Practical Sailing Formulas
for Rhumb-Line Tracks on an Oblate Earth

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Abstract Despite the fact that the mathematical basis for rhumb lines on the ellipsoidal Earth is well known, the Earth's oblateness is usually neglected in dead-reckoning sailing calculations. This leads to significant errors in the estimated position of a vessel over one day's sailing. This paper provides simple formulas that include the Earth's oblateness, which can be readily applied to sailing calculations. The accuracy of these formulas is assessed, and they are shown to be good to about 10 m over ranges up to 1000 km.

Key Words sailing formulas, dead reckoning, rhumb lines, loxodromes, ellipsoid

Introduction

The formulas for determining a ship's dead-reckoned position, given an initial known location, the vessel's course and speed, and the time elapsed, are referred to as the sailing formulas. These formulas are based on the assumption that the vessel is sailing a rhumb line, or loxodrome; that is, the vessel's track maintains a constant azimuth. Loxodromes must be distinguished from geodesics—the shortest path between two points across the Earth. Geodesics have important applications in geodesy, electromagnetic propagation (e.g., for LORAN), and air-route optimization, and a great deal of work has gone into accurate algorithms for their computation. Loxodromes have received less attention because they are normally used only over relatively short distances, and their navigational applications do not require extremely high accuracy.

Geodesics and loxodromes share the characteristic that if the Earth were a sphere, relatively simple, closed-form formulas would solve each of the problems. The Earth's true shape approximates an oblate spheroid. Since the Earth's flattening, f, is only about 1/300, the spherical-Earth approximation suffices for many low-precision applications. In the *American Practical Navigator* [1] (hereafter referred to as Bowditch) we find the statement that "for many navigational purposes, the earth is assumed to be a sphere, without intolerable error." The error in geodetic coordinates resulting from neglect of the Earth's oblateness is of order fd, where d is the distance from the initial point. This amounts to several kilometers over a distance of a thousand kilometers (approximately one-day's sailing), which is likely to be much smaller than errors due to effects such as current and wind. Figure 1 shows the growth in this error as a function of distance, from a starting point at latitude 45°S.

Despite the uncertainties in dead-reckoning from unpredictable influences, it seems unwise to add to the problem by ignoring a known effect which can be accurately accounted for. The effect of the Earth's oblateness on rhumb-line sailing calculations has been addressed—although infrequently–in the literature. An accurate formula for longitude, taking into account the Earth's oblateness, was published over a century ago [2]. The equation for longitude (as a function of latitude) is closely related to the problem of Mercator map projection [3]. Five decades ago, a scheme for applying mid-latitude formulas and "meridional parts" tables to rhumb lines on the oblate Earth was derived [4]. Formulas for distances along rhumb-line tracks, which take the Earth's oblateness into account, are available [5], and tables for latitude, longitude, and distance along a rhumb line on the ellipsoidal Earth have been published [6]. It has been suggested [7] that a table of "latitude parts" be used to deal with the oblateness effect on latitude. Other modifications to the sailing formulas have been proposed [8]. On a more fundamental level, a succinct account of the mathematical basis of loxodromic curves has recently been published [9].

Yet, none of these references provides a complete, convenient solution to the problem of sailing formulas for an oblate Earth. Some present a mathematical theory that is not straightforward to apply. Others approach rhumb-line sailing calculations from a very traditional point of view, which includes calculations of mid-latitudes or meridional parts—an obsolete and unnecessary approach given modern computing power. However, the primary deficiency is the lack of a closed-form latitude formula that takes into account the Earth's oblateness.

Therefore, someone who has to perform or program navigational calculations is still confronted with a confusing choice of algorithms (such as plane sailing, mid-latitude sailing, and Mercator sailing) none of which is entirely satisfactory. Some of these issues were discussed almost 40 years ago [10] and have yet to be resolved. As a result, in the treatment of the sailing calculations in Bowditch, the Earth's oblateness is taken into account (to 6th order!) only in Table 5 of meridional parts. Someone who needs accurate sailing formulas could use the formula on which Table 5 is based, but that solves only the longitude half of the problem. That formula (and Table 5) requires pre-computed latitude values, and the latitude formulas in Bowditch have no oblateness terms at all. So the potential accuracy of the longitude calculation is not realized in practice.

In this paper, I address the problem of rhumb-line sailing formulas from the beginning. The next section provides some fundamental relations from geodesy and casts the problem as the solution to a pair of differential equations. Then I solve these equations for the case of the spherical Earth. Up to this point the developments are conventional, and well-known exact solutions are obtained. In the fourth section of the paper I solve the differential equations for the more difficult case of an oblate Earth and obtain two new formulas that directly yield, respectively, the latitude and longitude of a vessel as a function of time. I also describe how to assess the accuracy of these, or other, sailing formulas, and I make recommendations on the best formulas to use. Although the paper's primary original contribution is a closed-form sailing formula for latitude that includes the Earth's oblateness, I have attempted a straightforward presentation of the entire problem that is intended to serve those who must perform real-world navigational calculations.

Throughout the development, the accuracy sought is such that, over a day's sailing, the error in computed position will not exceed the dimensions of a typical vessel or the uncertainty of a typical GPS fix. Expressed quantitatively, this amounts to an accuracy of a few tens of meters (about an arcsecond) over a distance of about 1000 km (= 640 nmi = 22.5 kn × 24 h). That is a relative accuracy of a few $\times 10^{-5}$, so the Earth's oblateness must be accounted for to about 1%. This is a fairly modest requirement and very elementary mathematics can be applied to it.

Statement of the Problem

Over a small area on the surface of the Earth, the following relations hold for a track of infinitesimal length dl, expressed in kilometers, which is at latitude ϕ with azimuth C (the latter measured eastward from north):

$$\begin{aligned} d\phi &= \frac{\cos C}{M} dl \\ d\lambda &= \frac{\sin C}{N \cos\phi} dl \end{aligned} \tag{1}$$

Here, $d\phi$ and $d\lambda$ are infinitesimal differences of geodetic latitude, ϕ, and longitude, λ, respectively, in radians, corresponding to dl. The quantities M and N are the radii of curvature of the Earth's surface in the meridian and the prime vertical, respectively, in kilometers. (The prime vertical is a plane containing the normal to the Earth's surface at the location of interest, perpendicular to the local meridian.) The above equations, along with discussions of radii of curvature, can be found in most texts on geodesy (e.g., [11]). The curvatures M and N are computed from the Earth's equatorial radius, a, in kilometers, and eccentricity of the Earth's ellipsoid, e:

$$\begin{aligned} M &= \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} \\ N &= \frac{a}{(1-e^2\sin^2\phi)^{1/2}} \end{aligned} \tag{2}$$

Here, e is related to the Earth's flattening, f, by $e^2 = 2f - f^2$ (in geodesy texts, e is referred to as the "first eccentricity"). The WGS-84 values are $a = 6378.137$ km and $f = 1/298.257223563$.

