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TRANSIMS REPORT SERIES

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and Traffic Flow Theory

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August 14, 1995



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LA-UR-95:2908

Title: PARTICLE HOPPING MODELS AND TRAFFIC FLOW THEORY

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Submitted to: Physical Review E
American Physical Society
August 14, 1995

Los Alamos
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Form No. 836 R5
ST 2629 10/91

Particle hopping models and traffic flow theory

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This version: August 17, 1995

This paper shows how particle hopping models fit into the context of traffic flow theory. Connections between fluid-dynamical traffic flow models, which derive from the Navier-Stokes-equations, and particle hopping models are shown. In some cases, these connections are exact and have long been established, but have never been viewed in the context of traffic theory. In other cases, critical behavior of traffic jam clusters can be compared to instabilities in the partial differential equations. Finally, it is shown how this leads to a consistent picture of traffic jam dynamics.

I. INTRODUCTION

Traffic jams have always been annoying. At least in the industrialized countries, the standard reaction has been to expand the transportation infrastructure to match demand. In this phase of fast growth, relatively rough planning tools were sufficient. However, in the last years most industrialized societies started to see the limits of such growth. In densely populated areas, there is only limited space available for extensions of the transportation system; and we face pollution and growing accident probabilities as the downsides of mobility. In consequence, planning is now turning to a fine-tuning of the existing systems, without major extensions of facilities. This is for example reflected in the United States by the Clean Air Act and by the ISTEA (Intermodal Surface Transportation and Efficiency Act) legislation. The former sets standards of air quality for urban areas, whereas the latter forces planning authorities to evaluate land use policies, intermodal connectivity, and enhanced transit service when planning transportation .

In consequence, planning and prediction tools with a much higher reliability than in the past are necessary. Due to the high complexity of the problems, analytical approaches are infeasible. Current approaches are simulation-based (e.g. [1-4]), which is driven by necessity, but largely enhanced by the widespread availability of computing power nowadays. Yet, also for computers one needs good models of the phenomena of interest: Just coding a perfect representation of reality into the computer is not possible because of limits of knowledge, limits of human resources for coding all these details, and because of limits of computational resources.

Practical simulation has to observe trade-offs between *resolution*, *fidelity*, and *scale* [5]. Resolution refers to the smallest entities (objects, particles, processes) resolved in a simulation, whereas fidelity means the degree of realism in modeling each of these entities. It is empirically well known, for example from fluid dynamics, that to a certain extent a low fidelity high resolution model (lattice gas automata [6,7]) can do as well as a high fidelity low resolution model (discretization of the Navier-Stokes-equations), or in short: Resolution can replace fidelity.

Current state-of-the-art traffic modeling, though, has a fixed unit of (minimal) resolution, and that is the individual traveler. Since one is aiming for rather large scales (for example the Los Angeles area consists of approx. 10 million potential travelers), it is rather obvious that one has to sacrifice fidelity to achieve reasonable computing times.

One important part of transportation modeling is road traffic. For example in Germany, road traffic currently

contributes more than 81% of all passenger and 52.7% of all freight transportation [8]. And despite widespread efforts, the share of road transportation is still increasing. In this sense, it makes sense to start with road traffic when dealing with transportation systems.

Putting these arguments together, one thing which is needed for transportation simulations is a *minimal* representation of road traffic. Particle hopping models clearly are candidates for this, and even if not, building a *minimal* theory of road traffic is certainly the right starting point.

This paper shows how particle hopping models fit into the context of traffic flow theory. It starts out with an historical overview of traffic flow theory (Section II), followed by a systematic review of fluid-dynamical models for traffic flow (Section III) starting from the Navier-Stokes-equations. Section IV defines different particle hopping models which are of interest in the context of traffic flow. Section V then shows the different connections between the fluid-dynamical traffic flow models and particle hopping models. In some cases, these connections are exact and have long been established, but have never been viewed in the context of traffic theory. In other cases, critical behavior of traffic jam clusters can be compared to instabilities in the partial differential equations. Finally, it is shown how this leads to a consistent picture of traffic jam dynamics (Section VI). A discussion of the consequences for traffic simulations (Section VII) serves as summary and discussion, and a collection of open questions (Section VIII) conclude the paper.

II. HISTORICAL OVERVIEW

Vehicular traffic has been a widely and thoroughly researched area in the 1950s and 60s. Vehicular traffic theory can be broadly separated into two branches: Traffic flow theory, and car-following theory. For a review of traffic theory, see, for example, one of [9–11].

Traffic flow theory is concerned with finding relations between the three fundamental variables of traffic flow, which are velocity v , density ρ , and current or throughput or flow j . Only two of these variables are independent since they are related through $j = \rho v$.

The first task of traffic flow theory historically was to search for time-independent relations between j , ρ and v , the so-called fundamental diagrams. The form of such a relation is, though, still debated in the traffic flow literature [12,13]. The problem stems mainly from the fact that reality measurements are done in non-stationary conditions. There, only short time averages make sense, and they usually show large fluctuations. I will at the end of the paper discuss how a dynamic, particle based description of traffic can account for these difficulties.

The second step of traffic flow theory was to introduce a dynamic, i.e. time-dependent description. This was achieved by a well-known paper from Lighthill and Whitham, published in 1955 [14]. This paper introduces a description based on the equation of continuity plus the assumption that flow (or velocity) depend on the density only, i.e. there is no relaxation time (instantaneous adaption).

Prigogine, Herman, and coworkers developed a kinetic theory for traffic flow [15]. They derived the Lighthill-Whitham situation as a limiting case of the kinetic theory. The kinetic theory anticipates many of the phenomena which come up in later work, but probably because the mathematics of working in this framework is fairly laborious, this theory has not been developed any further until recently [16].

Instead, in 1979, Payne replaced the assumption of instantaneous adaption in the Lighthill-Whitham theory by an equation for inertia, which is similar to a Navier-Stokes-equation [17]. Kühne, in 1984, added a viscosity term and initiated using the methods of nonlinear dynamics for analyzing the equations [18–21].

In a parallel development, Musha and Higuchi proposed the noisy Burgers equation as a model for traffic and backed that up by measurements of the power spectrum of traffic count data [22].

In the next section, these fluid-dynamical models will be put into a common perspective.

Car-following theory regards traffic from a more microscopic point of view: The behavior of each vehicle is modeled in relation to the vehicle ahead. As the definition indicates, this theory concentrates on single lane situations where a driver reacts to the movements of the vehicle ahead of him. Many car-following models are of the form

$$a(t + T) \propto \frac{v(t)^m}{[\delta x(t)]^l} \cdot \delta v(t) , \quad (1)$$

where a and v are the acceleration and velocity, respectively, of the car under consideration, δx is the distance to the car ahead, δv is the velocity difference to that car, and m and l are constants. T is a delay time between stimulus and response, which summarizes all delay effects such as human reaction time or time the car mechanics needs to react to input.

Other examples for car-following equations are $v(t + T) \propto \delta x$ [23,24] or $a(t) \propto V[\delta x(t)] - v(t)$ [25], where $V[\delta x]$ gives a preferred velocity as a function of distance headway.

Mathematically, parts of this theory are very similar to the treatment of atomic movements in crystals, and give results about the stability of chains of cars ("platoons") in follow-the-leader situations.

One of the achievements of traffic theory of that period was that relations between car-following models and static flow-density-relations can be derived.

Car-following theory will not be treated any further in this paper.

A more recent addition to the development of vehicular traffic flow theory are cellular automata (CA). Actually, the first proposition of such a model is from Gerlough in 1956 [26] and has been further extended by Cremer and coworkers [27,28]. They implemented fairly sophisticated driving rules and also used single-bit-coding with the goal of using the model in real-time traffic applications. The bit-coded implementation, though, made it too impractical for many traffic applications.

In 1992, CA models for traffic were brought into the statistical physics community. Biham and coworkers used a model with maximum velocity one for one- and for two-dimensional traffic [29]. One-dimensional here refers to roads etc., and includes multi-lane traffic. Two-dimensional traffic in the CA context usually means traffic on a 2-d grid, as a model for traffic in urban areas. Nagel and Schreckenberg introduced a model with maximum velocity five for one-dimensional traffic, which compared favorably with real world data [30]. Both approaches were further analyzed and extended in a series of subsequent papers, both for the one-dimensional [31-51] (see also [52]) and the two-dimensional (see, e.g., [53-55]) investigations.

The motivation here was twofold. The first was again the aiming for computational speed, but this time to make statistical analysis possible. The second motivation is that the model is still simple enough to be treated analytically, but distinctively different from other particle-hopping models. In addition, CA-methodology is planned to be used as a high-speed option in traffic projects in Germany [2] and in the United States [1].

From a theoretical point of view, the methodology of CA is placed between fluid-dynamical and car-following theories, and is helpful to further clarify the connections between these approaches. This paper aims at contributing to the first part, i.e. understanding and clarifying the relations between particle-hopping models and fluid-dynamical models for traffic flow.

III. FLUID-DYNAMICAL MODELS FOR TRAFFIC FLOW

This section reviews fluid-dynamical models for traffic flow. The models can broadly be distinguished into two classes: Models which do not include the effects of inertia and models which do. Models of the first class can be derived from the equation of continuity where velocity or current are defined as functions of the density only. Models of the second class are formally Navier-Stokes-equations, with a car-specific force term which takes into account that drivers want to drive at a certain desired speed. If one sets the time constant of this force term to zero, i.e. assuming *instantaneous adaption* to the surrounding density, then one recovers the models of the first class.

A. General equations

Papers on traffic flow theory usually start with stating the equations under consideration, without putting them into perspective. I will therefore in this paper attempt a more fundamental approach, similar to conventional fluid-dynamics. The precise presentation of most of these equations is necessary anyhow because the particle-hopping models presented later relate to these equations.

