Towards reduction of uncertainty in the operation of reservoir systems

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Outline

1. **Introduction**
   - Overview
   - Problem Setup

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2. **Proposed Framework**
   - Preliminaries
   - Generalized Polynomial Chaos
   - Stochastic Collocation Method

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   - Preliminaries
   - Generalized Polynomial Chaos
   - Stochastic Collocation Method
3. **Numerical Results**
   - gPC Example

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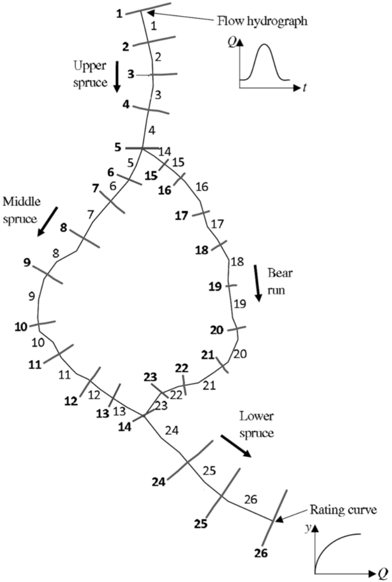
River Network System

- Input is flow hydrograph $Q(t)$
- River system flow dynamics determined by unsteady flow routing
- Nonlinear time-dependent system
- We use Performance Graphs approach to simulate (OSU Rivers)



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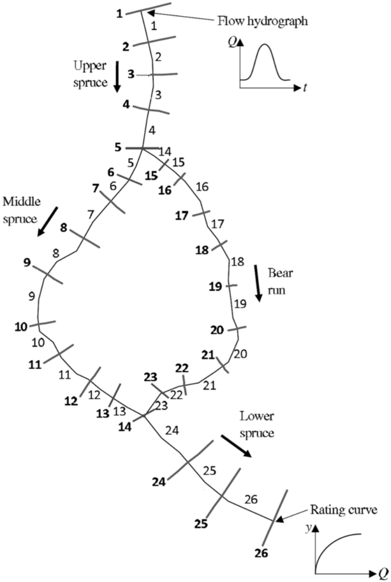

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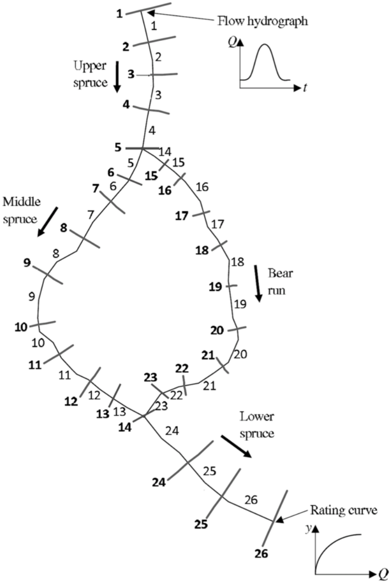

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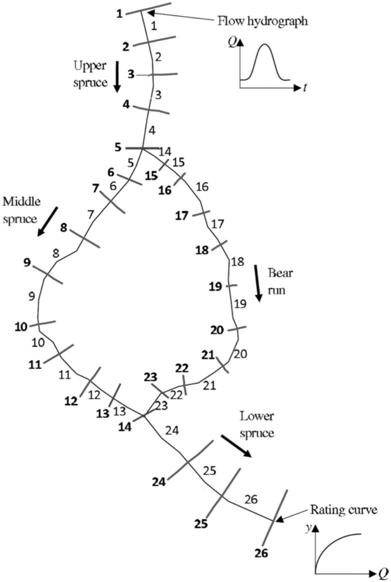