Equations (1) can be used to describe a vessel's motion along a rhumb line if we set $dl = Sdt$, where S is the vessel's speed and dt is an infinitesimal increment of time, t. For rhumb lines, the azimuth C is a constant. Equations (1) can then be re-cast as:

$$
\begin{aligned}
\frac{d\phi}{dt} &= S \cdot \frac{\cos C}{M(\phi(t))} \\
\frac{d\lambda}{dt} &= S \cdot \frac{\sin C}{N(\phi(t)) \cos \phi(t)}
\end{aligned}
\tag{3}
$$

where the dependence of M and N on latitude, and the dependence of latitude on time are indicated. Time, t, is expressed in hours, and the vessel's speed, S, is expressed in kilometers per hour (1 kn = 1.852 km/h exactly, using the International Nautical Mile). The angle C now represents the vessel's constant course. I have used metric units for distance and speed to avoid confusion between nautical miles and arcminutes; on an oblate Earth, nautical miles and arcminutes do not, in general, correspond.

Equations (3) are a simple example of a system of ordinary differential equations. The problem at hand is to find their solution, which will yield the latitude and longitude of the vessel as a function of time. The only boundary condition is that at $t = 0$, the vessel is at known position (ϕ_0, λ_0). The equations are coupled, but, fortunately, the coupling is only one-way: although ϕ appears in the equation for λ, λ does not appear in the equation for ϕ. This makes the solution of these equations much more tractable, since we can address the equation for ϕ first, separately, and can then apply its solution to the solution of the equation for λ.

The solutions to the differential equations (3) can be represented as integrals:

$$
\begin{aligned}
\phi &= \phi_0 + S \cos C \int \frac{dt}{M(\phi(t))} \\
\lambda &= \lambda_0 + S \sin C \int \frac{dt}{N(\phi(t)) \cos \phi(t)}
\end{aligned}
\tag{4}
$$

where the limits of integration in both cases run from 0 to t, and ϕ_0 and λ_0 represent the vessel's initial (t=0) latitude and longitude. (Formally, the limits of integration run from t=0 to t=T, where T represents the elapsed time in hours from t=0; but since there is no real distinction between t and T, we can replace T with t after the integral is evaluated.) The remainder of this paper is devoted to the evaluation of these two equations.

Special Solution—Spherical Earth

Before considering the general solution of equations (3), let us take up the special case of a spherical Earth. For a spherical Earth, the flattening, f, and eccentricity, e, are zero and $M = N = a$, a constant. Equations (3) then reduce to:

$$
\begin{aligned}
\frac{d\phi}{dt} &= \frac{S}{a} \cos C \\
\frac{d\lambda}{dt} &= \frac{S}{a \cos \phi} \sin C
\end{aligned}
\tag{5}
$$

and equations (4) become, for this case:

$$
\begin{aligned}
\phi &= \phi_0 + \frac{S}{a} \cos C \int dt \\
\lambda &= \lambda_0 + \frac{S}{a} \sin C \int \frac{dt}{\cos \phi}
\end{aligned}
\tag{6}
$$

Using the first equation in (5), we can change the integration variable in the longitude equation from dt to $d\phi$:

$$\lambda = \lambda_0 + \frac{\sin C}{\cos C} \int \frac{d\phi}{\cos\phi} = \lambda_0 + \tan C \int \sec\phi \, d\phi \tag{7}$$

where we are integrating over the latitude traversed from time 0 to t. Now both the latitude and longitude integrals can be easily evaluated. The solutions are:

$$\begin{aligned} \phi &= \phi_0 + \frac{S\,t}{a} \cos C \\ \lambda &= \lambda_0 + \tan C \left(\ln\left[\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right] - \ln\left[\tan\left(\frac{\pi}{4} + \frac{\phi_0}{2}\right)\right]\right) \end{aligned} \tag{8}$$

where t is the time elapsed, in hours, since the vessel left its initial position (ϕ_0, λ_0). All angles are in radians. In the equation for λ above, the $\ln[\tan(\pi/4 + \phi/2)]$ terms are the basis for the tables of meridional parts for a spherical Earth.

Obviously, the equation for λ has an indeterminacy for east-west courses. For east-west courses, or courses that are very nearly so, the following equation for λ can be used:

$$\lambda = \lambda_0 + \frac{S\,t}{a\cos\phi_0} \sin C \tag{9}$$

This equation is based on the condition that, for these courses, the latitude ϕ never departs significantly from ϕ_0. When the course is exactly east-west, $|\sin C| = 1, \phi = \phi_0$, and equation (9) is exact.

General Solution—Oblate Earth

In the more general case, equations (3) do not have exact closed-form analytic solutions. Various approximate solutions are, however, possible.

Equations (3) can be solved numerically. A numerical integration can be performed to evaluate the integrals in (4). Obviously the longitude integral must be simultaneously evaluated with the latitude integral, so that a current value for ϕ is available at each step. (The latitude integral could be evaluated by itself.) This is actually a fairly simple procedure. For example, FORTRAN code of about 50 statements will perform a simple predictor-corrector integration on both equations. If the computations are done in double-precision arithmetic, step sizes such that $S\,dt$ is a few kilometers result in numerical errors of a few meters or less after integrations of a thousand kilometers. The numerical error can be tested by simple forward-backward integration tests. Since this procedure provides a nearly exact solution (more sophisticated numerical integrators could be applied to the problem if more accuracy was required) and is readily implemented on even PC-class computers, we could declare the problem solved. However, closed-form analytic formulas are more convenient, and they facilitate applications in which the sensitivity of the formulation to any of its parameters must be evaluated.

