ON THE THEORY OF CANONICAL PERTURBATIONS AND ITS APPLICATIONS TO EARTH ROTATION: A SOURCE OF INACCURACY IN THE CALCULATION OF ANGULAR VELOCITY

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ABSTRACT. When the dynamical equations, written in terms of variable "constants," are demanded to be symplectic, these "constants" make conjugated pairs and are called Delaunay elements, in the orbital case, or Serret-Andoyer elements, in the rotational case. These sets of elements share a feature not readily apparent: in certain cases, the standard equations render them non-osculating. Non-osculating orbital elements parametrise instantaneous conics **not** tangent to the orbit. The non-osculating i, may differ much from the physical inclination of the orbit, given by the osculating i. Similarly, in the case of rotation, non-osculating Serret-Andoyer variables yield correct orientation angles for the body figure but not for the instantaneous spin axis. As a result, the Kinoshita-Souchay theory (which tacitly employs non-osculating Serret-Andoyer elements) gives correct results for the figure axis but needs corrections for the rotation axis.

1. KEPLER AND EULER

In orbital dynamics, a Keplerian conic, emerging as an undisturbed two-body orbit, is regarded as a sort of "elementary motion," so that all the other available motions are conveniently considered as distortions of such conics, distortions implemented through endowing the orbital constants C_j with their own time dependence. Points of the orbit can be contributed by the "elementary curves" either in a non-osculating fashion, as in Fig. 1, or in the osculating way, as in Fig. 2.

The disturbances, causing the evolution of the motion from one instantaneous conic to another, are the primary's oblateness, the gravitational pull of other bodies, the atmospheric and radiation-caused drag, and the non-inertiality of the reference system.

Similarly, in rotational dynamics, a complex spin can be presented as a sequence of configurations borrowed from a family of some elementary rotations. The easiest possibility here will be to employ in this role the Eulerian cones, i.e., the loci of the rotational axis, corresponding to non-perturbed spin states. These are the simple motions exhibited by an undeformable free top with no torques acting thereupon.[1] Then, to implement a perturbed motion, we shall have to

[1] Here one opportunity will be to employ in the role of "elementary" motions the non-circular Eulerian cones described by the actual triaxial top, when this top is unforced. Another opportunity will be to use, as "elementary" motions, the circular Eulerian cones described by a dynamically symmetrical top (and to treat its actual triaxiality

go from one Eulerian cone to another, just as in Fig. 1 and 2 we go from one Keplerian ellipse

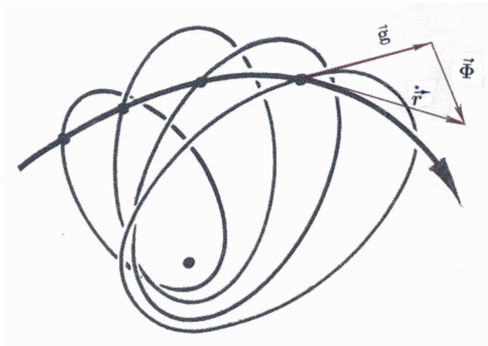

Figure 1: The perturbed trajectory is a set of points belonging to a sequence of confocal instantaneous ellipses. The ellipses are **not** supposed to be tangent, nor even coplanar to the orbit at the intersection point. As a result, the physical velocity $\dot{\mathbf{r}}$ (tangent to the trajectory at each of its points) differs from the Keplerian velocity $\tilde{\mathbf{g}}$ (tangent to the ellipse). To parametrise the depicted sequence of non-osculating ellipses, and to single it out of all the other such sequences, it is suitable to employ the difference between $\dot{\mathbf{r}}$ and $\vec{\boldsymbol{g}}$, expressed as a function of time and six (non-osculating) orbital elements: $\vec{\boldsymbol{\Phi}}(t, C_1, ..., C_6) = \dot{\mathbf{r}}(t, C_1, ..., C_6) - \tilde{\mathbf{g}}(t, C_1, ..., C_6)$. Since $\dot{\mathbf{r}} = \frac{\partial \tilde{\mathbf{r}}}{\partial t} + \sum_{j=1}^{6} \frac{\partial C_j}{\partial t} \dot{C}_j = \vec{\boldsymbol{g}} + \sum_{j=1}^{6} \frac{\partial C_j}{\partial t} \dot{C}_j$, then the difference $\vec{\boldsymbol{\Phi}}$ is simply the convective term $\sum \left(\partial \vec{\boldsymbol{r}} / \partial C_j\right) \dot{C}_j$ which emerges whenever the instantaneous ellipses are being gradually altered by the perturbation (and the orbital elements become time-dependent). In the literature, $\vec{\boldsymbol{\Phi}}(t, C_1, ..., C_6)$ is called the gauge function or gauge velocity or, simply, gauge.