Incorporating Uncertainty

Incorporate uncertainty into inputs of the system

- We propose to add a stochastic component to input variables
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- Representation of stochastic outputs in the form of a polynomial in a random variable (easy to sample)
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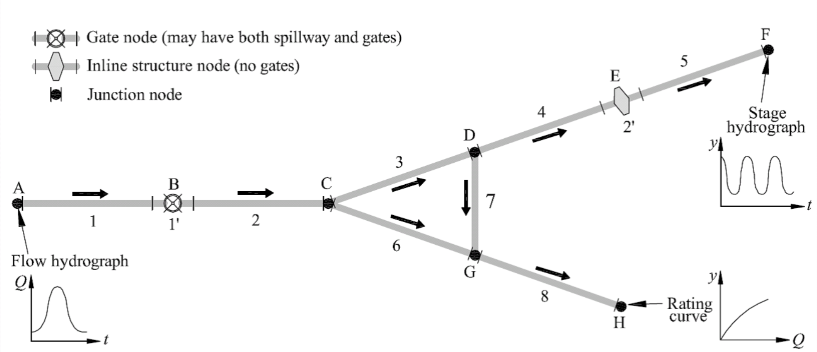
  - compute failure probabilities
- Polynomials computed using uncoupled, deterministic forward simulations with different parameters (embarrassingly parallel)

Simple Network System

Consider this simple network system



The (nonlinear) relationship between variables $\vec{X} = [y_{d_1}, \ldots, Q_{d_8}]$ is given by the following map

$$
\begin{bmatrix}
x & - & - & - & - & - & - & - & x & - & - & - & - & - & - & - & x & - & - & - & - & - & - & - \\
- & x & - & - & - & - & - & - & - & x & - & - & - & - & - & - & - & x & - & - & - & - & - & - \\
- & - & x & - & - & - & - & - & - & - & x & - & - & - & - & - & - & - & x & - & - & - & - & - \\
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- & - & - & - & - & - & - & - & - & - & - & - & x & - & - & - & - & - & - & x & - & - & - & - \\
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- & x & - & - & - & x & - & - & - & - & - & - & - & x & - & - & - & - & - & - & - & x & - & - \\
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- & - & x & - & - & - & x & - & - & - & - & - & - & - & x & - & - & - & - & - & - & - & x & - \\
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x & x & - & - & - & - & - & - & - & x & - & - & - & - & - & - & x & x & - & - & - & - & - & -
\end{bmatrix}
\begin{bmatrix}
y_{d_1} \\ y_{d_2} \\ y_{d_3} \\ y_{d_4} \\ y_{d_5} \\ y_{d_6} \\ y_{d_7} \\ y_{d_8} \\ Q_{u_1} \\ Q_{u_2} \\ Q_{u_3} \\ Q_{u_4} \\ Q_{u_5} \\ Q_{u_6} \\ Q_{u_7} \\ Q_{u_8} \\ Q_{d_1} \\ Q_{d_2} \\ Q_{d_3} \\ Q_{d_4} \\ Q_{d_5} \\ Q_{d_6} \\ Q_{d_7} \\ Q_{d_8}
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\begin{bmatrix}
a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \\ a_{16} \\ a_{17} \\ a_{18} \\ a_{19} \\ a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{24}
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Problem Inputs

- Inputs: Q_{u_1}, y_5, Rating Curve, gate positions
- Assume uncertainty envelope around Q_{u_1} prediction (representing flow discharges upstream of reach 1)

$$Q_{u_1}(t) = \overline{Q}_{u_1}(t) + \tilde{Q}_{u_1}(t)$$

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Questions one could ask

- What is the resulting pdf of Q_{d_8}?
  (propagation of uncertainty)
- What choice of gate positions cause Q_{d_8} to behave as desired?
  (on average)
- What choice of gate positions cause hydropower production $h(\vec{X})$ to behave as desired?
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- What choice of gate positions minimize risk of flooding?
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- Are there choices of gate positions that lead to robust predictions of Q_{d_8} or $h(\vec{X})$ or flood volumes?
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Mathematical Framework

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Find gate positions $g(t)$ such that one or more of the following are minimized

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$$J_2 = Var(Q_{d_8})$$

$$J_3 = Prob[Q_{d_8} < \text{tolerance}]$$

or similar for $\vec{X}_i$ or $h(\vec{X})$ or FV.

