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Flood Damage Reduction Analysis

Technical Reference

Uncertainty Estimates for Graphical (Non-Analytic) Frequency Curves

October 2014 CPD-72a

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14. ABSTRACT
One of the features of the Hydrologic Engineering Center's Flood Damage Reduction Analysis (HEC-FDA) software is to use graphical frequency curves and include the uncertainty about those curves. This technical reference manual to HEC-FDA describes the methods used in computing the uncertainty about graphical frequency curves.

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Glossary/Abbreviations

Glossary

CDF	Cumulative Distribution Function of a random variable, defining probability
$f(y)$	PDF for variable Y, derived from the CDF (frequency curve), $F(Y)$
$F(Y)$	CDF for variable Y, referred to as the estimated flow or stage-frequency curve
$F(Y_p)$	non-exceedance probability associated with quantile Y_p
$F(Y_j)$	non-exceedance probability associated with the *jth* ordered observation.
$F(y)$	frequency curve
$IB_x(a,b)$	regularized incomplete beta function
m	event rank
M	mean of frequency curve
M_Y	mean of the uncertainty distribution at quantile Y_p
n	number of years of record (sample size); record length; equivalent years of record
n_u, n_l	equivalent record length for asymptotic uncertainty at upper and lower "match" point
p	non-exceedance probability
PDF	Probability Density Function of a random variable, defining probability
quantile	value of variable having a specified non-exceedance probability, p
S	standard deviation of frequency curve
S_u, S_l	standard deviation of uncertainty distribution at upper and lower "match" point
S_Y	standard deviation of uncertainty distribution at quantile Y_p
Y_p	flow or stage quantile at non-exceedance probability p
Y_u, Y_l	quantiles that become the upper and lower "match" points
Z_p	Standard Normal deviate at non-exceedance probability, p

Abbreviations

CDF	cumulative distribution function
EM	USACE Engineer Manual
ER	USACE Engineer Regulation
ERL	equivalent record length
HEC	Hydrologic Engineering Center
HEC-FDA	Flood Damage Reduction Analysis software
HEC-RAS	River Analysis System software
IID	independent and identically distributed
PDF	probability density function
USACE	U.S. Army Corps of Engineers

CHAPTER 1

Introduction

Welcome to the Hydrologic Engineering Center's (HEC) Flood Damage Reduction Analysis (HEC-FDA) computer software. HEC-FDA is a tool for formulating and evaluating flood damage reduction plans. Thesoftware is designed to (a) follow federal and U.S. Army Corps of Engineers (USACE) methods of risk analysis, (b) analyze the economic and hydrologic aspects of flood damage, (c) evaluate damage reduction plans based on damage analysis, project performance by analysis years, and equivalent annual damage, and (d) report results in a variety of output tables and plots. Additional details about the HEC-FDA software are available in the User's Manual (HEC, 2014).

1.1 Purpose

This Technical Reference describes the methods HEC-FDA uses to compute uncertainty around graphical (non-analytic) frequency curves developed for regulated flow or stage, or for flow-frequency curves estimated from hypothetical events. The recommended method involves the application of order statistics to compute the uncertainty distribution around the frequency curve.

1.2 Computation Procedures

USACE requires the use of risk analysis procedures for formulating and evaluating flood damage reduction measures (ER 1105-2-101; USACE, 2006). Uncertainty in discharge-exceedance probability, stage-discharge, and damage-stage functions are quantified and incorporated into economic and engineering performance analyses of alternatives. The process applies a Monte Carlo simulation, which is a numerical-analysis procedure that computes the expected value of damage while explicitly accounting for the uncertainty in the basic hydrologic, hydraulic and economic relationships used to determine flood inundation damage. HEC has developed the HEC-FDA software to assist in analyzing flood damage reduction plans using these procedures. Refer to Appendices E and G of the HEC-FDA User's Manual (HEC, 2014) for descriptions of the analysis process used by HEC-FDA. Engineering Manual (EM) 1110-2-1619 (USACE, 1996) provides a more detailed descriptions of the analysis procedures used by HEC-FDA.

1.3 Exceedance Probability Functions

Economics and performance analyses require an exceedance probability function to be defined (assigned) for each plan, analysis year, stream, and damage reach. In order to perform a flood damage analysis that considers flood events of all sizes, a relationship between flood magnitude and the probability of exceeding that magnitude is needed. This relationship is the exceedance probability function, also known as a frequency curve. Flood event magnitude might be defined in terms of discharge (flow) or stage. Exceedance probability functions are defined in the **Exceedance Probability Functions with Uncertainty** dialog box (Figure 1.1) of the HEC-FDA software.

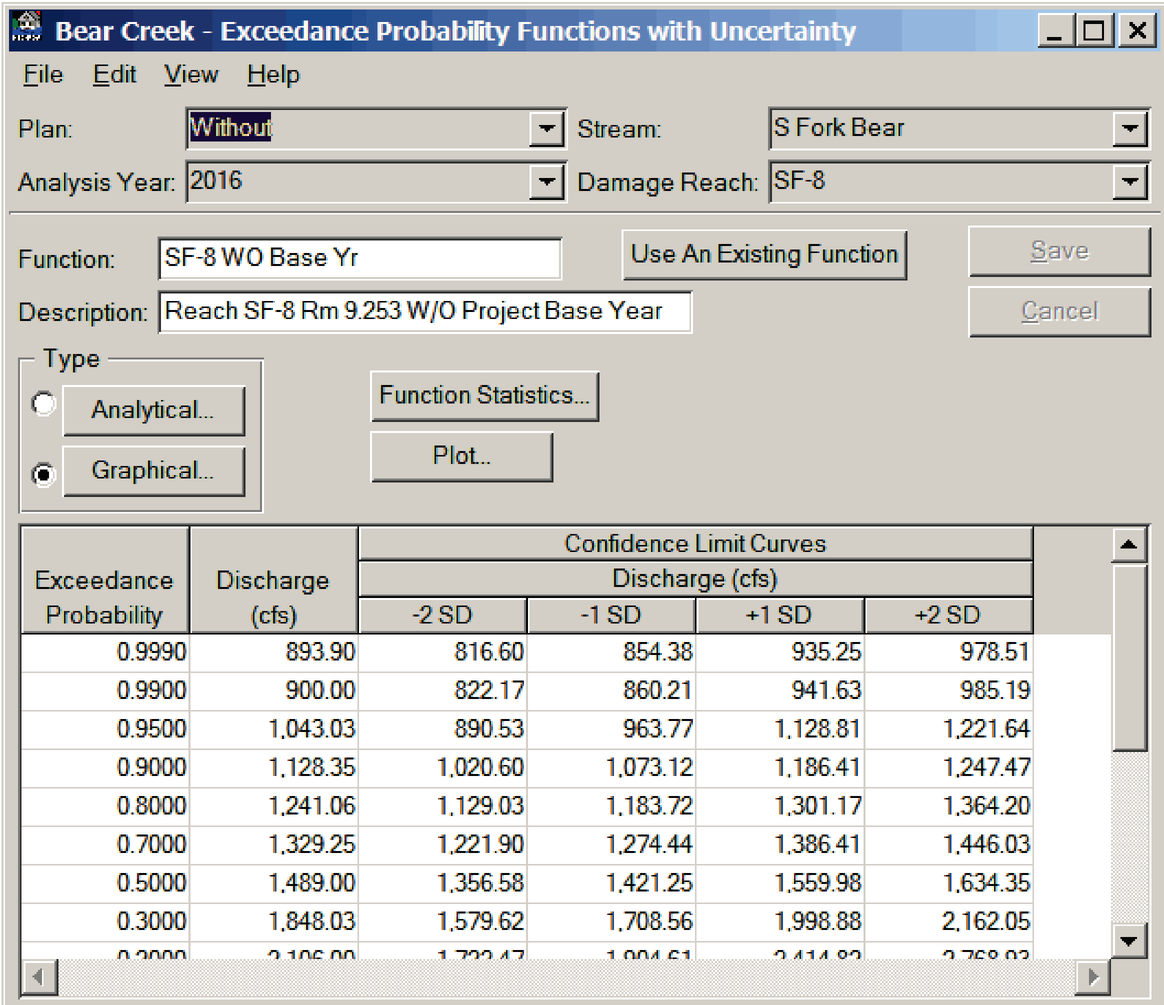

Figure 1.1 HEC-FDA - Exceedance Probability Functions with Uncertainty Dialog Box

The same functions may be used for several reaches, plans, and analysis years but not different streams. The user may retrieve a graphical exceedance probability function from the water surface profiles, enter the data manually, or copy an existing exceedance probability function. The exceedance probabilities can be copied from one plan to another plan. An exceedance probability (frequency) function can be either analytical (discharge-probability) or graphical (discharge- or stage-probability).

1.4 Graphical Exceedance Probability Functions

If the function does not fit the Log Pearson Type III distribution, a graphical probability function is needed. The graphical approach is typically applicable for regulated flows and stages, for which an analytical probability distribution such as Log Pearson Type III or Log Normal are unsuited. The graphical probability function is defined by ordered pairs of exceedance probability versus flow or stage. The uncertainty in the graphical probability function is based on Equivalent Record Length (ERL) as the defining parameter, and is computed by an approximation of the order statistics approach currently termed "Less Simple". Figure 1.2 shows the **Probability Function - Type Graphical** dialog box from the HEC-FDA software, which is where the user will enter the graphical exceedance probability function.

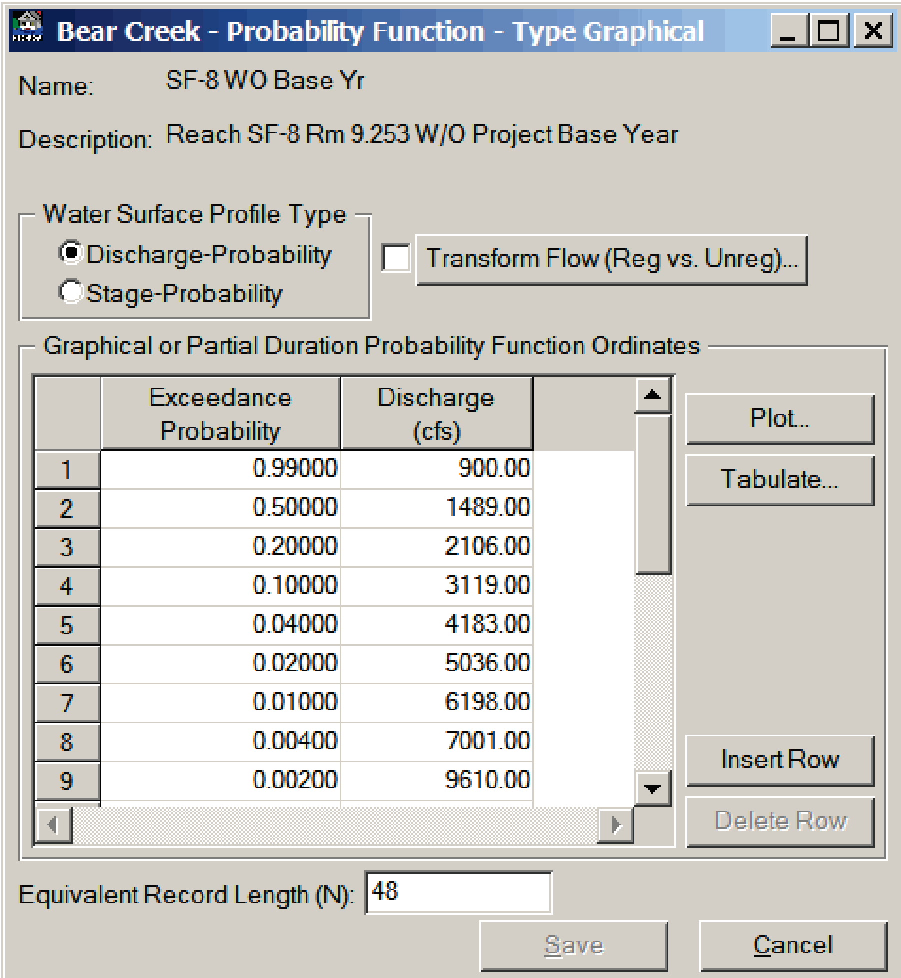

Figure 1.2 HEC-FDA - Probability Function - Type Graphical Dialog Box

A graphical probability function (discharge- or stage-frequency) is defined by specifying the discharge- or stage-probability ordinates and the ERL that describes the known function. When entering data to define graphical exceedance-probability functions, it is important to use enough points to describe the full range and resolution of the function. (The number and distribution of the points affect the uncertainty computation results.) Once the function is specified, ordered values are interpolated from the function based upon the ERL, and a standard error is determined at each quantile using the "Less Simple" approximation to the order statistics method. The method determines standard error of the values (or quantiles) from the relationship of each of the estimates to adjacent points. The distribution of errors is then assumed to be Normal around the specified function.

CHAPTER 2

Uncertainty Estimation for Graphical Frequency Curves

2.1 Uncertainty Estimation

2.1.1 General

A frequency curve or exceedance-probability function is a form of the cumulative distribution function (CDF) of a random variable (sometimes called the complement CDF). A graphical or non-analytic frequency curve is estimated either by plotting ordered observations against empirical frequency estimates (plotting positions) and fitting a smooth line, or by estimating hypothetical values with specified frequencies. Graphical curves are generally used for annual peak stages or regulated flows, which are not well described by analytical probability distributions such as the Log Pearson Type III distribution. Computation of the uncertainty due to limited record length for a graphical frequency curve is made more difficult by the lack of an analytical probability distribution.

A frequency analysis is based on the assumption that annual peak values are independent and identically distributed (IID) observations of a single random variable, and the number of values available affects the quality of the estimate. This assumption also allows uncertainty in a frequency curve to be estimated using several methods. One method is based on order statistics and the binomial distribution, described here, and another is based on a Monte Carlo simulation that repeatedly samples and re-estimates the frequency curve based on the defined years of record. (This approach is described in other documentation.) No assumption need be made concerning the analytic form of the frequency curve. Under these circumstances the statistic derived to estimate uncertainty is termed non-parametric, or distribution-free. Note that the procedures outlined in Bulletin 17B (IACWD, 1982) or EM 1110-2-1619, Section 4 (USACE, 1996), should be followed if an analytic distribution such as the Log Pearson Type III distribution can be used to approximate the frequency curve.

The order statistic approach is limited to calculating uncertainty in the estimated frequency curve for the range of observed data or, alternatively, the likely range of observations based on the equivalent record length. For example, when twenty years of data are available, the largest event likely to have occurred is near the 5% exceedance probability event, and the order statistics approach will be able to estimate the confidence limits (uncertainty distribution) only for quantiles ranging approximately from 0.25 to 0.75, as will be shown in the upcoming discussion. The uncertainty estimates beyond the range of data are extrapolated using asymptotic approximations of the uncertainty distributions. The order statistic estimates and the asymptotic estimates of uncertainty are matched up at the limits of the observed data. In order to make use of this combined uncertainty description, the standard deviations at each quantile computed from the order statistics estimates and the asymptotic approximations are paired with a Normal distribution as the final description of uncertainty around a graphical frequency curve.