Since the Earth's oblateness is small, series expansions in oblateness converge rapidly and allow a point of attack. As a first step, we can expand M^{-1} and N^{-1}:

$$\begin{aligned} M^{-1} &= \frac{1}{a(1-e^2)} \left(1 - \tfrac{3}{2}e^2 \sin^2\phi + \cdots\right) \\ N^{-1} &= \frac{1}{a} \left(1 - \tfrac{1}{2}e^2 \sin^2\phi + \cdots\right) \end{aligned} \tag{10}$$

The omitted high-order terms in the above expansions are not negligible for high-precision geodesy but, as we shall see, the truncated series is adequate for our accuracy requirements.

Now we can substitute the above expression for M^{-1} into the basic equation for latitude, from equations (4):

$$\phi = \phi_0 + \frac{S\cos C}{a(1-e^2)} \int \bigl(1 - \tfrac{3}{2}e^2 \sin^2 \phi(t)\bigr) dt \tag{11}$$

To perform the integration, we need to know ϕ as a function of t which, unfortunately, is the function we are seeking. However, note that $\phi(t)$ contributes only weakly to the integrand, since $e^2 \ll 1$. That means that a reasonably good approximation to $\phi(t)$, denoted $\phi'(t)$, might allow us to proceed (the prime does *not* imply differentiation). One such approximation, similar to the first of equations (8), is:

$$\begin{aligned} \phi'(t) &= \phi_0 + \frac{S\,t\cos C}{M_0} \\ \text{with} \qquad d\phi' &= \frac{S\cos C}{M_0} dt \end{aligned} \tag{12}$$

where $M_0 = M(\phi_0)$ is the radius of curvature of the Earth's surface in the meridian at the initial latitude. Substituting these relations into equation (11),

$$\begin{aligned} \phi &= \phi_0 + \frac{S\cos C}{a(1-e^2)} \frac{M_0}{S\cos C} \int \bigl(1 - \tfrac{3}{2}e^2 \sin^2 \phi'\bigr) d\phi' \\ &= \phi_0 + \frac{M_0}{a(1-e^2)} \int \bigl(1 - \tfrac{3}{2}e^2 \sin^2 \phi'\bigr) d\phi' \\ &= \phi_0 + \frac{M_0}{a(1-e^2)} \left[\phi' - \frac{3}{2}e^2 \left(\frac{\phi'}{2} - \frac{1}{4}\sin 2\phi' \right) \right]_{\phi'(0)}^{\phi'(t)} \end{aligned}$$

Since $\phi'(0) = \phi_0$, the equation for latitude as a function of time becomes

$$\begin{aligned} \phi &= \phi_0 + \frac{M_0}{a(1-e^2)} \left[\bigl(1 - \frac{3}{4}e^2\bigr)(\phi' - \phi_0) + \frac{3}{8}e^2(\sin 2\phi' - \sin 2\phi_0) \right] \\ \text{where} \qquad \phi' &= \phi_0 + \frac{S\,t\cos C}{M_0} \\ M_0 &= \frac{a(1-e^2)}{(1-e^2\sin^2\phi_0)^{3/2}} \end{aligned} \tag{13}$$

All angles are in radians, all distances are in kilometers, and the time interval, t, is in hours. The vessel's speed, S, is expressed in km/h = kn × 1.852.

We can obtain the equation for longitude through a similar development, using the expansion for N^{-1} from equations (10). Again, the integral is straightforward and the result is:

$$\begin{aligned} \lambda = \lambda_0 + \frac{M_0}{a}\tan C \Bigl[&(1-\tfrac{1}{2}e^2) \left(\ln\left[\tan\left(\frac{\pi}{4} + \frac{\phi'}{2}\right)\right] - \ln\left[\tan\left(\frac{\pi}{4} + \frac{\phi_0}{2}\right)\right] \right) \\ &+ \tfrac{1}{2}e^2(\sin\phi' - \sin\phi_0) \Bigr] \end{aligned} \tag{14}$$

where ϕ' and M_0 have the same meanings as in equations (13).

Equations (13) and (14) are new and, as we shall see later, both meet our accuracy requirements. However, there is a better formula for λ. The approach is given in [3] and [6]. If we return to the original equation for λ, equation (4), we can change the variable of integration from dt to $d\phi$ (just as we did in the spherical-Earth case) by using the relation $d\phi = ScosC\, dt/M$ from equations (3). Then we have

$$\begin{aligned} \lambda &= \lambda_0 + \tan C \int \frac{M(\phi)d\phi}{N(\phi)\cos\phi} \\ &= \lambda_0 + \tan C \int \frac{(1-e^2)d\phi}{(1-e^2\sin^2\phi)\cos\phi} \end{aligned} \tag{15}$$

The latter integral occurs in the construction of Mercator projection maps (not surprisingly) and its evaluation is detailed in [12], pp. 113–114:

$$\begin{aligned} \lambda = \lambda_0 + \tan C \Bigg(& \ln\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right] + \frac{e}{2}\ln\left[\frac{1-e\sin\phi}{1+e\sin\phi}\right] \\ & - \ln\left[\tan\left(\frac{\pi}{4}+\frac{\phi_0}{2}\right)\right] - \frac{e}{2}\ln\left[\frac{1-e\sin\phi_0}{1+e\sin\phi_0}\right]\Bigg) \end{aligned} \tag{16}$$

Thus to use equation (16) for longitude, we must first evaluate equation (13) for the latitude, ϕ.

Note that if the Earth were a sphere, e would be zero and $M_0 = a$, so equations (13), (14), and (16) reduce to the spherical-Earth equations (8).

Just as in the spherical-Earth case, the longitude equations (14) or (16) are indeterminate for east-west courses. For those courses, or ones very close to east-west, the following equations can be used:

$$\begin{aligned} \phi &= \phi' \\ \lambda &= \lambda_0 + \frac{S\,t\sin C}{N_0\cos\left(\frac{1}{2}(\phi_0+\phi')\right)} \\ \text{where} \qquad N_0 &= \frac{a}{(1-e^2\sin^2\phi_0)^{1/2}} \end{aligned} \tag{17}$$

and where ϕ' is from (13) (in this case it will not be much different from ϕ_0). Here, $N_0 = N(\phi_0)$ is the radius of curvature of the Earth's surface in the prime vertical at the initial latitude. These formulas can be used for courses C within about a half-degree (0.01 radian) of 90° or 270°, or when the excursion in latitude will not exceed a few tens of kilometers.

