LONG-TERM EVOLUTION OF ORBITS ABOUT A PRECESSING OBLATE PLANET: 1. THE CASE OF UNIFORM PRECESSION[1]

MICHAEL EFROIMSKY
US Naval Observatory, Washington DC 20392, USA, e-mail: me@usno.navy.mil

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Abstract. It was believed until very recently that a near-equatorial satellite would always keep up with the planet's equator (with oscillations in inclination, but without a secular drift). As explained in Efroimsky and Goldreich [*Astronomy* & *Astrophysics* (2004) Vol. 415, pp. 1187–1199], this misconception originated from a wrong interpretation of a (mathematically correct) result obtained in terms of non-osculating orbital elements. A similar analysis carried out in the language of osculating elements will endow the planetary equations with some extra terms caused by the planet's obliquity change. Some of these terms will be non-trivial, in that they will not be amendments to the disturbing function. Due to the extra terms, the variations of a planet's obliquity may cause a secular drift of its satellite orbit inclination. In this article we set out the analytical formalism for our study of this drift. We demonstrate that, in the case of uniform precession, the drift will be extremely slow, because the first-order terms responsible for the drift will be short-period and, thus, will have vanishing orbital averages (as anticipated 40 years ago by Peter Goldreich), while the secular terms will be of the second order only. However, it turns out that variations of the planetary precession make the first-order terms secular. For example, the planetary nutations will resonate with the satellite's orbital frequency and, thereby, may instigate a secular drift. A detailed study of this process will be offered in a subsequent publication, while here we work out the required mathematical formalism and point out the key aspects of the dynamics.

Key words: near-equatorial satellites of oblate planets, contact orbital elements

1. Physical Motivation and the Statement of Purpose

Ward (1973, 1974) noted that the obliquity of Mars may have suffered large-angle motions at long time scales. Later, Laskar and Robutel (1993) and Touma and Wisdom (1994) demonstrated that these motions may have been chaotic. This would cause severe climate variations and have major consequences for development of life.

It is a customary assumption that a near-equatorial satellite of an oblate planet would always keep up with the planet's equator (with only small

[1]In this article, as well as in (Efroimsky 2004), we use the word "precession" in its most general sense which embraces the entire spectrum of changes of the spin-axis orientation – from the long-term variations down to the Chandler Wobble down to nutations and to the polar wonder.

oscillations of the orbit inclination) provided the obliquity changes are sufficiently slow (Goldreich, 1965; Kinoshita, 1993). As demonstrated in Efroimsky and Goldreich (2004), this belief stems from a calculation performed in the language of non-osculating orbital elements. A similar analysis carried out in terms of osculating elements will contain hitherto overlooked extra terms entailed by the planet's obliquity variations. These terms (emerging already in the first order over the precession-caused perturbation) will cause a secular angular drift of the satellite orbit away from the planetary equator.

The existence of Phobos and Deimos, and the ability of Mars to keep them close to its equatorial plane during obliquity variations sets constraints on the obliquity variation amplitude and rate. Our eventual goal is to establish such constraints. If the satellites' secular inclination drifts are slow enough that the satellites stay close to Mars' equator during its obliquity changes through billions of years, then the rigid-planet non-dissipative models used by Ward (1973, 1974), Laskar and Robutel (1993), and Touma and Wisdom (1994) will get a totally independent confirmation. If the obliquity-change-caused inclination drifts are too fast (fast enough that within a billion or several billions of years the satellites get driven away from Mars' equatorial plane), then the inelastic dissipation and planetary structure must play a larger role than previously assumed.

Having this big motivation in mind, we restrict the current article to building the required mathematical background: we study the obliquity-variation-caused terms in the planetary equations, calculate their secular components and point out the resonant coupling emerging between a satellite's orbiting frequency and certain frequencies in the planet axis' precession. A more thorough investigation of this interaction will be left for our next paper.

2. Mathematical Preliminaries: Osculating Elements vs Orbital Elements

Whenever one embarks on integrating a satellite orbit and wants to take into account direction variations of the planet's spin, it is most natural to carry out this work in a co-precessing coordinate system. This always yields orbital elements which are defined in the said frame and, therefore, ready for immediate physical interpretation by a planet-based observer. A well camouflaged pitfall of this approach lies in that these orbital elements may come out non-osculating, i.e., that the instantaneous ellipses (or hyperbolae) parametrised by these elements will not be tangent to the physical trajectory as seen in the frame of reference.

2.1. THE OSCULATION CONDITION AND ALTERNATIVES TO IT

An instantaneous orbit is defined by a set of six parameters. Systems of planetary equations for these parameters may be derived either through the variation-of-parameters (VOP) method or via the Hamilton-Jacobi one. The latter approach, though fine and elegant, lacks the power instilled into the direct VOP technique: it cannot account for the gauge invariance of the N-body problem (Efroimsky, 2002a,b), important feature intiantely connected with some general concepts in ODE (Newman and Efroimsky, 2003). The Hamilton-Jacobi technique implicitly fixes the gauge and thus leaves the internal symmetry heavily veiled (Efroimsky and Goldreich, 2003).

As well known, a solution

$$\vec{\mathbf{r}} = \vec{\mathbf{f}}(C_1, \ldots, C_6, t) \tag{1}$$

to the reduced two-body problem

$$\ddot{\vec{\mathbf{r}}} + \frac{Gm}{r^2}\frac{\vec{\mathbf{r}}}{r} = 0 \tag{2}$$

is a Keplerian ellipse or hyperbola parametrised with some set of six independent orbital elements C_i which are constants in the absence of disturbances.

In the presence of disturbances, each body becomes subject to a total perturbing force $\Delta\vec{\mathbf{F}}$:

$$\ddot{\vec{\mathbf{r}}} + \frac{Gm}{r^2}\frac{\vec{\mathbf{r}}}{r} = \Delta\vec{\mathbf{F}}, \tag{3}$$

Solving the above equation of motion by the VOP method implies the use of (1) as an ansatz,

$$\vec{\mathbf{r}} = \vec{\mathbf{f}}(C_1(t), \ldots, C_6(t), t), \tag{4}$$

the function $\vec{\mathbf{f}}$ being the same as in (1), and the "constants" C_i now being endowed with a time dependence of their own. Insertion of (4) into (3) is insufficient for determining the six functions $C_i(t)$. To furnish a solution, three more equations are needed. Since the age of Lagrange it has been advised in the literature to employ, for this purpose, the conditions of osculation,

$$\sum_i \frac{\partial\vec{\mathbf{f}}}{\partial C_i}\frac{\mathrm{d}C_i}{\mathrm{d}t} = 0, \tag{5}$$

the imposition whereof ensures that, in the disturbed case, the physical velocity

$$\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t} = \frac{\partial\vec{\mathbf{f}}}{\partial t} + \sum_i \frac{\partial\vec{\mathbf{f}}}{\partial C_i}\frac{\mathrm{d}C_i}{\mathrm{d}t} \tag{6}$$

is expressed by the same function $\vec{\mathbf{g}}(C_1, \ldots, C_6, t)$ as in the two-body configuration:

$$\vec{\mathbf{g}} = \frac{\partial \vec{\mathbf{f}}}{\partial t}. \tag{7}$$

Condition (5) being arbitrary, its choice affects only mathematical developments, and not the physical orbit. This is merely a matter of ambiguous parametrisation. Together, (5) and (3), with ansatz (4) inserted therein, yield the following equation of the elements' evolution:

$$\sum_j [C_n\, C_j] \frac{\mathrm{d}C_j}{\mathrm{d}t} = \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \Delta \vec{\mathbf{F}}, \tag{8}$$

$[C_n\, C_j]$ being the matrix of Lagrange brackets introduced as

$$[C_n\, C_j] \equiv \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \frac{\partial \vec{\mathbf{g}}}{\partial C_j} - \frac{\partial \vec{\mathbf{f}}}{\partial C_j} \frac{\partial \vec{\mathbf{g}}}{\partial C_n}. \tag{9}$$

So defined, the brackets depend neither on the time evolution of C_i nor on the choice of supplementary conditions, but solely on the functional form of $\vec{\mathbf{f}}(C_{1,\ldots,6}, t)$ and $\vec{\mathbf{g}} \equiv \partial \vec{\mathbf{f}}/\partial t$.

In case we decide to relax the Lagrange constraint and to substitute it by

$$\sum_i \frac{\partial \vec{\mathbf{f}}}{\partial C_i} \frac{\mathrm{d}C_i}{\mathrm{d}t} = \vec{\mathbf{\Phi}}(C_{1,\ldots\ldots,6}, t), \tag{10}$$

$\vec{\mathbf{\Phi}}$ being some arbitrary function of time and parameters C_i (but, for the reasons of sheer convenience, not of time derivatives of C_i), then instead of (8) we shall get, for $n = 1, \ldots, 6$,

$$\sum_j \left([C_n\, C_j] + \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \frac{\partial \vec{\mathbf{\Phi}}}{\partial C_j} \right) \frac{\mathrm{d}C_j}{\mathrm{d}t} = \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \Delta \vec{\mathbf{F}} - \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \frac{\partial \vec{\mathbf{\Phi}}}{\partial t} - \frac{\partial \vec{\mathbf{g}}}{\partial C_n} \vec{\mathbf{\Phi}}. \tag{11}$$

This equation, derived in Efroimsky (2002b), gives us an opportunity to use, in an arbitrary gauge $\vec{\mathbf{\Phi}}$, the Lagrange brackets (9) defined in terms of the unperturbed functions $\vec{\mathbf{f}}$ and $\vec{\mathbf{g}}$. Expression (11) is the most general form of the planetary equations, in terms of the disturbing force. To get the generic form in terms of the Lagrangian disturbance, begin with the two-body Lagrangian $\mathcal{L}_o(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t) = \dot{\vec{\mathbf{r}}}^2/2 - U(\vec{\mathbf{r}}, t)$, momentum $\vec{\mathbf{p}} = \dot{\vec{\mathbf{r}}}$, and Hamiltonian $\mathcal{H}_o(\vec{\mathbf{r}}, \vec{\mathbf{p}}, t) = \vec{\mathbf{p}}^2/2 + U(\vec{\mathbf{r}}, t)$. Their perturbed counterparts will look, respectively:

$$\mathcal{L}(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t) = \mathcal{L}_o + \Delta\mathcal{L} = \frac{\dot{\vec{\mathbf{r}}}^2}{2} - U(\vec{\mathbf{r}}, t) + \Delta\mathcal{L}(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t), \tag{12}$$

$$\vec{\mathbf{p}} = \dot{\vec{\mathbf{r}}} + \frac{\partial \Delta\mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}}, \tag{13}$$

$$\mathcal{H}(\vec{\mathbf{r}}, \vec{\mathbf{p}}, t) = \vec{\mathbf{p}}\,\dot{\vec{\mathbf{r}}} - \mathcal{L} = \frac{\vec{\mathbf{p}}^{\,2}}{2} + U + \Delta\mathcal{H}, \tag{14}$$

$$\Delta\mathcal{H}(\vec{\mathbf{r}}, \vec{\mathbf{p}}, t) \equiv \mathcal{H}(\vec{\mathbf{r}}, \vec{\mathbf{p}}, t) - \mathcal{H}_o(\vec{\mathbf{r}}, \vec{\mathbf{p}}, t) \equiv -\Delta\mathcal{L} - \frac{1}{2}\left(\frac{\partial \Delta\mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}}\right)^2. \tag{15}$$

The Euler–Lagrange equations, written for the perturbed Lagrangian, will read:

$$\ddot{\vec{\mathbf{r}}} = -\frac{\partial U}{\partial \vec{\mathbf{r}}} + \Delta\vec{\mathbf{F}}, \tag{16}$$

the new term being the disturbing force:

$$\Delta\vec{\mathbf{F}} \equiv \frac{\partial \Delta\mathcal{L}}{\partial \vec{\mathbf{r}}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Delta\mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}}\right). \tag{17}$$

Substitution of (17) into (11) then yields

$$\sum_j \left([C_n\, C_j] + \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \frac{\partial}{\partial C_j} \left(\frac{\partial \Delta\mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} + \vec{\Phi} \right) \right) \frac{\mathrm{d}C_j}{\mathrm{d}t} = \frac{\partial}{\partial C_n} \left[\Delta\mathcal{L} + \frac{1}{2} \left(\frac{\partial \Delta\mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \right)^2 \right]$$
$$- \left(\frac{\partial \vec{\mathbf{g}}}{\partial C_n} + \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \frac{\partial}{\partial t} + \frac{\partial \Delta\mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \frac{\partial}{\partial C_n} \right) \left(\vec{\Phi} + \frac{\partial \Delta\mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \right). \tag{18}$$

The latter not only reveals the convenience of the generalised Lagrange gauge

$$\vec{\boldsymbol{\Phi}} = -\frac{\partial \Delta\mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}}, \tag{19}$$

(which reduces to $\vec{\boldsymbol{\Phi}} = 0$ for velocity-independent perturbations), but also demonstrates how the Hamiltonian variation comes into play: the sum in square brackets is equal to $-\Delta\mathcal{H}$. When C_i are chosen as the Kepler or Delaunay variables, (18) entails the gauge-invariant versions of the Lagrange or Delaunay planetary equations (see Appendix 1 to Efroimsky and Goldreich, 2003).