Requirements

- Need a framework that allows fast computation of statistics and/or failure probabilities of select variables in the solution of the system. (or nonlinear/non-smooth functions of select variables)
- To take advantage of an already developed flow dynamics model based on the HPG/VPG approach, we will incorporate the proposed uncertainty framework non-intrusively into OSU Rivers.

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Generalized Polynomial Chaos

- Our approach is to explicitly model the random space (via random variables and processes) and perform a generalized Polynomial Chaos (gPC) representation
- This method uses an orthogonal polynomial expansion in random space to represent the stochastic input quantities as well as the solutions to the system.
- Convergence of polynomial chaos methods can be shown to be exponential in the number of basis functions.

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Example

To illustrate the idea of our approach, we present the following example. Consider the quantity Q_{u_1} above.

- We may assume that the uncertainty around the prediction $\overline{Q}_{u_1}(t)$ is relative with magnitude 10%.
- We introduce the standard random variable (hereafter referred to as a *germ*) $\xi \sim Beta(\alpha, \beta)$ with support $[-1, 1]$ then

$$Q_{u_1} = \overline{Q}_{u_1} + 0.1\xi\overline{Q}_{u_1} \tag{1}$$

- Equation (1) represents a polynomial chaos expansion of the random input.
- If $\alpha = \beta$, F is a symmetric beta distribution centered around mean 0; for the special case $\alpha = \beta = 0$ this is simply a uniform distribution.

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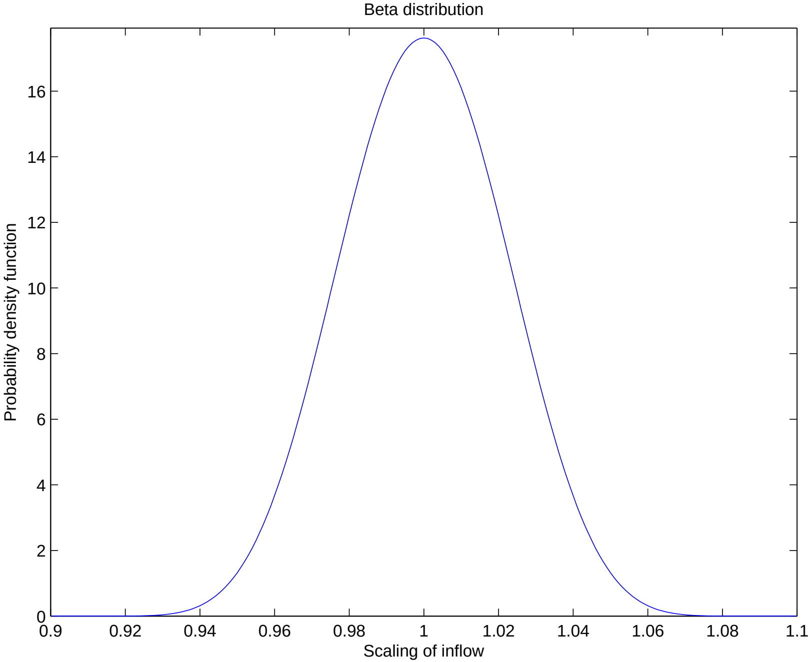
Beta distribution
Probability density function
0
2
4
6
8
10
12
14
16
0.9
0.92
0.94
0.96
0.98
1
1.02
1.04
1.06
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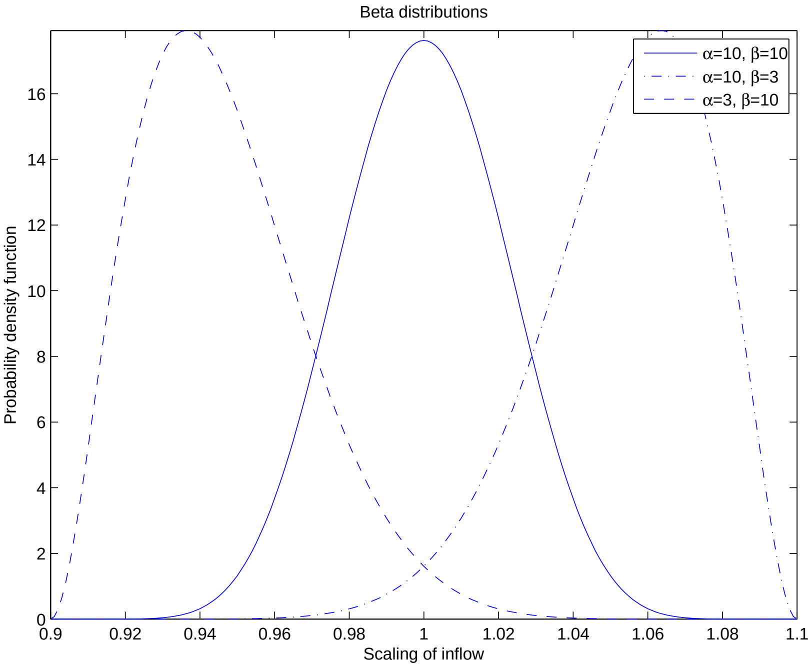
1.1
Scaling of inflow