Obviously all of the latitude and longitude formulas in this section become indeterminate very close to the poles. However, rhumb lines are not used for polar navigation.

Accuracy of the Formulas

Equation (13) for latitude and equation (16) for longitude were compared with two formulas found in the literature. Bowditch does not contain a formula for latitude adequate for this exercise. A formula from [5], which is transcendental in latitude, was adapted to the purpose by solving for the latitude iteratively. (Neither [6], [7], nor [9] contain closed-form formulas for latitude.) Once accurate latitudes were obtained, the longitude formula from Bowditch for "Mercator sailing" could be used (the formula is given in the explanation to Table 5). An adjustment also had to be made for the fact that both [5] and Bowditch use nautical miles and arcminutes (along the equator) interchangeably, which does not conform to the definition of the International Nautical Mile used in this paper.

Once all that was done, the comparison between the previously-published formulas and equations (13) and (16) from this paper was quite satisfactory. For example, for rhumb lines starting at a latitude of 40°

and extending to 1000 km along a variety of courses, the total difference was less than 6 m (0.2 arcsec). Even at a starting latitude of 60°, differences in position reached only about 10 m (0.3 arcsec) at 1000 km from the starting point. These tests also showed that the differences were almost entirely due to the latitude equation. Since longitude is evaluated as a function of latitude, the difference in latitude is propagated into longitude. The longitude formulas, taken by themselves (that is, with the same latitudes as input), were found to be equivalent to a high degree of accuracy—less than a centimeter difference over the rhumb lines tested.

Another way to evaluate the error in the formulas for latitude and longitude is to compare their results with the results of the numerical integration described at the beginning of the previous section. Figures 2, 3, and 4 show, respectively, the total error in position, as a function of position relative to the starting point, for several formula combinations given in this paper. The total error in position is the length of the vector connecting the position computed from the formulas to the "true position" from the numerical integration. In each figure, the XY-plane represents an area on the Earth's surface, and the error in position at each point in the XY-plane is represented by the Z height of the function surface shown. The starting point (ϕ_0, λ_0) is located at the center of the XY-plane (the error surface always touches it, since the error is zero there). Error surfaces for starting points at latitude 25°N and 50°N are shown for each formula combination. Figure 2 shows the error resulting from the use of equations (13) and (14); Figure 3 shows the error resulting from the use of equations (13) and (16); and Figure 4 shows the error resulting from the use of equations (17). Despite the apparent continuity of the surfaces, courses that were exactly east-west were avoided in the calculations so that the degeneracy near $C = 90°$ or $270°$ in equations (14) and (16) was not encountered.

There are several things to note about these figures. Each error surface is east-west symmetric and is not a function of the longitude of the starting point. The error surfaces for the corresponding southern latitudes are simply north-south reflections of the surfaces shown. The equations (13)-(16) combination is clearly superior to the (13)-(14) combination. Equations (17) are obviously not suitable for general use, but they do have a corridor a few tens of kilometers wide along an east-west line in which they are actually quite good. Thus they are appropriate for the east-west courses which cause the indeterminacy in equations (14) and (16). For the equations (13)-(14) combination, as well as the (13)-(16) combination, along each course, the error increases approximately linearly with distance from the starting point (a contour plot would show this more obviously). The rate of increase of error with distance becomes steeper at higher latitudes.

An algorithm defined by equations (13) and (16) for non-east-west courses and equations (17) for east-west courses is recommended. "East-west courses" are defined for this purpose as those with a latitude excursion of less than 15 km. This algorithm results in errors of 10 m (0.3 arcsec) or less for 1000 km rhumb lines starting within the three-quarters of the Earth's surface that is within 50° of the equator. Even for 1500 km rhumb lines starting at a latitude of 70°, the maximum error is still only about 30 m (1 arcsec). The error surface for this algorithm is shown in Figure 5 for a starting point at latitude 45°S; compare it to Figure 1, where the oblateness of the Earth is neglected entirely, but notice that the Z scale in Figure 5 is only 1/100 that of Figure 1.

Conclusion

The oblateness of the Earth should be accounted for in the sailing formulas for both latitude and longitude. Relatively simple, closed-form formulas have been presented which include the Earth's oblateness and which have good accuracy for navigational applications. Equation (13) for latitude, which is new, and equation (16) for longitude are recommended for computing the dead-reckoned position of a vessel sailing a rhumb line, provided the course is not east-west. For east-west courses, or those very close to east-west, the much simpler equations (17) can be used. Over most of the surface of the Earth, these formulas provide positions to an accuracy of about 10 m (0.3 arcsec) or better for rhumb lines extending to 1000 km; only at high latitudes are the errors larger, and then only by small factors. This accuracy is such that, over a day's sailing, the error in the dead-reckoned position contributed by these formulas will be less than the linear dimensions of a typical vessel or the uncertainty of a typical GPS fix.