2.2. GOLDREICH (1965)

The earliest attempts to describe satellite motion about the precessing and nutating Earth were undertaken by Brouwer (1959), Proskurin and Batrakov (1960), and Kozai (1960). In 1965, Goldreich accomplished a ground-breaking work that marked the beginning of studies of the Martian satellite dynamics. He started out with two major assertions. One was that the Martian satellites had either been formed in the equatorial plane or been brought therein very long ago. The second was that Mars has experienced, through its history, a uniform precession. While the former proved to be almost certainly correct (Murison, 1988), the validity of the latter remains model-dependent. While, in the simplest approximation, the planetary precession is uniform, a more involved analysis, carried out by Ward (1973, 1974), Laskar and Robutel (1993), and Touma and Wisdom (1994), offers evidence of strongly non-uniform, perhaps even chaotic, variations of the Martian obliquity at long time scales. It should be said, though, that the analysis presented by these authors was model-dependent. In particular, it was performed in the approximation of the planet being a nondissipative rigid rotator. Since the orbits of both satellites are located $<2°$ from the equator, the two assertions contradict each other, unless there exists a mechanism constraining satellite orbits within the vicinity of the primary's equator. (Otherwise, as Goldreich noted in his paper, "the present low inclinations of these satellites' orbits would amount to an unbelievable coincidence.") In quest of such a mechanism, Goldreich (1965) investigated the evolution of the Kepler elements of a satellite in a reference system co-precessing with the planet. He followed the traditional VOP scheme, i.e., assumed a two-body setting as an undisturbed problem and then treated the inertial forces, emerging in the co-precessing frame, as perturbation (along with another perturbation caused by the oblateness of the planet). Below, we present a brief summary of Goldreich's results, with only minor comments.

Goldreich began by applying formulae (12–17) to motion in a coordinate system attached to the planet's centre of mass and precessing (but not spinning) with the planet. In this system, the equation of motion includes inertial forces and, therefore, reads:

$$\begin{aligned}\ddot{\vec{\mathbf{r}}} &= -\frac{\partial U}{\partial \vec{\mathbf{r}}} - 2\vec{\boldsymbol{\mu}} \times \dot{\vec{\mathbf{r}}} - \dot{\vec{\boldsymbol{\mu}}} \times \vec{\mathbf{r}} - \vec{\boldsymbol{\mu}} \times (\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}) \\ &= -\frac{\partial U_0}{\partial \vec{\mathbf{r}}} - \frac{\partial \Delta U}{\partial \vec{\mathbf{r}}} - 2\vec{\boldsymbol{\mu}} \times \dot{\vec{\mathbf{r}}} - \dot{\vec{\boldsymbol{\mu}}} \times \vec{\mathbf{r}} - \vec{\boldsymbol{\mu}} \times (\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}), \qquad (20)\end{aligned}$$

with dots denoting time derivatives in the co-precessing frame and $\vec{\boldsymbol{\mu}}$ standing for the coordinate system angular velocity relative to an inertial frame.[1] Here the physical (i.e., not associated with inertial forces) potential $U(\vec{\mathbf{r}})$ consists of the (reduced) two-body part $U_o(\vec{\mathbf{r}}) \equiv -GM\vec{\mathbf{r}}/r^3$ and a term

[1]Be mindful that $\vec{\boldsymbol{\mu}}$, though being a precession rate relative to an inertial frame, is a vector defined in the precessing frame. For details see section 8.6 in Marsden and Ratiu (2003) or section 27 in Arnold (1989). In this frame:

$$\vec{\boldsymbol{\mu}} = \hat{\boldsymbol{e}}_1 \frac{di_p}{dt} + \hat{\boldsymbol{e}}_2 \frac{dh_p}{dt} \sin i_p + \hat{\boldsymbol{e}}_3 \frac{dh_p}{dt} \cos i_p,$$

angles i_p and h_p being the inclination and the longitude of the node of the planetary equator of date relative to that of epoch; unit vector $\hat{\boldsymbol{e}}_3$ being orthogonal to the equator of date; and $\hat{\boldsymbol{e}}_1$ being aimed along the line of the moving equator's ascending node on the plane of the equator of epoch.

$\Delta U(\vec{\mathbf{r}})$ caused by the planet's oblateness. The overall disturbing force on the right-hand side of the above equation is generated, according to (17), by

$$\Delta\mathcal{L}(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t) = -\Delta U(\vec{\mathbf{r}}) + \dot{\vec{\mathbf{r}}} \cdot (\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}) + \tfrac{1}{2}(\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}) \cdot (\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}). \tag{21}$$

Since in this case

$$\frac{\partial \Delta\mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}},$$

then

$$\vec{\mathbf{p}} = \dot{\vec{\mathbf{r}}} + \frac{\partial \Delta\mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} = \dot{\vec{\mathbf{r}}} + \vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}} \tag{22}$$

and, therefore, the appropriate Hamiltonian variation will look:

$$\begin{aligned} \Delta\mathcal{H} &= -\left[\Delta\mathcal{L} + \frac{1}{2}\left(\frac{\partial \Delta\mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}}\right)^2\right] \\ &= -[-\Delta U + \vec{\mathbf{p}} \cdot (\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}})] = \Delta U - (\vec{\mathbf{r}} \times \vec{\mathbf{p}}) \cdot \vec{\boldsymbol{\mu}} = \Delta U - \vec{\mathbf{J}} \cdot \vec{\boldsymbol{\mu}}. \end{aligned} \tag{23}$$

This way, $\Delta\mathcal{H}$ becomes expressed through quantities defined in the co-precessing frame: the satellite's orbital momentum vector $\vec{\mathbf{J}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ and the precession rate $\vec{\boldsymbol{\mu}}$.

Goldreich employed the above expression in the role of a disturbing function R in the planetary equations:

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial(-\Delta\mathcal{H})}{\partial\omega} - \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial(-\Delta\mathcal{H})}{\partial\Omega}, \tag{24}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial(-\Delta\mathcal{H})}{\partial i}, \tag{25}$$

where

$$-\Delta\mathcal{H} \equiv R = R_{\text{oblate}} + R_{\text{inertial}} \tag{26}$$

consists, according to (23), of two inputs:[2]

$$R_{\text{oblate}}(v) \equiv -\Delta U = \frac{G\,m\,J_2}{2}\frac{\rho^2}{r^3}[1 - 3\sin^2 i\,\sin^2(\omega + v)], \tag{27}$$

[2]Our formula (27) slightly differs from the one employed by Goldreich (1965), because here we use the modern definition of J_2:

$$U = -\frac{\mu}{r}\left\{1 - \sum_{m=2}^{\infty} J_m\left(\frac{\rho}{r}\right)^m P_m(\sin\alpha)\right\},$$

α being the satellite's latitude in the planet-related coordinate system. The coefficient J used by Goldreich (1965) differs from our J_2 by a constant factor: $J = (3/2)J_2\rho^2/a^2$.

and

$$R_{\text{inertial}} \equiv \vec{\mathbf{J}} \cdot \vec{\boldsymbol{\mu}} = \sqrt{G\,m\,a(1-e^2)} \quad \vec{\mathbf{w}} \cdot \vec{\boldsymbol{\mu}}. \tag{28}$$

Here $m \equiv (m_{\text{primary}} + m_{\text{secondary}})$. The mean motion is, as ever, $n \equiv (Gm)^{1/2} a^{-3/2}$, while ρ stands for the mean radius of the primary, v denotes the true anomaly, and

$$r = a \frac{1-e^2}{1+e\,\cos\,v} \tag{29}$$

is the instantaneous orbital radius. In the right-hand side of (28) it was assumed that the angular momentum is connected with the orbital elements through the well-known formula

$$\vec{\mathbf{J}} \equiv \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \sqrt{G\,m\,a(1-e^2)}\vec{\mathbf{w}} \tag{30}$$

where

$$\vec{\mathbf{w}} = \hat{\mathrm{x}}_1\,\sin\,i\,\sin\,\Omega - \hat{\mathrm{x}}_2\,\sin\,i\,\cos\,\Omega + \hat{\mathrm{x}}_3\,\cos\,i$$

is a unit vector normal to the instantaneous ellipse, expressed through unit vectors $\hat{\mathrm{x}}_1, \hat{\mathrm{x}}_2, \hat{\mathrm{x}}_3$ associated with the co-precessing frame x_1, x_2, x_3 (the axes x_1 and x_2 lie in the planet's equatorial plane of date, x_1 pointing along the fiducial line wherefrom the longitude of the ascending node of the satellite orbit Ω is measured). The aforewritten expression for $\vec{\mathbf{w}}$ evidently yields:

$$R_{\text{inertial}} = \sqrt{Gma(1-e^2)}(\mu_1\,\sin\,i\sin\Omega - \mu_2\sin i\cos\Omega + \mu_3\,\cos\,i), \tag{31}$$

while (27) may be, in the first approximation, substituted with its secular part, i.e., with its average over the orbital period:

$$\langle R_{\text{oblate}} \rangle = \frac{n^2 J_2}{4} \rho^2 \frac{3\,\cos^2\,i - 1}{(1-e^2)^{3/2}}, \tag{32}$$

the averaging having been carried out through the medium of formula (112) from the Appendix, with (29) inserted. With the aid of (31) and (32), the planetary equations (25) and (26) will simplify to:

$$\frac{\mathrm{d}i}{\mathrm{d}t} = -\mu_1\,\cos\Omega - \mu_2\,\sin\,\Omega, \tag{33}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -\frac{3}{2} n J_2 \left(\frac{\rho}{a}\right)^2 \frac{\cos\,i}{(1-e^2)^2} + O\left(\frac{|\vec{\mu}|}{J_2 n}\right). \tag{34}$$

The latter results in the well-known node-precession formula,

$$\Omega = \Omega_o - \frac{3}{2} n J_2 \left(\frac{\rho}{a}\right)^2 \frac{\cos\,i}{(1-e^2)^2} t. \tag{35}$$

Its insertion into the former entails

$$i = -\frac{\mu_1}{\chi}\cos(-\chi t + \Omega) + \frac{\mu_2}{\nu}\sin(-\chi t + \Omega_o) + i_o, \tag{36}$$

where

$$\chi \equiv \frac{3}{2} n J_2 \left(\frac{\rho}{a}\right)^2 \frac{\cos i}{(1 - e^2)^2}. \tag{37}$$

In (Goldreich 1965), equation (36) was the main result, its derivation being valid for wobble which is slow ($|\vec{\mu}| \ll J_2 n$) and close to uniform ($|\dot{\vec{\mu}}|/|\vec{\mu}| \ll J_2 n$).