Beta distributions
α=10, β=10
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Generalizations

- Continuous distributions that can be handled directly: Uniform, Beta, Gaussian, Gamma
- Discrete distributions: Poisson, Binomial, Hypergeometric
- Choice of distribution determines corresponding orthogonal polynomials (e.g., Gaussian pdf defines Hermite orthogonality)
- Other distributions require non-linear transformations of the above, or manual construction of orthogonal polynomials
- Random inputs can be random variables or random processes (time-dependent), e.g., represented by a Karhunen-Loeve (KL) expansion
- Any number of independent random inputs may be used, each with their own distribution

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Stochastic Galerkin Method

- In a Galerkin method, we seek to determine the coefficients of a gPC expansion of each component of the solution vector

$$\vec{X} = [y_{d_1}, \ldots, Q_{d_8}]$$

- To do this, one may take a Galerkin projection of the original system, but with these expansions substituted in for the solution quantities
- The resulting integrals can be computed analytically due to the polynomial basis representation
- This approach in general leads to a large coupled system of equations for the gPC coefficients
  (e.g., intrusive method: changes the system to be solved)
- This new system must be discretized in space and time

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- An alternative is to numerically approximate the integrals
- Consider Q_{d_8}: its representation in terms of a degree P expansion

$$Q_{d_8}^P(t,\xi) = \sum_{i=0}^{P} v_i(t)\phi_i(\xi)$$

  where $\phi_i(\xi)$ are the basis functions
  (Jacobi polynomials in the case of a Beta distribution of inputs).
- Each gPC expansion coefficient is given by

$$v_i(t) = \mathbb{E}[Q_{d_8}(t,\xi)\phi_i(\xi)]$$

  i.e., the expected value with respect to F of the true solution times the (normalized) basis function

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Stochastic Collocation Method with Gaussian Quadrature

- The computation of the integral in

$$\mathbb{E}[Q_{d_8}(t,\xi)\phi_i(\xi)] = \int Q_{d_8}(t,\xi)\phi_i(\xi)dF(\xi)$$

can be performed efficiently via Gaussian quadrature.

- Gaussian quadrature applies to functions which can be represented as $g(\xi)W(\xi)$ where $g(\xi)$ is well-approximated by a polynomial.
- Then the nodes ξ_j of the quadrature rule are the roots of an orthogonal polynomial in the support of F

$$\mathbb{E}[Q_{d_8}(t,\xi)\phi_i(\xi)] \approx \sum_{j=1}^{N} w_j Q_{d_8}(t,\xi_j)\phi_i(\xi_j)$$

- Stochastic Collocation thus requires only deterministic system solutions for the fixed values $\{\xi_j\}_{j=1}^{N}$ of the random variable ξ