2.1.2 Order Statistics Method

The order statistic approach relies on a straightforward application of the Binomial distribution (Mood, 1963), which describes the probability of some number of "successes" occurring in a number of independent trials, given the probability of success in one trial. For this application, the probability that a flow or stage quantile Y_p (the value corresponding to a ***non-exceedance*** probability p) exceeds a particular ordered sample value is calculated, based on the likelihood that the value would exceed some fraction of a sample $(1 - p)$. The sample of interest is the observations of the random variable. Assume that the observations of flow or stage are sorted in increasing order as Y_j, j =1, 2, 3, ... n, where Yj is the jth smallest observation ($Y_j \leq Y_{j+1}$) and n is the number of years of record (sample size). The uncertainty around the frequency curve at non-exceedance probability p is defined using the probability:

$$Prob[Y_p \geq Y_j] = P[j^{th} \text{ ordered observation} \leq Y_p] = P[j \text{ or more observations} \leq Y_p]$$

If quantile Y_p is greater than ordered observation Y_j, then it must also be greater than all ordered observations less than Y_j, or at least j observations. The Binomial Distribution (Equation 1) provides the probability of exactly x successes in n trials, as follows:

$$\text{Prob[exactly x successes in n trials]} = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!\,(n-x)!} p^x\,(1-p)^{n-x} \quad (1)$$

where p is the probability of "success" in a single trial. In this case, a "success" is an observation less than quantile Y_p, with probability also defined as $F(Y_p)$. However, the probability of exactly j successes is not the total probability needed. The probability of ***j or more*** successes is needed to represent all observations < Y_p, and requires a summation of Equation 1 from $x = j$ to n.

$$\text{Prob}[Y_p \geq Y_j] = \sum_{i=j}^{i=n} \binom{n}{i} p^i (1-p)^{n-i} \quad (2)$$

where $\boldsymbol{p = P[Y \leq Y_p] = F(Y_p)}$ is the non-exceedance probability associated with the quantile of interest. Performing this computation for every observation Y_j builds a CDF of the uncertainty distribution around quantile Y_p, as shown in Figure 2.1. Per standard definitions, the probability density function (PDF) is the first derivative of the CDF.

Note the probability computations described here do not consider the actual values of the observations, but rather their relative magnitude (rank) within the sample. Therefore the computed probability estimates $P[Y_p \geq Y_j]$ are associated with and assigned to the appropriate ordered value Y_j. This important characteristic means the computed $P[Y_p \geq Y_j]$ estimates can be provided only for the range of the Y_j. This limitation is depicted in Figure 2.2, which shows that the computed PDF around each quantile of the frequency curve is truncated at the value of the smallest and the largest observation. Therefore the uncertainty distribution cannot be defined beyond the smallest or largest observation. In other words, if only twenty observations exist,

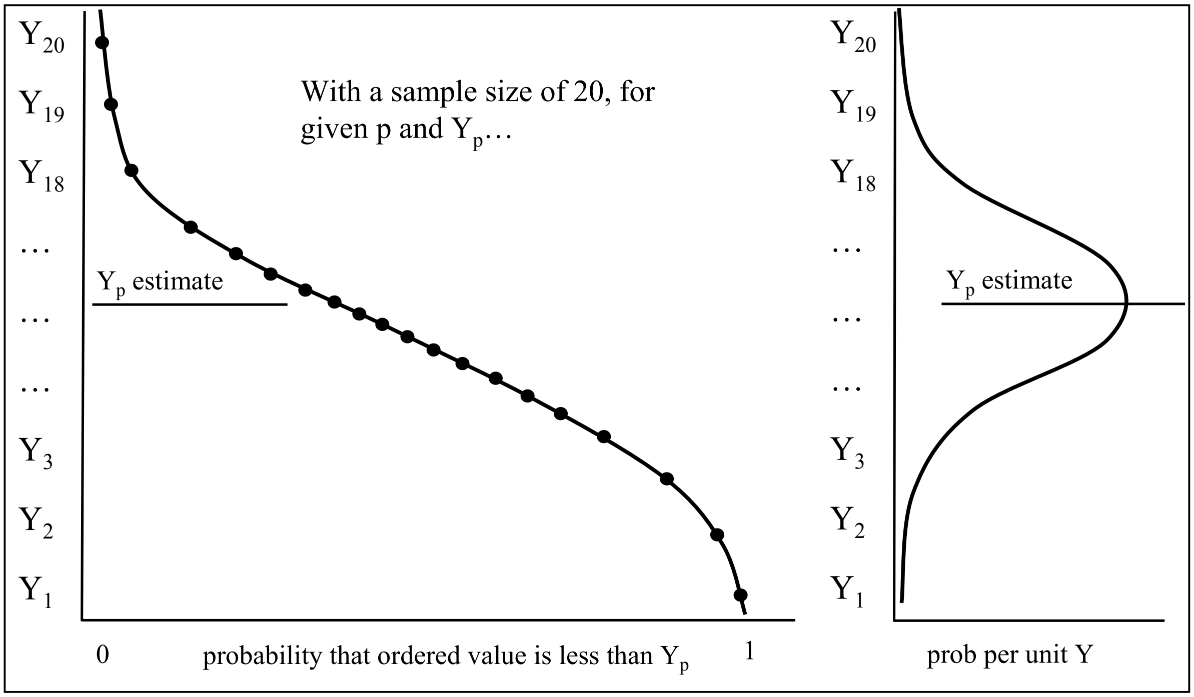

Figure 2.1 Construction of Uncertainty Distribution around Quantile Y_p

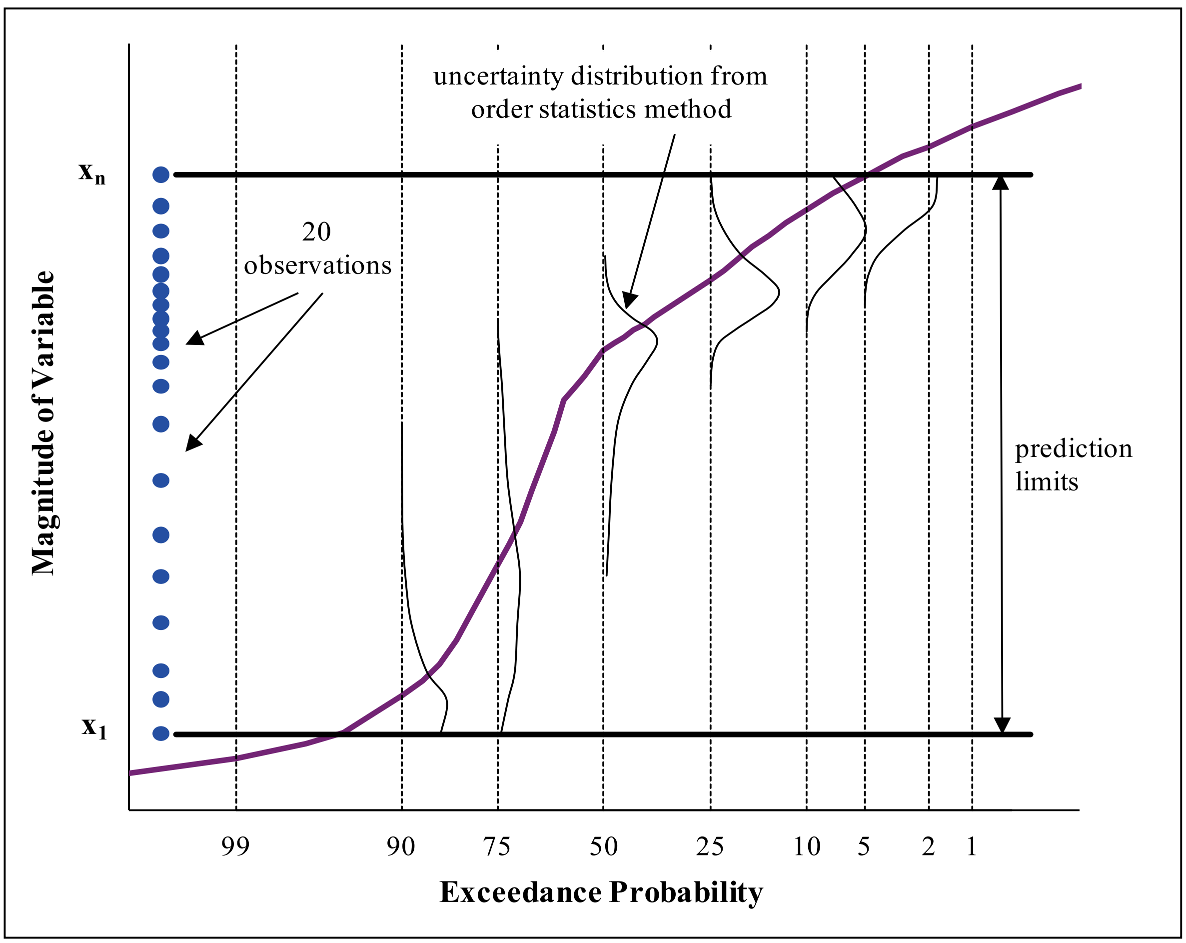

Figure 2.2 Uncertainty PDFs around Frequency Curve from Order Statistics Method

then the maximum value for any uncertainty distribution is the largest out of twenty observations and the minimum value is the smallest.

Figure 2.2 also shows that the uncertainty distributions are even less complete around quantiles in the tails of the frequency curve, limited by the minimum and maximum ranked observations. Thus, uncertainty estimates developed by the order statistics method can be used only for a limited range of the frequency curve for which the PDFs are complete. Figure 2.2 shows that when twenty observations exist, the uncertainty distributions are completely defined in approximately the range of 0.25 to 0.75. (To apply this method carefully, a minimum bound must be chosen to specify how much of the PDF must be defined to allow use of its description of the quantile uncertainty.) Methods to extend the uncertainty description to the distribution tails with various approximations are discussed in Section 2.1.5.

Despite this limitation in constructing the uncertainty PDFs, the order statistics estimate is valuable because it is sensitive to local changes in the slope of the frequency curve. When the observed values are closer together (the frequency curve slope is flatter), the uncertainty distribution is narrower, as seen in Figure 2.2. This result means that the method is able to account for the local variability of a random variable at different quantiles. For example, a regulated frequency curve may have a very small slope over a significant range in probability, indicating very little variability in stage or release for more or less frequent events. The order statistics approach will correctly predict very little uncertainty in the estimated frequency curve over this range of probabilities.

Another benefit of the order statistics estimate of the uncertainty distribution is that the computed PDF is appropriately asymmetrical. Figure 2.2 shows that where the frequency curve changes from a steep slope to a flatter slope (near the 50% chance exceedance quantile), the uncertainty

PDF shown reaches downward farther than it reaches upward. This asymmetry is relevant, and a good feature of the method. Unfortunately it is difficult to maintain in the final uncertainty description due to the previously mentioned limited range of the PDF at each quantile.

The computation of the Binomial Distribution and its summation is rather inconvenient when the factorial terms get large. A more convenient and equivalent expression involves the regularized incomplete Beta Function $IB_x(a,b)$ (Press, 1992) in Equation 3.

$$IB_x(a,b) = \frac{1}{B(a,b)}\int_o^x z^{a-1}\ (1-z)^{b-1}dz = \sum_{i=a}^{a+b-1} \frac{(a+b-1)!}{i!\,(a+b-1-i)!}x^i(1-x)^{a+b-1-i} \qquad (3)$$

$B(a,b)$ is the Beta Function, and the summation expression results from integration by parts. By performing the substitutions $a = j$, $b = n - j + 1$, and $x = p$, the incomplete Beta Function resolves to the Binomial Distribution, as shown in Equation 4.

$$IB_p(j, n-j+1) = \sum_{i=j}^{i=n} \frac{n!}{i!\,(n-i)!}p^i(1-p)^{n-i} = \sum_{i=j}^{i=n} \binom{n}{i} p^i(1-p)^{n-i} \qquad (4)$$

Therefore,

$$P[Y_p \geq Y_j] = IB_p(j, n-j+1) \quad (5)$$

This expression gives the probability that quantile Y_p is greater than or equal to the j^{th} ordered observation Y_j, or more specifically, that $F(Y_p)$, the non-exceedance probability associated with Y_p, is greater than or equal to $F(Y_j)$, the non-exceedance probability associated with the j^{th} ordered observation.

2.1.3 Simple Example of Binomial Distribution and Order Statistics for Uncertainty

A simple example of the Binomial Distribution (Equation 1) assumes that there are five years of observed peak annual stages. Interest is in computing the uncertainty in the non-exceedance probabilities assigned to the five peak stages. The estimated non-exceedance probability corresponding to the largest observed event (rank equals 5) is a relative frequency of non-exceedance, captured by the Weibull plotting position as:

$$F(Y_p) = p = m/(n+1) = 5/6 = 0.833$$

where m is the event rank and n is the number of years of record. Given that the largest event of the five is therefore the estimate of the p = 0.833 quantile, $Y_{0.833}$, the probability that the *actual* quantile exceeds the third ranked observation, $Y_j = Y_3$, can be computed using Equation 2, which determines the probability that *at least 3* of 5 observations are less than Y_p. This computation is shown in Table 2.1, with the binomial result for three, four and five observations, and their sum.

Table 2.1 Estimated Likelihood that Quantile $Y_{0.833}$ Exceeds Ordered Observation Y_3

Number of Observations < Y_p	*i*	Binomial Coefficient	p^i	$(1 - p)^{n-i}$	** $P[Y_{0.833}>Y_3]$
exactly 3	3	5!/(3!2!) = 10.0	0.833^3	0.167^2	0.161
exactly 4	4	5!/(4!1!) = 5.0	0.833^4	0.167^1	0.402
exactly 5	5	5!/(5!0!) = 1.0	0.833^5	0.167^0	0.402
					Sum = 0.965

**product of Columns 2, 3, and 4

Consequently, there is a 0.965 probability, or 96.5% chance, that the stage corresponding to the 0.833 non-exceedance probability, $Y_{0.833}$, exceeds the third-ranked event Y_3, because it exceeds either three, four, or five of the events.

2.1.4 Equivalent Years of Record, Synthetic Sample

The preceding discussion refers to estimating the uncertainty in a frequency curve developed from a sample of ordered observations. However, the theory can be extended to a frequency curve estimated in another way, for which a synthetic sample of equivalent size (length) is generated. For example, assume that the information used to develop a graphical flow frequency curve from hypothetical events is worth twenty years of equivalent record length. (Perhaps the frequency curve was developed using frequency-based precipitation events and a rainfall-runoff model.) Weibull plotting positions for the twenty years can be calculated using $p = m/(n + 1)$ for $m = 1, 2, \ldots, n$ and $n = 20$ and the corresponding values of Y_j are then determined (interpolated) from the resulting graphical frequency curve. These Y_j can be used as a synthetic sample, as previously described, to calculate confidence limits with the order statistics method.