Despite a warning issued by Goldreich in his paper, this result has often been misinterpreted and, therefore, misused in publications devoted to satellites and rings of wobbling planets, as well as in the literature on orbits about tumbling galaxies.

In (36) 'i' stands for the inclination defined in co-precessing axes associated with the planet's equator, and therefore (36) clearly demonstrates that, in the course of obliquity changes, this inclination oscillates about zero, with no secular shift accumulated. Does this necessarily mean that the satellite *orbit*, too, oscillates about the equatorial plane, without a secular deviation therefrom? Most surprisingly, the answer to this question is negative. The reason for this is that the so-calculated orbital elements, though defined in the co-precessing frame, are *not osculating* therein. In other words, in the frame where the elements are introduced, the instantaneous ellipses parametrised by these elements are not tangent to the physical orbit as seen in this frame.

This circumstance was emphasised yet by Goldreich, who noticed that formula (30) normally (i.e., when employed in an inertial frame) connects the osculating elements defined in that frame with the angular momentum $\vec{\mathbf{r}} \times \vec{\mathbf{p}}$ defined in the same inertial frame (i.e., with $\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}}$). Since, in the above calculation, the frame is not inertial (and, therefore, the angular momentum is different from $\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}}$ but is equal to $\vec{\mathbf{r}} \times \vec{\mathbf{p}} = \vec{\mathbf{r}} \times (\dot{\vec{\mathbf{r}}} + \vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}})$), the orbital elements returned by (30) cannot be osculating in this frame.[3] On these grounds, Goldreich warned the reader of the peculiar nature of the elements used in his integration.

To this we would add that it is not at all evident that the inertial-forces-caused alteration of the planetary equations should be achieved through

[3]Were these elements osculating in the frame wherein they had been defined, then formula (30) would read: $\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}} = \sqrt{G\,m\,a\,(1 - e^2)}\,\vec{\mathbf{w}}$, i.e., would connect the elements with the velocity in that frame. In reality, though, it reads: $\vec{\mathbf{r}} \times \vec{\mathbf{p}} = \sqrt{G\,m\,a\,(1 - e^2)}\,\vec{\mathbf{w}}$, i.e., connects the elements with the momentum $\vec{\mathbf{p}} = \dot{\vec{\mathbf{r}}} + \vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}$ which happens to coincide with the satellite's velocity relative to the inertial axes. This situation was formulated by Goldreich in the following terms: the orbital elements emerging in the above derivation are defined in the co-precessing frame but are osculating in the inertial one. This illustrative metaphor should not, though, be overplayed: the fact that the elements emerging in Goldreich's computation return the inertial-frame-related velocity does not mean that this inclination may be interpreted as that relative to the invariable plane. (The elements were introduced in the co-precessing frame!)

amending the disturbing function with the momentum-dependent variation of the negative Hamiltonian, $\vec{\mathbf{J}} \cdot \vec{\boldsymbol{\mu}}$. While the common fallacy identifies the disturbing function with the negative Hamiltonian perturbation, in reality this rule-of-thumb works (and yields elements that are osculating) only for disturbances dependent solely upon positions, but not upon velocities (i.e., for Hamiltonian perturbations dependent only upon coordinates, but not upon momenta). Ours is not that case and, therefore, more alterations in the planetary equations are needed to account for the frame precession, if we wish these equations to render osculating elements. However, if one neglects this circumstance and simply amends the disturbing function with $\vec{\mathbf{J}} \cdot \vec{\boldsymbol{\mu}}$, then the planetary equations will give some elements different from the osculating ones. It will then become an interesting question as to whether such elements will or will not coincide with those rendered by (29) when this formula is used in non-inertial frames.

All these subtle issues get untangled in the framework of the gauge formalism. Application of this formalism to motions in non-inertial frames of references was presented in Efroimsky and Goldreich (2004). The main results proven there are the following:

1. If one attempts to account for the inertial forces by simply adding the term $\vec{\mathbf{J}} \cdot \vec{\boldsymbol{\mu}}$ to the disturbing function, with no other alterations made in the planetary equations, then these equations indeed do render quantities that may be interpreted as some orbital elements (i.e., as parameters of some instantaneous conics). These elements are *not* osculating and, therefore, the instantaneous conics parameterised by these elements are not tangent to the physical orbit. Hence, these elements cannot, generally, be attributed a direct physical interpretation,[4] except in the situations when their deviation from the osculating elements remains sufficiently small.
2. By a remarkable coincidence, these non-osculating elements turned out to be identical with those emerging in formula (30). This coincidence was implicitly taken for granted by Goldreich (1965), which reveals his truly incredible scientific intuition.
3. To build up a system of planetary equations that render the osculating elements of the orbit as seen in the co-precessing coordinate system, one has not only to add $\vec{\mathbf{J}} \cdot \vec{\boldsymbol{\mu}}$ to the disturbing function, but also to amend each of these equations with several extra terms. Some of those terms are of order $(|\vec{\boldsymbol{\mu}}|/(J_2 n))^2$, some others are of order $|\dot{\vec{\boldsymbol{\mu}}}|/(|\vec{\boldsymbol{\mu}}| J_2\, n)$. Most importantly, some terms are of the first order in the precession-caused perturbation $|\vec{\boldsymbol{\mu}}|/(J_2 n)$,

[4]When the orbit evolution is suffciently slow, the observer can attribute some physical meaning to elements of the osculating conic. For example, whenever an observer talks about the inclination or the eccentricity of a perturbed orbit, he naturally implies those of the osculating ellipse or hyperbola.

which means right away that the non-osculating elements used by Goldreich (1965) differ from the osculating ones already in the first order. While more comprehensive account on this topic, with the resulting equations, will be offered in the end of this section, here we shall touch upon only one question which gets immediately raised by the presence of such first-order differences. This question is: what are the averages of these differences? Stated alternatively, do the secular components of the said non-osculating elements differ considerably from those of the osculating ones? Goldreich (1965) stated, without a proof, that the secular components differ only in high orders over the velocity-dependent part of the perturbation. In our paper we shall probe the limits for this assumption.

2.3. BRUMBERG AND KINOSHITA

A development, part of which was similar to that of Goldreich (1965), was independently carried out by Brumberg et al. (1971) who studied the orbits of artificial lunar satellites in a coordinate system co-precessing with the Moon. In that article, too, the non-osculating nature of the resulting orbital variables did not go unnoticed. The authors called these variables "contact elements" and stated (though never proved) that these variables do not return the correct value of the velocity but that of the momentum. Later, one of these authors rightly noted in his book (Brumberg, 1992) that the contact elements differ from the osculating ones already in the first order over the velocity-dependent part of the perturbation. In subsection 1.1.3 of that book, he unsuccessfully tried to derive analytical transformations interconnecting these sets of variables.[5]

[5]Contrary to the author's statement, formula (1.1.41) in (Brumberg 1992) is not rigorous, but is valid only to first order. (To make it rigorous, one should substitute everywhere, except in the denominators, $\vec{\mathbf{r}}$ with $\vec{\mathbf{r}} - \partial R/\partial \vec{\mathbf{r}}$.) Besides, the author did not demonstrate his derivation of formula (1.1.43) from (1.1.42). (In Brumberg's book the mean anomaly is denoted with l, not with M.) Most importantly, the qualitative reasoning presented by the author in the paragraph preceding formula (1.1.43) is un-rigorous and essentially incorrect. The cause of this is that the author compares the planetary equations for contact elements, written in a precessing frame, with the equations for osculating ones, written in an inertial frame, instead of comparing two such systems (for contact and for osculating elements) both of which are written in a precessing frame. This makes a big difference because, as we already explained above, transition to a precessing frame does not simply mean addition of an extra term to the disturbing function.

Despite all these mathematical irregularities, the averaged system of planetary equations (1.1.44), proposed by Brumberg for the first-order secular perturbations, turns out to be correct in the limit of uniform precession. Just as in the preceding subsection we had a reason to praise the unusual intuition of Goldreich, so here we have to pay tribute to the excellent intuition of Brumberg, intuition which superseded his mathematics.

A similar attempt was undertaken in a very interesting article by Ashby and Allison (1993). Though the authors succeeded in many other points, their attempt to derive formulae for such a gauge transformation was not successful.[6]

The setting, considered by Goldreich (1965) in the context of Martian satellites and by Brumberg et al. (1971) in the context of circumlunar orbits, later emerged in the article by Kinoshita (1993) who addressed the satellites of Uranus.

Kinoshita's treatment of the problem was based on the following mathematical construction. Denote satellite's positions and velocities in the inertial and in the co-precessing axes with $\{\vec{\mathbf{r}}\,', \vec{\mathbf{v}}\,'\}$ and with $\{\vec{\mathbf{r}}, \vec{\mathbf{v}}\}$, correspondingly.[7] Interconnection between them will be implemented by an orthogonal matrix $\hat{A}$,

$$\vec{\mathbf{r}} = \hat{A}\vec{\mathbf{r}}\,', \quad \vec{\mathbf{v}} \equiv \dot{\vec{\mathbf{r}}} = \dot{\hat{A}}\vec{\mathbf{r}}\,' + \hat{A}\dot{\vec{\mathbf{r}}}\,' = \dot{\hat{A}}\hat{A}^{-1}\vec{\mathbf{r}} + \hat{A}\vec{\mathbf{v}}\,' = -\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}} + \hat{A}\vec{\mathbf{p}}\,', \tag{38}$$

with $\vec{\boldsymbol{\mu}}$ being the precession rate as seen in the co-precessing coordinate system, and the inertial velocity $\vec{\mathbf{v}}\,'$ being identical to the inertial momentum $\vec{\mathbf{p}}\,'$. Kinoshita suggested to interpret this interconnection as a canonical transformation between variables $\{\vec{\mathbf{r}}\,', \vec{\mathbf{p}}\,'\}$ and $\{\vec{\mathbf{r}}, \vec{\mathbf{p}}\}$, implemented by a generating function

$$F_2 = \vec{\mathbf{p}} \cdot \hat{A}\vec{\mathbf{r}}\,' = (\hat{A}^T\vec{\mathbf{p}}) \cdot \vec{\mathbf{r}}\,'. \tag{39}$$

This choice of generating function rightly yields

$$\vec{\mathbf{r}} = \frac{\partial F_2}{\partial \vec{\boldsymbol{p}}} = \hat{A}\vec{\mathbf{r}}\,', \tag{40}$$

while the interconnection between momenta will look:

$$\vec{\mathbf{p}}\,' = \frac{\partial F_2}{\partial \vec{\mathbf{r}}\,'} = \hat{A}^T\vec{\mathbf{p}}, \quad \text{i.e.,} \quad \vec{\mathbf{p}} = (\hat{A}^T)^{-1}\vec{\mathbf{p}}\,' = \hat{A}\vec{\mathbf{p}}\,', \tag{41}$$

whence $\vec{\boldsymbol{p}} = \vec{\mathbf{v}} + \vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}$. The Hamiltonian in precessing axes will read

$$\begin{aligned} H(\vec{\mathbf{r}}, \vec{\boldsymbol{p}}) &= H^{\text{inert}}(\vec{\mathbf{r}}\,', \vec{\mathbf{p}}\,') + \frac{\partial F_2}{\partial t} \\ &= H^{\text{inert}}(\vec{\mathbf{r}}\,', \vec{\mathbf{p}}\,') + \vec{\mathbf{p}}\,' + \vec{\mathbf{p}} \cdot \dot{\hat{A}}\vec{\mathbf{r}}\,' \\ &= H^{\text{inert}}(\vec{\mathbf{r}}\,', \vec{\mathbf{p}}\,') - \vec{\mathbf{p}} \cdot (\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}) \\ &= H^{\text{inert}}(\vec{\mathbf{r}}\,', \vec{\mathbf{p}}\,') - (\vec{\mathbf{r}} \times \vec{\mathbf{p}}) \cdot \vec{\boldsymbol{\mu}}. \end{aligned} \tag{42}$$

[6] To carry out the gauge transformation, the authors used a set of intermediate variables $\{Q^k_{(o)}, P^k_{(o)}\}$ which were canonical and, at the same time, osculating. As follows from the theorem proven by Efroimsky & Goldreich (2003), these variables are nonexistent when the perturbation depends upon velocities.