One might use the standard fluid-dynamical conservation equations for mass and momentum as a starting point for a fluid-dynamical description of traffic:

$$\partial_t \rho + \partial_x (\rho v) = 0 \quad (2)$$

and

$$\frac{dv}{dt} \equiv \partial_t v + v \cdot \partial_x v = F/m, \quad (3)$$

where ρ is the density and v the velocity. d/dt is the individual (Lagrangian) derivative, F is the force acting on mass m . The first equation (of continuity) describes mass conservation; the second one (momentum conservation) describes the fact that the momentum of a point of mass may only be changed by a force. Obviously, for traffic, F has to include vehicle and driving dynamics.

B. Fluctuations

A standard first step in fluid-dynamics [56] is to assume that v and ρ fluctuate statistically around average values $\langle v \rangle$ and $\langle \rho \rangle$, i.e.

$$v = \langle v \rangle + v', \quad \langle v' \rangle = 0 \quad (4)$$

and

$$\rho = \langle \rho \rangle + \rho', \quad \langle \rho' \rangle = 0. \quad (5)$$

In this case, one only assumes that $\langle v \rangle$ and $\langle \rho \rangle$ fluctuate *slowly* in space and time; for the general subtleties of hydrodynamical theory see, e.g., [57]. Inserting these relations and subsequent averaging over the whole equations (e.g. $\langle \partial_x [(\rho + \rho')(\langle v \rangle + v')] \rangle = \partial_x \langle \rho \rangle \langle v \rangle + \partial_x \langle \rho' v' \rangle$) yields

$$\partial_t \langle \rho \rangle + \partial_x \langle \rho \rangle \langle v \rangle + \partial_x \langle \rho' v' \rangle = 0 \quad (6)$$

and

$$\partial_t \langle v \rangle_L + \langle v \rangle_L \partial_x \langle v \rangle_L + \frac{1}{2} \partial_x \langle v' v' \rangle = \langle F/m \rangle . \quad (7)$$

One often parameterizes averaged fluctuations by the corresponding gradient [56] $\langle v' A' \rangle \approx -\alpha \partial_x \langle A \rangle$, which leads to the set of equations

$$\begin{aligned} \partial_t \rho + \partial_x (\rho v) &= D \partial_x^2 \rho \\ \partial_t v + v \partial_x v &= \nu \partial_x^2 v + F/m , \end{aligned} \quad (8)$$

where, according to convention, the averaging brackets have been omitted, and the diffusion coefficient D as well as the (kinematic) viscosity ν are assumed to be independent of x and t . It should be noted that similar diffusion terms can also be obtained from other arguments.

C. Lighthill-Whitham-theory and kinematic waves

If one assumes that the velocity is a function of density only ($v = f(\rho)$), then the momentum equation is no longer necessary. This corresponds to instantaneous adaption; the particles (or cars) carry no memory. Using without loss of generality the current $j(\rho) \equiv \rho v(\rho)$, and setting in addition $D = 0$, one obtains

$$\partial_t \rho + j'(\rho) \partial_x \rho = 0 \quad (9)$$

(Lighthill-Whitham-equation [14]), where $j' = dj/d\rho$. j' will, in this paper, always mean the derivative of j , even when the prime in connection with other letters can denote fluctuations. For a review of this theory, see, e.g., [14,58].

The equation can be solved by the ansatz $\rho(x, t) = \rho(x - ct)$ with

$$c = j'(\rho) . \quad (10)$$

This allows the solution of the characteristics: A region with density ρ travels with constant velocity $c = j'(\rho)$, and the resulting straight line in space-time is called characteristic. When $j(\rho)$ is convex, i.e. $j'' < 0$, then for regions of decreasing density ($\rho(x_1) > \rho(x_2)$ for $x_1 < x_2$) the characteristics separate from each other. On the other hand, in regions of increasing density, the characteristics come closer and closer together. When two characteristics touch each other, a density discontinuity appears at this place (a front), which moves with velocity

$$c = \frac{j(x_2) - j(x_1)}{\rho(x_2) - \rho(x_1)} = \frac{\delta j}{\delta \rho} . \quad (11)$$

Note that formally the fluid-dynamical description has broken down here.

An illustrative example is a queue, such as at a red light. When the light turns green, the outflow front quickly smoothes out, whereas the inflow front remains steep.

Note that usually at maximum flow $c = j' = 0$. Structures which operate at maximum flow do not move in space.

Leibig [59] gives results how a random initial distribution of density steps in a closed system evolves towards two single steps according to the Lighthill-Whitham-theory.

D. Lighthill-Whitham with dissipation

Adding dissipation to the Lighthill-Whitham-equation leads to

$$\partial_t \rho + j'(\rho) \partial_x \rho = D \partial_x^2 \rho . \quad (12)$$

The solution of this equation is again a non-dispersive wave with phase and group velocity j' . The difference is that D introduces dissipation (damping) of the wave: The amplitude decays as e^{-Dk^2} , where k is the wavenumber. This reflects the intuitively reasonable effect that traffic jams should tend to dissolve under homogeneous and stationary conditions.

E. The nonlinear diffusion (Burgers) equation

For a further development, $j(\rho)$ has to be specified. Since we are mostly interested in the behavior of traffic near maximum throughput, we start by choosing the simplest mathematical form which yields a "well-behaved" maximum:

$$j(\rho) = v_{max} \rho(1 - \rho) , \quad (13)$$

which, in traffic science, is called the Greenshields-model (see [10]). v_{max} is, in principle, a free parameter, but it has an interpretation as the maximum average velocity for $\rho \rightarrow 0$.¹ The maximum current j_{max} is reached at $\rho(j_{max}) = 1/2$.

The result is the equation

$$\partial_t \rho + v_{max} \partial_x \rho - 2v_{max} \rho \partial_x \rho = D \partial_x^2 \rho . \quad (14)$$

Musha and coworkers [22] have shown that by introducing a linear transformation of variables

$$x = v_{max} t' - x' , \quad t = t' , \quad (15)$$

one obtains

$$\partial_{t'} \rho + 2 v_{max} \rho \partial_{x'} \rho = D \partial_{x'}^2 \rho , \quad (16)$$

which is the (deterministic) Burgers equation.

This equation has been investigated in great detail by Burgers [60] as the simplest non-linear diffusion equation. The stationary solution is a uniform density $\rho(x, t) = const$. A single disturbance from this state evolves over time into a characteristic triangular structure with amplitude $\sim t^{-1/2}$, width $\sim t^{1/2}$, bent to the right such that the right side of the disturbance becomes discontinuous, and moving to the right with velocity $c = j' = 2 \rho v_{max}$.

When interpreting this for traffic jams, one has to re-transform the coordinates. Jams can then move *both* to the left or to the right (with velocities between v_{max} and $-v_{max}$), and the discontinuous front develops at the inflow side of

¹Mathematicians would set $v_{max} = 1$; traffic scientists use $1 - \rho/\rho_{jam}$ for the term in parenthesis. ρ_{jam} is the density of vehicles in a jam.

the jam, i.e. where the vehicles enter the jam. One sees that this solution is just the solution of the characteristics, with a dissipating diffusion term added—as should be expected because of $D > 0$.

Some other versions of the Burgers equation are relevant for traffic and have been investigated thoroughly [61–63]:

Noisy Burgers equation: Adding a Gaussian noise term η to the equation (i.e. $\langle \eta(x, t)\eta(x', t') \rangle = \eta_0 \delta(x-x') \delta(t-t')$) leads to the noisy Burgers equation

$$\partial_t \rho + 2 v_{max} \rho \partial_x \rho = D \partial_x^2 \rho + \eta. \quad (17)$$

This equation does no longer reach a stationary state.

Generalized Burgers equation: The nonlinearity of the Burgers equation can be generalized:

$$\partial_t \rho = \sum_{\beta} b_{\beta} \partial_x^{\beta} \rho + D \partial_x^2 \rho. \quad (18)$$

Generalized Burgers equations with arbitrary β have been investigated [62,61].

F. Including momentum

The equations so far do not explain the spontaneous phase separation into relatively free and rather dense regions which is observed in real traffic. To obtain this, one has to include the effect of momentum: One can neither accelerate instantaneously to a desired speed nor slow down without delay. It becomes necessary to include the momentum equation. Here, one has to specify the force term F/m , which describes acceleration and slowing down. At least two properties are usually incorporated: relaxation towards desired speed, and interaction with other cars.

A first order approximation for the relaxation term is [18,17]

$$\frac{1}{\tau} (V(\rho) - v), \quad (19)$$

where $V(\rho)$ is the desired average speed as a function of density. This choice yields exponential relaxation towards the desired speed. The function $V(\rho)$ has to be specified externally, for example from measurements.

A commonly used interaction term [64,65,18,17] is

$$-\frac{c_0^2}{\rho} \partial_x \rho, \quad (20)$$

where c_0 is treated as constant. In traffic, a typical value for c_0 is 15 km/h [19]. The meaning is that one tends to reduce speed when the density increases, even when the local density is still consistent with the current speed. Formally, this term comes from the the pressure term of compressible flow $-(1/\rho)\partial_x p$, where p is the pressure due to thermal motion of the particles. Assuming an ideal gas ($p = \rho RT$) and isothermic behavior $T = const$, one obtains waves similar to sound waves as a solution of the linearized equations.² This leads to (20), where c_0 is the speed

²True sound waves, though, would assume the gas to behave adiabatic, i.e. $p \propto \rho^{\kappa}$.

of the "sound" waves. Note that sound waves move in *both* directions from a disturbance, which means that sound waves alone are not a good explanation for freeway start-stop-waves, contrary to what is written in [66].

A possible momentum equation for traffic therefore is [18]

$$\partial_t v + v \partial_x v = -\frac{c_0^2}{\rho} \partial_x \rho + \frac{1}{\tau} [V(\rho) - v] + \nu \partial_x^2 v, \quad (21)$$

and the system is closed together with an equation of continuity

$$\partial_t \rho + \partial_x (\rho v) = D \partial_x^2 \rho. \quad (22)$$

Usually, D is set to zero.

For this equation, the homogeneous solution $(v, \rho) \equiv (v_0, \rho_0)$ is unstable for densities near maximum flow for a suitable choice of parameters. Using the methods of nonlinear dynamics, Kühne and coworkers [18,21,20] went beyond linear stability analysis (see also [67,68]). One finds a multitude of stable or unstable fixpoints and limit cycles which suggest that traffic near maximum flow operates on a strange attractor. This can lead to quasi-periodic behavior, exactly as is observed in traffic measurements.