to another. Hence, similar to those pictures, a smooth "walk" over the instantaneous Eulerian cones may be osculating or non-osculating.

The physical torques, the actual triaxiality of the top, and the non-inertial nature of the reference frame will then be regarded as perturbations causing the "walk." The latter two perturbations depend not only upon the rotator's orientation but also upon its angular velocity.

2. DELAUNAY AND SERRET

In orbital dynamics, we can express the Lagrangian of the reduced two-body problem via the spherical coordinates $q_j = \{r, \varphi, \theta\}$, then calculate their conjugated momenta p_j and the Hamiltonian $\mathcal{H}(q, p)$, and then carry out the Hamilton-Jacobi procedure (Plummer 1918), to arrive to the Delaunay variables

$$\begin{gathered} \{Q_1, Q_2, Q_3; P_1, P_2, P_3\} \equiv \{L, G, H; l, g, h\} = \\ \{\sqrt{\mu a}, \sqrt{\mu a\,(1 - e^2)}, \sqrt{\mu a\,(1 - e^2)}\cos i; -M_o, -\omega, -\Omega\}, \end{gathered} \tag{1}$$

where μ denotes the reduced mass.

Similarly, in rotational dynamics one can define a spin state of a top by means of the three Euler angles $q_j = \psi, \theta, \varphi$ and their canonical momenta p_j, and then perform a canonical transformation to the Serret-Andoyer elements L, G, H, l, g, h. A minor technicality is that, historically, these variables were introduced by Serret (1866) in a manner slightly different from the set of canonical constants: while, for a free rotator, the three Serret-Andoyer variables G, H, h are constants, the other three, L, l, g do evolve in time (for the Serret-Andoyer Hamiltonian of

as another perturbation). The main result of our paper will be invariant under this choice.

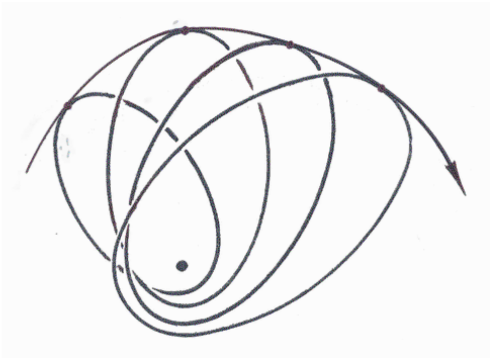

Figure 2: The perturbed trajectory is represented through a sequence of confocal instantaneous ellipses which are tangent to the trajectory at the intersection points, i.e., are osculating. Now, the physical velocity $\dot{\mathbf{r}}$ (which is tangent to the trajectory) will coincide with the Keplerian velocity $\tilde{\mathbf{g}}$ (which is tangent to the ellipse), so that their difference $\vec{\mathbf{\Phi}}(tC_1, ..., C_6)$ vanishes everywhere: $\vec{\mathbf{\Phi}}(t, C_1, ..., C_6) \equiv \dot{\vec{\boldsymbol{r}}}(t, C_1, ..., C_6) - \vec{\boldsymbol{g}}(t, C_1, ..., C_6) = \sum_{j=1}^{6} \frac{\partial C_j}{\partial t} \dot{C}_j = 0$. This equality, called Lagrange constraint or Lagrange gauge, is the necessary and sufficient condition of osculation.

a free top is not zero, but a function of l, L and G). This way, to make our analogy complete, we may carry out one more canonical transformation, from the Serret-Andoyer variables $\{L, G, H, l, g, h\}$ to "almost Serret-Andoyer" variables $\{L_o, G, H, l_o, g_o, h\}$, where L_o, l_o and g_o are the initial values of L, l and g. The latter set consists only of the constants of integration; the corresponding Hamiltonian becomes nil. Therefore, these constants are the true analogues of the Delaunay variables (while the conventional Serret-Andoyer set is analogous to the Delaunay set with M used instead of M_o.). The main result obtained below for the modified Serret-Andoyer set $\{L_o, G, H, l_o, g_o, h\}$ can be easily modified for the regular Serret-Andoyer set of variables $\{L, G, H, l, g, h\}$ (see the Appendix).