2.1.5 Asymptotic Approximations

As noted in Section 2.1.2, the order statistics method can compute an uncertainty PDF around a frequency curve quantile only within the range of the sample (real or synthetic), and so beyond the quantile range for which the PDFs are complete, approximations of uncertainty are required. The approximation differs depending on whether stage or flow is the focus.

In the case of a stage-frequency curve, the uncertainty in the stage quantile is determined based on the result that the Binomial Distribution is asymptotically Normal for a large *n*. Under this assumption, if one identifies the variance of uncertainty from the order statistics computation, the variance can then be paired with a Normal distribution PDF for quantile Y_p (with the best estimate of Y_p used as the mean). The variance results from a transformation of the sampling error in the estimate of the exceedance probability (from a non-parametric plotting position) to the uncertainty in the stage quantile, given as:

$$S_Y^2 = \frac{p(1-p)}{n\,f(y)^2} \tag{6}$$

where:

- S_Y = standard deviation of the uncertainty distribution around quantile Y_p
- p = non-exceedance probability of quantile Y_p
- n = record length
- $f(y)$ = the PDF for variable Y, derived from the frequency curve of interest $F(y)$

Note that the PDF $f(y)$ is the inverse of the slope of the frequency curve, which is a CDF $F(y)$ with probability on the *horizontal* axis.

The derivation of Equation 6 starts with the uncertainty in count X for the Binomial Distribution, with variance of X $S_{X,}^2 = p(1 - p)n$. Rather than count X, however, interest is in the sample proportion $p = X/n$ as an estimate of non-exceedance probability. When the constant $1/n$ is pulled out and squared, the variance of p $S_p^2 = X/n$ is simply the variance of X divided by n^2, so $S_p^2 = p(1 - p)n/n^2$ or $S_p^2 = p(1 - p)/n$.

The next step involves transforming from the distribution of p to the distribution of Y. Defining $p = F(y)$ (the frequency curve), an approximation to the uncertainty of quantile Y_p results from a first order Taylor expansion around the mean estimate of the frequency curve quantile y_m at non-exceedance probability $p_m = F(y_m)$, as shown in Figure 2.3:

$$\begin{aligned} p &= p_m + \frac{1}{slope}(y - y_m) = p_m + \left.\frac{dF(y)}{dY}\right|_{y_m}(y - y_m) \\ &= p_m + \frac{dF(y_m)}{dY}(y - y_m) \end{aligned} \tag{7}$$

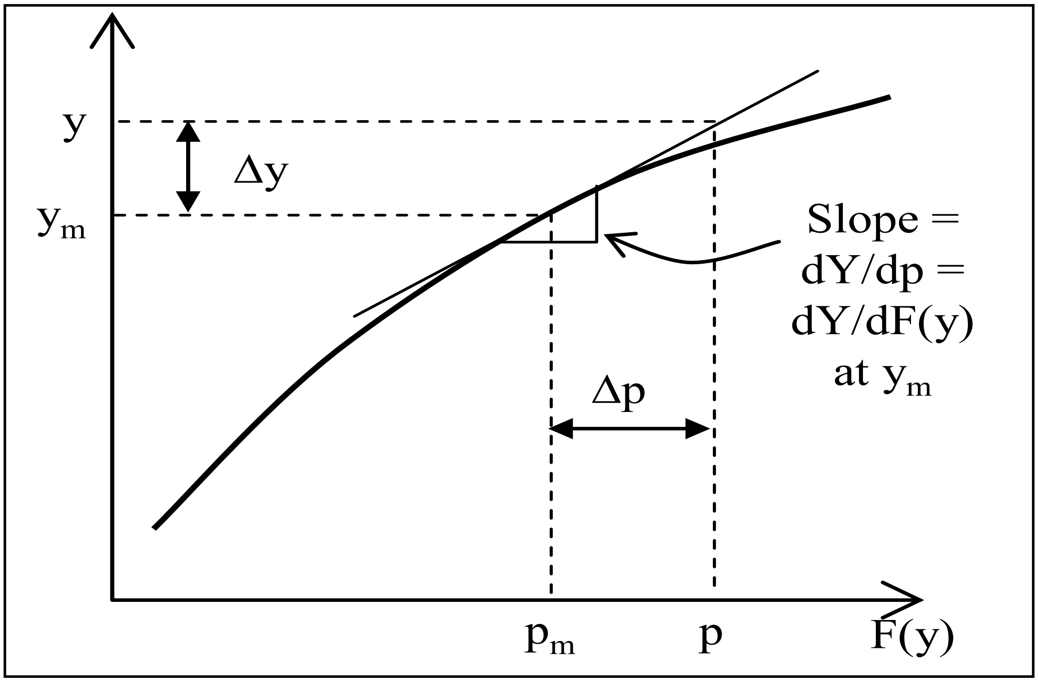

Figure 2.3 Taylor Expansion around Point *m* on Frequency Curve F(y)

Subtracting p_m from both sides, squaring both sides, and taking expected values creates a relationship between the variance of p, S_p^2, and the variance of Y, S_Y^2:

$$p - p_m = \frac{dF(y_m)}{dY}(y - y_m); \quad E(p - p_m)^2 = \frac{dF(y_m)^2}{dY}E(y - y_m)^2; \quad S_p^2 = \frac{dF(y)^2}{dY}S_Y^2 \qquad (8)$$

Finally, replacing $dF(y)/dY$ with $f(y)$, replacing $S_p{}^2$ with the Binomial Distribution $S_p{}^2 = p(1 - p)/n$, and rearranging yields Equation 6 as an expression of $S_Y{}^2$. Note that the sampling error (uncertainty) in probability due to record length, derived as S_p, is based on the non-exceedance probability p at a given stage or flow Y. This error has then been transposed to estimate S_Y, the error in the stage quantile Y_p, using the *local* slope of the frequency curve as represented by $f(y)$. This is a non-parametric estimate, independent of the *overall* shape of the frequency curve.

This approximation is particularly useful for extrapolating the uncertainty in stage frequency curve for stages corresponding to the channel overbank elevation. For these stages, the frequency curve is relatively flat and therefore the uncertainty in the frequency curve actually decreases with increasing deviation (distance) from the mean stage. This approximation captures this effect because the standard deviation is proportional to the local slope of the frequency curve (CDF with probability on the horizontal axis). Therefore, the uncertainty will be less where the frequency curve is flatter. The behavior of the standard deviation in this circumstance is very different from that for unregulated streamflow frequency curves where the uncertainty, and so the standard deviation, increase with distance from the mean flow.

Note that this approximation is based on the fact that the Binomial Distribution is asymptotically Normal for large samples. The sample sizes available for estimation of stage and flow frequency curves are not large. Further, use of the Normal distribution in place of the order statistics PDF unfortunately neglects the asymmetry present in the uncertainty distributions developed with order statistics. This asymmetry is visible in Figure 2.2.

In the case of a flow-frequency curve, the Equation 6 approximation is too conservative (too large) for application to flow frequency curves, which are approximately analytic for extreme probabilities, or to stage-frequency curves which do not have broad overbank areas. Consequently, as an alternative approach for near-analytic frequency curves, the order statistic estimation of uncertainty is extended using an approximation to the uncertainty in a Normal distribution (Kottegoda, 1980). Starting with finding the variance of quantile $Y_p = M + Z_p S$:

$$Var(Y_p) = Var(M + Z_p S) = Var(M) + Z_p^2\, Var(S) = \frac{S^2}{n} + Z_p^2 \frac{S^2}{2n} = S_Y^2 \quad (9)$$

$$S_Y^2 = \frac{S^2}{n} + Z_p^2 \frac{S^2}{2n} \quad (10)$$

where:

M = mean *of the frequency curve*
S = standard deviation *of frequency curve*
n = record length
p = non-exceedance probability
S_Y^2 = variance of the quantile Y_p estimate
Z_p is a normalized deviate computed in Equation 11 as:

$$Z_p = \frac{Y_p - M}{S} \quad (11)$$

Equation 10 is used as the uncertainty approximation for flow frequency curves, and is sometimes used to constrain the larger variance estimated by Equation 6 for stage frequency curves. M and S of the frequency curve can be computed using a numerical integration by the trapezoidal rule. To use this approximation of uncertainty in the tails of the frequency curve, the frequency curve itself must be defined for the full range of probability, meaning the curve's lower half (below the median) as well as the curve's upper half.

2.1.6 Plotting Position

As noted in Section 2.1.5, for hypothetical frequency curves for which a sample does not exist, this method involves generation of a synthetic sample of size n, where n is the equivalent years of record of the graphical frequency curve. The synthetic sample is generated by specifying the n plotting positions, or exceedance probabilities, and interpolating the flow or stage associated with those probabilities from the frequency curve. Numerous plotting position formulas have been proposed in the past. However, estimating the expected annual damage presumes that the expected value of the frequency curve (the expected probability estimate) is preserved in the process. Consequently, the Weibull plotting position is preferred because it is the mean estimate of the exceedance or non-exceedance probability, defined as $m/(n+1)$ for $m = 1, 2, .., n$.

2.2 Application of Order Statistics Approach to Estimated Frequency Curves

This section provides the steps for using statistics estimates and using asymptotic approximations to provide an uncertainty estimate that is paired with the Normal distribution.

2.2.1 Method

The order statistics approach provides estimates of uncertainty around a frequency curve *F(y)* for only a limited range. Beyond the range of the sample values (real or synthetic), the uncertainty distribution PDF is incomplete, as seen in Figure 2.2. An approach is suggested herein for using the order statistics estimates in the range for which the sample values are complete, and using the asymptotic approximations defined in Section 2.1.5 outside that range. In both cases, the uncertainty estimate is paired with the Normal distribution to obtain complete uncertainty distributions for the full range of frequency curve quantiles of interest.

The approach used is to compute the standard deviation of the uncertainty distribution PDF obtained with the order statistics approach (Equations 2 or 5), and pair it with a Normal distribution (using the frequency curve quantile as the mean). The Normal distribution was selected because it matches the asymptotic approximation used to extrapolate uncertainty estimates. Furthermore, the Normal distribution is convenient for use with Monte Carlo simulation. However, as noted in Section 2.1.2, the PDFs developed with the order statistics approach are frequently not symmetrical, especially when the slope of the frequency curve varies, while the Normal distribution is symmetrical. Therefore, this method becomes a poorer estimate as the slope of the frequency curve becomes more variable.

The steps followed in estimating this Normal uncertainty distribution are described below:

Step 1 - **Compute order statistics uncertainty PDFs.** From either Equation 2 or 5, compute the order statistics uncertainty PDF for each quantile Y_p of interest (based on specified frequency ordinates), using either the actual observations (the sample) or synthetic values interpolated from the frequency curve at the plotting positions corresponding to the appropriate record length (a synthetic sample). Note that while the order statistic approach can create the PDF of uncertainty for any quantile Y_p, a full PDF is not obtained at high and low quantiles because an exceedance probability estimate by Equation 2 or 5 is made only for each sample value Y_j. (In other words, Y_p can take any value of p, but the Y_j values are defined by the real or synthetic sample, and so the PDF can be truncated in one tail or the other.) Therefore, the calculation of uncertainty is limited by the range of observations or the record length as was described previously and shown in Figure 2.2.

Step 2 - **Calculate the mean M_Y and standard deviation S_Y of the order statistics uncertainty PDFs** from Step 1 for each specified quantile (or, each value of p.) The calculations are approximate because of the incomplete PDFs, as mentioned in Step 1 and Section 2.1.2. For the quantiles for which the uncertainty PDF is truncated by the data value range, the standard deviation of the uncertainty PDF will be underestimated, and the mean of the uncertainty PDF will tend toward to mean of the frequency curve, as discussed in Section 2.2.4 and shown in Figures 2.4 and 2.5, where the mean of the uncertainty PDF is depicted as

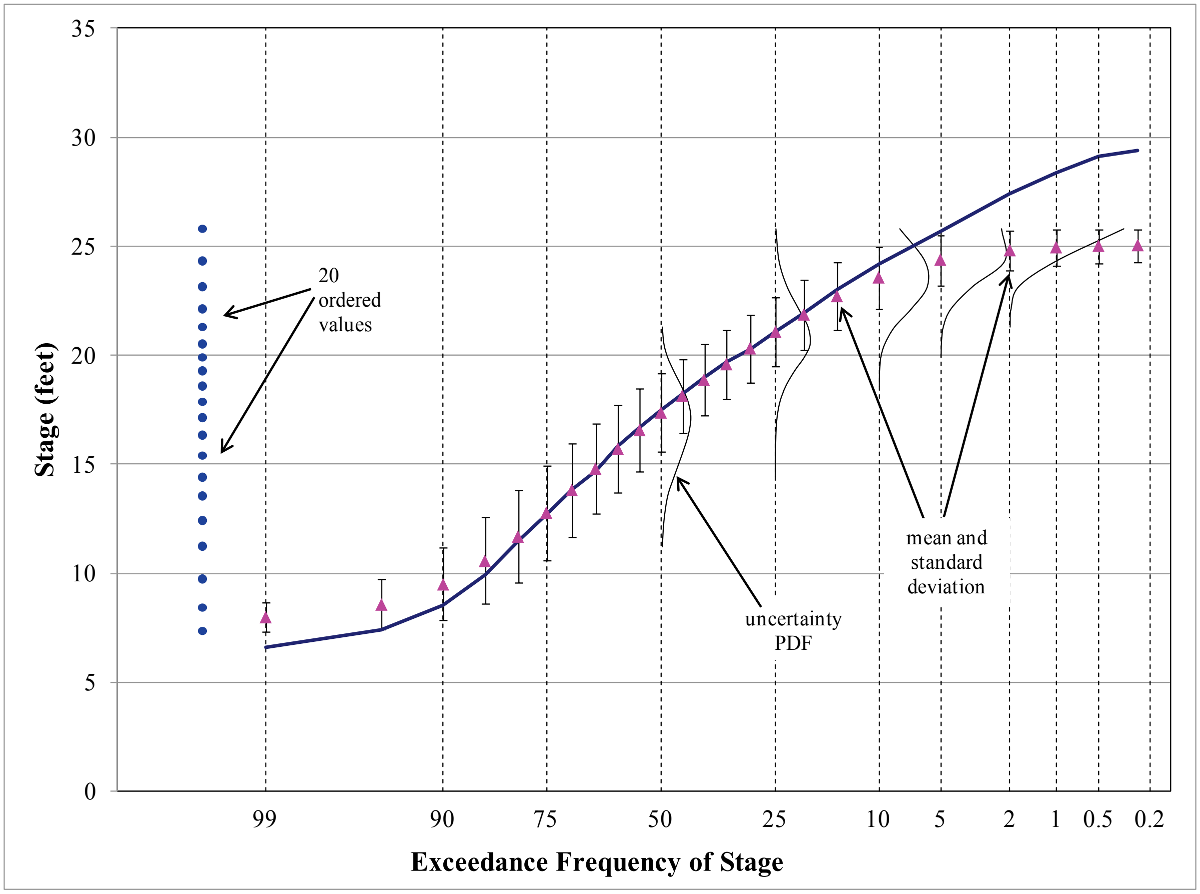

Figure 2.4 Mean and Standard Deviation of Uncertainty Distributions at Quantiles of Interest for ERL = 20

a triangle. Consequently, the mean and standard deviation computations are used only for quantiles with complete uncertainty PDFs, or quantiles for which a large percentage of the PDF exists (e.g., greater than 95 percent). In Figures 2.4 and 2.5, the incomplete uncertainty PDFs can be recognized by comparing the noting how the PDF mean (the triangle) diverges from the frequency curve (the line) due to truncation of the PDF.