[7] We use notations opposite to those in Kinoshita (1993), in order to conform with Goldreich (1965).

The Hamiltonian perturbation, caused by the inertial forces, is $-(\vec{\mathbf{r}} \times \vec{\mathbf{p}}) \cdot \vec{\boldsymbol{\mu}} = -\vec{\mathbf{J}} \times \vec{\boldsymbol{\mu}}$, vector $\vec{\mathbf{J}}$ being the orbital angular momentum as seen in the co-precessing frame. Comparing this with (23), we see that employment of the above canonical transformation is but another method of stepping on the same rake. In distinction from Goldreich (1965) and Brumberg et al. (1971), Kinoshita in his paper did not notice that he was working with non-osculating elements.

The problem with Kinoshita's treatment is that the condition of canonicity in some situations comes into contradiction with the osculation condition. In other words, canonicity sometimes implicitly contains a constraint which is sometimes different from the Lagrange constraint (5). This issue was comprehensively elucidated in the work Efroimsky and Goldreich (2003). The authors began with the reduced two-body setting and thoroughly re-examined the Hamilton-Jacobi procedure, which leads one from the spherical coordinates and the corresponding canonical momenta to the set of Delaunay variables. While, in the undisturbed two-body case, this procedure yields the Delaunay variables which are trivially osculating (and parameterise a fixed Keplerian ellipse or hyperbola), in the perturbed case the situation becomes more involved. According to the theorem proven in that paper, the resulting Delaunay elements are osculating (and parameterise a conic tangent to the perturbed trajectory) if the Hamiltonian perturbation depends solely upon positions, and not upon momenta (or, the same: if the Lagrangian perturbation depends upon positions but not upon velocities). Otherwise, the Delaunay elements turn out to be non-osculating (and parameterise the physical trajectory with a sequence of non-tangent conics). As one can see from the above equation, the Hamiltonian perturbation, caused by the inertial forces, depends upon the momentum, and this circumstance indicates the problem. This trap, in which many have fallen, is of a special importance in the general relativity, because the relativist corrections to the equations of motion are velocity-dependent.[8]

[8] In an interesting article (Chernoivan & Mamaev 1999), the authors addressed the two-body problem on a curved background. The curvature entailed a velocity-dependent relativist correction, which was treated as a perturbation. After carrying out the Hamilton-Jacobi development, the authors arrived at canonical variables analogous to the Delaunay elements. Orbit integration in terms of these variables would be as correct as in terms of any others. The problems began when the authors used these elements to arrive at some conclusions regarding the perihelion precession. Those conclusions need to be reconsidered, because they were rendered on the basis of Delaunay elements that were non-osculating. Similar comments may be made about the work by Richardson & Kelly (1988) who addressed, using a Hamiltonian language, the two-body problem in the post-Newtonian approximation.

3. Planetary Equations

In this section we briefly spell out some results obtained by Efroimsky and Goldreich (2004) and use these results to derive the Lagrange-type planetary equations (60–65) for osculating elements in a coordinate system co-precessing with an oblate primary.

3.1. PLANETARY EQUATIONS FOR CONTACT ELEMENTS

Above, in subsection 2.1, we provided a very short account of the gauge formalism. Expression (18), presented there, is the most general form of the planetary equations for an arbitrary set of six independent orbital elements, written in terms of an arbitrary disturbance of the Lagrangian.

When the elements C_i are chosen to be the Keplerian or Delaunay sets of variables, we arrive at the gauge-invariant versions of the Lagrange or Delaunay planetary equations, correspondingly. They are written down in Appendix to the paper (Efroimsky and Goldreich 2003). The interplay between gauge freedom and the freedom of frame choice is explained at length in Section 3 of Efroimsky and Goldreich (2004) which addresses orbits about a precessing planet. It is demonstrated in that work that, if one chooses to describe the motion in terms of the non-osculating elements that were introduced in a co-precessing frame and were defined in the generalised Lagrange gauge[9] (19), then the corresponding Hamiltonian perturbation will read:

$$\Delta\mathcal{H}^{(\mathrm{cont})} = -[R_{\mathrm{oblate}}(\vec{\mathbf{f}}) + \vec{\boldsymbol{\mu}} \cdot (\vec{\mathbf{f}} \times \vec{\mathbf{g}})], \tag{43}$$

while the planetary equations (18) acquire the form

$$[C_r\ C_i]\frac{\mathrm{d}C_i}{\mathrm{d}t} = \frac{\partial(-\Delta\mathcal{H}^{(\mathrm{cont})})}{\partial C_r},$$

or

$$[C_r\ C_i]\frac{\mathrm{d}C_i}{\mathrm{d}t} = \frac{\partial}{\partial C_r}[R_{\mathrm{oblate}}(\vec{\mathbf{f}}) + \vec{\boldsymbol{\mu}} \cdot (\vec{\mathbf{f}} \times \vec{\mathbf{g}})], \tag{44}$$

where $\vec{\mathbf{f}}$ and $\vec{\mathbf{g}}$ stand for the undisturbed (two-body) functional expressions of the position and velocity, respectively, via the time and the chosen set of orbital elements:

[9] It is an absolutely crucial point that choice of a gauge and choice of a reference frame are two totally independent procedures. In each frame one has an opportunity to choose among an infinite variety of gauges.

$$\vec{\mathbf{r}} = \vec{\mathbf{f}}\,(C_1, \ldots, C_6, t)$$
$$\vec{\mathbf{v}} = \vec{\mathbf{g}}(C_1, \ldots, C_6, t) \equiv \frac{\partial}{\partial t}\vec{\mathbf{f}}(C_1, \ldots, C_6, t) \tag{45}$$

(so that $\vec{\mathbf{f}}\,(C_1(t), \ldots, C_6(t), t)$ and $\vec{\mathbf{g}}(C_1(t), \ldots, C_6(t), t)$ become the ansatz for solving the disturbed problem). For $R_{\rm oblate}$ in (43)–(44), one can employ, dependent upon the desired degree of rigour, either the exact expression (27) or its orbital average (32).

In the generalised Lagrange gauge (19) the canonical momentum becomes:

$$\vec{\mathbf{p}} = \dot{\vec{\mathbf{r}}} + \frac{\partial\,\Delta\mathcal{L}}{\partial\dot{\vec{\mathbf{r}}}} = \vec{\mathbf{g}} + \vec{\mathbf{\Phi}} + \frac{\partial\,\Delta\mathcal{L}}{\partial\dot{\vec{\mathbf{r}}}} = \vec{\mathbf{g}}, \tag{46}$$

which means that its functional dependence upon the time and the chosen set of orbital elements is the same as in the unperturbed case where both the velocity and the momentum were simply equal to $\vec{\mathbf{g}}(C_1, \ldots, C_6, t)$. This also means that $\Delta\mathcal{H}^{(\rm cont)}$ coincides with Goldreich's $\Delta\mathcal{H}$ given by (27).

We see that the generalised Lagrange gauge (19) singles out the same set of non-osculating elements which showed up in the studies by Goldreich (1965) and Brumberg, et al. (1971), – the set of "contact elements." This is why in (43) the Hamiltonian perturbation was written with the superscript "cont."

In (44) the Lagrange-bracket matrix is defined in the unperturbed two-body fashion (9) and can, therefore, be trivially inverted. Hence, when the elements are chosen as the Keplerian ones, the appropriate equations look like the customary Lagrange-type equations (i.e., like (25) and (26) above), with the disturbing function given by (26) or, the same, by (43):

$$\frac{{\rm d}a}{{\rm d}t} = \frac{2}{n\,a}\frac{\partial\,(-\Delta\mathcal{H}^{(\rm cont)})}{\partial M_o}, \tag{47}$$

$$\frac{{\rm d}e}{{\rm d}t} = \frac{1-e^2}{n\,a^2\dot{e}}\frac{\partial\,(-\Delta\mathcal{H}^{(\rm cont)})}{\partial M_o} - \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\frac{\partial\,(-\Delta\mathcal{H}^{(\rm cont)})}{\partial\omega}, \tag{48}$$

$$\frac{{\rm d}\omega}{{\rm d}t} = \frac{-\cos i}{n\,a^2\,(1-e^2)^{1/2}\sin i}\frac{\partial\,(-\Delta\mathcal{H}^{(\rm cont)})}{\partial i} + \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\frac{\partial\,(-\Delta\mathcal{H}^{(\rm cont)})}{\partial e}, \tag{49}$$

$$\frac{{\rm d}i}{{\rm d}t} = \frac{-\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial\,(-\Delta\mathcal{H}^{(\rm cont)})}{\partial\omega} - \frac{1}{n\,a^2\,(1-e^2)^{1/2}\sin\,i}\frac{\partial\,(\Delta\mathcal{H}^{(\rm cont)})}{\partial\Omega}, \tag{50}$$

$$\frac{{\rm d}\Omega}{{\rm d}t} = \frac{1}{na^2(1-e^2)^{1/2}\sin\,i}\frac{\partial\,(-\Delta\mathcal{H}^{(\rm cont)})}{\partial i}, \tag{51}$$

$$\frac{\mathrm{d}M_o}{\mathrm{d}t} = \frac{1-e^2}{na^2e}\frac{\partial\left(-\Delta\mathcal{H}^{(\mathrm{cont})}\right)}{\partial e} - \frac{2}{n\,a}\frac{\partial\left(-\Delta\mathcal{H}^{(\mathrm{cont})}\right)}{\partial a}, \tag{52}$$

3.2. PLANETARY EQUATIONS FOR OSCULATING ELEMENTS

When one introduces elements in the precessing frame and also demands that they osculate in this frame (i.e., makes them obey the Lagrange gauge $\vec{\mathbf{\Phi}} = 0$) then the Hamiltonian variation reads:[10]

$$\Delta\mathcal{H}^{(\mathrm{osc})} = -[R_{\mathrm{oblate}}(v) + \vec{\boldsymbol{\mu}}\cdot(\vec{\mathbf{f}}\times\vec{\mathbf{g}}) + (\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}})\cdot(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}})], \tag{53}$$

while equation (18) becomes:

$$[C_n\ C_i]\frac{\mathrm{d}C_i}{\mathrm{d}t} = -\frac{\partial\,\Delta\mathcal{H}^{(\mathrm{osc})}}{\partial C_n} + \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial C_n}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial C_n}\right)$$

$$- \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial C_n}\right) - (\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}})\frac{\partial}{\partial C_n}(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}). \tag{54}$$

To ease the comparison of this equation with (44), it is convenient to split the expression for $\Delta\mathcal{H}^{(\mathrm{osc})}$ in (53) into two parts:

$$\Delta\mathcal{H}^{(\mathrm{osc})} = -\left[R_{\mathrm{oblate}}(\vec{\mathbf{f}}, t) + \vec{\boldsymbol{\mu}}\cdot(\vec{\mathbf{f}}\times\vec{\mathbf{g}})\right] \tag{55}$$

and

$$-(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}\,)\cdot(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}\,), \tag{56}$$

and then to group the latter part with the last term on the right-hand side of (54):

$$[C_n\ C_i]\frac{\mathrm{d}C_i}{\mathrm{d}t} = -\frac{\partial\,\Delta\mathcal{H}^{(\mathrm{cont})}}{\partial C_n} + \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial C_n}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial C_n}\right) - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial C_n}\right)$$

$$+ (\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}})\frac{\partial}{\partial C_n}(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}). \tag{57}$$