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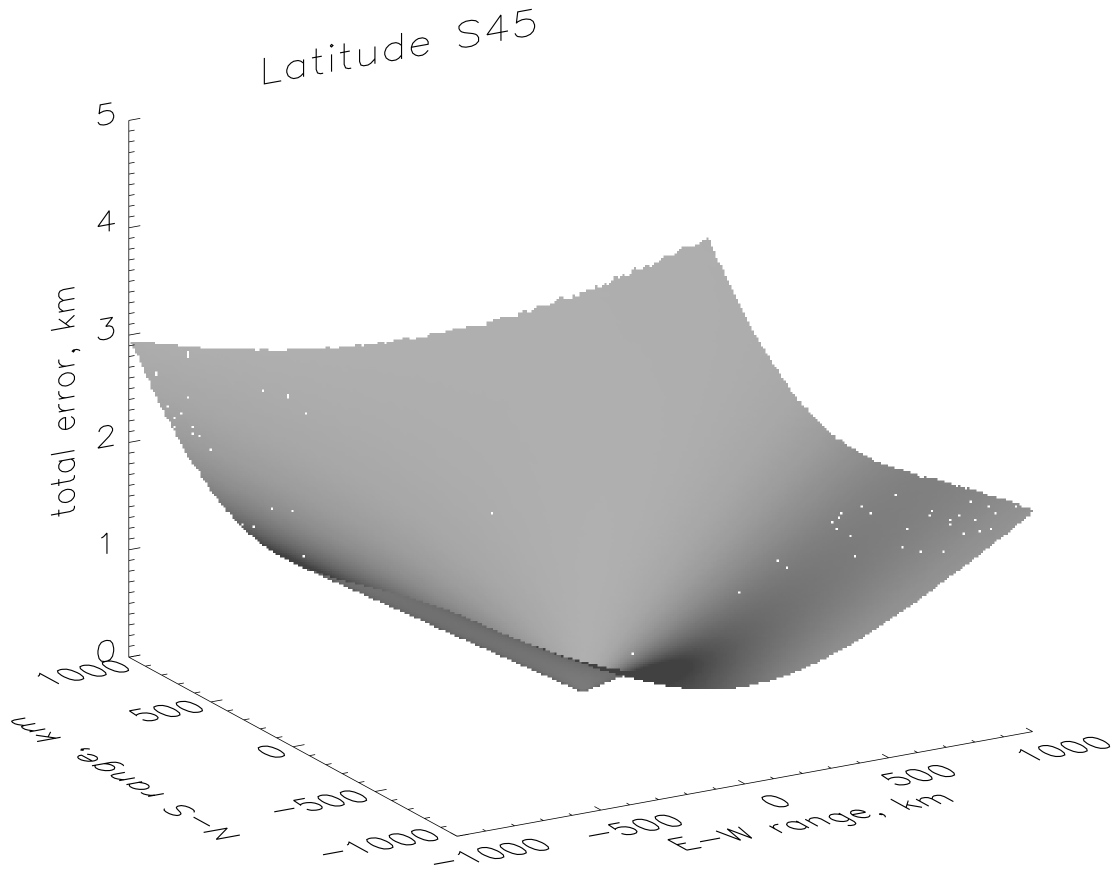

Figure 1 Total error in position as a function of position, resulting from neglect of the Earth's oblateness in the sailing formulas. The starting point for the calculations, 45°S, is at the center of the XY-plane, and for this calculation, the spherical Earth was given a radius equal to the Gaussian mean radius ($=\sqrt{MN}$) at that latitude.

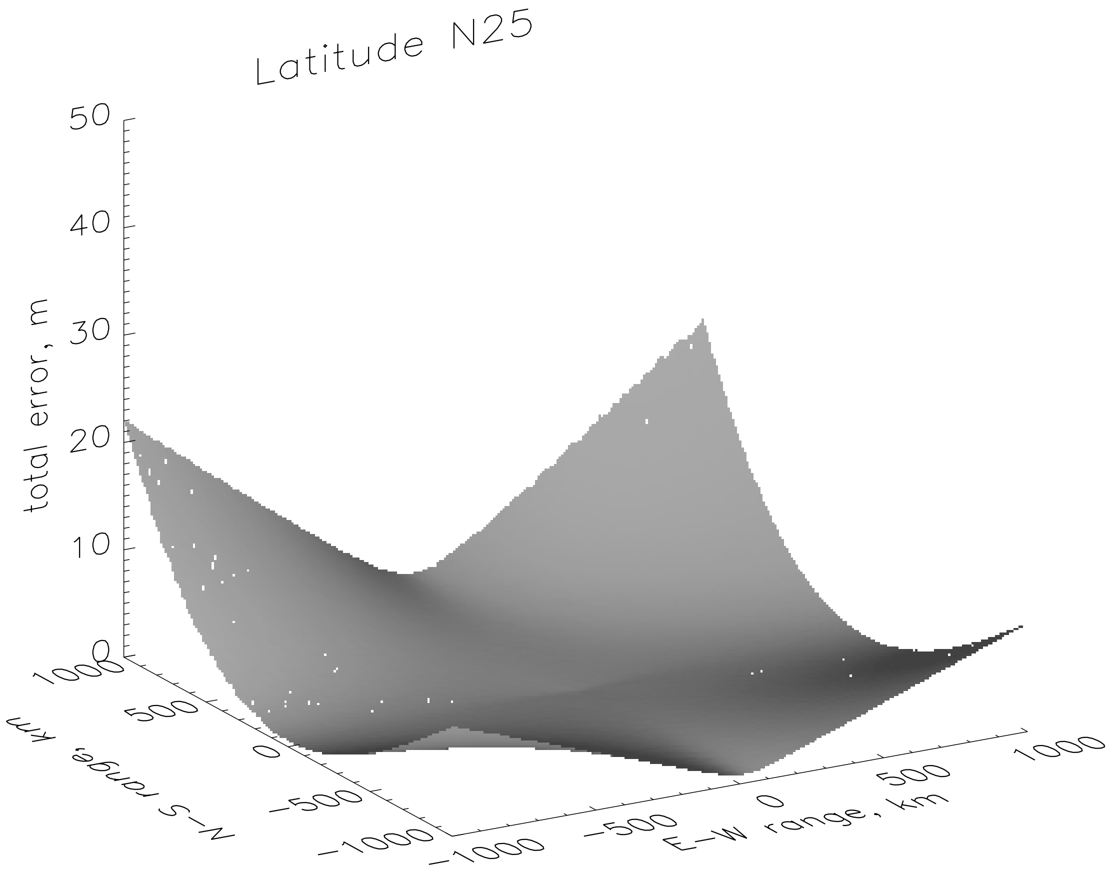

Figure 2(a) Total error in position as a function of position, resulting from the use of equations (13) and (14) as sailing formulas. The starting point for the calculations, 25°N, is at the center of the XY-plane.