The mean and standard deviation for the uncertainty distribution PDFs around quantile Y_p (computed from either Equation 2 or 5) are estimated using trapezoidal rule integration. By the definition, the mean (expected value) and variance are defined by:

$$M = E(X) = \int x \cdot f(x)\, dx \qquad S^2 = E(X - M)^2 = \int (x - M)^2 \cdot f(x)\, dx$$

Solving these integrals across the range of observations (as opposed to defined probabilities) with the trapezoidal rule leads to:

$$M_Y = \sum_{i=2}^{n} \tfrac{1}{2}\left(Y_{i-1} + Y_i\right) \cdot Prob\left[Y_{i-1} \le Y \le Y_i\right]$$

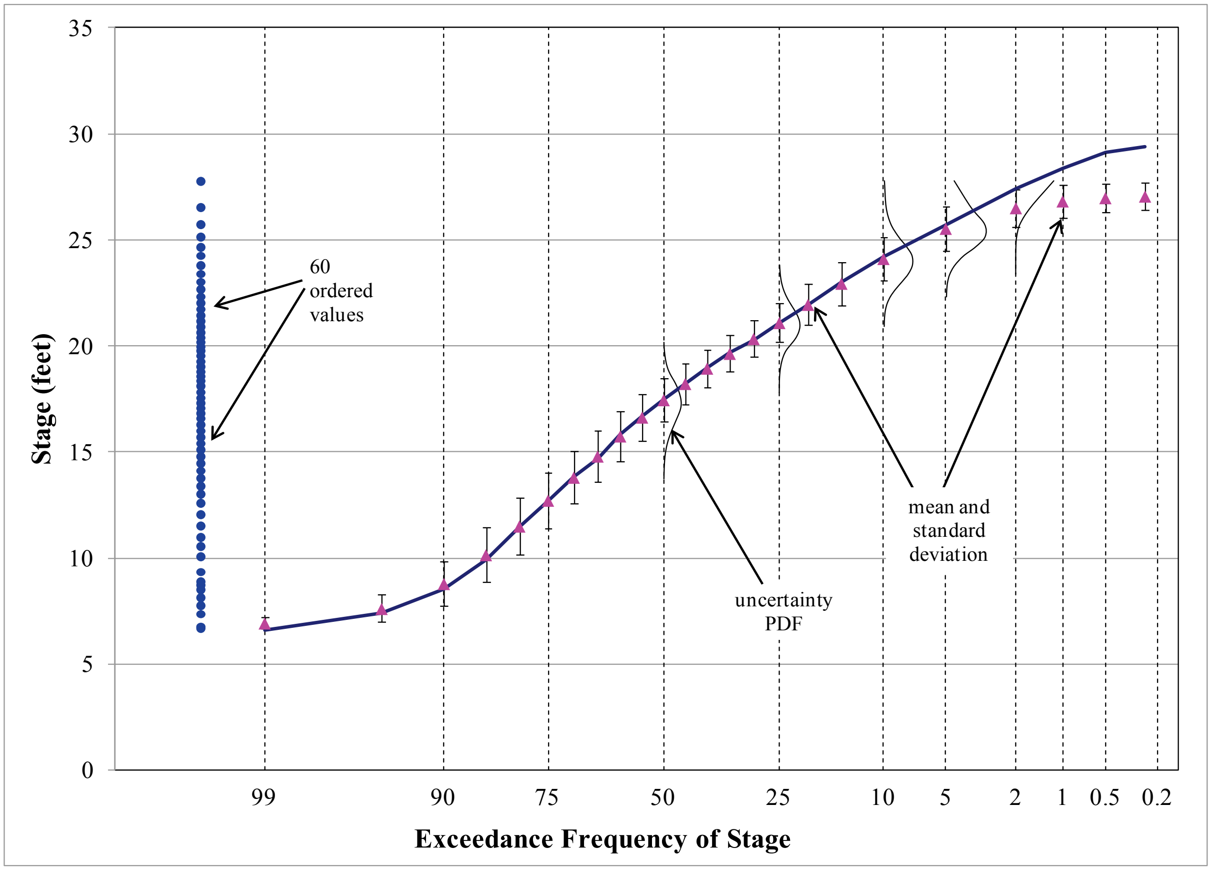

Figure 2.5 Mean and Standard Deviation of Uncertainty Distributions at Quantiles of Interest, for ERL = 60

$$S_Y^2 = \sum_{i=2}^{n} \tfrac{1}{2}\left((Y_{i-1} - M_Y)^2 + (Y_i - M_Y)^2\right) \cdot Prob\left[Y_{i-1} \le Y \le Y_i\right]$$

where Prob[$Y_{i-1} \le Y \le Y_i$] can be computed with $\dfrac{P[Y_p \ge Y_{i-1}] - P[Y_p \ge Y_i]}{P[Y_p \ge Y_1] - P[Y_p \ge Y_n]}$

The denominator of the expression represents the portion of the uncertainty distribution captured by the range of observations Y_1 to Y_n, which, due to the limitation of the order statistics computation, is less than one. Evaluating the expression with the Incomplete Beta Function as in Equation 5 produces Equations 12 and 13:

$$M_Y = \sum_{i=2}^{n} {}^{1}\!/_{2}\,(Y_{i-1} + Y_i)\frac{IB_p\,(i-1, n-i+2) - IB_p\,(i, n-i+1)}{IB_p(1,n) - IB_p\,(n,1)} \qquad (12)$$

$$S_Y{}^2 = \sum_{i=2}^{n} {}^{1}\!/_{2}\,((Y_{i-1} - M_Y)^2 + (Y_i - M_Y)^2)\frac{IB_p\,(i-1, n-i+2) - IB_p\,(i, n-i+1)}{IB_p(1,n) - IB_p\,(n,1)} \qquad (13)$$

where M_Y and S_Y are estimates of the sample mean and standard deviation of the uncertainty distribution PDF computed in Step 1 for any quantile.

Step 3 - **Determine the quantile range for which standard deviations S_Y are usable**. As stated above, because the uncertainty PDF developed by the order statistics approach is incomplete at high and low quantiles, it is not used explicitly. Instead, an approximate uncertainty distribution is estimated around each quantile of the frequency curve by the Normal distribution, using the standard deviation of the uncertainty PDF computed in Step 2. Again, this standard deviation is only an adequate estimate for a limited range of frequency curve quantiles for which the order statistics PDF is complete, or nearly so. As will be shown in Example 1 (Section 2.2.4), when the PDF is incomplete, the mean of the PDF diverges from the quantile value and the standard deviation is underestimated.

One approach to determine the range of quantiles for which the order statistics PDF and its standard deviation are adequate is to define the quantiles for which "most" of the PDF is generated. For example, if more than 95% of the PDF is formed using the real or synthetic sample of values, its standard deviation will be used; and if less than 95% of the PDF is formed, the asymptotic approximation of standard deviation will be used. As described in the discussion of Equations 12 and 13, the portion of the uncertainty PDF formed by the sample values Y_1 to Y_n is equal to Prob[$Y_1 \leq Y_p \leq Y_n$]. Equation 14 below defines this probability, and so the portion of the PDF formed.

$$Prob[Y_1 \leq Y_p \leq Y_n] = P[Y_p \geq Y_1] - [Y_p \geq Y_n]$$

$$= \sum_{i=1}^{n} \binom{n}{i} p^i (1-p)^{n-i} - \sum_{i=1}^{n} \binom{n}{n} p^n (1-p)^{n-n}$$

$$= \left(1 - \sum_{i=0}^{0} \binom{n}{i} p^i (1-p)^{n-i}\right) - \sum_{i=n}^{n} \binom{n}{n} p^n (1-p)^{n-n}$$

$$= (1 - p^0 (1-p)^{n-0}) - p^n (1-p)^{n-n}$$

$$Prob[Y_1 \leq Y_p \leq Y_n] = 1 - (1-p)^n - p^n$$ **OR**, using the Incomplete Beta function,

$$Prob[Y_1 \leq Y_p \leq Y_n] = IB_p(1, n) - IB_p(n, 1) \quad (14)$$

The portion of the PDF formed at a given quantile will eventually fall below 95% for quantiles toward either end of the frequency curve. The farthest quantiles Y_p having greater than 95% of the PDF define the points where the asymptotic approximations of standard deviation is equated or "matched" to that computed in Step 2. These points are defined as Y_u (upper) and Y_l (lower), the quantile range referenced above, and S_u and S_l are the corresponding standard deviations at those quantiles.

Step 4 - **Compute *n* for asymptotic approximation of S_Y at upper and lower match points, Y_u and Y_l.** The Normal distribution replacement for the uncertainty PDF mentioned in Section 2.2.1 uses the standard deviations computed in Step 2. This replacement uncertainty distribution requires standard deviations (S_Y) for the entire frequency curve, not just the range for which the order statistics PDFs are complete. The asymptotic approximations of S_Y for either stage-frequency curves (Equation 6) or flow-frequency curves (Equation 10) are equated or matched to the order statistic estimates of S_Y at the upper and lower "match points", which were defined as Y_u and Y_l. Matching is accomplished by rearranging either Equation 6 or Equation 10 to determine an equivalent record length n for a given S_Y from the asymptotic approximation. Using Equation 6, for quantile values greater than Y_u, the upper match point, this equivalent record length n_u becomes:

$$n_u = \frac{p(1-p)}{S_u^2\, f(Y_u)^2} \tag{15}$$

where p is the non-exceedance probability for quantile Y_u, and $f(Y_u)$ is the value of the probability density function (PDF) at that quantile, defined by the inverse of the slope of the frequency curve *F(Y)*. (Note that the frequency curve has probability on the horizontal axis and Y on the vertical axis, making its slope the inverse of that of a standard CDF.) Correspondingly, the order statistics and asymptotic approximation estimates are matched at Y_l (for use with quantiles less than Y_l, the lower match point) to compute n_l:

$$n_l = \frac{p(1-p)}{S_l^2\, f(Y_l)^2} \tag{16}$$

Equation 6 can now be used to extrapolate the estimates of uncertainty using the computed values of n_u and n_l.

A similar procedure is used to obtain equivalent record lengths from Equation 10 for flow frequency curves. For quantile values greater than Y_u this becomes:

$$n_u = \frac{S^2}{S_u^2}\left(1 + \frac{Z_u^2}{2}\right) \qquad where: \quad Z_u = \frac{Y_u - M}{S} \tag{17}$$

where M is the mean of the frequency curve and S is the standard deviation of the frequency curve. Correspondingly, the order statistics and asymptotic approximation estimates are matched for quantiles less than Y_l using:

$$n_l = \frac{S^2}{S_l^2}\left(1 + \frac{Z_l^2}{2}\right) \qquad where: \quad Z_l = \frac{Y_l - M}{S} \tag{18}$$

2.2.2 Source of the Frequency Curve, and Estimate Equivalent Record Length

Estimation of frequency curves involves various kinds of information. Ideally, sufficient gaged observations would be available to estimate the frequency curve accurately, yet this is rarely the case. If no gaged observations are available, rainfall-runoff analysis can be used to develop a synthetic frequency curve from frequency-based precipitation events. An equivalent length of record can be chosen for this frequency curve from Table 2.2, reproduced here from EM 1110-2-1619 (Table 4-5; USACE, 1996). The order statistics method is used to estimate the uncertainty in the frequency curve given the equivalent years of record as described in Section 2.1.1.

Table 2.2 Equivalent Record Length Guidelines (Table 3-1; USACE, 1996)

	Method of Frequency Function Estimation	Equivalent Record Length
1	Analytical distribution fitted with long-period gaged record available at site	Use systematic record length
2	Analytical distribution fitted for long-period gage on stream, drainage area within twenty percent	Use 90 to 100 percent of record length
3	Analytical distribution fitted for long-period gage within watershed, model calibrated to gage-based curve	Use 50 to 90 percent of record length
4	Regional discharge-probability function parameters	Average years from regional study
5	Rainfall-runoff-routing model calibrated to several events recorded at short-interval event gauge in watershed	Use twenty to thirty years record length
6	Rainfall-runoff-routing model with regional model parameters (no rainfall-runoff-routing model calibration)	Use fifteen to twenty-five years record length
7	Rainfall-runoff-routing model with handbook or textbook model parameters	Use ten to fifteen years record length

Note: Based on judgment to account for the quality of any data used in the analysis, for the degree of confidence in models, and for previous experience with similar studies.

In some cases, a mixture of gaged observations and hypothetical events is used to estimate the frequency curve. When the gage record length is sufficient only to estimate the frequency curve for relatively high frequency (smaller) events, the hypothetical events are used to extend the frequency curve to include low-frequency (larger, more rare) events. The confidence and corresponding uncertainty in the frequency curve differ for the region of the curve defined by the observed or the hypothetical events.

2.2.3 Limitations

Extrapolation of a frequency curve needs to be constrained by the characteristics of the field conditions. This is also true of uncertainty distributions. Typical situations where the field conditions need to be considered are applications to regulated frequency curves. Regulated frequency curves should approach the unregulated frequency curves as the effect of the regulation diminishes. The uncertainty distributions around these curves should become equivalent as these two frequency curves become equivalent.

2.2.4 Example 1, Uncertainty Computation for a Stage-Frequency Curve

The estimation of the uncertainty distribution for a stage-frequency curve will be used to demonstrate the procedure outlined herein. The frequency curve developed for the location is shown in Figure 2.6 and tabulated in Table 2.3. Also listed in Table 2.3 is the mean and standard deviation for the frequency curve, which was computed using trapezoidal rule integration.