To summarise this section, in both cases we start out with

$$\dot{q} = \frac{\partial \mathcal{H}^{(o)}}{\partial p}, \quad \dot{p} = -\frac{\partial \mathcal{H}^{(o)}}{\partial q} \quad . \tag{2}$$

q and p being the coordinates and their conjugated momenta, in the orbital case, or the Euler angles and their momenta, in the rotation case. Then we switch, via a canonical transformation

$$\begin{aligned} q &= f(Q, P, t) \\ p &= \chi(Q, P, t), \\ \text{to } \dot{Q} &= \frac{\partial \mathcal{H}^*}{\partial P} = 0, \; \dot{P} = -\frac{\partial \mathcal{H}^*}{\partial Q} = 0, \; \mathcal{H}^* = 0, \end{aligned} \tag{3}$$

where Q and P denote the set of Delaunay elements, in the orbital case, or the (modified, as explained above) Serret-Andoyer set $\{L_o, G, H, l_o, g_o, h\}$, in the case of rigid-body rotation. This scheme relies on the fact that, for an unperturbed Keplerian orbit (and, similarly, for an undisturbed Eulerian cone) its six-constant parametrisation may be chosen so that:

1. the parameters are constants and, at the same time, are canonical variables $\{Q, P\}$ with a zero Hamiltonian: $\mathcal{H}^*(Q, P) = 0$;

2. for constant Q and P, the transformation equations (3) are mathematically equivalent to the dynamical equations (2).

3. WHEN DO THE ELEMENTS COME OUT NON-OSCULATING?

3.1 General-type motion

Under perturbation, the "constants" Q, P begin to evolve so that, after their substitution into

$$\begin{aligned} q &= f\,(Q(t), P(t), t) \\ p &= \chi(Q(t), P(t), t) \end{aligned}$$

(f and χ being the same functions as in (3)), the resulting motion obeys the disturbed equations

$$\dot{q} = \frac{\partial\left(\mathcal{H}^{(o)} + \Delta\mathcal{H}\right)}{\partial p}, \quad \dot{p} = -\frac{\partial\left(\mathcal{H}^{(o)} + \Delta\mathcal{H}\right)}{\partial q} \quad . \tag{4}$$

We also want our "constants" Q and P to remain canonical and to obey

$$\dot{Q} = \frac{\partial\,(\mathcal{H}^* + \Delta\mathcal{H}^*)}{\partial P}, \quad \dot{P} = -\frac{\partial\,(\mathcal{H}^* + \Delta\mathcal{H}^*)}{\partial Q} \tag{5}$$

$$\text{where } \mathcal{H}^* = 0 \text{ and } \Delta\mathcal{H}^*\,(Q, Pt) = \Delta\mathcal{H}\,(q(Q, P, t), p(Q, P, t), t)\,. \tag{6}$$

Above all, an optimist will expect that the perturbed "constants" $C_j \equiv Q_1, Q_2, Q_3, P_1, P_2, P_3$ (the Delaunay elements, in the orbital case, or the modified Serret-Andoyer elements, in the rotation case) will remain osculating. This means that the perturbed velocity will be expressed by the same function of $C_j(t)$ and t as the unperturbed one used to. Let us check to what extent this optimism is justified. The perturbed velocity reads

$$\dot{q} = g + \Phi \tag{7}$$

$$\text{where } g(C(t), t) \equiv \frac{\partial q(C(t), t)}{\partial t} \tag{8}$$

is the functional expression for the unperturbed velocity; and

$$\Phi(C(t), t) \equiv \sum_{j=1}^{6} \frac{\partial q(C(t), t)}{\partial C_j}\dot{C}_j(t) \tag{9}$$

is the convective term. Since we chose the "constants" C_j to make canonical pairs (Q, P) obeying (5 - 6), with vanishing $\mathcal{H}^*$, then insertion of (5) into (9) will result in

$$\Phi = \sum_{n=1}^{3} \frac{\partial q}{\partial Q_n}\dot{Q}_n(t) + \sum_{n=1}^{3} \frac{\partial q}{\partial P_n}\dot{P}_n(t) = \frac{\partial \Delta\mathcal{H}(q, p)}{\partial p}. \tag{10}$$