Earlier work [64,17] has analyzed the same equation without viscosity ($\nu = 0$).

G. Discussion of fluid-dynamical approaches

Fluid-dynamical models have been used in traffic science for a long time, with considerable success. But they have shortcomings. Some of the major points are:

- (i) One has to give externally the relation between speed or current and density. This is unsatisfying in terms of the development of a theory. But an even more intricate problem is that there is no agreement on a functional form of the speed-density relation; it is even under discussion if this relation is at all continuous [13,69].
- (ii) Temperature parameterizes the random fluctuations of particles around their mean speed. For gases, fluctuations and therefore temperature increase with density. For granular media, fluctuations *decrease* with density (i.e. inside a jam)—it has been claimed that exactly this inverse temperature effect is responsible for clustering [70]. In this way, assuming isothermic instead of adiabatic behavior as done for the momentum equation seems only half the way one has to go. Helbing [71] discusses this further.
- (iii) Helbing [71] also discusses excluded volume to take into account the spatial extension of vehicles.

Nonetheless, fluid-dynamical approaches [65,18,21,20] give, for the first time, systematic insight into traffic near maximum flow beyond simple extrapolation of light and dense traffic results. These results will be further discussed near the end of this paper.

IV. DEFINITIONS OF PARTICLE HOPPING MODELS

This section defines several particle hopping models which are candidate models for traffic. They all have in common that they are defined on a lattice of, say, length L , where L is the number of sites, and that each site can be either

empty, or occupied by exactly one particle. Also, in all models particles can only move in one direction. The number of particles, N , is conserved except at the boundaries.

A. The Stochastic Traffic Cellular Automaton (STCA)

The Stochastic Traffic Cellular Automaton (STCA), which has been treated in a series of papers [30,38,47,41–45,48], is defined as follows. Each particle (= car) can have an integer velocity between 0 and v_{max} . The complete configuration at time-step t is stored, and the configuration at time-step $t+1$ is computed from that (parallel or synchronous update). For each particle, the following steps are done in parallel:

- Find number of empty sites ahead (= gap) at time t .
- If $v > gap$ (too fast), then slow down to $v := gap$. [rule 1]
- Else if ($v < gap$) (enough headway) and $v < v_{max}$, then accelerate by one: $v := v + 1$. [rule 2]
- **Randomization:** If after the above steps the velocity is larger than zero ($v > 0$), then, with probability p , reduce v by one. [rule 3]
- **Particle propagation:** Each particle moves v sites ahead. [rule 4]

The randomization condenses three different properties of human driving into one computational operation: Fluctuations at maximum speed, over-reactions at braking, and retarded (noisy) acceleration.

Note that, because of integer arithmetic, relations like $v > gap$ and $v \geq gap + 1$ are equivalent.

When the maximum velocity of this model is set to one ($v_{max} = 1$), then the model becomes much simpler: For each particle, do in parallel:

- If site ahead was free at time t , move, with probability p , to that site.

Since the STCA shows different behavior for $v_{max} \geq 2$ than for $v_{max} = 1$, I will distinguish them using STCA/1 and STCA/2, respectively.

B. The cruise control limit of the STCA (STCA-CC)

In the so-called cruise control limit of the STCA [46], fluctuations at free driving, i.e. at maximum speed and undisturbed by other cars, are set to zero. Algorithmically, this amounts to the following: For all cars do in parallel:

- A vehicle is stationary when it travels at maximum velocity v_{max} and has free headway: $gap \geq v_{max}$. Such a vehicle just maintains its velocity.
- Else (i.e. if a vehicle is not stationary) the standard rules of the STCA are applied.

Both acceleration and braking still have a stochastic component. The stochastic component of braking is realistic, but it is irrelevant for the results presented here.

C. The deterministic limit of the STCA (CA-184)

One can take the deterministic limit of the STCA by setting the randomization probability p equal to zero, which just amounts to skipping the randomization step. It turns out that, together with maximum velocity $v_{max} = 1$, this is equivalent [61] to the cellular automaton rule 184 in Wolfram's notation [72], which is why I will use the notation CA-184/1 and CA-184/2.

Most work using CA models for traffic is based on this model. Biham and coworkers [29] have introduced it for traffic flow, with $v_{max} = 1$. Other authors base further results on it [49,32,37,34], also for two-lane traffic [35]. Vilar and coworkers [33] use it with $v_{max} = 5$. It is also the basis of the two-dimensional CA models for traffic [53,54].

D. The cruise control version for the CA-184 (CA-184-CC)

Takayasu and Takayasu [37] introduced a model which amounts to a deterministic cruise control situation for CA-184/1. This may not be obvious from the rules, but it will become clear from the dynamic behavior summarized later. Since they use only maximum velocity $v_{max} = 1$, the rules are short: For all particles do in parallel:

- A particle with velocity one is moved one site ahead when the site ahead is free ($gap \geq 1$).
- A particle at rest ($v = 0$) can only move when $gap \geq 2$.

Generalizations to maximum velocity larger than one are straightforward, but do not seem to lead to additional insight.

E. The Asymmetric Stochastic Exclusion Process (ASEP)

The probably most-investigated particle hopping model is the Asymmetric Stochastic Exclusion Process (ASEP). It is defined as follows:

- Pick one particle randomly. [rule 1]
- If the site to the right is free, move the particle to that site. [rule 2]

Note the close relationship to STCA/1 and CA-184/1. In CA-184/1, all particles are moved *synchronously* according to the ASEP propagation rule. In STCA/1 (with $p = 1/2$), *half of* all particles are moved synchronously according to the ASEP propagation rule. It was already noted earlier [61] that going from ASEP to CA-184/1 produces very different dynamics; in this paper I will show that re-introducing the randomness via the randomization in STCA again leads to different results.

In order to compare the ASEP with the other, synchronously updated models, one has to note that, in the ASEP, *on average* each particle is updated once after N single-particle updates. A time-step (also called update-step or iteration) is therefore completed after N single-particle updates ($= N$ attempted hops).

One can also define higher velocities for the ASEP by simply replacing ASEP-rule 2 by STCA/2-rules 1, 2, and 4. In such a case, each particle has to remember its velocity v from the last move.

Reducing the noise for the ASEP could be done using techniques described by Wolf and Kertesz [73], i.e. by putting a counter on each particle and move it only after k trials. Taking the limit $k \rightarrow \infty$ reduces the ASEP again to the CA-184 process, i.e. the deterministic limit if the STCA and the ASEP are the same.

V. PARTICLE HOPPING MODELS, FLUID DYNAMICS, AND CRITICAL EXPONENTS

Both for the ASEP/1 and for the CA-184/1, fluid-dynamical limits and critical exponents are well known (see, e.g., [63,61,74,62]). The most straightforward way to put the concept of critical exponents into the context of traffic flow is to consider "disturbances" (i.e. jams) of length x and ask for the time t to dissolve them. For example, one would intuitively assume that a queue of length x at a traffic light which just turned green would need a time t proportional to x until everybody is in full motion. By this argument, the dynamic exponent z , defined by $t \sim x^z$, should be one.

Yet, there may be more complicated cases. Imagine again a queue at a traffic light just turned green but this time also some fairly high inflow at the end of the queue. The jam-queue itself will start moving backwards, clearing its initial position in time $t \sim x$. However, the dissolving of the jam itself may be governed by different rules.

A. ASEP/1

The classic stochastic asymmetric exclusion process corresponds to the noisy Burgers equation with $\beta = 2$ [61]. More precisely, the particle process corresponds to a diffusion equation $\partial_t \rho + \partial_x j = D \partial_x^2 \rho + \eta$ with a current [61,75] of $j = \rho(1 - \rho)$. Inserting this yields

$$\partial_t \rho + \partial_x \rho - \partial_x \rho^2 = D \partial_x^2 \rho + \eta, \quad (23)$$

which is exactly the Lighthill-Whitham-Greenshields case with noise and diffusion described earlier. In other words, the ASEP/1 particle hopping process and the Lighthill-Whitham-theory (plus noise plus diffusion), specialized to the case of the Greenshields flow-density relation, describe the same behavior.

In the steady state, this model shows kinematic waves (= small jams), which are produced by the noise and damped by diffusion (Fig. 1). These non-dispersive waves move forwards ($c = j' = 1 - 2\rho > 0$) for $\rho < 1/2$ and backwards ($c < 0$) for $\rho > 1/2$ (Fig. 2). At $\rho = 1/2$, the wave velocity is exactly zero ($c = 0$), and this is the point of maximum throughput [76]. When traffic would be modeled by the ASEP, then one could detect maximum traffic flow by standing on a bridge: Jam-waves moving in flow direction indicate too low density (cf. Fig. 1), jam-waves moving against the flow direction indicate too high density.

This is one of the cases where clearing a site follows a different exponent than dissolving a disturbance.³ As long as $\rho \neq 1/2$, a disturbance of size x moves with speed $c \neq 0$ and therefore clears the initial site in time $t \sim c \cdot x \sim x^1$, i.e. with dynamical exponent $z = 1$. In order to see how the disturbance itself dissolves, one therefore transforms into the coordinate system of the wave velocity. This has to be done after separating between the average density $\langle \rho \rangle_L$ and the fluctuations ρ' : One first inserts $\rho = \langle \rho \rangle_L + \rho'$ and obtains

³Note that technically, all these remarks are only valid for *small* disturbances. The problem is that if one is no longer close to the steady state, one sees transient behavior which may be different [61].

$$\partial_t \rho' + (1 - 2\langle \rho \rangle_L) \partial_x \rho' - 2\rho' \partial_x \rho' = D \partial_x^2 \rho' + \eta. \quad (24)$$

When transforming this into the moving coordinate system $x' = x + (1 - 2\langle \rho \rangle_L) \cdot t$, one obtains the classic noisy Burgers equation and therefore the anomalous KPZ exponent $z = 3/2$ in the moving coordinate system.