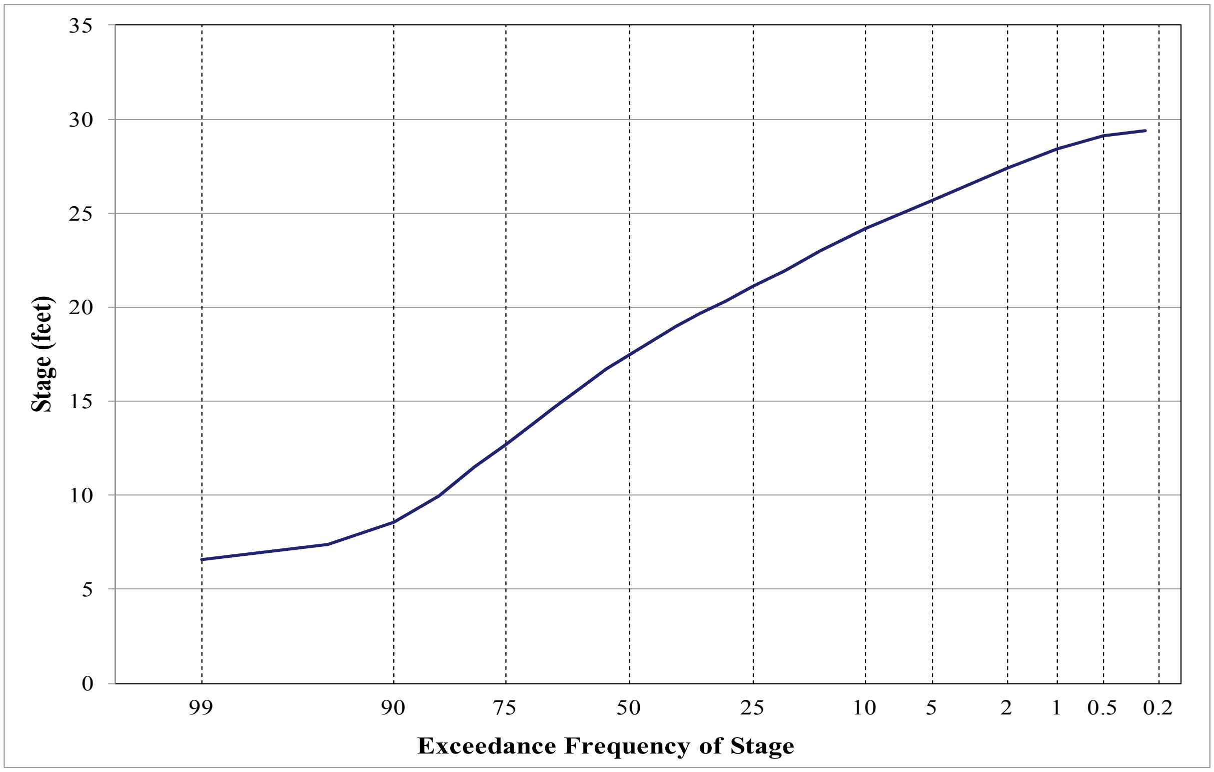

Figure 2.6 Example 1 Stage Frequency Curve

Table 2.3 Example 1 Stage Frequency Curve

Stage, (feet)	Exceedance Probability		Stage, (feet)	Exceedance Probability
6.60	0.9900		19.00	0.4000
7.40	0.9500		19.70	0.3500
8.55	0.9000		20.30	0.3000
9.95	0.8500		21.10	0.2500
11.50	0.8000		21.95	0.2000
12.70	0.7500		23.00	0.1500
13.85	0.7000		24.20	0.1000
14.70	0.6500		25.70	0.0500
15.80	0.6000		27.40	0.0200
16.70	0.5500		28.40	0.0100
17.50	0.5000		29.10	0.0050
18.25	0.4500		29.40	0.0025

Frequency Curve Moments: mean = 16.98 feet; standard deviation = 5.60 feet

Assuming that the frequency curve was estimated using information that is worth twenty years of record (equivalent record length, ERL), 20 ordered values were obtained from the frequency curve using the Weibull plotting position $p = m / (n + 1)$ where $n = 20$ as shown in Table 2.4. These values are used as a synthetic sample in the computation of the uncertainty distribution, following the steps outlined in Section 2.2.1.

Table 2.4 Synthetic Sample of Size ERL = 20 from Stage Frequency Curve

Event Rank	Stage (feet)	Exceedance Probability		Event Rank	Stage, (feet)	Exceedance Probability
1	7.34	0.9524		11	17.86	0.4762
2	8.43	0.9048		12	18.57	0.4286
3	9.74	0.8571		13	19.27	0.3810
4	11.23	0.8095		14	19.90	0.3333
5	12.42	0.7619		15	20.51	0.2857
6	13.55	0.7143		16	21.30	0.2381
7	14.41	0.6667		17	22.13	0.1905
8	15.39	0.6190		18	23.16	0.1429
9	16.33	0.5714		19	24.32	0.0952
10	17.13	0.5238		20	25.80	0.0476

Convert stages to meters, multiply by 0.3048

Step 1 - The uncertainty distribution for each quantile of interest (specified frequency ordinates) was computed using the order statistics estimates, either Equation 2 or 5. As an example, Figure 2.7 shows the computed uncertainty CDF for the 30% exceedance quantile, with exceedance probability around the quantile displayed on the "floating" horizontal axis.

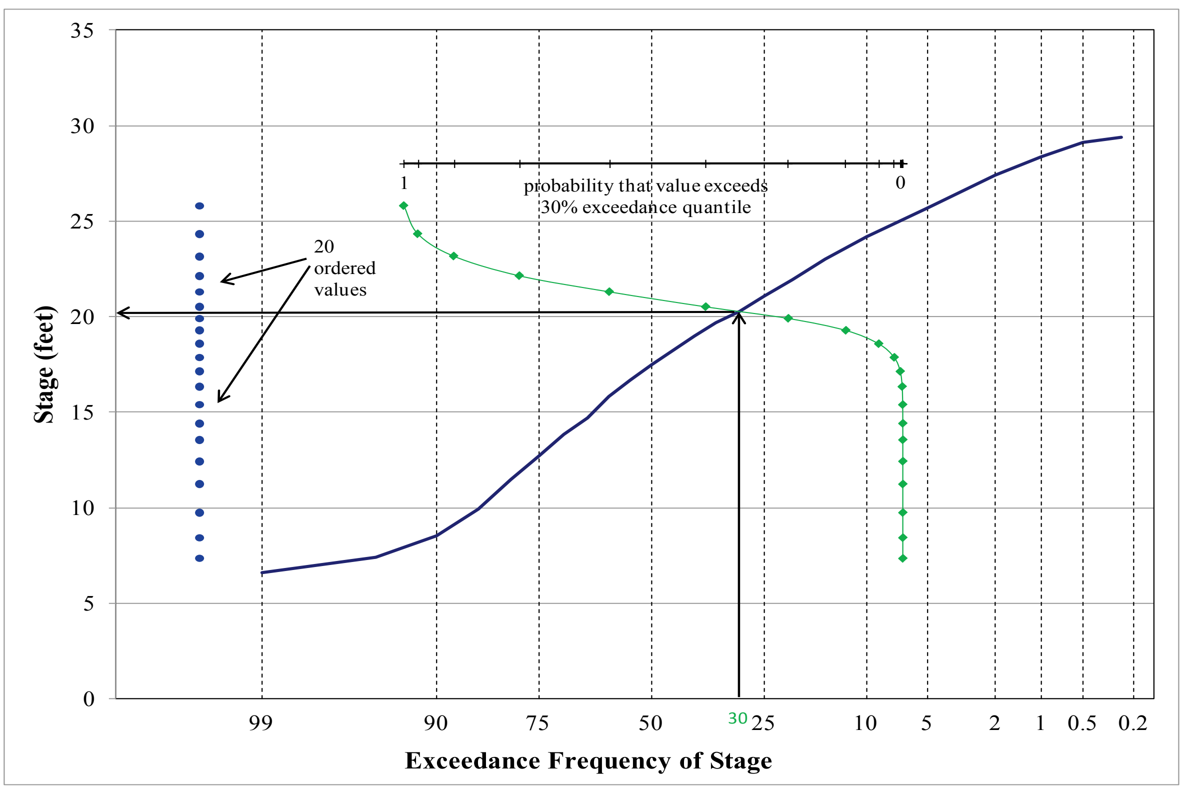

Figure 2.7 Computed Uncertainty Distribution CDF around 30% Exceedance Quantile, Example 1

(The CDF is displayed here because it is the raw result of the order statistics computation, however other figures instead display the uncertainty PDF.)

Step 2. The mean and standard deviation of the uncertainty distribution at each quantile of interest were computed via Equations 12 and 13 and are shown in Figure 2.4 and Table 2.5. For several quantiles, the uncertainty distribution is also displayed in Figure 2.4 in PDF form. (The PDF should be imagined as coming out of the page on a Z-axis.) Note again that for the

Table 2.5 Mean and Standard Deviations of Uncertainty Distributions Computed via Order Statistics for Example 1

Exceedance Probability	Non-Exceedance Probability	Stage Quantile	PDF Mean	PDF Standard Deviation	Percent of PDF Formed = Prob($Y_1 \leq Y_p \leq Y_n$)
0.9900	0.0100	6.60	8.04	0.678	18.2
0.9500	0.0500	7.40	8.61	1.160	64.2
0.9000	0.1000	8.55	9.53	1.646	87.8
0.8500	0.1500	9.95	10.59	1.967	96.1
0.8000	0.2000	11.50	11.70	2.126	98.8
0.7500	0.2500	12.70	12.79	2.167	99.7
0.7000	0.3000	13.85	13.82	2.139	99.9
0.6500	0.3500	14.70	14.80	2.076	100.0
0.6000	0.4000	15.80	15.71	1.994	100.0
0.5500	0.4500	16.70	16.56	1.900	100.0
0.5000	0.5000	17.50	17.37	1.800	100.0
0.4500	0.5500	18.25	18.13	1.706	100.0
0.4000	0.6000	19.00	18.86	1.628	100.0
0.3500	0.6500	19.70	19.58	1.580	100.0
0.3000	0.7000	20.30	20.30	1.567	99.9
0.2500	0.7500	21.10	21.06	1.580	99.7
0.2000	0.8000	21.95	21.86	1.591	98.8
0.1500	0.8500	23.00	22.70	1.552	96.1
0.1000	0.9000	24.20	23.55	1.415	87.8
0.0500	0.9500	25.70	24.36	1.150	64.2
0.0200	0.9800	27.40	24.79	0.924	33.2
0.0100	0.9900	28.40	24.93	0.836	18.2
0.0050	0.9950	29.10	24.99	0.790	9.5
0.0025	0.9975	29.40	25.02	0.766	4.9

more extreme quantiles, the PDFs are incomplete, limited by the range of the data (the synthetic sample) used to generate them. As a result, the means and standard deviations of those uncertainty distribution PDFs are underestimated, demonstrating the reason this computation is used only for quantiles for which at least 95% of the PDF is generated. For comparison, Figure 2.5 displays a similar computation with an ERL of sixty years, having sixty points to define the uncertainty. Note that the points are not only closer together but span a wider range, allowing complete PDFs to be generated completely for more extreme quantiles.

Step 3. Table 2.5 also displays the percentage of the uncertainty PDF generated at each quantile, computed with Equation 14. At the lower end of the frequency curve, the most extreme quantile with percentage-of-PDF greater than 95% is the 15% non-exceedance probability, quantile $Y_l = 9.95$, having $S_l = 1.967$. At the upper end of the frequency curve, that quantile is the 85% non-exceedance probability, $Y_u = 23.0$, having $S_u = 1.552$. These upper and lower points will be used to "match" the order statistics computation with the asymptotic approximation to be used beyond those points, by computing the appropriate *n* for each match point.

Step 4. Calculation of n_u and n_l for upper and lower match points, used for computation of the asymptotic approximation, is performed using Equations 15 and 16. Application of the equations requires an estimate of the probability density function *f(Y)* at each point, computed as the inverse of the slope of the stage-frequency curve *F(Y)*. From Table 2.5, the frequency curve stage quantiles are used to compute an approximation to this slope. The inverse slopes at quantile $Y_l = 9.95$ and at quantile $Y_u = 23.0$ are:

$$f_Y(Y_l) \approx 1/\frac{11.5 - 8.55}{0.2 - 0.1} = 0.0339$$

$$f_Y(Y_u) \approx 1/\frac{24.2 - 21.95}{0.9 - 0.8} = 0.0444$$

The equivalent record lengths are obtained using $S_l = 1.967$ and non-exceedance $p = 0.15$ in Equation 16:

$$n_l = \frac{0.15(1 - 0.15)}{1.967^2(0.0339)^2} = 28.67$$

and using $S_u = 1.552$ and non-exceedance $p = 0.85$ in Equation 15:

$$n_u = \frac{0.85(1 - 0.85)}{1.552^2(0.0444)^2} = 26.85$$

A straight-forward application of Equation 6 to compute standard deviation, paired with the Normal distribution, can now be used to extrapolate the uncertainty distribution. Results are displayed in Table 2.6.

Step 5. Equation 10 is used to constrain the variance computed by Equation 6. Calculation of n_u and n_l for computation of the asymptotic approximation is performed using Equations 17 and 18. The equivalent record lengths are obtained using $S_l = 1.967$ and $Y_l = 9.95$ in Equation 18:

$$n_l = \frac{5.60^2}{1.967^2}\left(1 + \frac{\left(\frac{9.95 - 16.98}{5.60}\right)^2}{2}\right) = 14.49$$

and using $S_u = 1.552$ and $Y_u = 23.0$ in Equation 17:

Table 2.6 Standard Deviations of Uncertainty Distributions Computed via Order Statistics, Equation 6, and Equation 10 for Example 1

Exceedance Probability	Non-Exceedance Probability	Stage Quantile	PDF Standard Deviation			Final Standard Deviation
			Order Statistics	Equation 6	Equation 10	
0.9900	0.0100	6.60	0.678	0.335	2.423	0.335
0.9500	0.0500	7.40	1.160	0.883	2.307	0.883
0.9000	0.1000	8.55	1.646	1.430	2.147	1.430
0.8500	**0.1500**	**9.95**	**1.967**	**1.967**	**1.967**	**1.967**
0.8000	0.2000	11.50	2.126	-	-	2.126
0.7500	0.2500	12.70	2.167	-	-	2.167
0.7000	0.3000	13.85	2.139	-	-	2.139
0.6500	0.3500	14.70	2.076	-	-	2.076
0.6000	0.4000	15.80	1.994	-	-	1.994
0.5500	0.4500	16.70	1.900	-	-	1.900
0.5000	0.5000	17.50	1.800	-	-	1.800
0.4500	0.5500	18.25	1.706	-	-	1.706
0.4000	0.6000	19.00	1.628	-	-	1.628
0.3500	0.6500	19.70	1.580	-	-	1.580
0.3000	0.7000	20.30	1.567	-	-	1.567
0.2500	0.7500	21.10	1.580	-	-	1.580
0.2000	0.8000	21.95	1.591	-	-	1.591
0.1500	**0.8500**	**23.00**	**1.552**	**1.552**	**1.552**	**1.552**
0.1000	0.9000	24.20	1.415	1.560	1.671	1.560
0.0500	0.9500	25.70	1.150	1.769	1.836	1.769
0.0200	0.9800	27.40	0.924	2.145	2.039	2.039
0.0100	0.9900	28.40	0.836	1.434	2.164	1.434
0.0050	0.9950	29.10	0.790	0.935	2.254	0.935
0.0025	0.9975	29.40	0.766	0.633	2.293	0.633

$$n_u = \frac{5.60^2}{1.552^2}\left(1 + \frac{\left(\frac{23.0 - 16.98}{5.60}\right)^2}{2}\right) = 20.54$$

Application of Equation 10 to compute standard deviation was used to constrain values computed using Equation 6. Results are displayed in Table 2.6. The resulting 90% confidence limits associated with the uncertainty calculation (at ±1.645 standard deviations for the assumed Normal distribution) are shown in Figure 2.8 and contained in Table 2.7 for the entire frequency curve. (Table 2.7 also contains curves for ±2 standard deviations, which is a common computation and spans about 95% confidence.) An additional step is often needed to ensure that the confidence interval edges do not decrease at the tails of the frequency curve. In this example, the +1.645 standard deviation (0.05 confidence estimate), at 0.0025 exceedance probability, was originally computed to be 30.44 feet (9.278 meters). This value was increased to 30.76 feet (9.376 meters) so that the upper confidence curve did not decrease beyond the 0.005 exceedance probability.