One other option is to fully absorb the $O(|\vec{\boldsymbol{\mu}}|^2)$ term into $\Delta\mathcal{H}$, i.e., to introduce the amended "Hamiltonian"

$$\text{“}\Delta\mathcal{H}\text{”} = -[R_{\mathrm{oblate}}(v) + \vec{\boldsymbol{\mu}}\cdot(\vec{\mathbf{f}}\times\vec{\mathbf{g}}) + \frac{1}{2}(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}})\cdot(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}})] \tag{58}$$

[10] Just as $\Delta\mathcal{H}^{(\mathrm{cont})}$ in (43), this Hamiltonian variation is still equal to

$$-\left[R\,(\vec{\mathbf{f}}, t) + \vec{\boldsymbol{\mu}}\cdot\vec{\mathbf{J}}\right] = -\left[R\,(\vec{\mathbf{f}}, t) + \vec{\boldsymbol{\mu}}\cdot(\vec{\mathbf{f}}\times\vec{\mathbf{p}})\right].$$

However, the canonical momentum now is different from $\vec{\mathbf{g}}$ and reads as: $\vec{\boldsymbol{p}} = \vec{\mathbf{g}} + (\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}})$.

and to write down the equations like this:

$$[C_n\ C_i]\frac{\mathrm{d}C_i}{\mathrm{d}t} = -\frac{\partial\,``\Delta\mathcal{H}"}{\partial C_n} + \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial C_n}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial C_n}\right) - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial C_n}\right). \tag{59}$$

For C_i being chosen as the Keplerian elements, inversion of the Lagrange brackets will yield the following Lagrange-type system:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\left[\frac{\partial\,(-``\Delta\mathcal{H}")}{\partial M_o} - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial M_o}\right)\right], \tag{60}$$

$$\begin{aligned}\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{na^2e}\left[\frac{\partial\,(-``\Delta\mathcal{H}")}{\partial M_o} - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial M_o}\right)\right]\\ &\quad - \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\left[\frac{\partial\,(-``\Delta\mathcal{H}")}{\partial\omega} + \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial\omega}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial\omega}\right)\right.\\ &\quad \left. - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial\omega}\right)\right],\end{aligned} \tag{61}$$

$$\begin{aligned}\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{-\cos i}{na^2(1-e^2)^{1/2}\sin i}\left[\frac{\partial\,(-``\Delta\mathcal{H}")}{\partial i} + \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial i}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial i}\right)\right.\\ &\quad \left. - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial i}\right)\right] + \frac{(1-e^2)^{1/2}}{na^2e}\left[\frac{\partial\,(-``\Delta\mathcal{H}")}{\partial e} + \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial e}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial e}\right)\right.\\ &\quad \left. - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial e}\right)\right],\end{aligned} \tag{62}$$

$$\begin{aligned}\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\left[\frac{\partial\,(-``\Delta\mathcal{H}")}{\partial\omega} + \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial\omega}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial\omega}\right)\right.\\ &\quad \left. - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial\omega}\right)\right] - \frac{1}{na^2(1-e^2)^{1/2}\sin i}\left[\frac{\partial\,(-``\Delta\mathcal{H}")}{\partial\Omega}\right.\\ &\quad \left. + \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial\Omega}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial\Omega}\right) - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial\Omega}\right)\right],\end{aligned} \tag{63}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2(1-e^2)^{1/2}\sin\, i}\left[\frac{\partial\,(-``\Delta\mathcal{H}")}{\partial i} + \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial i}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial i}\right)\right.$$
$$\left. - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial i}\right)\right], \tag{64}$$

$$\frac{\mathrm{d}M_o}{\mathrm{d}t} = -\frac{1-e^2}{n\,a^2\,e}\left[\frac{\partial\,(-``\Delta\mathcal{H}")}{\partial e} + \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial e}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial e}\right) - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial e}\right)\right]$$
$$-\frac{2}{n\,a}\left[\frac{\partial\,(-``\Delta\mathcal{H}")}{\partial a} + \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial a}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial a}\right) - \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial a}\right)\right], \tag{65}$$

terms $\vec{\boldsymbol{\mu}}\cdot\left((\partial\vec{\mathbf{f}}/\partial M_o)\times\vec{\mathbf{g}} - (\partial\vec{\mathbf{g}}/\partial M_o)\times\vec{\mathbf{f}}\right)$ being omitted in (60–61), because these terms vanish identically (see the Appendix). In equations (59–65), "$\Delta\mathcal{H}$" is given by (58). With (27–28) and (31) taken into account, it will look like this:

$$\begin{aligned}
``\Delta\mathcal{H}" &= -\left[R_{\text{oblate}}(v) + \vec{\boldsymbol{\mu}}\cdot(\vec{\mathbf{f}}\times\vec{\mathbf{g}}) + \frac{1}{2}\left(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}\right)\cdot\left(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}\right)\right]\\
&= \frac{G\,m\,J_2}{2}\frac{\rho^2}{r^3}\left[1-3\,\sin^2 i\,\sin^2(\omega+v)\right] + \sqrt{G\,m\,a(1-e^2)}\,\vec{\omega}.\cdot\vec{\boldsymbol{\mu}}\\
&\quad + \frac{1}{2}\left(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}\right)\cdot\left(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}\right)\\
&= \frac{G\,m\,J_2}{2}\frac{\rho^2}{a^3}\left(\frac{1+e\,\cos v}{1-e^2}\right)^3\left[1-3\,\sin^2 i\,\sin^2(\omega+v)\right]\\
&\quad + \sqrt{G\,m\,a(1-e^2)}\;\;(\mu_1\sin i\,\sin\,\Omega - \mu_2\sin\,i\,\cos\,\Omega + \mu_3\cos\,i)\\
&\quad + \frac{1}{2}\left(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}\right)\cdot\left(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}\right).
\end{aligned} \tag{66}$$

To compute the secular parts of the elements, one can use, on the right-hand side of (60–65), the orbital averages (denoted with the $\langle\ldots\rangle$ symbol). The averaged "Hamiltonian" will look:

$$\begin{aligned}
\langle``\Delta\mathcal{H}"\rangle &= -\left[\langle R_{\text{oblate}}\rangle + \vec{\boldsymbol{\mu}}\cdot(\vec{\mathbf{f}}\times\vec{\mathbf{g}}) + \frac{1}{2}\langle\left(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}\right)\cdot\left(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{f}}\right)\rangle\right]\\
&= \frac{G\,m\,J_2}{4}\frac{\rho^2}{a^3}\frac{3\,\cos^2 i-1}{(1-e^2)^{3/2}} + \sqrt{G\,m\,a(1-e^2)}
\end{aligned}$$

$$\times\,(\mu_1 \sin i \sin \Omega - \mu_2 \sin i \cos \Omega + \mu_3 \cos i)$$
$$+\frac{1}{2}\langle\left(\vec{\mu}\times\vec{\mathbf{f}}\right)\cdot\left(\vec{\mu}\times\vec{\mathbf{f}}\right)\rangle. \tag{67}$$

The expression for $\vec{\mathbf{f}}\times\vec{\mathbf{g}}$ is true-anomaly-independent and, therefore, does not need to be bracketed with the averaging symbols $\langle\cdots\rangle$. The expression for $\left(\vec{\mu}\times\vec{\mathbf{f}}\right)\cdot\left(\vec{\mu}\times\vec{\mathbf{f}}\right)$ through the orbital elements is too cumbersome, and here we do not write it down explicitly. (See formula (188) in the Appendix to our preprint Efroimsky, 2004.) When we permit ourselves to neglect the $O(|\vec{\mu}|^2)$ inputs, all three functions, $\Delta\mathcal{H}^{(\mathrm{osc})}$, $\Delta\mathcal{H}^{(\mathrm{cont})}$, and "$\Delta\mathcal{H}$" coincidard so do their averages. In this approximation, they all are equal to:

$$\Delta\mathcal{H}^{(\mathrm{eff})} \equiv \langle\text{``}\Delta\mathcal{H}\text{''}\rangle \approx \langle\Delta\mathcal{H}^{(\mathrm{osc})}\rangle\langle\Delta\mathcal{H}^{(\mathrm{cont})}\rangle = -\left[\langle R_{\mathrm{oblate}}\rangle + \vec{\mu}\cdot(\vec{\mathbf{f}}\times\vec{\mathbf{g}})\right]$$
$$= \frac{G\,m\,J_2}{4}\frac{\rho^2}{a^3}\frac{3\,\cos^2 i - 1}{(1-e^2)^{3/2}} + \sqrt{G\,m\,a(1-e^2)}$$
$$\times\,(\mu_1 \sin i \sin \Omega - \mu_2 \sin i \cos \Omega + \mu_3 \cos i). \tag{68}$$

Two important issues should be dwelt upon at this point. First, we would remind that the function $\Delta\mathcal{H}^{(\mathrm{cont})}$, given by expression (43), yields the correct functional form of the Hamiltonian only in the case where we express the Hamiltonian through the contact elements (and calculate these through (44) or (47–52)). In the currently considered case of osculating elements, this $\Delta\mathcal{H}^{(\mathrm{cont})}$ is *not* the correct expression for the Hamiltonian. The correct functional dependence of the Hamiltonian upon the osculating elements, $\Delta\mathcal{H}^{(\mathrm{osc})}$, is given by formula (53). Though in this dynamical problem the Hamiltonian is unique, its functional dependencies through the contact and through the osculating elements differ from one another, one being $\Delta\mathcal{H}^{(\mathrm{cont})}$ as in (43), and the other being $\Delta\mathcal{H}^{(\mathrm{osc})}$ as in (53). As for the function "$\Delta\mathcal{H}$" rendered by (58), it is not really a Hamiltonian, but is simply a convenient mathematical entity. In the approximation, where $O(|\vec{\mu}|^2)$ terms are neglected, there is no difference between these three functions. Despite this, the $O(|\vec{\mu}|)$ and $O(|\vec{\mu}|^2)$ terms do stay in equations (59–65) for osculating elements.

Second, we would comment on our use of expressions (30–31) in our derivation of (66–68). Above, in subsections 2.2 and 3.1, the use of (28) and (31) was based on formula (30) which interconnected $\vec{\mathbf{J}} = \vec{\mathbf{r}}\times\vec{\mathbf{p}} = \vec{\mathbf{f}}\times\left(\dot{\vec{\mathbf{r}}}+\vec{\mu}\times\vec{\mathbf{f}}\right)$ with contact elements a, e, i, and Ω. As demonstrated in Efroimsky and Goldreich (2004), in that frame $\dot{\vec{\mathbf{r}}} = \vec{\mathbf{g}} - \vec{\mu}\times\vec{\mathbf{f}}$. Hence, formula (30) interconnects the contact elements with $\vec{\mathbf{J}} = \vec{\mathbf{f}}\times\vec{\mathbf{g}}$. In the present subsection, we use formula (30) in its usual capacity, i.e., to interconnect $\vec{\mathbf{J}} = \vec{\mathbf{r}}\times\dot{\vec{\mathbf{r}}} = \vec{\mathbf{f}}\times\vec{\mathbf{g}}$ with osculating elements a, e, i, and Ω. It may seem confusing that, though in both cases this formula can be written down in the same way,

$$\vec{\mathbf{f}} \times \vec{\mathbf{g}} = \sqrt{G\, m\, a(1 - e^2)}\,\vec{\mathbf{w}}, \tag{69}$$

its meaning is so different. The clue to understanding this difference lies in the fact that in one case $\dot{\vec{\mathbf{r}}} = \vec{\mathbf{g}} - \vec{\boldsymbol{\mu}} \times \vec{\mathbf{f}}$ (and, therefore, the elements are contact), while in the other case $\dot{\vec{\mathbf{r}}} = \vec{\mathbf{g}}$ (which makes the elements osculating). For more details see Efroimsky and Goldreich (2004).