So the canonicity demand is incompatible with osculation. In other words, whenever a momentum-dependent perturbation is present, we still can use the ansatz (4) for calculation of the coordinates and momenta, but can no longer use (12) for calculating the velocities. Instead, we must use (11). Application of this machinery to the case of orbital motion is depicted on Fig.1. Here the constants $C_j = (Q_n, P_n)$ parametrise instantaneous ellipses which, for nonzero Φ, are **not** tangent to the trajectory. (For more details see Efroimsky & Goldreich (2003).) In the case of orbital motion, the situation will be similar, except that, instead of the instantaneous Keplerian conics, one will deal with instantaneous Eulerian cones (i.e., with the loci of the rotational axis, corresponding to non-perturbed spin states).

3.2 Orbital motion

In the orbital-motion case, osculation means the following. Let the unperturbed position be given, in some fixed Cartesian frame, by vector function $\vec{\boldsymbol{f}}$:

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{f}}\,(C_1, ..., C_6, t)\,, \vec{\boldsymbol{r}} \equiv \{x, y, z\}. \tag{11}$$

Employing this functional ansatz also under disturbance, we get the perturbed velocity as

$$\begin{aligned} \dot{\vec{\boldsymbol{r}}} &= \vec{\boldsymbol{g}}\,(C_1(t), ..., C_6(t), t) + \vec{\boldsymbol{\Phi}}\,(C_1(t), ..., C_6(t), t) & (12)\\ \text{where } \vec{\boldsymbol{g}} &\equiv \frac{\partial \vec{\boldsymbol{f}}}{\partial t} \quad \text{and} \quad \vec{\boldsymbol{\Phi}} \equiv \sum_{j=1}^{\infty} \frac{\partial \vec{\boldsymbol{f}}}{\partial C_j} \dot{C}_j. & (13) \end{aligned}$$

The osculation condition is a convenient (but totally arbitrary!) demand that the perturbed velocity $\dot{\vec{\boldsymbol{r}}}$has the same functional dependence upon t and C_j as the unperturbed velocity $\vec{\boldsymbol{g}}$:

$$\begin{aligned} \vec{\boldsymbol{r}}\,(C_1(t), ..., C_6(t), t) &= \vec{\boldsymbol{f}}\,(C_1(t), ..., C_6(t), t)\,, \\ \dot{\vec{\boldsymbol{r}}}(C_1(t), ..., C_6(t), t) &= \vec{\boldsymbol{g}}\,(C_1(t), ..., C_6(t), t)\,. \end{aligned}$$

or, equivalently, that the so-called Lagrange constraint is satisfied:

$$\sum_{j=1}^{\infty} \frac{\partial \vec{\boldsymbol{f}}}{\partial C_j} \dot{C}_j = \vec{\boldsymbol{\Phi}}\,(C_1(t), ..., C_6(t), t) \,\text{where}\, \vec{\boldsymbol{\Phi}} = 0. \tag{14}$$

Fulfilment of these expectations, however, should in no way be taken for granted, because the Lagrange constraint (14) and the canonicity demand (5 - 6) are now two independent conditions whose compatibility is not guaranteed. As shown in Efroimsky (2002a,b), this problem has gauge freedom, which means that any arbitrary choice of the gauge function $\vec{\boldsymbol{\Phi}}\,(C_1(t), ..., C_6(t), t)$ will render, after substitution into (11 - 12), the same values for $\vec{\boldsymbol{r}}$ and $\dot{\vec{\boldsymbol{r}}}$as were rendered by Lagrange's choice (14).[2] As can be seen from (10), **the assumption, that the "constants" Q and P are canonical, fixes the non-Lagrange gauge**

$$\sum_{j=1}^{\infty} \frac{\partial \vec{\boldsymbol{f}}}{\partial C_j} \dot{C}_j = \vec{\boldsymbol{\Phi}}\,(C_1(t), ..., C_6(t), t) \,\text{where}\, \vec{\boldsymbol{\Phi}} = \frac{\partial \Delta \mathcal{H}}{\partial \vec{\boldsymbol{p}}}. \tag{15}$$