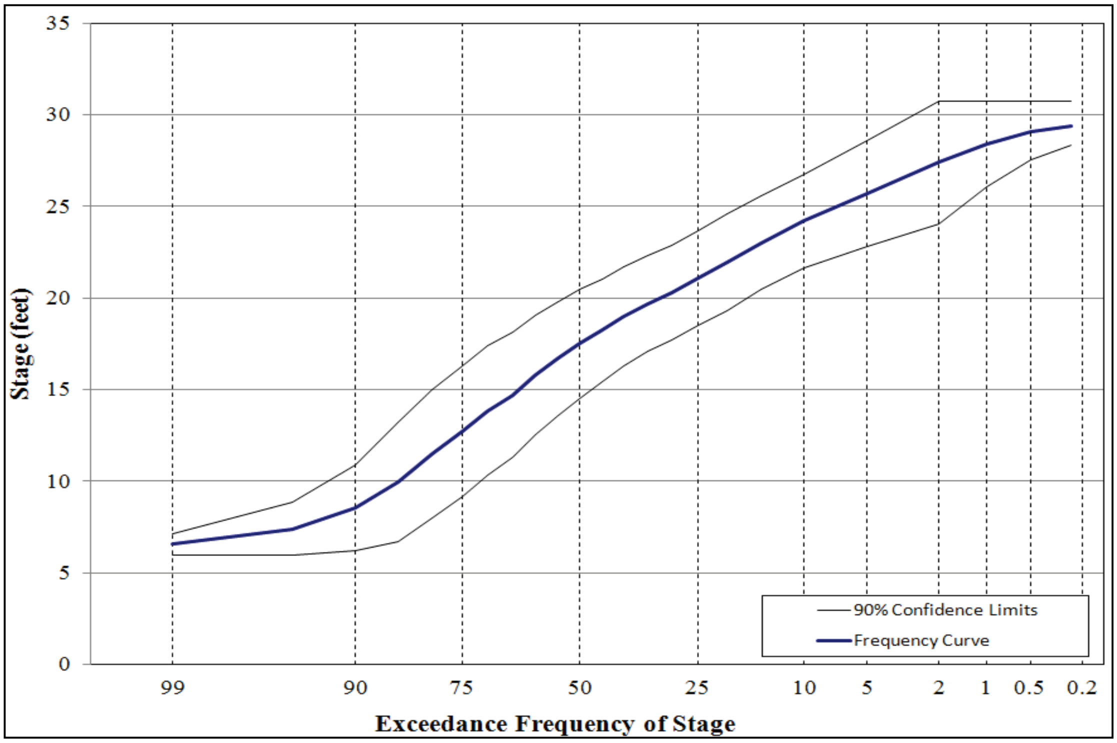

Figure 2.8 Example 1 Stage-Frequency Curve with Confidence Limits

2.2.5 Example 2, Uncertainty Computation for a Regulated Flow-Frequency Curve

The estimation of the uncertainty distribution for a regulated flow-frequency curve will be used to demonstrate the procedure outlined herein. The frequency curve developed for the location is shown in Figure 2.9 and tabulated in Table 2.8. Also shown in Table 2.8 are the mean and standard deviation for the frequency curve as computed using trapezoidal rule integration. The frequency curve was based on several sources of information, including a long inflow time-series, historic information, observed regulated flows and model simulations. The equivalent record length from this information was estimated to be about 120 years (ERL = 120). The ordered values are obtained from the frequency curve using the Weibull plotting position and shown in Table 2.9. These values are used as a synthetic sample in the computation of the uncertainty distribution, following the steps outlined in Section 2.2.1.

Step 1. The uncertainty distribution for each quantile of interest was computed using the order statistics estimate from either Equation 2 or 5. As an example, Figure 2.10 shows the computed uncertainty CDF for the 30% exceedance quantile, with exceedance probability around the quantile displayed on the "floating" horizontal axis. (The CDF is displayed here because it is the raw result of the order statistics computation, however other figures instead display the uncertainty PDF.)

Step 2. The mean and standard deviations of the uncertainty distribution at each quantile of interest were computed by using Equations 12 and 13 and are shown in Figure 2.11 and Table 2.10. For several quantiles, the uncertainty distribution is also displayed in PDF form

Table 2.7 Example 1 Stage Frequency Curve with Confidence Limits

Exceedance Probability	**Stage** (feet)	**90% Confidence Interval + 1.645 Std Error**	**90% Confidence Interval - 1.645 Std Error**	**95% Confidence Interval + 2 Std Error**	**95% Confidence Interval - 2 Std Error**
0.9900	6.60	7.15	5.95	7.27	5.63
0.9500	7.40	8.85	5.95	9.17	5.63
0.9000	8.55	10.90	6.20	11.41	5.69
0.8500	**9.95**	**13.19**	**6.71**	**13.88**	**6.02**
0.8000	11.50	15.00	8.00	15.75	7.25
0.7500	12.70	16.26	9.14	17.03	8.37
0.7000	13.85	17.37	10.33	18.13	9.57
0.6500	14.70	18.12	11.28	18.85	10.55
0.6000	15.80	19.08	12.52	19.79	11.81
0.5500	16.70	19.83	13.57	20.50	12.90
0.5000	17.50	20.46	14.54	21.10	13.90
0.4500	18.25	21.06	15.44	21.66	14.84
0.4000	19.00	21.68	16.32	22.26	15.74
0.3500	19.70	22.30	17.10	22.86	16.54
0.3000	20.30	22.88	17.72	23.43	17.17
0.2500	21.10	23.70	18.50	24.26	17.94
0.2000	21.95	24.57	19.33	25.13	18.77
0.1500	**23.00**	**25.55**	**20.45**	**26.10**	**19.90**
0.1000	24.20	26.77	21.63	27.32	21.08
0.0500	25.70	28.61	22.79	29.24	22.16
0.0200	27.40	30.75	24.05	31.48	23.32
0.0100	28.40	30.76	26.04	31.48	25.53
0.0050	29.10	30.76	27.56	31.48	27.23
0.0025	29.40	30.76	28.36	31.48	28.13

in Figure 2.10. Note again that for the more extreme quantiles, the PDFs are incomplete. As a result the means and standard deviations of those uncertainty distributions are underestimated, demonstrating why this computation is used only for quantiles ranging from 0.025 to 0.975 for this frequency curve with ERL equal to 120.

Step 3. Table 2.10 also displays the percentage of the uncertainty PDF generated at each quantile, computed with Equation 14. At the lower end of the frequency curve, the most extreme quantile with percent-of-PDF greater than 95% is the 2.5% non-exceedance probability, $S_l = 0.0159$, corresponding to a base-10-log flow of $Y_l = 3.2250$. At the upper end, the quantile is the 97.5% non-exceedance probability, with $S_u = 0.0894$ corresponding to a base-10-log flow of $Y_u = 4.0896$. These points will be used to "match" the order statistics computation with the asymptotic approximation to be used beyond those points, by computing the appropriate n at each match point.

Step 4. Calculation of n_u and n_l at the upper and lower match points for computation of the asymptotic approximation is performed using Equations 17 and 18. The equivalent record lengths obtained using $S_l = 0.0159$ and $Y_l = 3.2250$ in Equation 18 is:

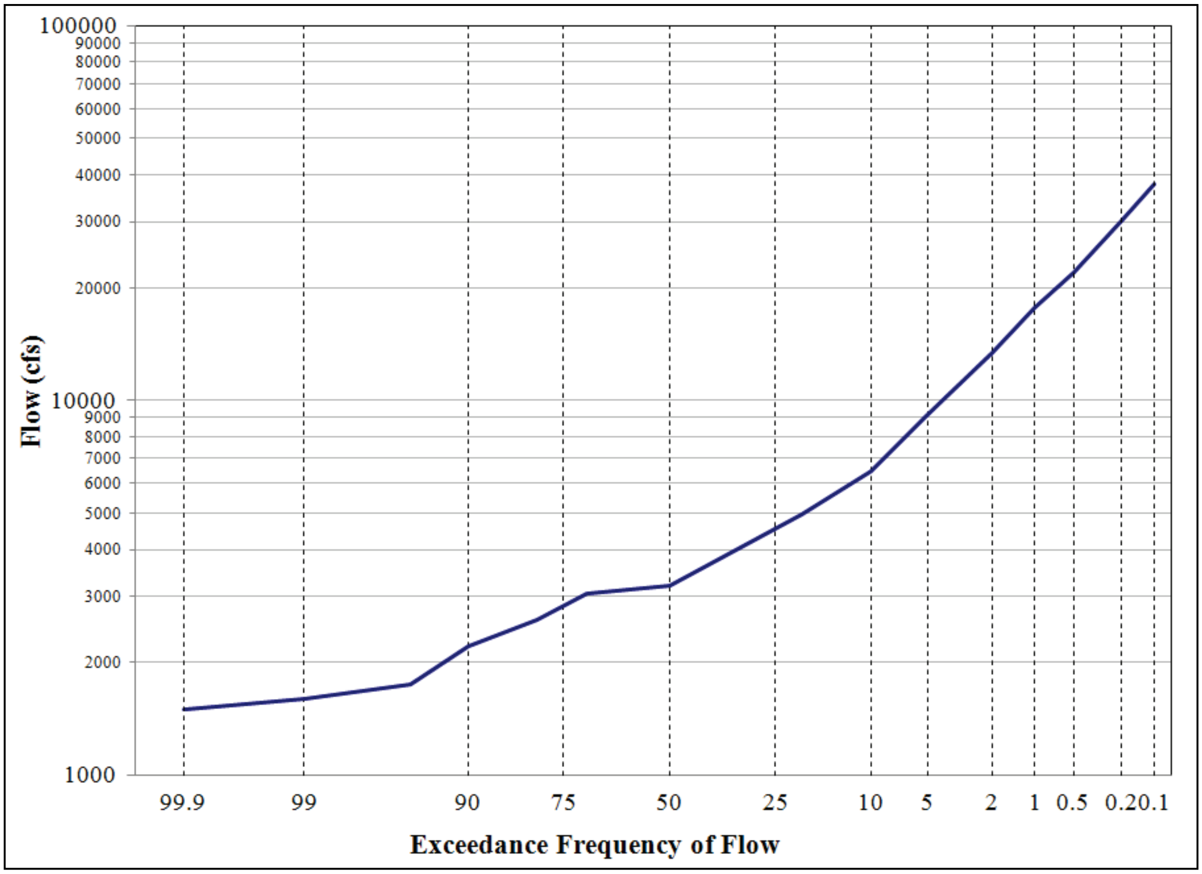

Figure 2.9 Example 2 Flow Frequency Curve

Table 2.8 Example 2 Flow Frequency Curve

Flow (cfs)	**Flow (Log Transform)**	**Exceedance Probability**
1,500	3.1761	0.999
1,600	3.2041	0.990
1,750	3.2430	0.950
2,200	3.3424	0.900
2,600	3.4150	0.800
3,050	3.4843	0.700
3,200	3.5051	0.500
4,950	3.6946	0.200
6,500	3.8129	0.100
9,200	3.9638	0.050
13,400	4.1271	0.020
17,600	4.2455	0.010
21,900	4.3404	0.005
30,200	4.4800	0.002
37,700	4.5763	0.001

Frequency Curve Integral Moments (Log Transform): mean = 3.564, std dev = 0.206

Table 2.9 Synthetic Sample of Size ERL=120 from Flow Frequency Curve

Event Rank	Flow (Log Transform)	Exceedance Probability		Event Rank	Flow (Log Transform)	Exceedance Probability
1	3.2015	0.9917		100	3.7211	0.1736
2	3.2153	0.9835		101	3.7299	0.1653
3	3.2248	0.9752		102	3.7390	0.1570
4	3.2320	0.9669		103	3.7484	0.1488
5	3.2379	0.9587		104	3.7582	0.1405
6	3.2428	0.9504		105	3.7684	0.1322
7	3.2627	0.9421		106	3.7790	0.1240
8	3.2812	0.9339		107	3.7901	0.1157
9	3.2980	0.9256		108	3.8018	0.1074
10	3.3134	0.9174		109	3.8149	0.0992
11	3.3278	0.9091		110	3.8352	0.0909
12	3.3411	0.9008		111	3.8569	0.0826
13	3.3492	0.8926		112	3.8803	0.0744
14	3.3564	0.8843		113	3.9059	0.0661
15	3.3632	0.8760		114	3.9340	0.0579
16	3.3697	0.8678		115	3.9654	0.0496
17	3.3760	0.8595		116	4.0000	0.0413
18	3.3820	0.8512		117	4.0408	0.0331
19	3.3877	0.8430		118	4.0911	0.0248
20	3.3933	0.8347		119	4.1608	0.0165
~	~	~		120	4.2724	0.0083