4. Comparison of the Orbital Calculations, Performed in Terms of the Contact Elements, With those Performed in Terms of the Osculating Elements: The Simplest Approximation

As explained in Section 2, it follows from equations (24–25) that an initially small inclination remains so in the course of the oblate primary's precession. Whether this famous result may be interpreted as keeping of satellites in the near-equatorial zone of a precessing planet will depend upon how well the non-osculating (contact) inclination emerging in (24–37) approximates the physical, osculating, inclination rendered by (60–65).

4.1. THE AVERAGED PLANETARY EQUATIONS

Comparing equations (60–65) for osculating elements with equations (47–52) for contact elements, we immediately see that they differ already in the first order over the precession rate $\vec{\boldsymbol{\mu}}$ and, therefore, the values of the contact elements will differ from those of their osculating counterparts in the first order, too. A thorough investigation of this difference would demand numerical implementation of both systems and would be extremely time consuming. Meanwhile, we can get some preliminary estimates by asking the following, simplified, question: how will the secular, i.e., averaged over an orbital period, components of the contact and orbital elements differ from one another? To answer this question, we perform the following approximations:

(1). In equations (60–65), we substitute both the R_{oblate} term in the Hamiltonian and the $\vec{\boldsymbol{\mu}}$-dependent terms with their averages (so that, for example, the R_{oblate} term will be now substituted with $\langle R_{\text{oblate}} \rangle$ expressed via (32)).

(2). We neglect the terms of order $\vec{\boldsymbol{\mu}}^2$. This way, we restrict the length of time scales involved. (Over sufficiently long times even small terms may accumulate to a noticeable secular correction.) However, we can now benefit from the approximate equality (68).

So truncated and averaged system of Lagrange-type equations will read:

$$\frac{da}{dt} = \frac{2}{na}\left[-\left\langle \dot{\vec{\boldsymbol{\mu}}}\left(\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial M_o}\right)\right\rangle\right] \tag{70}$$

$$\frac{de}{dt} = \frac{1-e^2}{n\,a^2\,e}\left[-\left\langle \dot{\vec{\boldsymbol{\mu}}}\left(\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial M_o}\right)\right\rangle\right] - \frac{(1-e^2)^{1/2}}{n\,a^2\,e}$$
$$\left[\left\langle \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial \omega} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial \omega}\right)\right\rangle - \left\langle \dot{\vec{\boldsymbol{\mu}}}\left(\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial \omega}\right)\right\rangle\right], \tag{71}$$

$$\frac{d\omega}{dt} = \frac{-\cos i}{na^2(1-e^2)^{1/2}\sin i}\left[\frac{\partial(-\Delta\mathcal{H}^{(\mathrm{eff})})}{\partial i} + \left\langle \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial i} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial i}\right)\right\rangle\right.$$
$$\left.-\left\langle \dot{\vec{\boldsymbol{\mu}}}\left(\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial i}\right)\right\rangle\right] + \frac{(1-e^2)^{1/2}}{na^2e}\left[\frac{\partial(-\Delta\mathcal{H}^{(\mathrm{eff})})}{\partial e} + \left\langle \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial e} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial e}\right)\right\rangle\right.$$
$$\left.-\left\langle \dot{\vec{\boldsymbol{\mu}}}\left(\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial e}\right)\right\rangle\right], \tag{72}$$

$$\frac{di}{dt} = \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\left[\left\langle \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial \omega} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial \omega}\right)\right\rangle - \left\langle \dot{\vec{\boldsymbol{\mu}}}\left(\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial \omega}\right)\right\rangle\right]$$
$$-\frac{1}{na^2(1-e^2)^{1/2}\sin i}\left[\frac{\partial(-\Delta\mathcal{H}^{(\mathrm{eff})})}{\partial \Omega} + \left\langle \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial \Omega} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial \Omega}\right)\right\rangle\right.$$
$$\left.-\left\langle \dot{\vec{\boldsymbol{\mu}}}\left(\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial \Omega}\right)\right\rangle\right], \tag{73}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2(1-e^2)^{1/2}\sin i}\left[\frac{\partial(-\Delta\mathcal{H}^{(\mathrm{eff})})}{\partial i} + \left\langle \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial i} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial i}\right)\right\rangle\right.$$
$$\left.-\left\langle \dot{\vec{\boldsymbol{\mu}}}\left(\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial i}\right)\right\rangle\right], \tag{74}$$

$$\frac{dM_o}{dt} = -\frac{1-e^2}{na^2e}\left[\frac{\partial(-\Delta\mathcal{H}^{(\mathrm{eff})})}{\partial e} + \left\langle \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial e} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial e}\right)\right\rangle\right.$$

$$-\left\langle \dot{\vec{\boldsymbol{\mu}}}\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial e}\right)\right\rangle\Bigg] - \frac{2}{na}\left[\frac{\partial\left(-\Delta\mathcal{H}^{(\mathrm{eff})}\right)}{\partial a} + \left\langle \vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial a}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial a}\right)\right\rangle \right.$$
$$\left. -\left\langle \dot{\vec{\boldsymbol{\mu}}}\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial a}\right)\right\rangle\right], \tag{75}$$

the Hamiltonian here being approximated by (68), and the angular brackets signifying orbital averaging. In equations (70), (71) and (73) we took into account that the averaged Hamiltonian (68) bears no dependence upon M_o and ω. (This, though, will not be the case for the exact, v-dependent, Hamiltonian given by (53) and (27)!)

Calculation of $\left((\partial\vec{\mathbf{f}}/\partial C_j)\times\vec{\mathbf{g}} - (\partial\vec{\mathbf{g}}/\partial C_j)\times\vec{\mathbf{f}}\right)$ and $-\dot{\vec{\boldsymbol{\mu}}}\left(\vec{\mathbf{f}}\times\partial\vec{\mathbf{g}}/\partial C_j\right)$ takes pages of algebra. A short synopsis of this calculation is offered in the Appendix (while a detailed calculation is presented in the Appendix in Efroimsky, 2004). Here follows the outcome:

$$\vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial a}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial a}\right) = \frac{3}{2}\mu_\perp\sqrt{\frac{G\,m\,(1-e^2)}{a}}, \tag{76}$$

$$\vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial e}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial e}\right) = -\mu_\perp\frac{n\,a^2\,(3e+2\,\cos\,v+e^2\,\cos\,v)}{(1+e\,cos\,v)\sqrt{1-e^2}}, \tag{77}$$

$$\vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial\omega}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial\omega}\right) = -2\mu_\perp\frac{n\,a^2\sqrt{1-e^2}}{1+e\,\cos\,v}e\,\sin\,v, \tag{78}$$

$$\begin{aligned}\vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial\Omega}\times\vec{\mathbf{g}} - \vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial\Omega}\right) = \mu_1 & \left[\frac{na^2\sqrt{1-e^2}}{1+e\cos v}\sin i\{\cos\Omega\cos[2(\omega+v)]\right.\\ & -\sin\Omega\cos i\sin[2(\omega+v)]\}\\ & +\frac{na^2\sqrt{1-e^2}}{1+e\cos v}e\sin i\{\cos\Omega\cos(v+2\omega)\\ & \left.-2\sin\Omega\cos i\sin(\omega+v)\cos\omega\}\right]\\ & +\mu_2\left[\frac{na^2\sqrt{1-e^2}}{1+e\cos v}\sin i\{\sin\Omega\cos[2(\omega+v)]\right.\\ & +\cos\Omega\cos i\sin[2(\omega+v)]\}\\ & +\frac{na^2\sqrt{1-e^2}}{1+e\cos v}e\sin i\{\sin\Omega\cos(v+2\omega)\end{aligned}$$

$$
\begin{aligned}
&+2\cos\Omega\cos i\sin(\omega+v)\cos\omega\}\Bigg]\\
&+\mu_3\left[\frac{na^2\sqrt{1-e^2}}{1+e\cos v}\sin^2 i\sin[2(\omega+v)]\right.\\
&\left.+\frac{na^2\sqrt{1-e^2}}{1+e\cos v}2e\left\{-\sin v+\sin^2 i\cos\omega\sin(\omega+v)\right\}\right],
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
\vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial i}\times\vec{\mathbf{g}}-\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial i}\right)=\mu_1\frac{na^2\sqrt{1-e^2}}{1+e\cos v}&\\
\times\{[(-\sin\Omega\cos i\cos 2\omega-\cos\Omega\sin 2\omega)&\\
\times\left(\cos^2 v-\sin^2 v\right)+2\sin v\cos v&\\
\times(\sin\Omega\cos i\sin 2\omega-\cos\Omega\cos 2\omega)]&\\
+e[(-\sin\Omega\cos i\cos 2\omega-\cos\Omega\sin 2\omega)\cos v&\\
+\left(\sin\Omega\cos i\sin 2\omega-2\cos\Omega\cos^2\omega\right)\sin v]\}&\\
+\mu_2\frac{na^2\sqrt{1-e^2}}{1+e\cos v}\{[(\cos\Omega\cos i\cos 2\omega&\\
-\sin\Omega\sin 2\omega)\left(\cos^2 v-\sin^2 v\right)+2\sin v\cos v&\\
\times(-\cos\Omega\cos i\Omega\sin 2\omega-\sin\Omega\cos 2\omega)]&\\
+e[(\cos\Omega\cos i\,\Omega\cos 2\omega-\sin\Omega\sin 2\omega)\cos v&\\
+\left(-2\sin\Omega\cos^2\omega+\cos\Omega\sin 2\omega\cos i\right)\sin v]\}&\\
+\mu_3\frac{na^2\sqrt{1-e^2}}{1+e\cos v}\left\{\sin i\left[\cos 2\omega\left(\cos^2 v-\sin^2 v\right)\right.\right.&\\
-2\sin 2\omega\sin v\cos v]+e[\sin i\cos 2\omega\cos v&\\
-\sin i\sin 2\omega\sin v]\},&
\end{aligned}
\tag{80}
$$

$$
\vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\mathbf{f}}}{\partial M_o}\times\vec{\mathbf{g}}-\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial M_o}\right)=0,
\tag{81}
$$

$$
-\dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial a}\right)=\frac{3}{2}\dot{\mu}_\perp an(t-t_o)\sqrt{1-e^2},
\tag{82}
$$

$$
-\dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\mathbf{f}}\times\frac{\partial\vec{\mathbf{f}}}{\partial e}\right)=-\dot{\mu}_\perp a^2\frac{(1-e^2)}{1+e\cos v}\sin v.
\tag{83}
$$

$$-\dot{\vec{\mu}} \cdot \left(\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial \omega} \right) = -\dot{\mu}_{\perp} a^2 \frac{(1-e^2)^2}{(1+e\cos v)^2}, \tag{84}$$

$$\begin{aligned} -\dot{\vec{\mu}} \cdot (\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial \Omega}) = a^2 \frac{(1-e^2)^2}{(1+e\cos v)^2} \{ & \dot{\mu}_1 [\cos\Omega \cos(\omega+v) \\ & - \sin\Omega \sin(\omega+v)\cos i] \sin(\omega+v) \sin i. \\ & + \dot{\mu}_2 [\sin\Omega \cos(\omega+v) + \cos\Omega \sin(\omega+v)\cos i] \\ & \times \sin(\omega+v)\sin i \\ & + \dot{\mu}_3 \left[\cos^2(\omega+v) + \sin^2(\omega+v)\cos^2 i\right] \}, \end{aligned} \tag{85}$$

$$\begin{aligned} -\dot{\vec{\mu}} \cdot \left(\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial i} \right) = a^2 \frac{(1-e^2)^2}{(1+e\cos v)^2} \times \{ & \dot{\mu}_1 [-\cos\Omega \sin(\omega+v) \\ & - \sin\Omega \cos(\omega+v)\cos i] \sin(\omega+v). \\ & + \dot{\mu}_2 [-\sin\Omega \sin(\omega+v) \\ & + \cos\Omega \cos(\omega+v)\cos i] \sin(\omega+v) \\ & + \dot{\mu}_3 \sin(\omega+v)\cos(\omega+v)\sin i \} \end{aligned} \tag{86}$$

$$-\dot{\vec{\mu}} \cdot \left(\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{f}}}{\partial M_o} \right) = -\dot{\mu}_{\perp} a^2 \sqrt{(1-e^2)}, \tag{87}$$

with μ_1, μ_2, μ_3 being the Cartesian components of the precession rate, as seen in the co-precessing frame, and $\mu_{\perp}$ being the component of the precession rate, aimed in the direction of the angular momentum; it is given by (120).