It is easy to show (Efroimsky & Goldreich 2003; Efroimsky 2004b) that this same non-Lagrange gauge simultaneously guarantees fulfilment of the momentum-osculation condition:

$$\begin{aligned} \vec{\boldsymbol{r}}\,(C_1(t), ..., C_6(t), t) &= \vec{\boldsymbol{f}}\,(C_1(t), ..., C_6(t), t)\,, \\ \vec{\boldsymbol{p}}\,(C_1(t), ..., C_6(t), t) &= \vec{\boldsymbol{g}}\,(C_1(t), ..., C_6(t), t)\,. \end{aligned}$$

Any gauge different from (15), will prohibit the canonicity of the elements. In particular, for momentum-dependent $\Delta\mathcal{H}$, the choice of osculation condition $\vec{\boldsymbol{\Phi}} = 0$ would violate canonicity.

For example, an attempt of a Hamiltonian description of orbits about a precessing oblate primary will bring up the following predicament. On the one hand, it is most natural and convenient to define the Delaunay elements in a co-precessing (equatorial) coordinate system. On the other hand, these elements will not be osculating in the frame wherein they were introduced, and therefore their physical interpretation will be difficult, if at all possible. Indeed, instantaneous ellipses on Fig.1 may cross the trajectory at whatever angles (and may be even perpendicular

[2] Physically, this simply means that $\dot{\vec{\boldsymbol{r}}}$ on Fig.1 can be decomposed into $\vec{\boldsymbol{g}}$ and $\vec{\boldsymbol{\Phi}}$ in a continuous variety of ways. Mathematically, this freedom reflects a more general construction that emerges in the ODE theory. (Newman & Efroimsky 2003)

thereto). Thence, their orbital elements will not describe the real orientation or shape of the physical trajectory (Efroimsky & Goldreich 2004; Efroimsky 2004a).

For the first time, non-osculating elements obeying (16) implicitly emerged in (Goldreich 1965) and then in Brumberg et al (1971), though their exact definition in terms of gauge freedom was not yet known at that time. Both authors noticed that these elements were not osculating. Brumberg (1992) called them "contact elements." The osculating and contact variables coincide when the disturbance is velocity-independent. Otherwise, they differ already in the first order of the velocity-dependent perturbation. Luckily, in some situations their secular parts differ only in the second order (Efroimsky 2004), a fortunate circumstance anticipated yet by Goldreich (1965).

3.3 Rotational motion

In rotational dynamics, the situation of an axially symmetric unsupported top at each instant of time is fully defined by the three Euler angles $q_n = \theta, \phi, \psi$ and their derivatives $\dot{q}_n = \dot{\theta}, \dot{\phi}, \dot{\psi}$. The time dependence of these six quantities can be calculated from three dynamical equations of the second order and will, therefore, depend upon the time and six integration constants:

$$\begin{aligned} q_n &= f_n\left(S_1, ..., S_6, t\right), \\ \dot{q}_n &= \mathrm{g}_n\left(S_1, ..., S_6, t\right), \end{aligned}$$

the functions g_n and f_n being interconnected via $\mathrm{g}_n \equiv \partial f_n/\partial t$, for $n = 1, 2, 3 = \psi, \theta, \phi$.