In other words, a disturbance four times as big as another one, $x' = 4x$, needs $t' \sim x' = 4x \sim 4t$, i.e. four times as much time to clear the site, but $t' \sim x'^{3/2} = (4x)^{3/2} \sim 8t$, i.e. 8 times as much time until the jam-structure itself is no longer visible in the noise. A precise treatment of this uses, e.g., correlations between tagged particles [63].

The drawback of this model with respect to traffic flow is that it does neither have a regime of lamellar flow nor “real”, big jams. Because of the random sequential update, vehicles with average speed \bar{v} fluctuate severely around their average position given by $\bar{v}t$. As a result, they always “collide” with their neighbors, even at very low densities, leading to “mini-jams” everywhere. This is clearly unrealistic for light traffic.

Actually, this fact is also visible in the speed-density-diagram. Using the Greenshields flow-density relation, one obtains

$$v = \frac{q}{\rho} \propto 1 - \rho. \quad (25)$$

This is in contrast to the observed result that, at low densities, speed is nearly independent of density (practically no interaction between vehicles).

B. ASEP/2

Judging from space-time plots, changing the maximum velocity in the update from $v_{max} = 1$ to $v_{max} \geq 2$ does not change the universality class [48] (see Figs. 1 and 2). It skews the flow-density-relation towards lower densities, but does not lead to other phenomenological behavior.

C. CA-184/1

As explained above, the CA-184/1 is the deterministic limit of the ASEP/1. But taking away the noise from the particle update completely changes the universality class [61]. The model now corresponds to the non-diffusive, non-noisy equation of continuity with a linear flow

$$j(\rho) = \frac{1}{2} - \left| \rho - \frac{1}{2} \right|. \quad (26)$$

This yields a linear (except at $\rho = 1/2$) Burgers equation:

$$\partial_t \rho + \text{sign}\left[\frac{1}{2} - \rho\right] \partial_x \rho = 0 \quad (27)$$

Therefore, the dynamic exponent z is equal to 1 [61].

More precisely, the following happens: The outflow of a jam in this model always operates at flow $j_{out} = j_{max} = 1/2$ and density $\rho_{jmax} := \rho(j_{max}) = 1/2$. The time t until a jam of length x dissolves therefore obeys the average relation $t \propto x (j_{max} - j_{in})$, where j_{in} is the average inflow to the jam. Since $j \propto \rho$ for $\rho \leq \rho_{jmax}$, one can write that as

$$t \propto x [\rho_{jmax} - \rho(j_{in})]. \quad (28)$$

This means that for $\rho < \rho_{jmax}$, the critical exponent z is indeed one, but at $\rho = \rho_{jmax}$, t diverges. This effect is also visible when disturbing the system from its stationary state: The transient time t_{trans} until the system is again stationary scales as $t_{trans} \sim \rho_{jmax} - \rho$ [40].

The scaling law (28) is actually also true for $\rho > \rho_{jmax}$, albeit for a different reason. Here, some jams are never sorted out, and they survive forever while traveling backwards with the kinematic wave velocity $c = -1$. A disturbance, however, makes the model non-stationary only for a short time: Usually, it just shifts another wave or part of it to a new wave originated by the disturbance (Fig. 4). If one would look at the space-time-pattern of all *non-stationary* (“disturbed”, “damaged” [77]) particles, one would again recover $t \sim x$, i.e. the dynamic exponent $z = 1$.

D. CA-184/2

Using a maximum velocity higher than one does not change the general behavior. The only changes are in the constants of the equations. One finds $j_{max} = v_{max}/(v_{max} + 1)$, $\rho_{jmax} = 1/(v_{max} + 1)$, $j(\rho \leq \rho_{jmax}) = \rho v_{max}$, and $j(\rho \geq \rho_{jmax}) = 1 - \rho$ [40].

Two observations are important at this point:

(i) Many papers in the physics literature [29,31–36] use this model for their investigations. Also the 2d-grid models (see, e.g., [53–55]) essentially use this driving model, although the two-dimensional interactions seem to change the flow-density relationship [88]. The CA-184 model lacks at least two features which are, as I will argue later, important with respect to reality: (a) Bi-stability: Laminar flow above a certain density becomes unstable, but can exist for long times. (b) Stochasticity: CA-184 is completely deterministic, i.e. a certain initial condition always leads to the same dynamics. Real traffic, however, is stochastic, that is even identical initial conditions will lead to different outcomes, and a model should be capable of calculating some distribution of outcomes (by using different random seeds).

(ii) The fluid-dynamical model behind the cell transmission model [78], which is a discretization of the Lighthill-Whitham-theory, is similar to Eqs. 26 and 27, especially with respect to the range of physical phenomena which are represented. The only difference is that the j - ρ -relation of Ref. [78] has a flat portion at maximum flow instead of the single peak of Eq. 26. That means that low density and high density traffic behave similarly to CA-184, but traffic at capacity has a regime where waves do not move at all.

Using other j - ρ -relations in discretized Lighthill-Whitham-models will lead to other relations for the wave speeds, but the range of physical phenomena (backwards or forwards moving waves) which can be represented will always resemble CA-184; especially, neither the bi-stability nor the stochasticity can be represented.

No fluid-dynamical limits for the other particle hopping models are known. Yet, results for the jam dynamics for the cruise control situations [37,46] offer valuable insights, which will be described in the following.

The important new feature of the cruise control version of CA-184 is a bi-stability [37]. Using $v_{max} = 1$ in this section ($v_{max} > 1$ does not seem to offer additional insight), this bi-stability occurs for $\rho_{c1} = 1/3 \leq \langle \rho \rangle_L := N/L \leq \rho_{c2} = 1/2$, where some initial conditions lead to laminar flow but others lead to traffic including jams. ($\langle \cdot \rangle_L$ means the average over the whole (closed) system of length L .) Takayasu and Takayasu found the following:

(i) Starting with maximally spaced particles and initial velocity one, one finds stable configurations with flow $\langle j \rangle_L = \langle \rho \rangle_L v_{max} = \langle \rho \rangle_L$ for low densities $\langle \rho \rangle_L \leq 1/2 =: \rho_{c2}$. For high densities $\langle \rho \rangle_L > \rho_{c2}$, a jam phase appears for *all* initial conditions since not all particles can keep $gap \geq 1$. Once a jam has been created, all particles in the outflow of this jam have $gap = 2$. For $t \rightarrow \infty$, this dynamics reorganizes the system into jammed regions with density one and zero current, and laminar outflow regions with $\rho_{out} = 1/3$ and $j_{out} = 1/3$. Simple geometric arguments lead, for the whole system, to $\langle j \rangle_L = (1 - \langle \rho \rangle_L)/2$ and $\langle v \rangle_L = (1/\langle \rho \rangle_L - 1)/2$.

(ii) Starting, however, with an initial condition where all particles are clustered in a jam, this jam is only sorted out up to $\langle \rho \rangle_L \leq 1/3 =: \rho_{c1}$, leading to $\langle j \rangle_L = \langle \rho \rangle_L$ and $\langle v \rangle_L = 1$. For $\langle \rho \rangle_L > \rho_{c1}$, the initial jam survives forever, yielding $\langle j \rangle_L = (1 - \langle \rho \rangle_L)/2$ and $\langle v \rangle_L = (1/\langle \rho \rangle_L - 1)/2$. One observes that, for $\rho_{c1} < \langle \rho \rangle_L < \rho_{c2}$, this initial condition leads to a different final flow state than the initial conditions in (i). — Note that ρ_{c1} is equal to the outflow density ρ_{out} .

Starting from an arbitrary initial condition, the density-velocity relation converges to one of the above two types [37].

Note that up to before this section, all relations between j , v , and ρ were also locally correct, which is why averaging brackets were omitted. Now, this is no longer true. For example densities slightly above ρ_{c2} do not really exist on a local level; they are only possible as a global composition of regions with local densities $\rho = \rho_{c1}$ plus others with local densities $\rho = 1$.

Since the model is deterministic, one can calculate the behavior from the initial conditions. For any particle i with initial velocity zero one can determine the influence that particle has on following particles $i+1, i+2, \dots$. For particle $i+k$ not to be involved in the jam, one needs the average gap between i and k to be larger than two. This corresponds to a density between i and k of $\rho_{ik} < 1/(gap+1) = 1/3 = \rho_{c1}$. The sequence $(gap_{i+j})_j$ describes a random walk, which is positively (negatively) biased for $\rho > \rho_{c1}$ ($\rho < \rho_{c1}$), and unbiased at the critical point $\rho = \rho_c$ [37].

F. STCA-CC/1

The cruise control limit of the STCA is in some sense a mixture between the CA-184 and the full STCA. Since the STCA-CC has no fluctuations at free driving, the maximum flow one can reach is with all cars at maximum speed and $gap = v_{max}$. Therefore, one *can* manually achieve flows which follow, for $\rho \leq \rho_{c2}$, the same j - ρ -relationship as the CA-184, where ρ_{c2} now denotes the density of maximum flow of the deterministic model CA-184, i.e. $\rho_{c2} = 1/(v_{max} + 1)$.

Above a certain ρ_{c1} , these flows are unstable to small local perturbations. This density will turn out to be a “critical” density; for that reason I will use $\rho_c \equiv \rho_{c1}$. Many different choices for the local perturbation give rise to the same large scale behavior. The perturbed car eventually re-accelerates to maximum velocity. In the meantime, though, a following car may have come too close to the disturbed car and has to slow down. This initiates a chain reaction – an emergent traffic jam.

It is straightforward to see [46] that $n(t)$, the number of cars in the jam, follows a usually biased, absorbing random walk (RW), where $n(t) = 0$ is the absorbing state (jam dissolved): Every time a new car arrives at the end of the jam, $n(t)$ increases by one, and this happens with probability j_{in} , which is the inflow rate. Every time a car leaves the jam at the outflow side, $n(t)$ decreases by one, and this happens with probability j_{out} . When $j_{in} = j_{out}$, $n(t)$ follows an unbiased absorbing RW. $j_{in} \neq j_{out}$ introduces a bias or drift term $\propto (j_{in} - j_{out}) \cdot t$.