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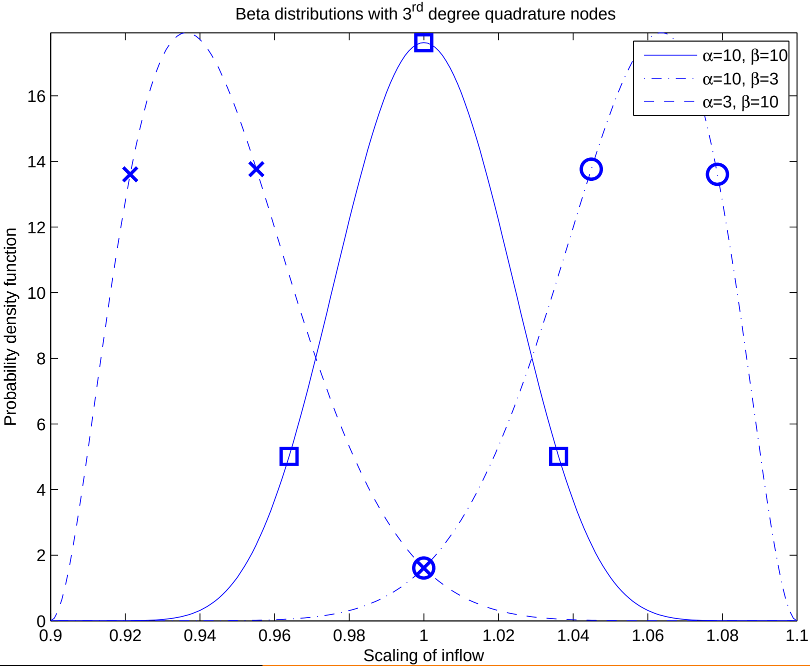
Beta distributions with 3rd degree quadrature nodes
α=10, β=10
α=10, β=3
α=3, β=10
Probability density function
Scaling of inflow
0
2
4
6
8
10
12
14
16
0.9
0.92
0.94
0.96
0.98
1
1.02
1.04
1.06
1.08
1.1

Comments on Stochastic Collocation

- System solutions at ξ_j can be recycled even if the input pdf changes
- Gaussian nodes are pre-determined (by choice of distribution) and thus simulations are independent and parallelizable!
- For large random dimension, sparse grids (Smolyak) are used to mitigate curse of dimensionality in quadrature
- Implemented in DAKOTA (Design Analysis Kit for Optimization and Terascale Applications) toolkit by Sandia National Laboratories
  - Open source, C++ software
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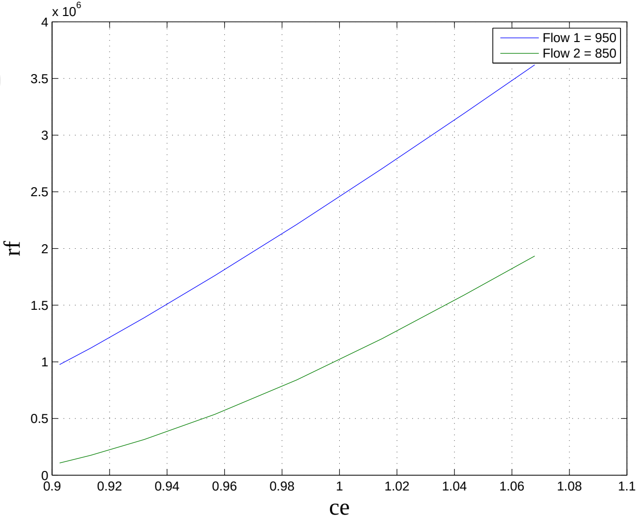
Outline

We build the gPC expansion (response function) for the flood volume in a single reach simulation using two different predictions of Q_{u_1}:

- Peak flows of 850 and 950
- 10% relative Beta uncertainty envelope
- FV given by FPGs
- Expansion is non-linear (yet low degree)