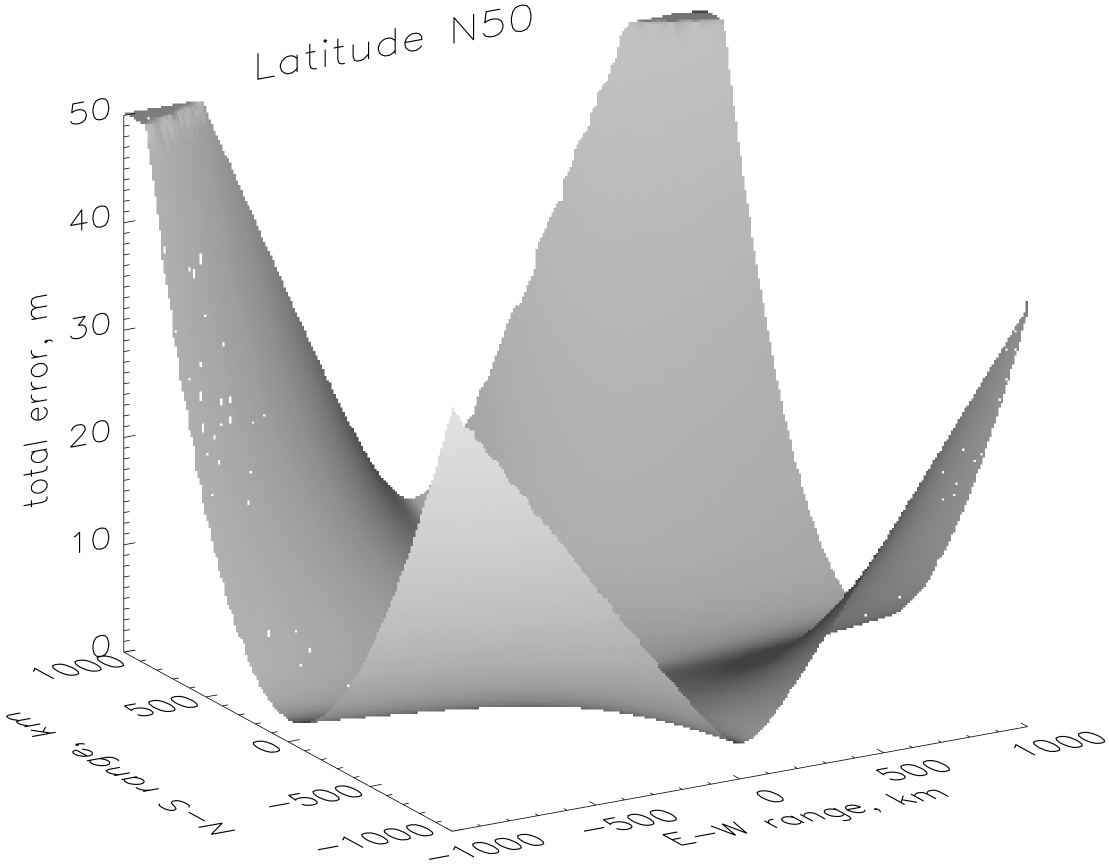

Figure 2(b) Same as Figure 2(a), but with the starting point at latitude 50°N. The error surface has been truncated at the top in the far northeast and northwest corners.

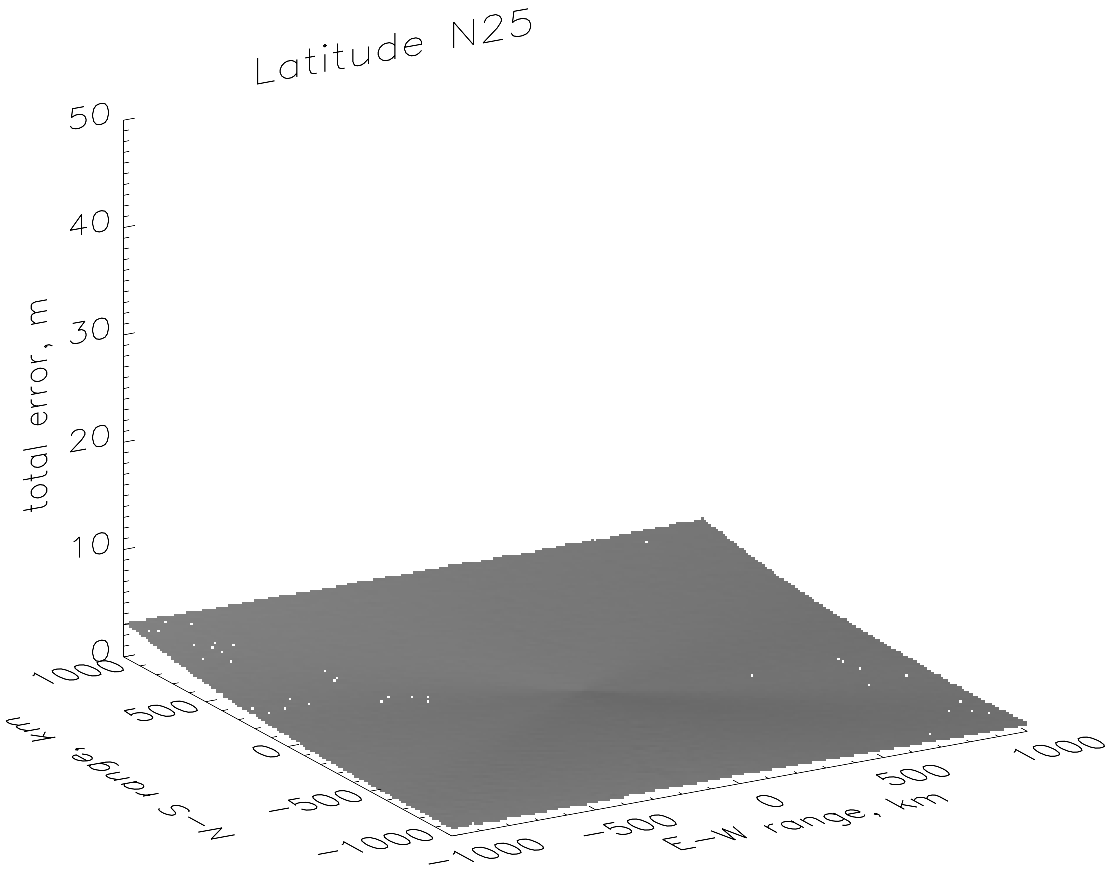

Figure 3(a) Similar to Figure 2(a), but the error shown results from the use of equations (13) and (16). The starting point is at latitude 25°N.