This picture is consistent with Takayasu and Takayasu's (TT's) observations for the CA-184-CC model. The main difference is that now *both* the inflow gaps and the outflow gaps form a random sequence. Another difference is conceptually: TT have looked at the transient time starting from initial conditions, whereas Nagel and Paczuski (NP) look at jams starting from a *single disturbance*. The latter leads to a cleaner picture of the traffic jam dynamics because it concentrates on the transition from laminar to start-stop-traffic which is observed in realistic traffic.

The statistics of such absorbing random walks can be calculated exactly. For the unbiased case one finds that

$$\langle n(t) \rangle \sim t^\eta, \quad P_{surv}(t) \sim t^{-\delta} \text{ and } \langle w(t) \rangle_{surv} \sim t^{\eta+\delta}, \quad (29)$$

where P_{surv} is the survival probability of a jam until time t , $w(t)$ means the width of the jam, i.e. the distance between the leftmost and the rightmost car in the jam. $\langle \cdot \rangle$ means the ensemble average over *all* jams which have been initiated, and $\langle \cdot \rangle_{surv}$ means the ensemble average over *surviving* jams. For the critical exponents, one finds as well from theory as from numerical simulations $\delta = 1/2$ and $\eta = 0$.

$\eta = 0$ re-confirms that, at the critical density ρ_c , jams in the average *barely survive*.

If one now uses j_{in} as order parameter, and, say, $P_{surv}(t)$ as control parameter, then we have a second order phase transition, where

$$P_{surv}(t) \begin{cases} = 0 & \text{for } j_{in} < j_{out} \text{ and } t \rightarrow \infty, \\ \sim t^{-\delta} & \text{for } j_{in} = j_{out} \text{ and } t \rightarrow \infty, \text{ and} \\ = const & \text{for } j_{in} > j_{out} \text{ and } t \rightarrow \infty. \end{cases} \quad (30)$$

For that reason, we call $j_c := j_{out}$ the critical flow, and the associated density $\rho_c := \rho(j_c)$ the critical density.

It is important to note that $j_{in} > j_{out}$ as a stable state is only possible due to the particular definition of the cruise control limit and in an open system. Deviations from the cruise control limit will be addressed later; let us now consider a *closed* system (periodic boundary conditions, i.e. traffic in a loop):

- For $\rho < \rho_c$ and arbitrary initial conditions, jams are ultimately sorted out. Then, every car has velocity $v = v_{max}$ and $gap \geq v_{max}$, is thus in the free driving regime as defined above.
- For $\rho_c \leq \rho \leq \rho_{c2}$, the long-time behavior depends on the initial conditions. For example, even in the extreme case of $\rho = \rho_{c2}$, the state where every car has velocity $v = v_{max}$ and $gap = v_{max}$ is stable and results in a flow of $j = v_{max} \rho_{c2}$. However, most other initial configurations will lead to jams, and for the limit of infinite system size, at least one of them never sorts out.
- For $\rho > \rho_{c2}$, all initial conditions lead to jams.

Note that this is again consistent with the results of Takayasu and Takayasu for the CA-184-CC system [37].

G. STCA-CC/2

Replacing maximum velocity $v_{max} = 1$ by $v_{max} \geq 2$ does not change the critical behavior, but it adds a complication [46]. Now, jam clusters can branch, with large jam-free holes in between branches of the jam. As a result, space-time plots of such jams now appear to show fractal properties, and in simulations at the critical density $w(t)$ does not follow any longer a clean scaling law, whereas $n(t)$ and P_{surv} still do.

The explanation of this is that the holes in the jam are large enough to cause logarithmic corrections to the width, but not large enough to make it completely fractal. More precisely, the hole size distribution $P_h(x)$, i.e. the probability to find a hole of size x in a given equal time cut (= jam configuration) scales as

$$P_h(x) \sim x^{-\tau_h}, \quad (31)$$

where both from a theoretical argument and from simulations $\tau_h = 2$. Yet, it is known that for $\tau_h \leq 2$ the fractal dimension for such a configuration is $D_f = \tau_h - 1$ (see, e.g., [79]). In this sense, such a traffic jam cluster operates at the "edge of fractality".

And such a hole size distribution causes logarithmic corrections to the width when $n(t)$ is given: $\langle w(t) \rangle_{surv} \sim \langle n(t) \rangle_{surv} (1 + c \log t)$.

H. STCA/1

For the STCA at $v_{max} = 1$, from visual inspection (Fig. 7) one finds that distinguishable jams do not exist here. Instead, the space-time plot looks much more like one from the ASEP.

This is confirmed by theoretical analysis. Schreckenberg and coworkers [47,48] have performed analytical calculations for the stationary state throughput j given ρ in a closed system using n -point correlations (n -cluster method) and found that for $v_{max} = 1$ this analysis is already exact for $n = 2$. This is no longer true for higher v_{max} . For the ASEP, for the same analysis, the mean-field approximation, i.e. $n = 1$, is already exact. The difference between the ASEP and the STCA/1 in this analysis is that in the STCA/1 one finds an effective repulsive force of range one between particles, caused by the parallel update. This helps to keep particles more equidistant than in the ASEP case, thus leading to a higher flow.

I. STCA/2

For $v_{max} \geq 2$, the n -cluster analysis does no longer lead to an exact solution, indicating that a different dynamical regime has now completely taken over. (In practice, though, the n -cluster analysis is already fairly close to simulation results for $n \simeq 5$.) Visual inspection of space-time plots confirms that the dynamics now is much more similar to the cruise control limit, i.e. to STCA-CC/2, than to the ASEP.

Yet, in contrast to the cruise control limit, here multiple jams exist simultaneously. Jams start spontaneously and independently of other jams because vehicles fluctuate even at maximum speed, as determined by a parameter $p_{free} \neq 0$.

The STCA displays a scaling regime near the density of maximum throughput ρ_{jam} , but with an upper cutoff at $t \simeq 10^4$ which was observed to depend on p_{free} [41]. One can attribute this cutoff to the non-separation of the time scales between disturbances and the emergent traffic jams [46]. As soon as p_{free} is different from zero, the spontaneous initiation of a new jam can terminate another one. Obviously, this happens more often when p_{free} is high, which explains why the scaling region gets longer when one reduces p_{free} . — Csányi and Kertész [80] observe that, for $p_{free} \gg 0$, percolation, defined by long-range connectivity, may actually occur only at much higher densities than ρ_{jam} .

A helpful analogy is droplet formation in a gas-liquid transition [81], where gas corresponds to the laminar phase and the droplets correspond to the jams. The gas always “tests” (in fluctuations) simultaneously at many positions if droplets can survive. When one neglects surface tension, then droplets *cannot* survive at sub-critical density, they *can* survive at supercritical density, and they *barely* survive exactly at the critical density, making macroscopic fluctuations maximal at this point. — Note that neglecting the surface tension of the droplets changes the nature of the phase transition from first order to second order.

VI. GOING BEYOND: TRAFFIC JAM DYNAMICS

All these results together put us into a position to draw a fairly consistent picture of traffic jam dynamics.

A. An intuitive starting point

Measurements of human driving behavior show that over a fairly large velocity range, *gap* is proportional to velocity. *gap* here is $\delta x - L$, where δx is the front-bumper-to-front-bumper distance (= distance headway) between two cars, and L is the length one car occupies (in the average) in a jam. This makes intuitively a lot of sense, since it reflects the fact that the *time gap* should be approximately the same as the delay time T which is needed between seeing the brake lights and actually starting to brake, and should therefore be largely independent of velocity (Pipes’ theory, cf. [11]).⁴

Field studies (cf. [11]) indeed confirm that the delay time is approximately constant for velocities between 15 and 40 miles per hour (between 24 and 64 km/h, data from the 1950s). This delay time consists of several components, including, e.g., reaction time or the time needed for actually pressing the brake pedal, and it is of the order of one second.

Therefore, we have $gap = T \cdot v$. Using the average relations $1/\delta x = \rho$ and $1/L = \rho_{jam}$ (density inside a jam), we obtain

$$v = \frac{1}{T} \left(\frac{1}{\rho} - \frac{1}{\rho_{jam}} \right). \quad (32)$$

Thus, for high density, for the current j one has

⁴Note that traffic science traditionally does not include some “security space” into the definition of L [11]; therefore “gap” and “time gap” are somewhat different here.

$$j_{high} = \rho v = \frac{1}{T} \left(1 - \frac{\rho}{\rho_{jam}} \right) \quad (33)$$

For low density, one can assume that there is some (average) v_{max} which is independent of gap for large enough spacings, and therefore, for low densities

$$j_{low} = \rho v_{max} \quad (34)$$

At j_{c2} and ρ_{c2} , these two curves intersect and thereby define the maximum flow according to this model. Assuming, say, $v_{max} = 120$ km/h, $T = 1.1$ sec, and $L = 7.5$ m, one obtains $\rho_{c2} \approx 1/45$ m and $j_{c2} \approx 2650$ vehicles per hour per lane, which is slightly above the highest 5-minute-averages which are obtained in reality (e.g. [13,82]).

This model is essentially equivalent to the CA-184/2 particle hopping model.

As a side remark, note that traffic security experts teach drivers that one should reach the position of the car ahead only after more than two seconds, which is independent of velocity and reflects the fact that time headway is approximately equal to time gap. But it is interesting to see that this would actually lead to a maximum current of 2 cars per second or 1800 cars per hour, much less than the up to 2400 cars per hour which are observed.

B. More realistic traffic jam dynamics

Yet, real traffic behaves differently. At high densities, we do not observe the homogeneous velocity $v = gap/T$ as predicted by the intuitive argument above, but relatively free flow which is inter-dispersed by start-stop-waves. This is confirmed by measurements of the j - ρ -relation, where, instead of lining up on a single curve, the measurements form a fairly scattered data cloud especially in the region of the flow maximum.

The explanation of this is as follows. Laminar traffic will always and at all densities, due to small fluctuations, have small disturbances which can develop into jams. The inflow to the jam decides if such a jam can become long-lived or not: Since the average outflow j_{out} is fixed by the driving dynamics, $j_{in} > j_{out}$ makes the jam (in the average) long-lived, $j_{in} < j_{out}$ not. $j_{in} = j_{out}$ defines a critical point, i.e. $j_c \equiv j_{out}$ and $\rho_c \equiv \rho_{out}$, where traffic jam clusters in the average barely survive, as e.g. quantified by $\langle n(t) \rangle = t^\eta$ with $\eta = 0$.