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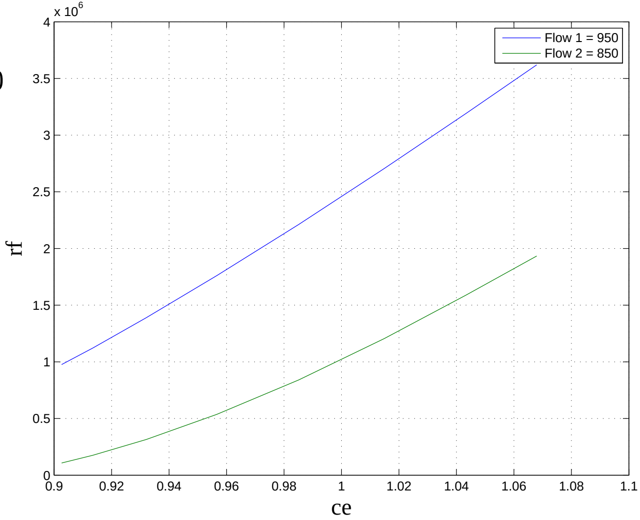

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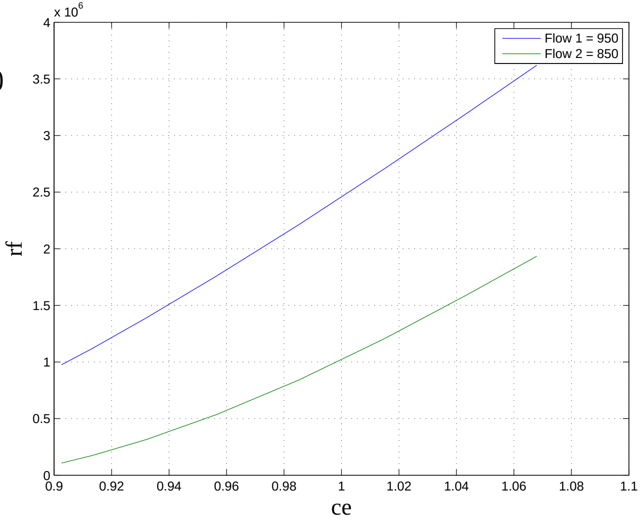

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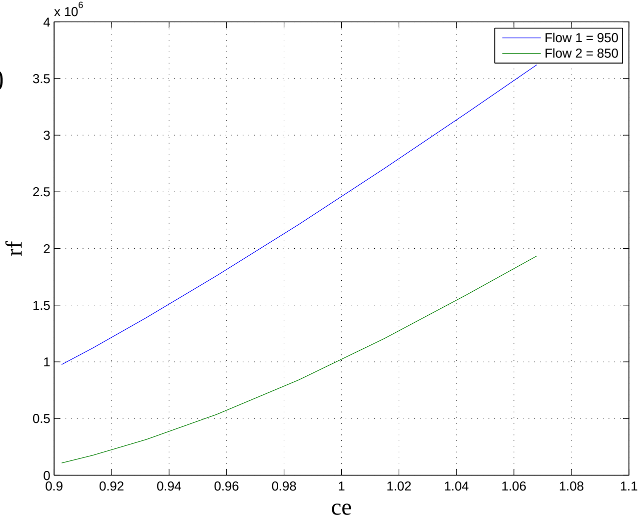