Under disturbance, the motion will be altered:

$$\begin{aligned} q_n &= f_n\left(S_1(t), ..., S_6(t), t\right), \\ \dot{q}_n &= \mathrm{g}_n\left(S_1(t), ..., S_6(t), t\right) + \Phi_n\left(S_1(t), ..., S_6(t), t\right), \\ \text{where} \quad \Phi_n & \quad \left(S_1(t), ..., S_6(t), t\right) \equiv \sum_{j=1}^{6} \frac{\partial f_n}{\partial S_j} \dot{S}_j. \end{aligned} \tag{16}$$

Now choose the "constants" S_j to make canonical pairs (Q, P) obeying (5 - 6), with $\mathcal{H}^*$ being zero for $(Q, P) = (L_o, G, H, l_o, g_o, h)$. Then insertion of (5) into (16) will result in

$$\Phi_n\left(S_1(t), ..., S_6(t), t\right) \equiv \sum \frac{\partial f_n}{\partial Q} \dot{Q} + \sum \frac{\partial f_n}{\partial P} \dot{P} = \frac{\partial \Delta \mathcal{H}(q, p)}{\partial p_n}, \tag{17}$$

so that the canonicity demand (5 - 6) violates the gauge freedom in a non-Lagrange fashion. This is merely a particular case of (10).

This yields two consequences. One is that, in the canonical formalism, calculation of the angular velocities via the elements must be performed not through the second equation of (16) but through the second equation of (16), with (17) substituted therein. This means, for example, that in Kinoshita (1977) formula (2.6) and equations (6.26 - 6.27) for the instantaneous spin axis orientation must be amended. (While equations (6.24 - 6.25) remain in force because they pertain to the body figure.) After this amendment is introduced, the Kinoshita formalism will yield correct direction angles not only for the body figure but also for the rotation axis. (Even though the non-osculating elements, employed in it, will lack the evident physical interpretation inherent in the osculating variables.) Importance of this improvement is dictated by the fact that at present not only the body figure but also the instantaneous axis of rotation are directly observed (Schreiber et al 2004).

The second consequence is that, if we wish to make our Serret-Andoyer variables osculating (so that the second equation of (16) could be used), the price to be payed for this repair will be the loss of canonicity. (Angular-velocity-dependent perturbations cannot be accounted for by merely amending the Hamiltonian!) The osculating elements will obey non-canonical dynamical equations.

To draw to a close, we would add that, under some special circumstances, the secular parts of contact elements may coincide in the first order with those of their osculating counterparts.[3] Whether this will be the case for the Earth or Mars remains to be investigated. This matter will be crucial for examining the validity of the presently available computations of the history of Mars' obliquity.

APPENDIX: The case of regular Serret-Andoyer variables

For the purpose of convenience, above we assumed that the variable elements (Q_n, P_n) reduce, in the unperturbed case, to integration constants. This way, in the orbital case we employed the modified Delaunay set (the one with M_o instead of M), while in the rotation case we used the modified Serret-Andoyer set (L_o, G, H, l_o, g_o, h) instead of the regular set (L, G, H, l, g, h).

Here we shall demonstrate how our formalism should be reformulated for the regular set of variables. As everywhere in this article, (q, p) will denote the coordinates (Cartesian, in the orbital case; or Eulerian, in the rotation case) and their conjugated momenta. They will depend upon the regular set of canonical elements (α, β). (In the orbital case, these will be the regular Delaunay parameters $(L, G, H, -M, -\omega, -\Omega)$; while in the rotation case they will be the regular Serret-Andoyer ones (L, G, H, l, g, h).) Finally, (Q, P) will be the "constants": the modified Delaunay set $(L, G, H, -M_o, -\omega, -\Omega)$ in the case of orbital motion, and the modified Serret-Andoyer set (L_o, G, H, l_o, g_o, h) in the case of spin. Since

$$q = q\left(\alpha(Q, P, t), \beta(Q, P, t), t\right) \tag{18}$$

then, evidently,

$$\dot{q} = \left(\frac{\partial q}{\partial t}\right)_{\alpha\beta} + \left(\frac{\partial q}{\partial \alpha}\right)_{\beta t}\frac{d\alpha}{dt} + \left(\frac{\partial q}{\partial \beta}\right)_{\alpha t}\frac{d\beta}{dt} = \left[\left(\frac{\partial q}{\partial t}\right)_{\alpha\beta} + \left(\frac{\partial q}{\partial \alpha}\right)_{\beta t}\left(\frac{\partial \alpha}{\partial t}\right)_{QP} + \left(\frac{\partial q}{\partial \beta}\right)_{\alpha t}\left(\frac{\partial \beta}{\partial t}\right)_{QP}\right]$$