All this is true for an open system, or a system which is large enough and where times are short enough so that the closed boundaries are not felt. In a closed system, the jams ultimately absorb all the excess density $\rho_{excess} = \rho - \rho_c$; as a result, all traffic between jams operates at ρ_c .

Average measurements of the long time behavior of traffic flow can therefore show at most $j_c = j_{out}$.

Again, the gas-liquid analogy (without surface tension) is helpful. Gas can also be super-cooled, e.g. by increasing the density while keeping the temperature constant. But this state is only meta-stable, and eventually droplets will form and, in a closed system, absorb excess density until the density surrounding the droplets is exactly at the critical point.

This dynamical picture explains the high variations in the short time measurements. Measuring at a fixed position in a situation like in Fig. 8, one can measure arbitrary combinations of supercritical laminar traffic, critical laminar traffic, jams, or traffic during acceleration or slowing down. The first accounts for flows higher than the average $j_{max} = j_c$. Combinations between jammed traffic and critical laminar traffic define the average curve and account for the straight trend between these two cases. Acceleration regions account for j below the average curve: Velocity

and therefore flow are lower here than they should be according to the density. In the same way deceleration regions account for j above the average curve.

This picture also makes precise the hysteresis argument of Treiterer and coworkers [83], also confirmed later [69,82]. These measurements confirm the idea that the traffic density can go above the critical point while still being laminar, similar to the gas which can be "super-cooled" by increasing the density. Yet, both for traffic and for super-cooled gases, this state is only meta-stable and eventually leads to a phase separation into jams and laminar flow. Quantitative evidence of this will be given in a separate paper (in preparation).

The picture is also consistent with recent results both in fluid-dynamical models and mathematical car-following models for traffic flow. In Ref. [65], traffic simulations using a fluid-dynamical model starting from nearly homogeneous conditions eventually form stable waves. The fluid-dynamical model one can, by usual linearization, find the parameters for the onset of instability. What is $\langle n(t) \rangle \sim t^\eta g(t|\rho - \rho_c|^{\nu'})$ in the particle hopping model becomes the amplitude $A(t) \sim e^{\lambda t}$ in the fluid-dynamical model, and at the onset of instability $A = \text{const}$ similar to $\langle n(t) \rangle = \text{const}$ (since $\eta = 0$). Therefore, the wave in the fluid-dynamical model corresponds to the *average* jam-cluster in the particle-hopping model.

Lee [84] explains the underlying mechanism for a model for granular media. He distinguishes "dynamic" from "kinematic" waves. Dynamic waves are found in the Kühne/Kerner/Konhäuser (KKK) equations (Eqs. 21 and 22) when the relaxation time $\tau > 0$; they are similar to sound waves in gases. Kinematic waves are found in the same equations when $\tau \rightarrow 0$, in which case the equations reduce to the Lighthill-Whitham case. The wave formation mechanism thus is that the instability first triggers the "sound" wave. The density inside the wave increases and outside the wave decreases until both densities are outside the instable range. Then the kinematic mode takes over.

Kurtze and Hong [68] make this more precise for the KKK equations: Below the critical density, the kinematic wave with wave velocity $c = j' = dj/d\rho$ is the only solution of the linearized equation, and this solution is stable. At the critical density, this solution bifurcates into two unstable solutions with wave velocity $c = j' \pm \epsilon$, where $\epsilon \rightarrow 0$ for $\rho \searrow \rho_c$, and ϵ is the equivalent of the speed of sound.

Bando and co-workers [25] also find the separation of traffic into laminar and jammed phases in a deterministic continuous mathematical car-following model.

VII. CONSEQUENCES FOR TRAFFIC SIMULATION MODELS

These findings have some fairly far-reaching implications for traffic simulation models.

First, particle hopping models, which seem at the first glance as a too rough approximation of reality, include the same range of dynamic phenomena as the most advanced fluid-dynamical models for traffic flow to date. Yet, particle hopping models offer some distinctive advantages for practical simulations. Particle hopping models are known to be numerically robust especially in complex geometries, and realistic road networks with all their interconnections etc. certainly qualify as such. Practical road network implementations of the fluid-dynamical theory are so far only using the Lighthill-Whitham equations, which are (without diffusion) marginally stable and can certainly be made stable by using a stable numerical discretization scheme.

Second, the fact that traffic jam behavior follows critical exponents leads to the expectation that *all* microscopic models which include jam formation should display the same critical exponents: The universality hypothesis of critical phenomena states that critical exponents are fairly robust against changes in microscopic rules. For the traffic

jam case, this is backed up by the fact that the exponents can be theoretically explained. The consequence for traffic simulation is that, as long as one expects certain simple aspects of traffic jam formation to be realistic enough for the problem under consideration, e.g. for large scale questions, *the simplest possible model will be sufficient for the task*, thus saving human and computational resources.

Third, the present results show that close-up car-following behavior is not the most important part to model. The important part is to model *deviations* from the optimal (smooth) behavior and in which way they lead to jam formation. Another important part, which seems far from obvious, is the *acceleration behavior*, especially when there are other cars ahead, since it is the acceleration behavior which mostly decides about maximum flow out of a jam (which may be a simple traffic light!). Note that this part of driving behavior is much more connected to physical properties of the vehicles; electric vehicles with its different acceleration capabilities might completely change traffic flow [5]. Therefore, investigations such as this paper *are* important for microscopic modeling as long as one does not have the perfect model of driving, or not the computational resources to run it.

Fourth, fast running and easy to implement particle hopping models can be very useful in interpreting measurements. Measurements such as for the traditional 5-minutes-averaged fundamental diagrams (flow vs. density vs. velocity) have increasingly recognized the fact that the dynamics around the measurement site has an extreme influence on the outcome of the measurements, thus making the results far from universal. This point will be further discussed in another paper (in preparation).

Fifth, particle hopping models are inherently microscopic, which allows to add individual properties to each car such as identity of travelers, route plan, engine temperature (for emission modeling) etc. These properties are imperative for the kind of traffic models being needed in current policy evaluation processes.

And last but not least, particle hopping models are stochastic in nature, thus producing different results when using different random seeds even when starting from identical initial conditions. At first, this is certainly considered a disadvantage from the point of view of policy makers of traffic engineers. However, the traffic system is inherently stochastic, and the variance of the outcomes is an important variable itself. How will we be able to distinguish reliable from unreliable predictions without knowing something about the range of possible outcomes? — Furthermore, there is reason to believe that the average over several stochastic runs will *not* be identical to a deterministic run. Imagine, for example, a case where in a deterministic model, a queue at one intersection has a back-spill which in the average *just* does not reach another intersection.⁵ In the stochastic model, the maximum length of this queue will, between different simulation runs, fluctuate around its average value, thus back-spilling into the other intersection in nearly 50% of all runs. Since this possibly disrupts traffic in this other intersection, this can cause long-range effects and network breakdown.

VIII. SOME OPEN QUESTIONS

Many open questions remain, though. For example:

What is the exact relation between average cluster growth in CA models, wave amplitude growth in fluid-dynamical models, droplet growth in the liquid-gas transition interpretation, and phase space portraits in car following models?

Is there a hydrodynamical limit for the STCA? If so, how can it be proven to be correct? Do critical exponents help

⁵By queue I mean a queue with spatial extension. This is different from the use of the word in queuing theory.

here?

What is the exact relation to granular media? Pöschel has both observed and simulated similar waves for sand falling down in a narrow tube [85]. He has also found in the simulations the bi-stability leading to laminar flow or to jam waves depending on the initial conditions. Lee and coworkers [86,84] have related these waves to a fluid-dynamical theory which is similar to the Kühne/Kerner/Konhäuser theory for traffic flow. Schäfer finds a similar phase transition as the one stressed in this paper for simulated granular flow, except that above the critical point, the flow is exactly zero [87]; supposedly, such a flow-density relation would also support the same overall dynamics.

What is the *minimal* ingredient for the instability which causes the traffic breakdown? Both the car-following models (CA and continuous) and the fluid-dynamical approach have produced the instability after adding an inertia term. Yet, Goldhirsch and Zanetti point out that an inverse temperature effect is responsible for the clustering [70].

Can one say more about universality than in the last section?

What can one-dimensional theory say about two-dimensional problems, such as they are regularly encountered for urban traffic problems? A series of papers (see, e.g., [53–55,88]) have used cellular automata techniques for building models for town traffic. These models use the CA-184/1 model for driving dynamics, but add elements for directional changes. Molera and coworkers have built a theory for their two-dimensional model [88], and their flow equation is essentially a 2-dimensional version of the Lighthill-Whitham equation *with a quadratic flow-density relation*. That means that adding stochastic directional changes would change the model from CA-184 type to the ASEP type.

What is the relation to $1/f$ -noise? Musha and Higuchi have measured $1/f$ -noise in the power spectrum of a car detector time series [22]. They explained this by a noisy Burgers equation, in a way though which differs from Krug's interpretation [76]. Nagel and Paczuski [46] have predicted a precise $1/f$ law for the power spectrum of the density time series, which was roughly confirmed by simulations for STCA-CC/2. Yet, Nagel and Herrmann find, using a continuous car-following model and following the traffic movement, a $1/f^\alpha$ law, with $\alpha \approx 1.3$ [89]. Car following is slightly different from the particle hopping models in this paper; but if the arguments in [46] were entirely correct, this should not matter. — Understanding $1/f$ noise behavior would be helpful because it would be much easier to measure in reality than, say, lifetime distributions [41,46].

What is the meaning to the ongoing discussion about the value of synchronous updating for explaining physical phenomena? Huberman and Glance [90] have re-issued the warning that parallel updating may produce artifacts and that usually stochastic asynchronous updating would be a better approximation of reality. Yet, for the traffic case, it is clear from this paper that the (synchronous) STCA produces a much better model for reality than the (asynchronous) ASEP. One would probably have to go to much higher spatial and temporal resolutions (and thus lose all the computational advantages) when one wanted to build a stochastically updating model of traffic.