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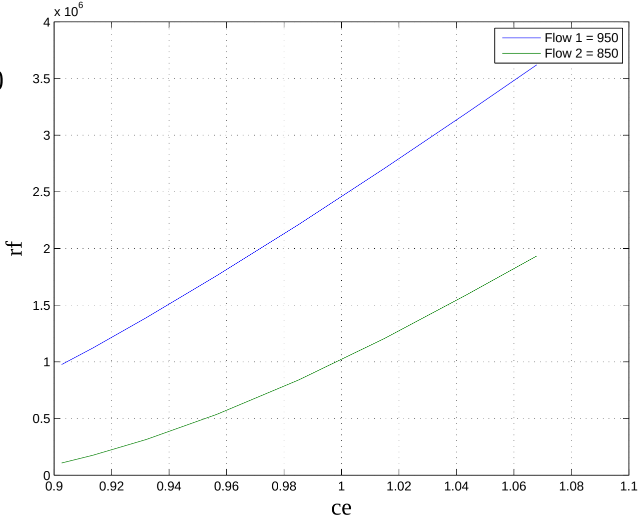
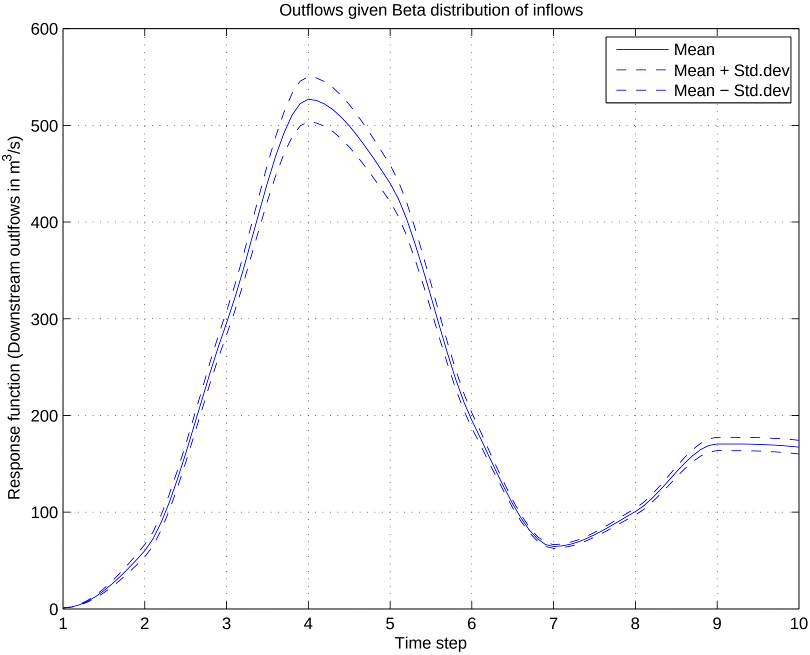
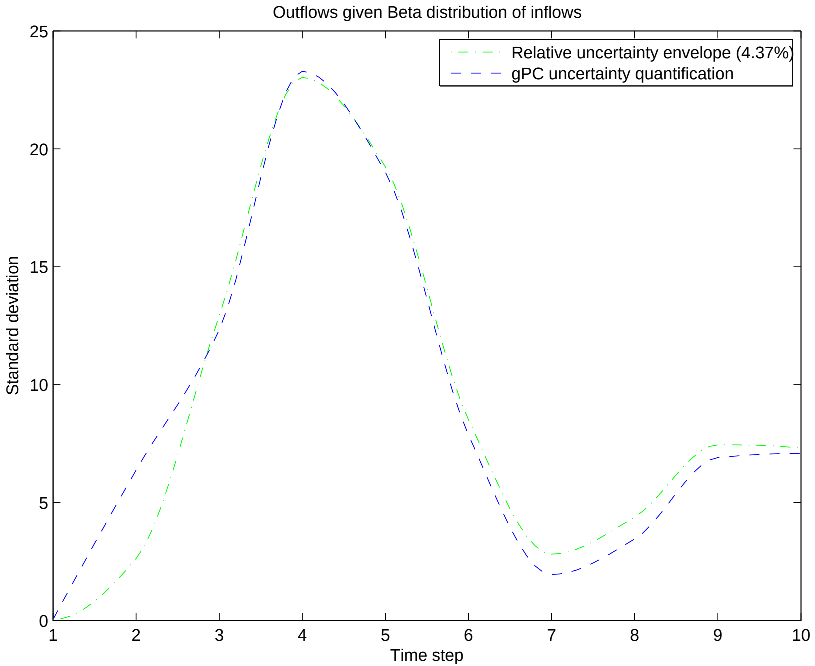



Outflows given Beta distribution of inflows
Mean
Mean + Std.dev
Mean − Std.dev
Response function (Downstream outflows in m3/s)
Time step
600
500
400
300
200
100
0
1
2
3
4
5
6
7
8
9
10


Outflows given Beta distribution of inflows
Relative uncertainty envelope (4.37%)
gPC uncertainty quantification
Standard deviation
Time step
0
5
10
15
20
25
1
2
3
4
5
6
7
8
9
10

Outline

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- Demonstrate on large-scale example
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  - OSU Rivers: NSGA-II Genetic Algorithm
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  - Hybrid approach (local vs global)
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