)\,Q(t) - q(t)\,U(t)\,\right], \tag{125}$$

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = -\,\alpha\,\left[\,q(t)\,Q(t) + p(t)\,P(t) + u(t)\,U(t)\,\right]\,\left[-q(t)\,P(t) + p(t)\,Q(t)\,\right], \tag{126}$$

where, at each time step, the following values of $q(t)$, $p(t)$ and $u(t)$ are to be used:

$$q(t) = \textstyle\sum_{j=1}^{\infty} N_j\,\sin\left(s'_j t + \delta_j\right), \tag{127}$$
$$p(t) = \textstyle\sum_{j=1}^{\infty} N_j\,\cos\left(s'_j t + \delta_j\right), \tag{128}$$
$$u(t) = \pm\sqrt{1 - q(t)^2 - p(t)^2}. \tag{129}$$

The resulting values of Q, P, U, obtained through this integration, will, at each time step, give us the angles I_p and h_p via formulae that follow from (122) and (123):

$$h_p = \arctan\frac{Q}{P} \qquad I_p = \arccos U, \tag{130}$$

It is evident from (123) that the variables $Q(t)$, $P(t)$ and $U(t)$ obey the constraint

$$Q(t)^2 + P(t)^2 + U(t)^2 = 1 \tag{131}$$

and therefore fulfilment of this constraint should be checked during integration. Deviation from it will indicate accumulation of errors. At each step, some attention will be needed also when the current value of $u(t)$ is evaluated (we mean the choice of sign in (129)).

It is straightforward from (123) that

$$\dot{I}_p = -\,\dot{U}\,\frac{1}{\sin I_p} = \alpha\,u\,\cos I_p\,\left(-\,q\,\cos h_p + p\,\sin h_p\right) + O(\sin^2 I_{\mathrm{orb}})$$
$$= \alpha\,\cos I_p\,\sin I_{\mathrm{orb}}\,\sin(h_p - \Omega_{\mathrm{orb}}) + O(\sin^2 I_{\mathrm{orb}}) \tag{132}$$

and

$$\dot{h}_p = \frac{\dot{Q}\cos h_p - \dot{P}\sin h_p}{\sin h_p \ \sin I_p} = \alpha u \left[\frac{\cos^2 I_p}{\sin I_p} \left(q \sin h_p + p \cos h_p \right) - u \cos I_p \right]$$
$$= -\alpha \cos I_p + O(\sin^2 I_{\mathrm{orb}}). \tag{133}$$

Differentiation of the latter, with the subsequent insertion of the former will result in:

$$\ddot{h}_p \ = \ \alpha^2 \ \sin I_{\mathrm{orb}} \ \sin I_p \ \cos I_p \ \sin(h_p \ - \ \Omega_{\mathrm{orb}}). \tag{134}$$

Finally, it should be stressed that the development by Brouwer and van Woerkom (1950) is limited in terms of precision and, therefore, in terms of the time span over which it remains valid. A more accurate and comprehensive development, with a validity span of tens of millions of years, was recently offered by Laskar (1988). At the future stages of our project, when developing a detailed physical model of the satellite motion, we shall employ Laskar's results.

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