ACKNOWLEDGMENTS

I thank A. Bachem, P. Bak, C. Barrett, S. Esipov, J. Lee, M. Leibig, H.J. Herrmann, M. Paczuski, S. Rasmussen, M. Rickert, J. Schäfer, M. Schreckenberg, and D.E. Wolf for discussions, hints and encouragement.

A discussion group at TSA-DO/SA (LANL) about microscopic traffic modeling, consisting of Chris Barrett, Steven Eubank, Steen Rasmussen, Jay Riordan, Murray Wolinsky, and me, helped clarify many issues.

Most of the ideas with respect to simulation are based on discussions with Chris Barrett and Steen Rasmussen,

reflecting work in progress which is only a small part published in [91].

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FIG. 1. Space-time plot for random sequential update, $v_{max} = 1$ (ASEP/1), and $\rho = 0.3$. Clearly, the kinematic waves are moving forwards. For $\rho > 1/2$, the kinematic waves would be moving backwards, and the plot would look similar to 2.

FIG. 2. Space-time plot for random sequential update, $v_{max} = 5$ (ASEP/5), and $\rho = 0.3$, showing that higher maximum velocity does not lead to a different appearance as long as one uses random sequential update.

FIG. 3. Space-time plot for CA-184/5 and subcritical density.

FIG. 4. Space-time plot for CA-184/5 and supercritical density.

FIG. 5. Space-time plot for STCA-CC/1, at supercritical density, with one disturbance. The jam first grows according to $n(t) \sim (j_{in} - j_{out}) \cdot t$. Eventually, via the periodic boundary conditions, the outflow reaches the jam as inflow, and $n(t)$ follows a random walk (apart from finite size effects).

FIG. 6. Space-time plot for parallel update (cruise control limit), $v_{max} = 5$, $\rho = 0.09$, i.e. slightly above critical. The flow is started in a deterministic, supercritical configuration, but from a single disturbance separates into a jam and a region of exactly critical density.—This is phenomenologically the same plot as Fig. 5 except that $v_{max} = 5$.

FIG. 7. Space-time plot for parallel update, $v_{max} = 1$.

FIG. 8. Space-time plot for parallel update, $v_{max} = 5$, $\rho = 0.09$ (i.e. slightly above $\rho(q_{max})$), starting from ordered initial conditions. The ordered state is meta-stable, i.e. "survives" for about 300 iterations until is spontaneously separates into jammed regions and into regions with $\rho = \rho(q_{max})$.