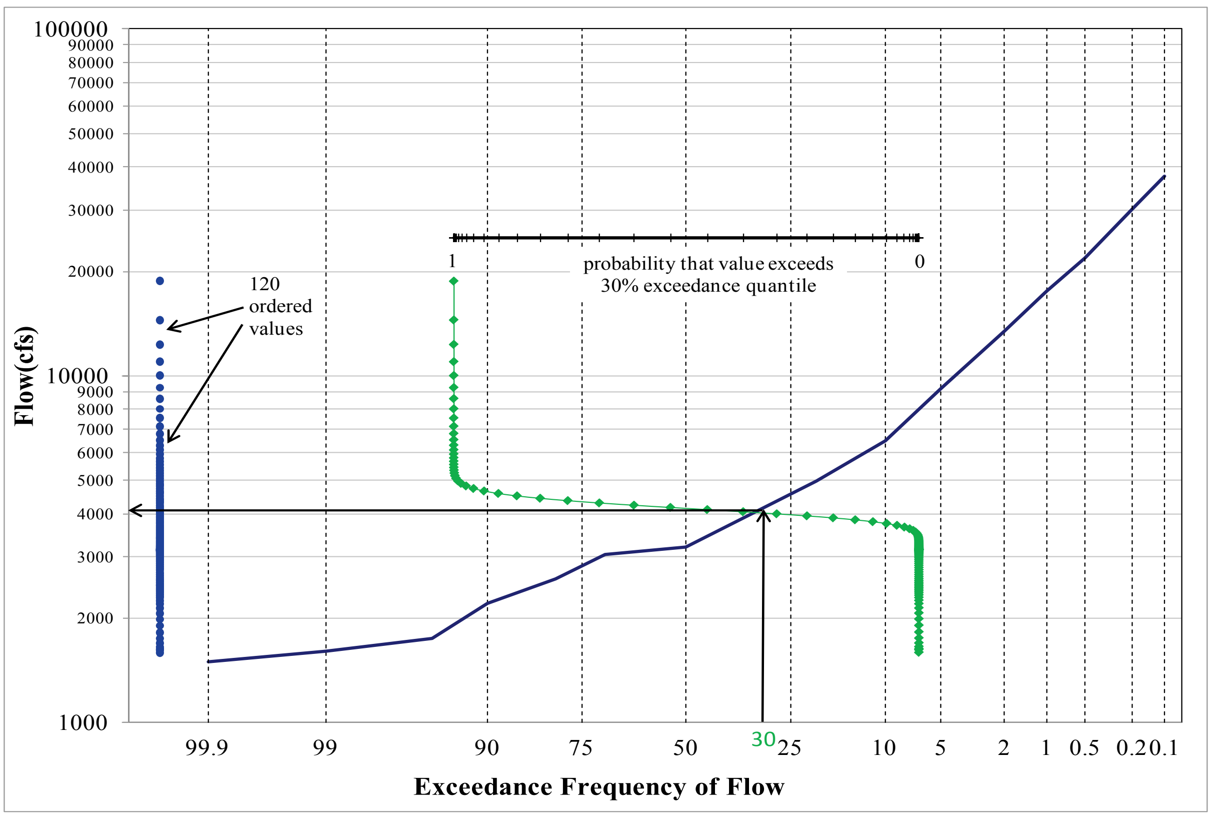

Figure 2.10 Computed Uncertainty Distribution CDF around 30% Exceedance Quantile for Example 2

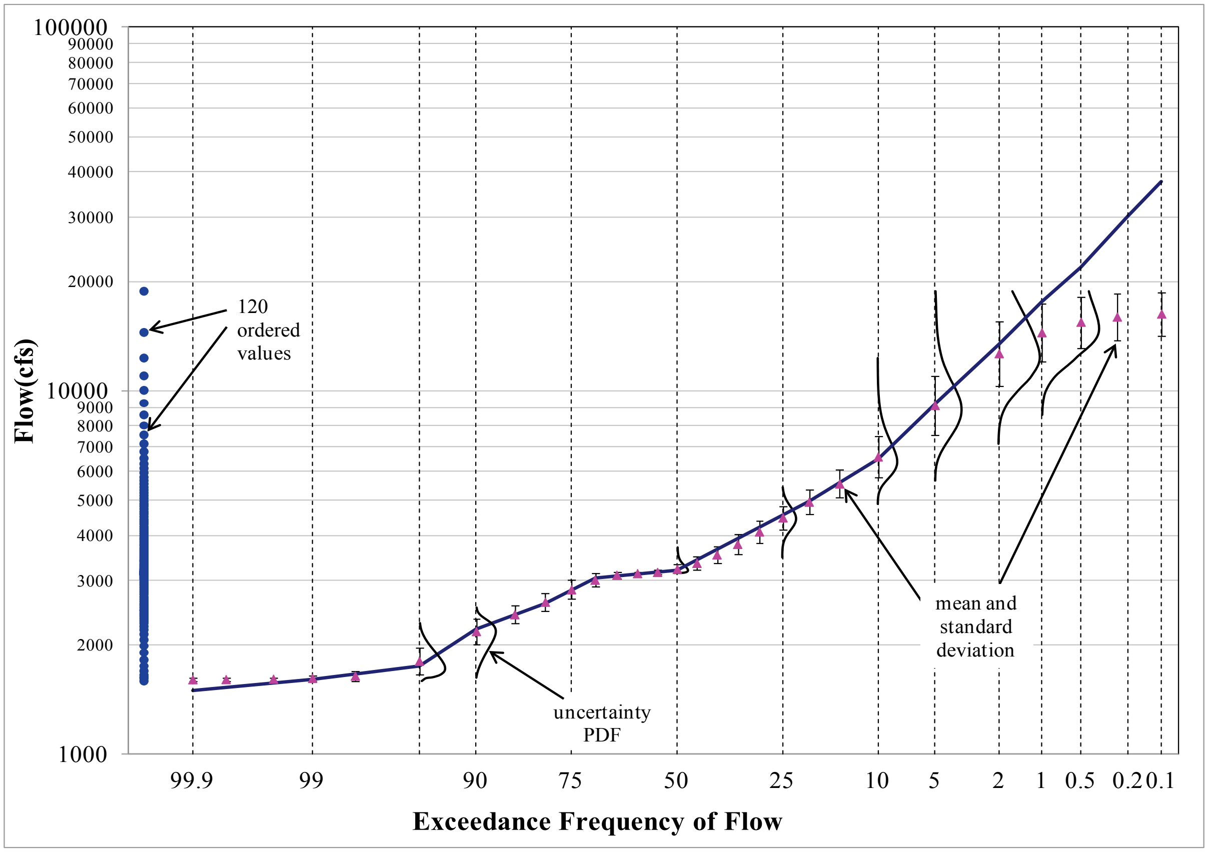

Figure 2.11 Mean and Standard Deviation of Uncertainty Distributions at Quantiles of Interest for Example 2

$$n_l = \frac{0.206^2}{0.0159^2}\left(1 + \frac{\left(\frac{3.2250 - 3.564}{0.206}\right)^2}{2}\right) = 395.14$$

and using S_u = 0.0894 and Y_u = 4.0896 in Equation 17 gives:

$$n_u = \frac{0.206^2}{0.0894^2}\left(1 + \frac{\left(\frac{4.0896 - 3.564}{0.206}\right)^2}{2}\right) = 22.59$$

These n-values are quite different from the actual record length of 120 years, and also quite different from each other. This difference occurs because the lower end of the frequency curve is quite flat, and so this method suggests a small uncertainty. The upper end of the curve is quite steep, suggesting a larger uncertainty. A straight-forward application of Equation 10 with n_l and n_u to compute standard deviation was used to extrapolate the uncertainty. Results are displayed in Table 2.11. The resulting 90% confidence interval (at plus/minus1.645 standard deviations for the Normal distribution) and 95% confidence interval (at plus/minus 2 standard deviations) associated with the uncertainty calculation are shown in Figure 2.12 and displayed in Table 2.12 for the entire frequency curve.

Table 2.10. Mean and Standard Deviations of Uncertainty Distributions Computed via Order Statistics for Example 2

Exceedance Probability	Non-Exceedance Probability	Quantile	Mean	Standard Deviation	Percent of PDF = $Prob(Y_1 \le Y_p \le Y_n)$
0.9990	0.0010	3.1761	3.2091	0.0073	11.313
0.9980	0.0020	3.1839	3.2098	0.0078	21.356
0.9950	0.0050	3.1950	3.2120	0.0089	45.201
0.9900	0.0100	3.2041	3.2158	0.0105	70.062
0.9750	**0.0250**	**3.2250**	**3.2288**	**0.0159**	**95.208**
0.9500	0.0500	3.2430	3.2606	0.0325	99.788
0.9000	0.1000	3.3424	3.3366	0.0349	100.000
0.8500	0.1500	3.3828	3.3831	0.0249	100.000
0.8000	0.2000	3.4150	3.4180	0.0251	100.000
0.7500	0.2500	3.4515	3.4509	0.0246	100.000
0.7000	0.3000	3.4843	3.4758	0.0171	100.000
0.6500	0.3500	3.4898	3.4886	0.0084	100.000
0.6000	0.4000	3.4951	3.4951	0.0054	100.000
0.5500	0.4500	3.5002	3.5016	0.0085	100.000
0.5000	0.5000	3.5051	3.5136	0.0165	100.000
0.4500	0.5500	3.5334	3.5349	0.0234	100.000
0.4000	0.6000	3.5622	3.5622	0.0259	100.000
0.3500	0.6500	3.5919	3.5917	0.0265	100.000
0.3000	0.7000	3.6232	3.6230	0.0272	100.000
0.2500	0.7500	3.6570	3.6570	0.0287	100.000
0.2000	0.8000	3.6946	3.6965	0.0327	100.000
0.1500	0.8500	3.7470	3.7471	0.0391	100.000
0.1000	0.9000	3.8129	3.8211	0.0559	100.000
0.0500	0.9500	3.9638	3.9605	0.0812	99.788
0.0250	**0.9750**	**4.0896**	**4.0751**	**0.0894**	**95.208**
0.0100	0.9900	4.2455	4.1600	0.0792	70.062
0.0050	0.9950	4.3404	4.1888	0.0698	45.201
0.0025	0.9975	4.4472	4.2029	0.0634	25.946
0.0010	0.9990	4.5763	4.2112	0.0590	11.313

The interesting aspect of this example is that the uncertainty about the frequency curve is appropriately quite small as the frequency curve flattens near the 50% chance exceedance probability event. This is indicated by the small standard deviation and the convergence of the 90% confidence limits in Figure 2.12. The example nicely demonstrates the capability of the order statistics method to sense and respond to the local changes in the slope of the frequency curve (i.e., the potential variability of flow values.)

2.2.6 Example 3, Use of Asymptotic Approximations in Place of Order Statistics Method

As an alternative to the order statistics approach to computing uncertainty around a graphical frequency curve, either of the two equations used to extrapolate the uncertainty beyond the useful range of the order statistics method can instead be used to estimate the uncertainty for the entire curve. Equation 6 approximates the standard deviation of the order statistics uncertainty

Table 2.11 Standard Deviations of Uncertainty Distributions from Order Statistics and Equation 10

Exceedance Probability	Non-exceedance Probability	Quantile	Standard Deviation (Order Statistics)	Standard Deviation (Equation 10)	Final Standard Deviation
0.9990	0.0010	3.1761	0.0073	0.0173	0.0173
0.9980	0.0020	3.1839	0.0078	0.0170	0.0170
0.9950	0.0050	3.1950	0.0089	0.0167	0.0167
0.9900	0.0100	3.2041	0.0105	0.0165	0.0165
0.9750	**0.0250**	**3.2250**	**0.0159**	**0.0159**	**0.0159**
0.9500	0.0500	3.2430	0.0325	-	0.0325
0.9000	0.1000	3.3424	0.0349	-	0.0349
0.8500	0.1500	3.3828	0.0249	-	0.0249
0.8000	0.2000	3.4150	0.0251	-	0.0251
0.7500	0.2500	3.4515	0.0246	-	0.0246
0.7000	0.3000	3.4843	0.0171	-	0.0171
0.6500	0.3500	3.4898	0.0084	-	0.0084
0.6000	0.4000	3.4951	0.0054	-	0.0054
0.5500	0.4500	3.5002	0.0085	-	0.0085
0.5000	0.5000	3.5051	0.0165	-	0.0165
0.4500	0.5500	3.5334	0.0234	-	0.0234
0.4000	0.6000	3.5622	0.0259	-	0.0259
0.3500	0.6500	3.5919	0.0265	-	0.0265
0.3000	0.7000	3.6232	0.0272	-	0.0272
0.2500	0.7500	3.6570	0.0287	-	0.0287
0.2000	0.8000	3.6946	0.0327	-	0.0327
0.1500	0.8500	3.7470	0.0391	-	0.0391
0.1000	0.9000	3.8129	0.0559	-	0.0559
0.0500	0.9500	3.9638	0.0812	-	0.0812
0.0250	**0.9750**	**4.0896**	**0.0894**	**0.0894**	**0.0894**
0.0100	0.9900	4.2455	0.0792	0.1103	0.1103
0.0050	0.9950	4.3404	0.0698	0.1234	0.1234
0.0025	0.9975	4.4472	0.0634	0.1384	0.1384
0.0010	0.9990	4.5763	0.0590	0.1567	0.1567

quite closely, and Equation 10 computes the standard deviation similar to a log Normal distribution. Both Equations 6 and 10 were used to compute standard deviation for the stage-frequency curve used in Example 1. (Table 2.3 presents the Example 1 frequency curve). Equation 6 requires an estimate of *f(x)* at each point on the frequency curve, computed as the inverse of the slope of the frequency curve (with probability on the horizontal axis). Equation 10 requires only the mean and standard deviation of the frequency curve. Table 2.3 shows that a mean of 16.98 feet (5.175 meters) and a standard deviation of 5.60 feet (1.707 meters) were computed for the frequency curve.

Twenty years of equivalent record length were used when applying first Equation 6 as an estimate of the standard deviation of uncertainty, and then Equation 10 as the estimate of that standard deviation. Results are presented in Table 2.13. The resulting 90% confidence interval at ±1.645 standard deviations, resulting from the generated uncertainty distributions, is shown in Figure 2.13 and presented in Table 2.14 for the entire frequency curve. The resulting 95%

Table 2.12 Example 2 Flow Frequency Curve with Confidence Limits

Exceedance Probability	Flow (cfs)	90% Confidence Interval + 1.645 Std Error	90% Confidence Interval - 1.645 Std Error	95% Confidence Interval + 2 Std Error	95% Confidence Interval - 2 Std Error
0.999	1,500	1,601	1,405	1,624	1,385
0.990	1,600	1,703	1,503	1,726	1,483
0.950	1,750	1,979	1,547	2,033	1,507
0.900	2,200	2,511	1,928	2,584	1,873
0.800	2,600	2,859	2,365	2,918	2,317
0.700	3,050	3,254	2,859	3,300	2,819
0.500	3,200	3,406	3,006	3,452	2,966
0.200	4,950	5,602	4,374	5,754	4,259
0.100	6,500	8,033	5,260	8,408	5,025
0.050	9,200	12,513	6,764	13,372	6,330
0.020	13,452	19,672	9,486	21,290	8,771
0.010	17,600	26,723	11,591	29,243	10,592
0.005	21,900	34,946	13,724	38,654	12,408
0.002	30,153	52,558	17,645	59,142	15,688
0.001	37,700	68,261	20,822	77,591	18,318