It may seem strange that the right-hand side of (77) does not vanish in the limit of $e \to 0$. The absurdity of this will be easily redeemed by the fact that this term shows up only in the equation for $\mathrm{d}\omega/\mathrm{d}t$ and, therefore, leads to no physical paradoxes in the limit of a circular orbit. However, for finite values of the eccentricity, this term contributes to the periapse precession.

Another seemingly calamitous thing is the divergence in (83). This divergence, however, entails no disastrous physical consequences, because the term (83) shows up only in the planetary equation for $\mathrm{d}M_o/\mathrm{d}t$ and simply leads to a steady shift of the initial condition M_o.

4.2 THE CASE OF A CONSTANT PRECESSION RATE

The situation might simplify very considerably if we could also assume that the precession rate $\vec{\mu}$ stays constant. Then in equations (70–75), we would take $\vec{\mu}$ out of the angular brackets and proceed with averaging the expressions $\left((\partial\vec{\mathbf{f}}/\partial C_j) \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times (\partial\vec{\mathbf{g}}/\partial C_j)\right)$ only (while all the terms with $\dot{\vec{\boldsymbol{\mu}}}$ will now vanish). It is, of course, well known that this is physically wrong, because the planetary precession has a continuous spectrum of frequencies (some of which are commensurate with the orbital frequency of the satellite).[11] Nevertheless, for the sake of argument let us go on with this assumption.

Averaging of (76) and (81) is self-evident. Averaging of (77–80) is lengthy and is presented in the Appendix of Efroimsky (2004). All in all, we get, for constant $\vec{\boldsymbol{\mu}}$:

$$\vec{\boldsymbol{\mu}} \cdot \left\langle \left(\frac{\partial \vec{\mathbf{f}}}{\partial a} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial a} \right) \right\rangle = \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial a} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial a} \right) = \frac{3}{2}\mu_\perp \sqrt{\frac{G m (1-e^2)}{a}}, \tag{88}$$

$$\vec{\boldsymbol{\mu}} \cdot \left\langle \left(\frac{\partial \vec{\mathbf{f}}}{\partial C_j} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial C_j} \right) \right\rangle = 0, \quad C_j = e, \Omega, \omega, i, M_o. \tag{89}$$

Since the orbital averages (89) vanish, then e will, along with a , stay constant for as long as our approximation remains valid. Besides, no trace of $\vec{\boldsymbol{\mu}}$ will be left in the equations for Ω and i . This means that, in the assumed approximation and under the extra assumption of constant $\vec{\boldsymbol{\mu}}$, the aforequoted analysis (24–37), offered by Goldreich (1965), will remain valid at time scales which are not too long. At longer scales (of order dozens to hundreds of millions of years) one has to take into consideration the back reaction of the short-period terms upon the secular ones (Laskar, 1990). Besides the latter issue, the problem with this approximation is that it ignores both the long-term evolution of the spin axis and the short-term nutations. For these reasons, this approximation will not be extendable to long periods of Mars' evolution. This puts forward the bigger question: what maximal amplitude of obliquity variations could Mars afford, to keep both its satellites so close to its equatorial plane?

Even in the unphysical case of constant $\vec{\boldsymbol{\mu}}$ the averaged equation (75) for the osculating M_o differs, already in the first order over $\vec{\boldsymbol{\mu}}$, from equation (49) for the contact M_o. In the realistic case of time-dependent precession, the averages of terms containing $\vec{\boldsymbol{\mu}}$ and $\dot{\vec{\boldsymbol{\mu}}}$ do not vanish (except $\vec{\boldsymbol{\mu}} \cdot \left((\partial\vec{\mathbf{f}}/\partial M_o) \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times (\partial\vec{\mathbf{g}}/\partial M_o)\right)$ which is identically nil). These terms

[11] The case of the Earth rotation and precession is comprehensively reviewed by Eubanks (1993). The Martian short-time-scale rotational dynamics is of an equal complexity, even though Mars lacks oceans and the coupling of its rotation with the atmospheric motions is weaker than in the case of the Earth. (Defraigne et al. 2003, Van Hoolst et al. 2000, Dehant et al. 2000).

show up in all equations (except in that for a) and influence the motion. They will be the key to our understanding the long-term satellite dynamics, including the secular drift of the orbit plane, caused by the precession $\vec{\boldsymbol{\mu}}$.

5. An Outline of a More Accurate Analysis: Resonances between the Planetary Nutation and the Satellite Orbital Frequency

Precession of any planet contains, in itself, a continuous spectrum of circular frequencies involved:[12]

$$\vec{\boldsymbol{\mu}}(t) = \int_o^{\infty} \Big[\vec{\boldsymbol{\mu}}^{(s)}(u) \sin{(ut)} + \vec{\boldsymbol{\mu}}^{(c)}(u) \cos{(ut)}\Big] \mathrm{d}u. \tag{90}$$

with some modes being more prominent than the others. Here we denote the angular frequency by u, because letters ω and Ω are already in use and stand for the angles.

For our present purposes, it will be advantageous to express the precession rate as function of the satellite's true anomaly:

$$\vec{\boldsymbol{\mu}}(v) = \int_0^{\infty} \Big[\vec{\boldsymbol{\mu}}^{(s)}(W) \sin(Wv) + \vec{\boldsymbol{\mu}}^{(c)}(W) \cos(Wv)\Big] \mathrm{d}W. \tag{91}$$

with W being the circular "frequency" related to the true anomaly v. Needless to say, $\vec{\boldsymbol{\mu}}(t), \vec{\boldsymbol{\mu}}(v), \vec{\boldsymbol{\mu}}(u)$, and $\vec{\boldsymbol{\mu}}(W)$ are different functions. However, we take the liberty of using the same notation $\vec{\boldsymbol{\mu}}(\cdots)$ because the argument will always reveal which of these functions we imply. Evidently, $\vec{\boldsymbol{\mu}}(v)$ is a short notation for $\vec{\boldsymbol{\mu}}(t(v))$. It is also possible to demonstrate that, under the assumption of vanishing eccentricity and slowly-changing semimajor axis,

$$\vec{\boldsymbol{\mu}}(W) \approx n\vec{\boldsymbol{\mu}}(u)|_{W=u/n}, \quad \text{and} \quad n \equiv (G\,m)^{1/2} a^{-3/2}. \tag{92}$$

If we now plug the real part of (91) into (76–88) and carry out averaging in accordance with formula (112) of the Appendix, we shall see that the secular parts of these $\vec{\boldsymbol{\mu}}$- dependent terms do not vanish. They reveal the influence of the planetary precession upon the satellite orbital motion. Especially interesting are the resonant contributions provided by the integer Ws, because these nutation modes are commensurate with the orbital motion of the satellite. Another important (though non-resonant) contribution will come from inte-

[12] A more honest analysis should take into account also the direct dependence of the planet's precession rate upon the instantaneous position(s) of its satellite(s): $\vec{\boldsymbol{\mu}} = \vec{\boldsymbol{\mu}}(t; a, e, \Omega, \omega, i, M_o)$. However, the back-reaction of the satellites upon the primary is known to be an effect of a higher order of smallness (Laskar, 2004), at least in the case of Mars; and therefore we shall omit this circumstance by simply assuming that $\vec{\boldsymbol{\mu}} = \vec{\boldsymbol{\mu}}(t) = \vec{\boldsymbol{\mu}}(t(v))$.

gration over the interval $0 < W < 1$, because this interval is "responsible" for the influence of the long-term obliquity variations upon the satellite orbit.

Insertion of (91) into (76–87) will result in extremely sophisticated integrals. For example, the term (76) will have the following orbital average:

$$\left\langle \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial a} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial a} \right) \right\rangle = \frac{3}{2} \sqrt{\frac{G m (1 - e^2)}{a}} \langle \mu_\perp \rangle$$
$$= \frac{3}{4\pi} \sqrt{\frac{Gm}{a}} \left(1 - e^2\right)^2 \int_0^\infty \mathrm{d}W \int_0^{2\pi} \mathrm{d}v \frac{\mu_\perp^{(s)}(W) \sin(Wv) + \mu_\perp^{(c)}(W) \cos(Wv)}{(1 + e \cos v)^2} \tag{93}$$

(the averaging rule (112) being employed). Evaluation of the two integrals emerging in this expression can, in principle, be carried out in terms of the hypergeometric functions, but the outcome will be hard to work with and hard to interpret physically. Even worse integrals will show up in the averages of (77–87).

To get an idea of how much the terms (76–87) contribute to the secular drift of the satellite orbits from the planet's equator, it may be good to first calculate these term's averages under the assumption of small eccentricity. Then, for example, (93) will simplify to

$$\left\langle \vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial a} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial a} \right) \right\rangle \approx \frac{3}{4\pi} \sqrt{\frac{Gm}{a}} \left(1 - e^2\right)^2 \int_0^\infty \mathrm{d}W$$
$$\times \int_0^{2\pi} \mathrm{d}v \Big\{ \mu_\perp^{(s)}(W) \sin(Wv) \left(1 - 2e \cos v + 3e^2 \cos^2 v + \cdots\right)$$
$$+ \mu_\perp^{(c)}(W) \cos(Wv) \left(1 - 2e \cos v + 3e^2 \cos^2 v + \cdots\right) \Big\} \tag{94}$$

This expression contains both non-resonant and resonant terms. The leading resonant term emerges from integration of $\mu_\perp^{(c)}(W) \cos(Wv)(-2e) \cos v$ and is of the first order in the eccentricity. This input describes the resonance at $W = 1$. Under the assumption of vanishing e and slowly-changing a, this resonance corresponds to the 1 : 1 commensurability between the orbital frequency and the consonant nutational mode. The next resonance will emerge from integration of $\left[\mu_\perp^{(s)}(W) \sin(Wv) + \mu_\perp^{(c)}(W) \cos(Wv)\right](3e^2) \cos^2 v$. It will produce resonance at $W = 2$ and will be of the second order in e. We also see that the non-resonant term is of the zeroth order over e (though it will also accept contributions of higher orders in e). This feature, though, will not be generic. For example, orbital averaging of expression (77) will yield a resonance already in the zeroth order over e . (This resonance will emerge due to the term $2 \cos v$ in the denominator, and will single out the mode $W = 1$.) At the same time, the non-resonant contribution will show up due to the term $3e$ in the denominator and will, therefore, be only of the first order in the eccentricity.

This topic will be addressed in our subsequent paper where we shall try to determine how much time is needed for these subtle effects to accumulate, enough to cause a substantial secular drift of the orbit plane.

6. Conclusions

In this article we have prepared an analytical launching pad for the research of long-term evolution of orbits about a precessing oblate primary. This paper is the first in a series and is technical, so we deliberately avoided making quantitative estimates, leaving those for the next part of our project.

The pivotal question emerging in the context of this research is whether the orbital planes of near-equatorial satellites will drift away from the planetary equator in the cause of the planet's obliquity changes. Several facts have been established in this regard.

First, the planetary equations for osculating elements of the satellite do contain terms responsible for such a drift. These terms contain inputs of first order and second order in $\vec{\mu}$, and of first order in $\dot{\vec{\mu}}$, where $\vec{\mu}$ is the precession rate of the primary.