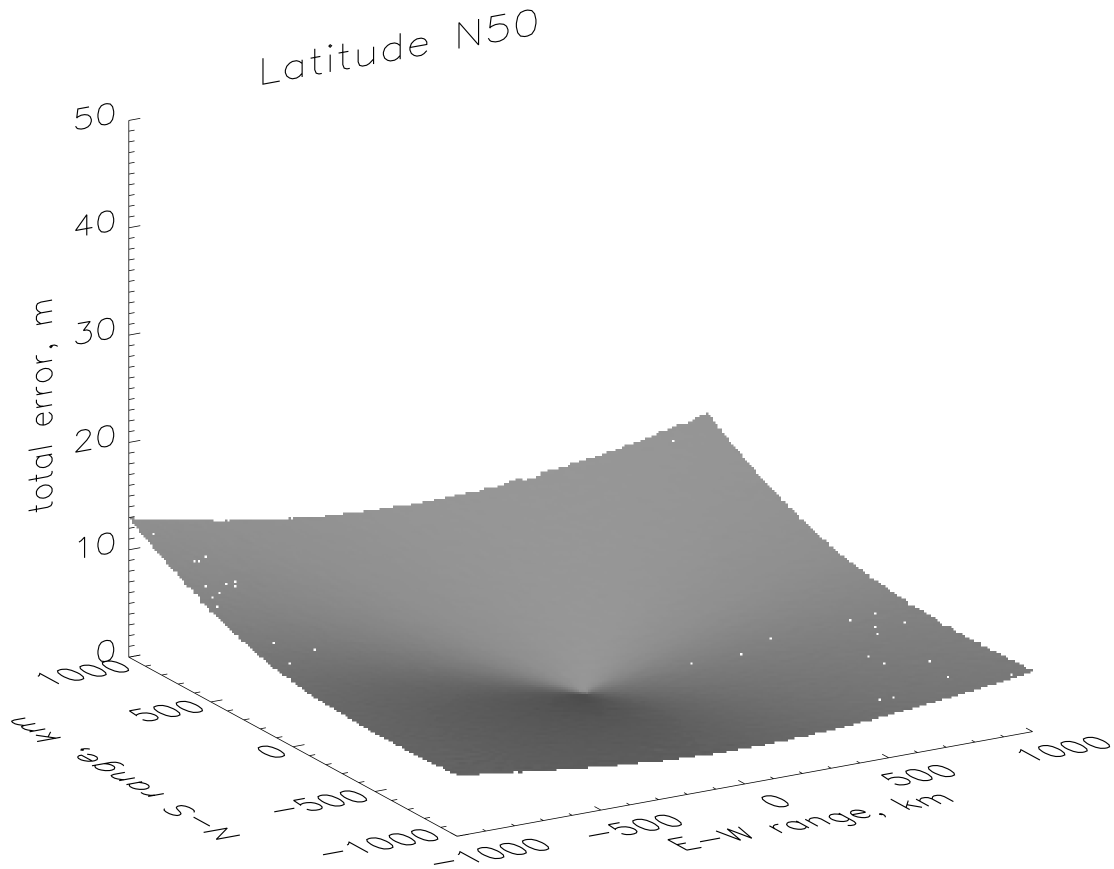

Figure 3(b) Same as Figure 3(a), but with the starting point at latitude 50°N.

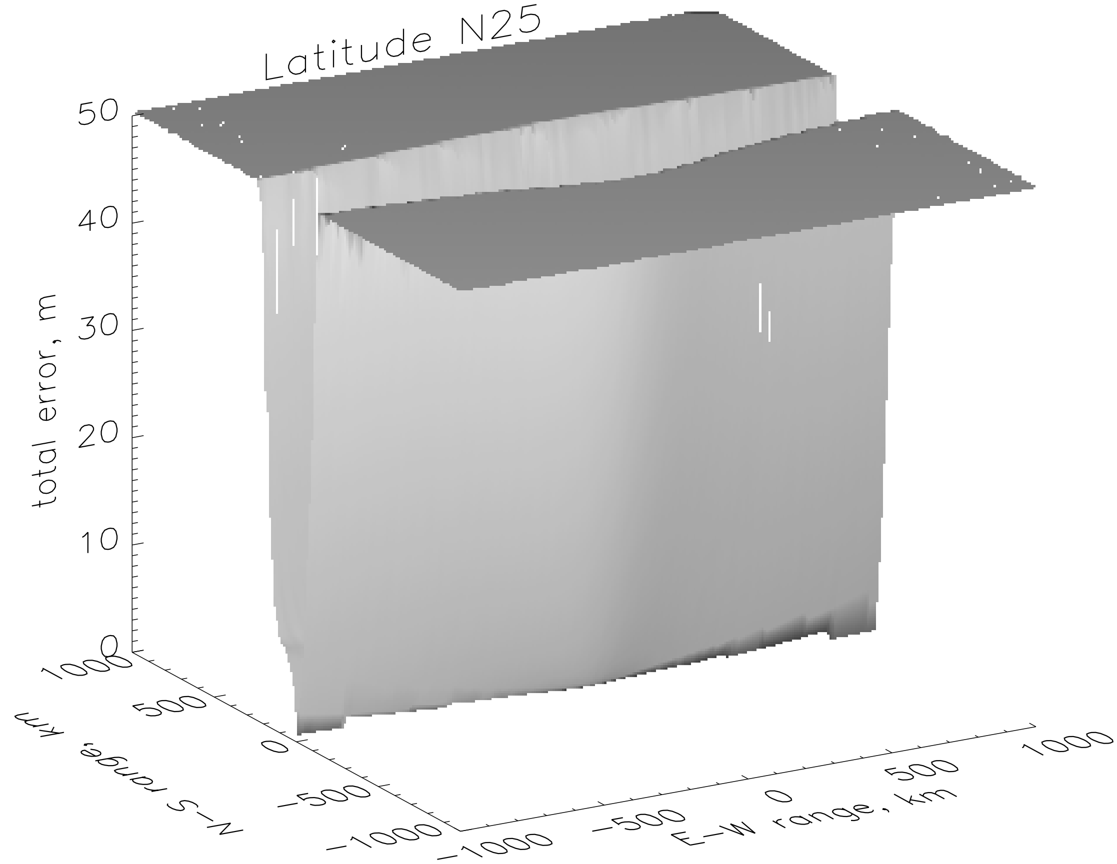

Figure 4(a) Similar to Figure 2(a), but the error shown results from the use of equations (17). The error surface has been truncated at the top so only errors less than 50 m are correctly indicated. The starting point is at latitude 25°N.