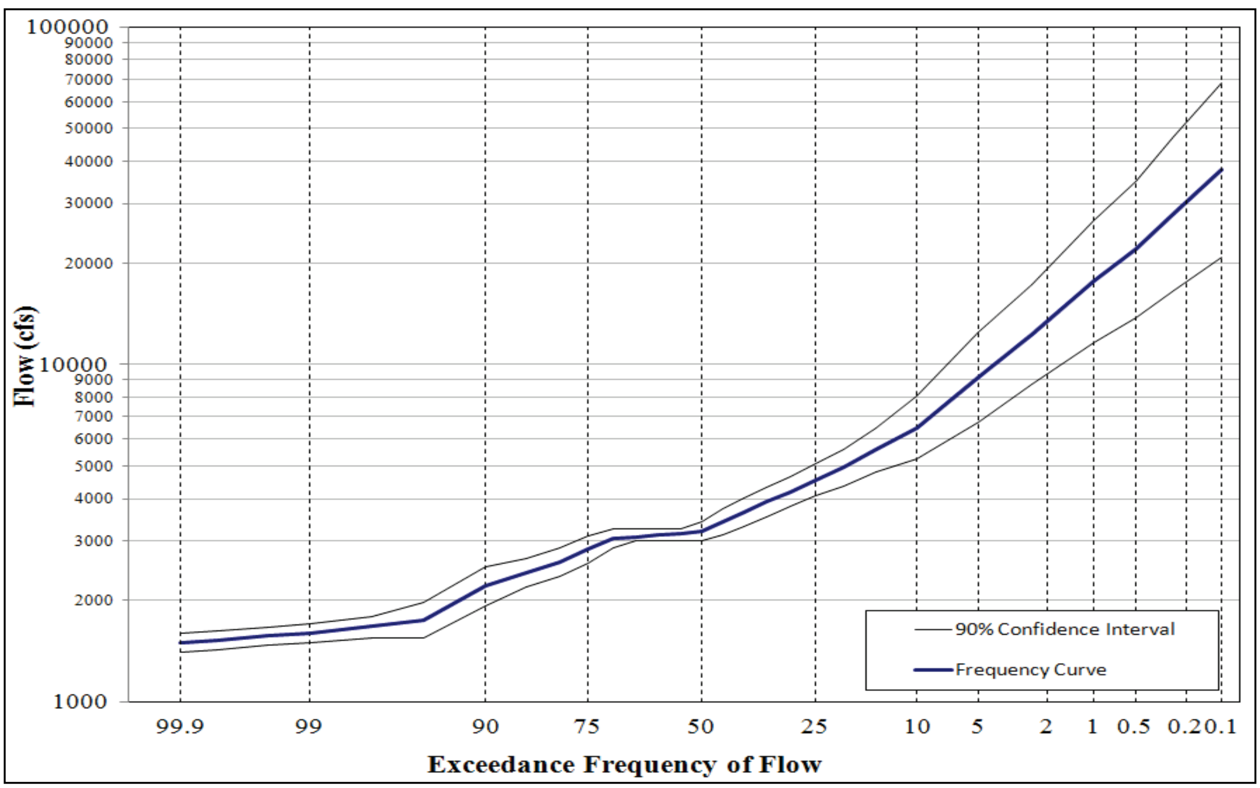

Figure 2.12 Example 2 Flow Frequency Curve with Confidence Limits

Table 2.13. Standard Deviations of Uncertainty Distributions Computed via Equation 6 and Equation 10 for Example 3

Exceedance Probability	Non-exceedance Probability	Quantile	f(y) Inverse Slope of F(y)	Standard Deviation (Equation 6)	Standard Deviation (Equation 10)
0.9900	0.0100	6.60	0.070	0.320	2.059
0.9500	0.0500	7.40	0.047	1.046	1.960
0.9000	0.1000	8.55	0.039	1.705	1.825
0.8500	0.1500	9.95	0.034	2.353	1.671
0.8000	0.2000	11.50	0.037	2.446	1.521
0.7500	0.2500	12.70	0.043	2.275	1.422
0.7000	0.3000	13.85	0.050	2.034	1.346
0.6500	0.3500	14.70	0.052	2.068	1.303
0.6000	0.4000	15.80	0.050	2.184	1.267
0.5500	0.4500	16.70	0.059	1.889	1.254
0.5000	0.5000	17.50	0.065	1.732	1.256
0.4500	0.5500	18.25	0.067	1.669	1.269
0.4000	0.6000	19.00	0.069	1.588	1.293
0.3500	0.6500	19.70	0.077	1.384	1.324
0.3000	0.7000	20.30	0.072	1.425	1.358
0.2500	0.7500	21.10	0.061	1.597	1.411
0.2000	0.8000	21.95	0.053	1.693	1.477
0.1500	0.8500	23.00	0.045	1.794	1.571
0.1000	0.9000	24.20	0.037	1.804	1.692
0.0500	0.9500	25.70	0.024	2.045	1.859
0.0200	0.9800	27.40	0.013	2.480	2.065
0.0100	0.9900	28.40	0.013	1.658	2.192
0.0050	0.9950	29.10	0.015	1.081	2.283
0.0025	0.9975	29.40	0.015	0.732	2.323

confidence interval at ±2 standard deviations is shown in Table 2.15. An additional step is often needed to ensure that the confidence intervals do not decrease at the tails of the frequency curve.

Figure 2.14 depicts the computed standard deviations for the order statistics method alone, order statistics with Equations 6 and 10 each used to extrapolate results beyond the usable range of the order statistics method, and also Equation 6 used alone and Equation 10 used alone to estimate standard deviations. Note that the two approaches that recognize the slope of the frequency curve (order statistics, order statistics extrapolated with Equation 6 and Equation 6 alone) decrease toward the extreme quantiles, and the approaches that does not recognize slope (order statistics extrapolated with Equation 10, and Equation 10 alone) increase toward the extreme quantiles.

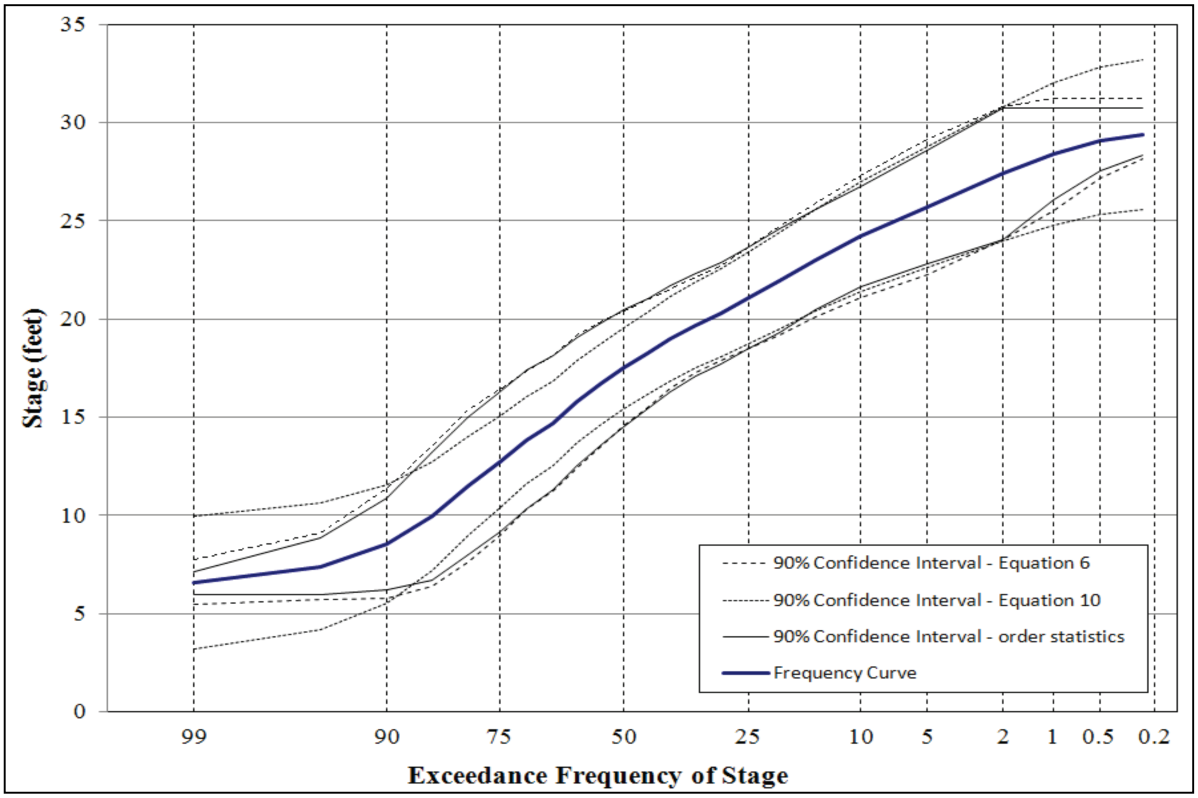

Figure 2.13 Example 3 Stage Frequency Curve with 90% Confidence Limits Computed at ±1.645 Standard Deviations via Equations 6 and 10

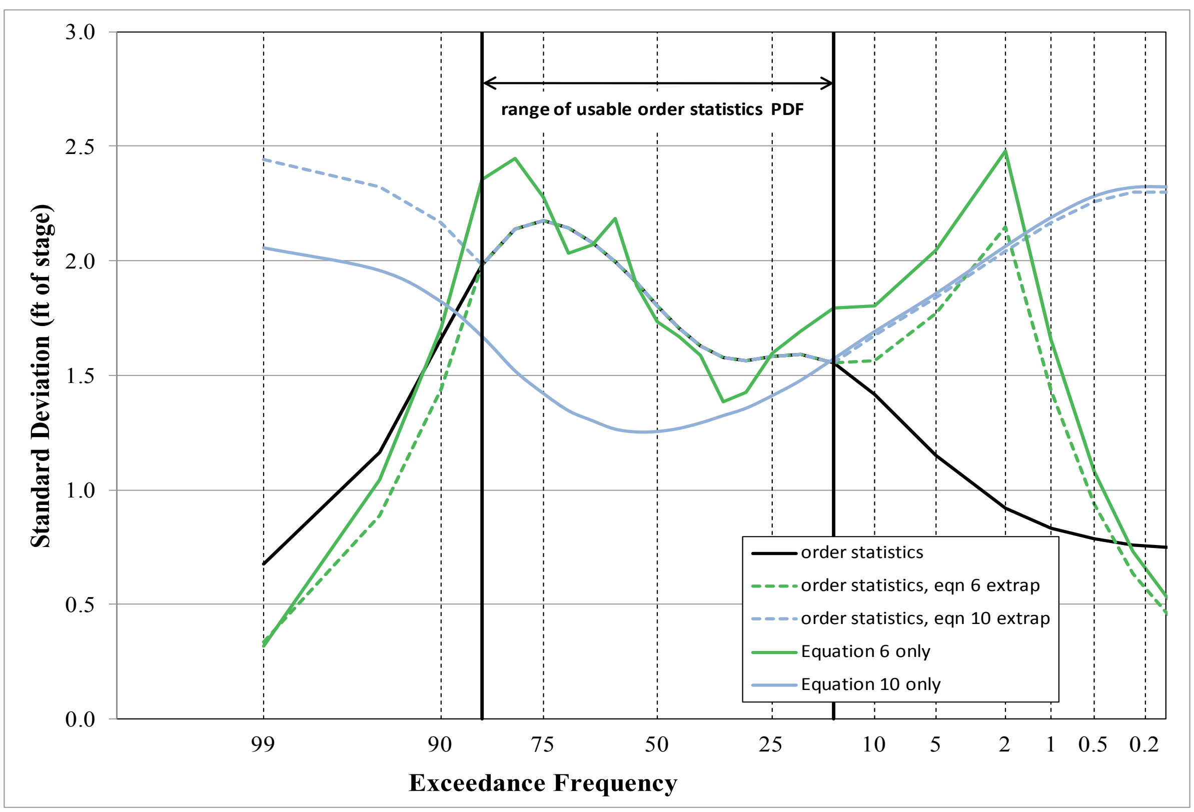

Figure 2.14 Standard Deviation of Uncertainty Distribution from Order Statistics Methods Using Equations 6 and 10 for Extrapolation, and using Equations 6 and 10 Alone

Table 2.14 Example 3 Stage-Frequency Curve (from Example 1) with 90% Confidence Intervals Computed via Equation 6 and Equation 10

		Computed via Equation 6		Computed via Equation 10	
Exceedance Probability	**Stage** (feet)	**90% Confidence Interval + 1.645 Std Error**	**90% Confidence Interval - 1.645 Std Error**	**90% Confidence Interval + 1.645 Std Error**	**90% Confidence Interval - 1.645 Std Error**
0.9900	6.60	7.72	5.48	9.99	3.21
0.9500	7.40	9.08	5.72	10.62	4.18
0.9000	8.55	11.35	5.75	11.55	5.55
0.8500	9.95	13.52	6.38	12.70	7.20
0.8000	11.50	15.38	7.62	14.00	9.00
0.7500	12.70	16.40	9.00	15.04	10.36
0.7000	13.85	17.35	10.35	16.06	11.64
0.6500	14.70	18.15	11.25	16.84	12.56
0.6000	15.80	19.17	12.43	17.88	13.72
0.5500	16.70	19.88	13.52	18.76	14.64
0.5000	17.50	20.40	14.60	19.57	15.43
0.4500	18.25	20.99	15.51	20.34	16.16
0.4000	19.00	21.54	16.46	21.13	16.87
0.3500	19.70	22.11	17.29	21.88	17.52
0.3000	20.30	22.72	17.88	22.53	18.07
0.2500	21.10	23.69	18.51	23.42	18.78
0.2000	21.95	24.74	19.16	24.38	19.52
0.1500	23.00	25.90	20.10	25.58	20.42
0.1000	24.20	27.29	21.11	26.98	21.42
0.0500	25.70	29.17	22.23	28.76	22.64
0.0200	27.40	30.79	24.01	30.80	24.00
0.0100	28.40	31.26	25.54	32.01	24.79
0.0050	29.10	31.26	27.20	32.86	25.34
0.0025	29.40	31.26	28.16	33.22	25.58

Table 2.15 Example 3 Stage-Frequency Curve (From Example 1) with 95% Confidence Intervals Computed via Equation 6 and Equation 10

		Computed via Equation 6		Computed via Equation 10	
Exceedance Probability	Stage (feet)	95% Confidence Interval + 2.0 Std Error	95% Confidence Interval - 2.0 Std Error	95% Confidence Interval + 2.0 Std Error	95% Confidence Interval - 2.0 Std Error
0.9900	6.60	7.97	5.15	10.72	2.48
0.9500	7.40	9.45	5.15	11.32	3.48
0.9000	8.55	11.95	5.15	12.20	4.90
0.8500	9.95	14.29	5.61	13.29	6.61
0.8000	11.50	16.22	6.78	14.54	8.46
0.7500	12.70	17.20	8.20	15.54	9.86
0.7000	13.85	18.10	9.60	16.54	11.16
0.6500	14.70	18.89	10.51	17.31	12.09
0.6000	15.80	19.89	11.71	18.33	13.27
0.5500	16.70	20.57	12.83	19.21	14.19
0.5000	17.50	21.03	13.97	20.01	14.99
0.4500	18.25	21.58	14.92	20.79	15.71
0.4000	19.00	22.09	15.91	21.59	16.41
0.3500	19.70	22.63	16.77	22.35	17.05
0.3000	20.30	23.24	17.36	23.02	17.58
0.2500	21.10	24.24	17.96	23.92	18.28
0.2000	21.95	25.34	18.56	24.90	19.00
0.1500	23.00	26.53	19.47	26.14	19.86
0.1000	24.20	27.96	20.44	27.58	20.82
0.0500	25.70	29.92	21.48	29.42	21.98
0.0200	27.40	31.52	23.28	31.53	23.27
0.0100	28.40	31.88	24.92	32.78	24.02
0.0050	29.10	31.88	26.79	33.67	24.53
0.0025	29.40	31.88	27.89	34.05	24.75

CHAPTER 3

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