Second, the first-order (but not the second-order) terms average out in the case of a constant precession rate, which means that in this case their effect will accumulate only over extremely long time scales. We would remind that the short-period terms of the planetary equations do exert back-reaction upon the secular ones. While in the artificial-satellite science, which deals with short intervals of time, the short-period terms may often be omitted in long-term astronomical computations (dozens of million years and higher), the accumulated influence of short-period terms must be taken into account. A simple explanation of how this should be done is offered in section 2 of Laskar (1990). Besides, over very large time intervals accumulation of the contribution from the secular second-order terms will be taking place.

Third, the first-order drift terms do *not* average out in the case of variable precession. Under these circumstances they become secular. This means, for example, that the turbulent history of Mars' obliquity–history which includes both long-term changes (Ward 1973, 1974, 1979; Laskar & Robutel 1993; Touma & Wisdom 1994) and short-term nutations (Dehant et al. 2000; Van Hoolst et al 2000; Defraigne et al 2004) – might have lead to a secular drift of the initially near-equatorial satellites. If that were the case, then the current, still near-equatorial, location of Phobos and Deimos may lead to restrictions upon the rate and amplitude of the Martian obliquity variations. To render a judgement on this topic, one should compute how quickly this drift accumulates. Very likely, this quest will demand heavy-duty numerics.

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Appendix: The Inertial Terms Emerging in the Planetary Equations

In this Appendix we explain how the $\vec{\boldsymbol{\mu}}$- and $\dot{\vec{\boldsymbol{\mu}}}$- dependent terms in (60–65) should be rewritten as functions of the true anomaly, and how these functions should be averaged over the orbital period. We present the averaging rule and provide only one example. Comprehensive calculations for all terms can be found in Efroimsky (2004).

A.1. THE BASIC FORMULAE

Formulae (60–65) contain the two-body unperturbed expressions for the position and velocity as functions of the time and the six Keplerian elements, (1) and (7). To find their explicit form, one can employ an auxiliary set of dextral perifocal coordinates $\vec{\mathbf{q}}$, with an origin at the gravitating centre, and with the first two axes located in the plane of orbit:

$$\vec{\mathbf{q}} = \{r \cos v,\; r \sin v,\; 0\}^{\mathrm{T}} = a\frac{1-e^2}{e+\cos v}\{\cos v,\; \sin v,\; 0\}^{\mathrm{T}}. \tag{97}$$

The corresponding velocities will read:

$$\dot{\vec{\mathbf{q}}} = \left\{-\frac{na \sin v}{\sqrt{1-e^2}},\; \frac{na(e+\cos v)}{\sqrt{1-e^2}},\; 0\right\}^{\mathrm{T}} = \frac{na}{\sqrt{1-e^2}}\{-\sin v,\; (e+\cos v), 0\}^{\mathrm{T}}. \tag{98}$$

the radius in (97) being a function of the major semiaxis a, the eccentricity e, and the true anomaly v:

$$r = a\frac{1-e^2}{1+e\,\cos\,v}, \tag{99}$$

the true anomaly v itself being a function of a, e, and of the mean anomaly $M \equiv M_o + \int_{t_o}^{t} n\,\mathrm{d}t$, where $n \equiv (Gm)^{1/2}a^{-3/2}$. Then, in the two-body setting, the position and velocity, related to some fiducial inertial frame, will appear as:

$$\begin{aligned}\vec{\mathbf{r}} &= \vec{\mathbf{f}}\,(C_1,\ldots,C_6,t) = \hat{\mathbf{R}}(\Omega,i,\omega)\,\vec{\mathbf{q}}(a,e,M_o,t),\\ \dot{\vec{\mathbf{r}}} &= \vec{\mathbf{g}}(C_1,\ldots,C_6,t) = \hat{\mathbf{R}}(\Omega,i,\omega)\,\dot{\vec{\mathbf{q}}}(a,e,M_o,t),\end{aligned} \tag{100}$$

with $\hat{\mathbf{R}}(\Omega,i,\omega)$ being the matrix of rotation from the orbital-plane-related axes q to some fixed Cartesian axes (x_1,x_2,x_3) in the fiducial inertial frame wherein the vectors $\vec{\mathbf{r}}$ and $\dot{\vec{\mathbf{r}}}$ are defined. The rotation is parameterised by the three Euler angles: inclination, i; the longitude of the node, Ω; and the argument of the pericentre, ω . Thence, as well known (see, e.g. Morbidelli (2002), subsection 1.2),

$$\hat{\mathbf{R}} = \begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i\\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i\\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{pmatrix} \tag{101}$$

insertion whereof, together with (97), into the first equation of (100) yields

$$f_1 = a\frac{1-e^2}{1+e\,\cos\,v}[\cos\Omega\,\cos(\omega+v) - \sin\,\Omega\,\sin(\omega+v)\cos\,i], \tag{102}$$

$$f_2 = a\frac{1-e^2}{1+e\,\cos\,v}[\sin\Omega\,\cos(\omega+v) + \cos\,\Omega\,\sin(\omega+v)\cos\,i], \tag{103}$$

$$f_3 = a\frac{1-e^2}{1+e\,\cos\,v}\sin(\omega+v)\sin\,i. \tag{104}$$

Similarly, substitution of (98) and (101) into the second equation of (100) entails:

$$\begin{aligned} g_1 = \frac{n\,a}{\sqrt{1-e^2}}&[-\cos\,\Omega\,\sin(\omega+v) - \sin\,\Omega\,\cos(\omega+v)\cos\,i\\ &+ e(-\cos\,\Omega\,\sin\omega - \sin\Omega\,\cos\omega\,\cos\,i)], \end{aligned} \tag{105}$$

$$\begin{aligned} g_2 = \frac{na}{\sqrt{1-e^2}}&[-\sin\,\Omega\,\sin(\omega+v) + \cos\,\Omega\,\cos(\omega+v)\cos\,i\\ &+ e(-\sin\,\Omega\,\sin\omega + \cos\Omega\,\cos\omega\,\cos\,i)], \end{aligned} \tag{106}$$

$$g_3 = \frac{na}{\sqrt{1-e^2}}\sin\,i\,[\cos(\omega+v) + e\,\cos\,\omega], \tag{107}$$

with the subscripts 1 , 2 ,3 denoting the x_1, x_2, x_3 components in the fiducial inertial frame wherein (100) is written.

A.2. THE AVERAGING RULE

From equations

$$\cos E = \frac{e + \cos v}{1 + e \cos v}, \quad \sin E = \frac{\sqrt{1-e^2} \sin v}{1 + e \cos v}, \tag{108}$$

it follows that

$$\frac{\partial E}{\partial v} = \frac{\sqrt{1-e^2}}{1 + e \cos v}. \tag{109}$$

From the first of formulae (108) and from the Kepler equation one can derive:

$$\frac{\partial M}{\partial E} = \frac{1-e^2}{1 + e \cos v}. \tag{110}$$

Together, (109) and (110) entail:

$$\frac{\partial M}{\partial v} = \frac{\partial M}{\partial E}\frac{\partial M}{\partial v} = \frac{(1-e^2)^{3/2}}{1 + e \cos v} \tag{111}$$

whence

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}M = \frac{(1-e^2)^{3/2}}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}v}{(1 + e \cos v)^2}.$$

Calculation of the integral shows that the right-hand side of the above equation is equal to unity, which means that the secular parts should be calculated through the following averaging rule:

$$\langle \cdots \rangle \equiv \frac{(1-e^2)^{3/2}}{2\pi}\int_0^{2\pi} \cdots \frac{\mathrm{d}v}{(1 + e \cos v)^2} \tag{112}$$

A.3. EXAMPLE: CALCULATION OF $\vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial a} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial a}\right)$

As evident from (97) and from the first equation of (100),

$$\begin{aligned} \frac{\partial \vec{\mathbf{f}}}{\partial a} &= \left(\frac{\partial \vec{\mathbf{f}}}{\partial a}\right)_v + \left(\frac{\partial \vec{\mathbf{f}}}{\partial v}\right)_a \left(\frac{\partial v}{\partial a}\right)_{t,e,M_o} = \frac{\vec{\mathbf{f}}}{a} + \frac{\partial \vec{\mathbf{f}}}{\partial t}\frac{\partial t}{\partial v}\left(\frac{\partial v}{\partial a}\right)_{t,e,M_o} \\ &= \frac{\vec{\mathbf{f}}}{a} + \vec{\mathbf{g}}\frac{\partial t}{\partial v}\left(\frac{\partial v}{\partial a}\right)_{t,e,M_o} \end{aligned} \tag{113}$$

and, therefore,

$$\frac{\partial \vec{\mathbf{f}}}{\partial a} \times \vec{\mathbf{g}} = \frac{1}{a} \vec{\mathbf{f}} \times \vec{\mathbf{g}}. \tag{114}$$

Similarly, from (98) and the second equation of (100) it ensues that

$$\begin{aligned}\frac{\partial \vec{\mathbf{g}}}{\partial a} &= \left(\frac{\partial \vec{\mathbf{g}}}{\partial a}\right)_v + \left(\frac{\partial \vec{\mathbf{g}}}{\partial v}\right)_a \left(\frac{\partial v}{\partial a}\right)_{t,e,M_o} = -\frac{\vec{\mathbf{g}}}{2a} + \frac{\partial \vec{\mathbf{g}}}{\partial t}\frac{\partial t}{\partial v}\left(\frac{\partial v}{\partial a}\right)_{t,e,M_o} \\ &= -\frac{\vec{\mathbf{g}}}{2a} + \left(-\frac{Gm}{|\vec{\mathbf{f}}|^3}\vec{\mathbf{f}}\right)\frac{\partial t}{\partial v}\left(\frac{\partial v}{\partial a}\right)_{t,e,M_o}\end{aligned} \tag{115}$$

wherefrom

$$\vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial a} = -\frac{1}{2a} \vec{\mathbf{f}} \times \vec{\mathbf{g}}. \tag{116}$$

In the undisturbed two-body problem, $\vec{\mathbf{f}} \times \vec{\mathbf{g}}$ is the angular momentum (per unit of the reduced mass) and is equal to $\sqrt{Gma(1-e^2)}\vec{\mathbf{w}}$, where the unit vector

$$\vec{\mathbf{w}} = \hat{\mathbf{x}}_1 \sin i \sin \Omega - \hat{\mathbf{x}}_2 \sin i \cos \Omega + \hat{\mathbf{x}}_3 \cos i \tag{117}$$

is normal to the instantaneous osculating ellipse, the unit vectors $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$ making the basis of the co-precessing coordinate system x_1, x_2, x_3 (the axes x_1 and x_2 belonging to the planet's equatorial plane).

Together, (114) and (116) give:

$$\begin{aligned}\frac{\partial \vec{\mathbf{f}}}{\partial a} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial a} &= \frac{3}{2a}\vec{\mathbf{f}} \times \vec{\mathbf{g}} = \frac{3}{2a}\sqrt{Gma(1-e^2)}\vec{\mathbf{w}} \\ &= \frac{3}{2}\sqrt{\frac{Gm(1-e^2)}{a}}[\hat{\mathbf{x}}_1 \sin i \sin \Omega - \hat{\mathbf{x}}_2 \sin i \cos \Omega \\ &\quad + \hat{\mathbf{x}}_3 \cos i]\end{aligned} \tag{118}$$

and, thereby,

$$\vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\mathbf{f}}}{\partial a} \times \vec{\mathbf{g}} - \vec{\mathbf{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial a}\right) = \frac{3}{2}\mu_\perp \sqrt{\frac{Gm(1-e^2)}{a}} \tag{119}$$

where

$$\mu_\perp \equiv \mu_1 \sin i \sin \Omega - \mu_2 \sin i \cos \Omega + \mu_3 \cos i. \tag{120}$$

Since, for constant $\vec{\boldsymbol{\mu}}$, (119) is v-independent, then in the uniform-precession case it will coincide with its orbital average.

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