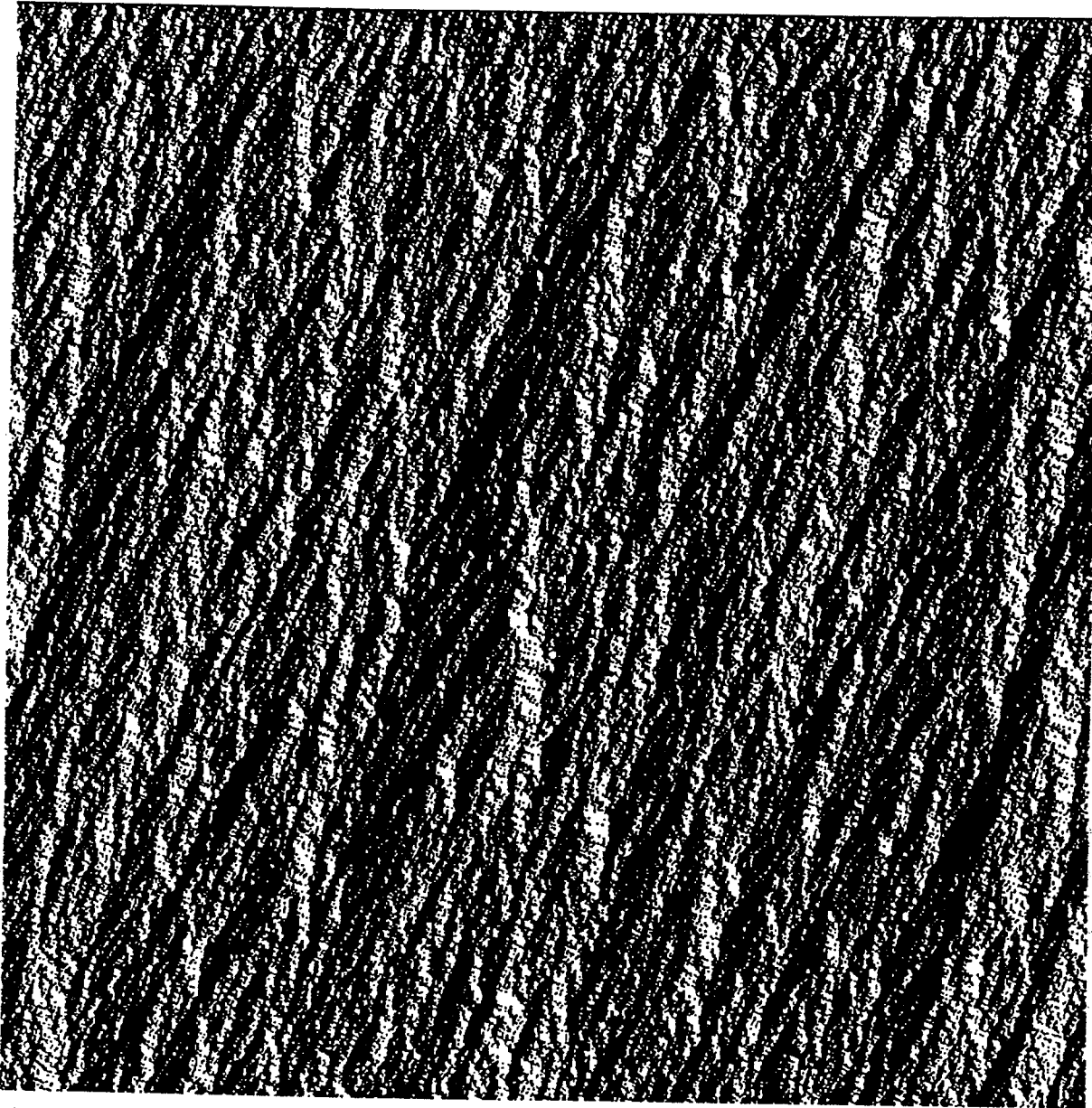
space (road) -->

<-- time

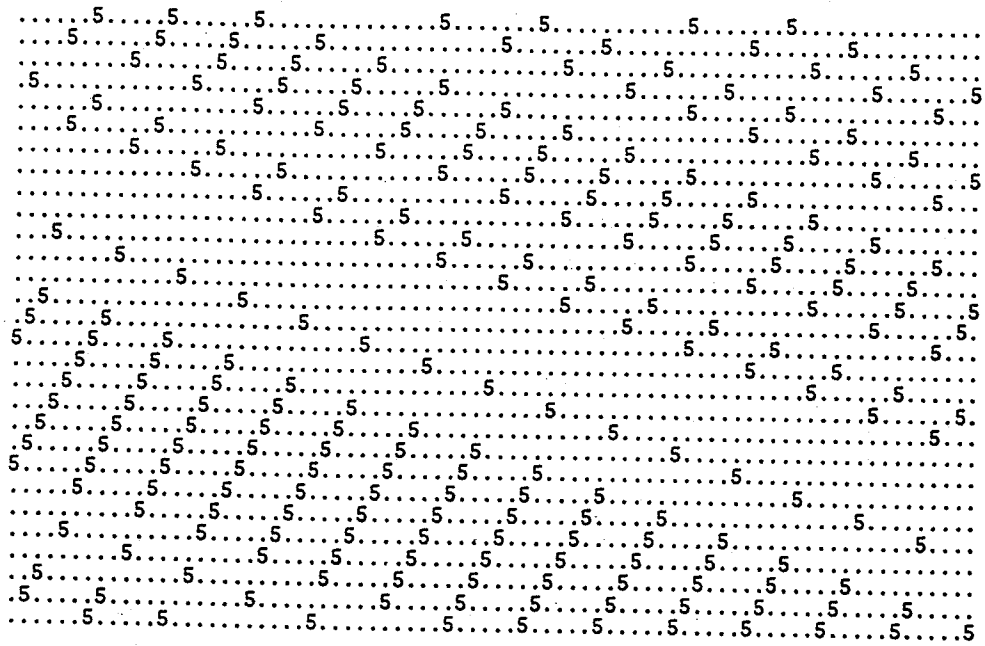


space (road) -->

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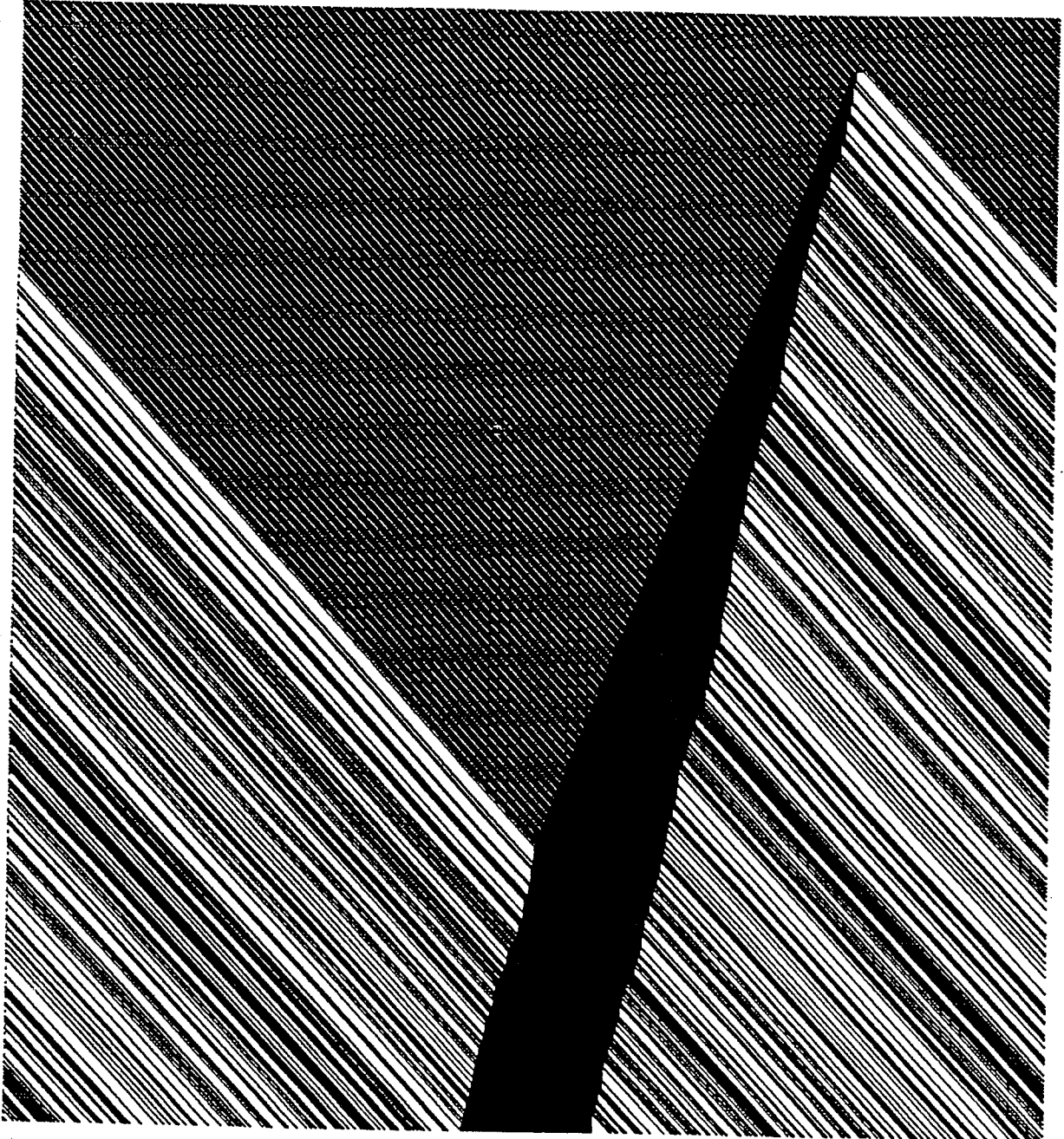


Nagel 1995 — Fig. 3



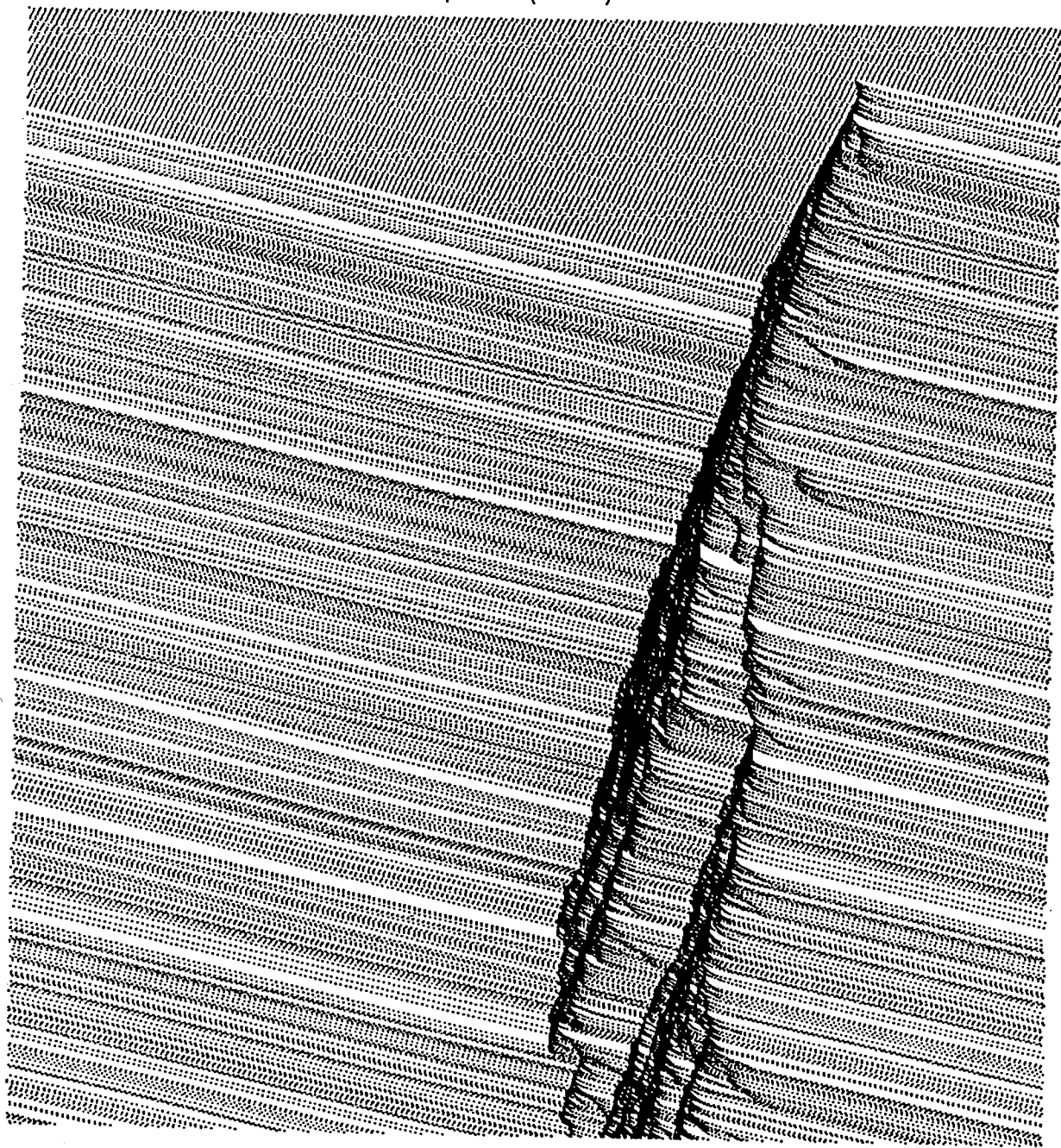
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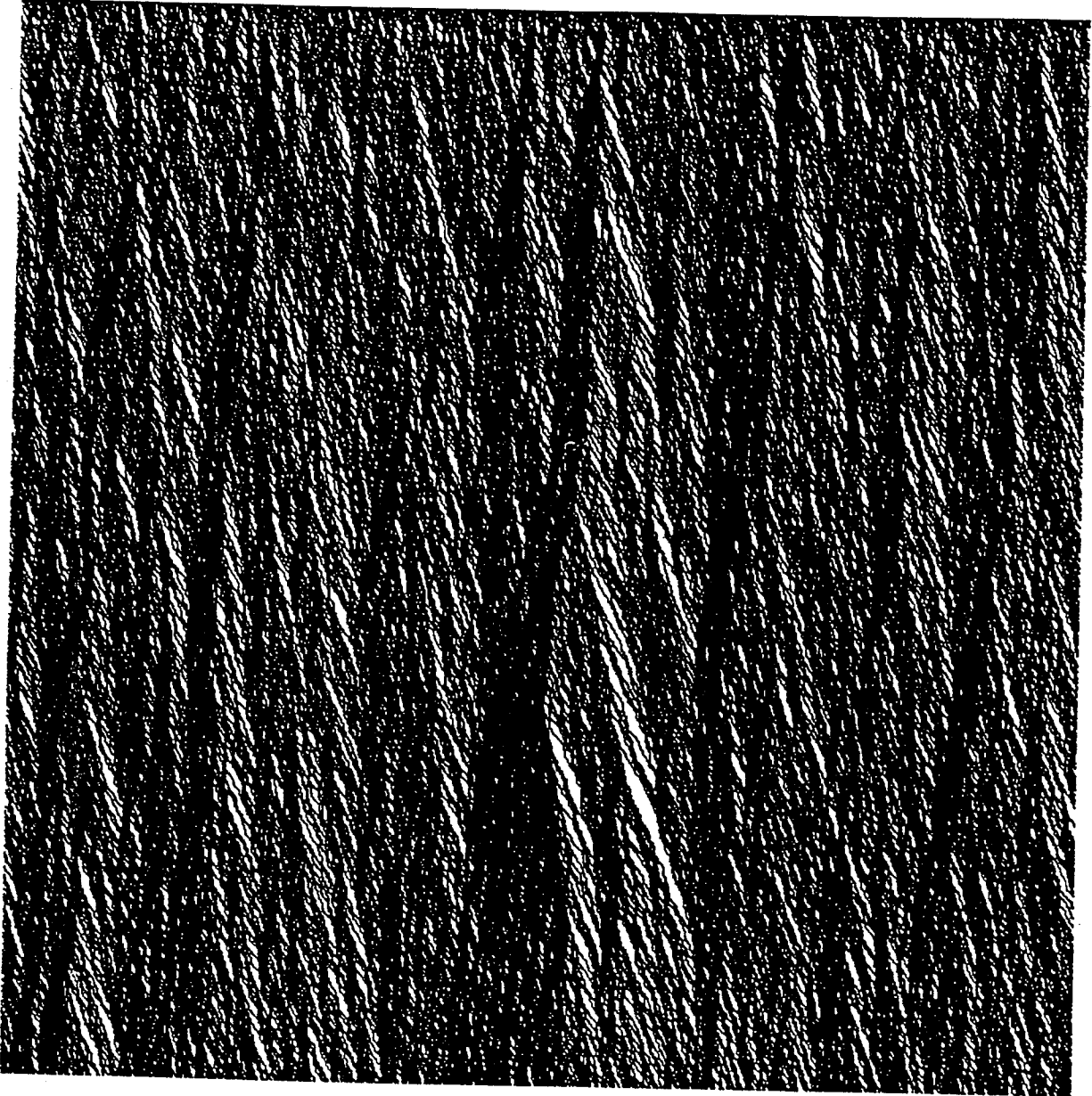
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