$$+ \left[\frac{\partial q}{\partial \alpha}\frac{\partial \alpha}{\partial Q} + \frac{\partial q}{\partial \beta}\frac{\partial \beta}{\partial Q}\right]\frac{dQ}{dt} + \left[\frac{\partial q}{\partial \alpha}\frac{\partial \alpha}{\partial P} + \frac{\partial q}{\partial \beta}\frac{\partial \beta}{\partial P}\right]\frac{dP}{dt}$$

$$\text{where } \left[\left(\frac{\partial q}{\partial t}\right)_{\alpha\beta} + \left(\frac{\partial q}{\partial \alpha}\right)_{\beta t}\left(\frac{\partial \alpha}{\partial t}\right)_{QP} + \left(\frac{\partial q}{\partial \beta}\right)_{\alpha t}\left(\frac{\partial \beta}{\partial t}\right)_{QP}\right] = \left(\frac{\partial q}{\partial t}\right)_{QP},$$

$$\left[\frac{\partial q}{\partial \alpha}\frac{\partial \alpha}{\partial Q} + \frac{\partial q}{\partial \beta}\frac{\partial \beta}{\partial Q}\right] = \left(\frac{\partial q}{\partial Q}\right)_{Pt}, \qquad \left[\frac{\partial q}{\partial \alpha}\frac{\partial \alpha}{\partial P} + \frac{\partial q}{\partial \beta}\frac{\partial \beta}{\partial P}\right] = \left(\frac{\partial q}{\partial P}\right)_{Qt}.$$

In our paper, when introducing osculation, we have in mind osculation relative to the constants $(Q, P) = (L_o, G, H, l_o, g_o, h)$. This osculation condition reads: $\dot{q} = (\partial q/\partial t)_{QP}$, so that the gauge function $\Phi \equiv \dot{q} - (\partial q/\partial t)_{QP} = (\partial q/\partial Q)_{Pt}\,\dot{Q} + (\partial q/\partial P)_{Qt}\,\dot{P}$ becomes the measure of non-osculation. When Kinoshita (1977), on the fourth page of his paper, referred to osculation, he implied osculation relative to the canonical elements $(\alpha, \beta) = (L, G, H, l, g, h)$. The corresponding osculation condition was: $\dot{q} = (\partial q/\partial t)_{\alpha\beta}$, while the appropriate measure of non-osculation, $\tilde{\Phi} \equiv \dot{q} - (\partial q/\partial t)_{\alpha\beta} = (\partial q/\partial \alpha)_{\beta t}\,\dot{\alpha} + (\partial q/\partial \beta)_{\alpha t}\,\dot{\alpha}$, was different from our Φ. By comparing the two expressions,

$$\dot{q} = (\partial q/\partial t)_{QP} + \Phi \text{and} \dot{q} = (\partial q/\partial t)_{\alpha\beta} + \tilde{\Phi}, \tag{19}$$

[3] In regard to orbital motions, this possibility was anticipated yet in 1965 by Peter Goldreich. As demonstrated by Efroimsky (2004a), this is true for constant rate of frame precession (but not for variable precession).

and by applying the first equation of (19) we see that

$$\tilde{\Phi} = \Phi - \left(\frac{\partial q}{\partial \alpha}\right)_{\beta t} \left(\frac{\partial \alpha}{\partial t}\right)_{QP} - \left(\frac{\partial q}{\partial \beta}\right)_{\alpha t} \left(\frac{\partial \beta}{\partial t}\right)_{QP} . \tag{20}$$

In particular, if we want all three sets, (q, p), (α, β), and (Q, P), to be canonical, then the non-osculation (in the sense of Kinoshita) will be:

$$\tilde{\Phi} = \frac{\partial \Delta \mathcal{H}}{\partial p} - \left(\frac{\partial q}{\partial \alpha}\right)_{\beta t} \left(\frac{\partial \alpha}{\partial t}\right)_{QP} - \left(\frac{\partial q}{\partial \beta}\right)_{\alpha t} \left(\frac{\partial \beta}{\partial t}\right)_{QP} . \tag{21}$$

It is this quantity that should be added to the partial derivatives of the Euler angles, in order to get the right values of the angular velocities. This way, the above formula will be the starting point for correcting equations (2.6) and (6.26 - 6.27) in the Kinoshita (1977) theory.

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