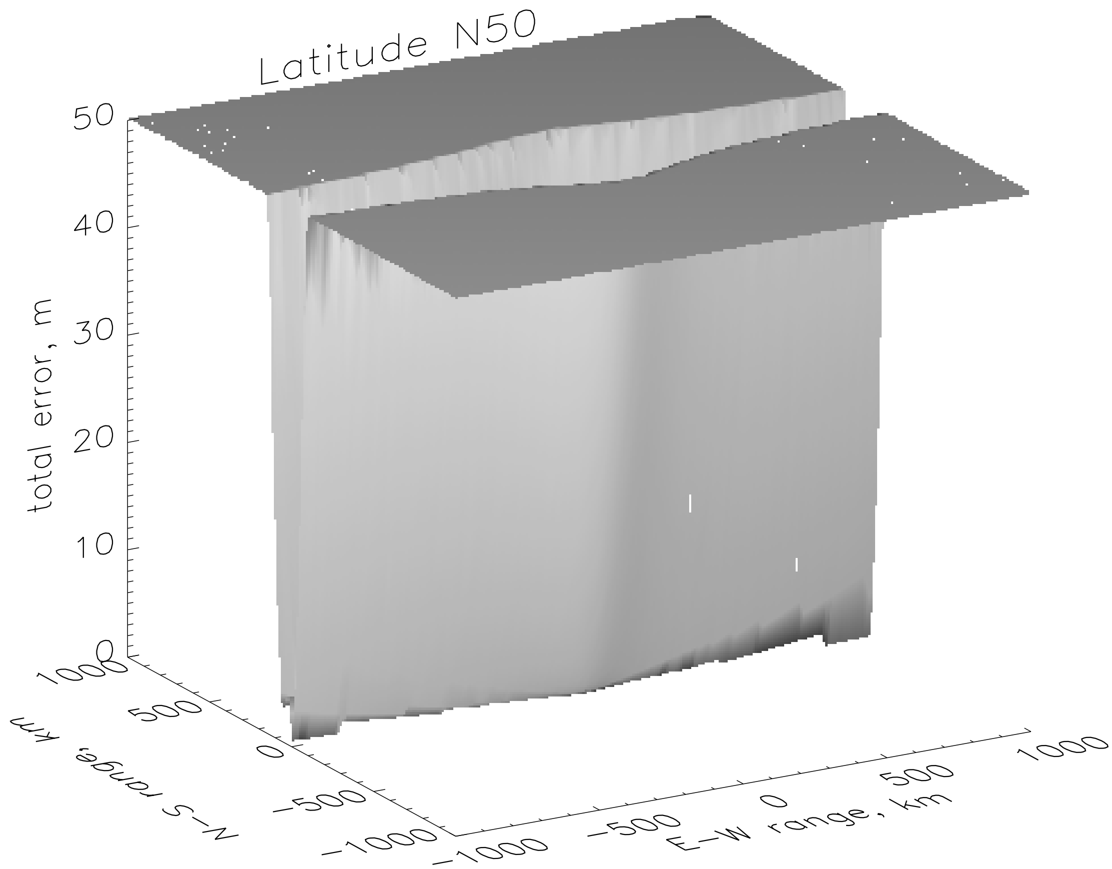

Figure 4(b) Same as Figure 4(a), but with the starting point at latitude 50°N.

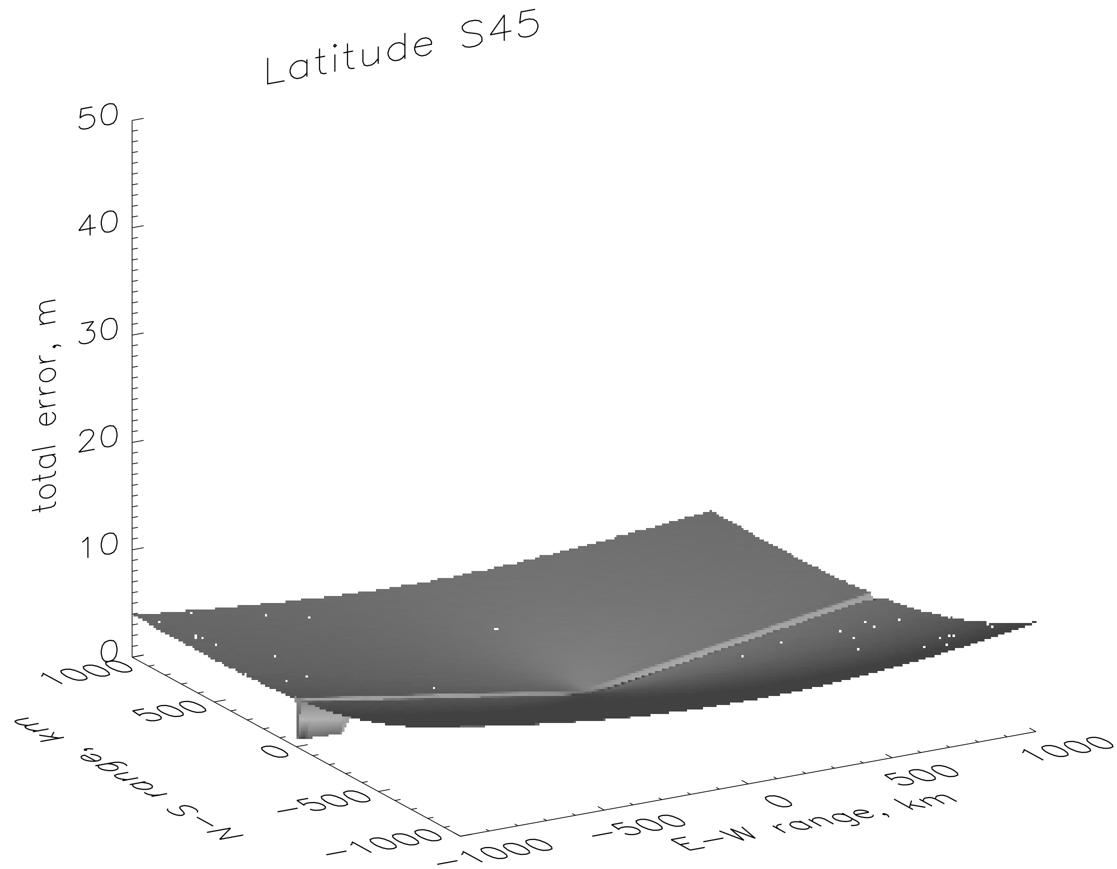

Figure 5 Total error in position as a function of position, resulting from the use of the recommeded sailing algorithm. The algorithm incorporates equations (13) and (16) for non-east-west courses and equations (17) for east-west courses, and the starting point for the calculations, 45°S, is at the center of the XY plane.