THE MOTION OF MARS' POLE. I. RIGID BODY PRECESSION AND NUTATION

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Received 5 March 1991; revised 20 June 1991

ABSTRACT

The total precession in longitude for a solid-core, rigid Mars model is found to be $\Delta\Psi = -7''.296 \pm 0''.021$ yr^{-1}. This precession includes the motion resulting from the extremely long-period (> 10 000 yr) nutation components that result from changes in Mars' orbit with time. These same very long-period nutation components also contribute a "precession" in latitude of $\Delta\epsilon = 0''.4255 \pm 0''.0012$ yr^{-1}. The nutation components in longitude with amplitudes larger than 0''.001 consist of eight solar nutation terms and one nutation each contributed by the motions of the nodes of Phobos and Deimos and a single nutation from the direct gravitational torque of Jupiter. The nutation components in obliquity consist of six solar terms and one nutation term each from the motion of the nodes of Phobos and Deimos. The amplitudes and periods of the solar nutation terms are in agreement with the nutation found by Reasenberg & King [J. Geophys. Res., 84, 6231 (1979)]. The solar precession rate also agrees with Reasenberg and King's value with the two small corrections resulting from the change in the orientation of Mars' orbit with time added to it. A single nutation in longitude driven by Jupiter is the only significant planetary contribution at the milliarcsecond level.

1. INTRODUCTION

Differences between the observed forced nutation component amplitudes of the Earth and predicted nutation amplitudes for a rigid Earth are a result of differences between the theoretical rigid structure of the Earth used in older models and the actual elastic Earth with a liquid core. However, except for the period of the Chandler wobble, the observations of the motion of the Earth's pole were not accurate enough to observe the effects of the elastic, liquid core Earth until the last 30 years. In more recent works, such as those by Wahr (1981a,b), the nutation resulting from an elastic, liquid core Earth are modeled as perturbations of the rigid Earth model nutation. This is the approach adopted for the 1980 IAU Theory of Nutation to determine the amplitude of the various nutational elements (Kaplan 1981). These perturbations result in modifications to the nutation amplitudes for the Earth from about 1% to 0.01% of the theoretical rigid nutation amplitudes or about 0''.019 for the largest term in the series. Since the 1950s the improvement in the measurement of the motion of the Earth's pole have made its observation a powerful probe of the structure of the Earth. The recent increase in the quality of the data available for the precession and nutation for the Earth has started a reevaluation of nutation theory for both rigid and elastic Earth models to improve their accuracies (e.g., Kinoshita & Souchay 1990; Zhu *et al.* 1990). The most recent work, such as done with *VLBI*, has made the information on the Earth's interior obtained from observations of precession, nutation, and polar motion of the same quality as can be determined from seismometry (Melchior 1986). This method can be used as a probe of the structure of other planets in the solar system provided that it is possible to observe the planet's orientation in space with high enough accuracy. This requirement can definitely be fulfilled for only one other planet in the solar system, Mars.

Mars, the fourth planet from the Sun, is the third in size of the terrestrial planets of the solar system. Its orbit brings Mars to within 0.53 AU of the Earth. Its surface is not covered by clouds like Venus and the gaseous giant planets, and, unlike Mercury, it is observable away from the Sun's glare. These properties make Mars the easiest planet in the solar system, aside from the Earth, to observe. Despite its small size [equatorial radius 3393.4 km and mass 6.42×10^{23} kg; Astronomical Almanac for the Year 1990 (1989)], Mars shows many Earthlike properties in its physical ephemeris. Mars has a sidereal day of 1.0260 Earth solar days, the inclination of its equator to the plane of its orbit of 25°.20, and a geometric oblateness of 0.0052 in comparison to the Earth's geometric oblateness of 0.0034.

It is, however, a mistake to assume that Mars has an internal composition that is the same as the Earth. First, Mars has a mean density that is only 61.9% of the mean density of the other terrestrial planets.

The inertia ratio is a measure of a planet's central condensation defined by

$$q \equiv C/\mathscr{M}R^2, \tag{1}$$

where C is the principal moment of inertia about the polar axis, $\mathscr{M}$ is the mass of the planet, and R is the equatorial radius of the planet. The estimates of q for Mars range from 0.3654 (Reasenberg 1977) to 0.3452 (Bills 1989) in comparison to 0.3335 for the Earth and 0.4 for a homogeneous sphere. The large value for the inertia ratio means that Mars is less centrally condensed than the Earth, but it is also small enough to show that Mars is not completely homogeneous. Mars may have a very small external magnetic field, the strength of which has not yet been accurately measured, but it has been established that Mars' magnetic field is much smaller than the Earth's. Other information on the interior structure of Mars is very sketchy. The only structural information that is certain is Mars' radius and its mean density. In addition the structure of the surface gravity field to twelfth degree and order has been published by Christensen & Balmino (1979), but the higher order structure of the field is only preliminary.

The seismic experiments included in the Viking probes were designed only as a preliminary survey to determine what instruments would be necessary on future missions to Mars. Also, the seismic package functioned properly on only one of the two landers (Anderson *et al.* 1977).

The less extreme central condensation of Mars and its small magnetic field are the reasons that some of the existing models of Mars picture a planet with a solid core (Binder & Davis 1973) rather than a liquid core as the Earth has. Most models, however, do use liquid cores, but the composition of the core is very much in question. Johnston & Toksoz (1977) and Okal & Anderson (1978) discuss the merits of various possible liquid core configurations. The two main types of core used in the models are either small and formed from molten Fe or large and containing a molten iron compound such as FeS. If the core is composed of Fe, it needs to be, proportionately, much smaller than the Earth's core to be consistent with the smaller central condensation of Mars and to account for the small magnetic field. A core composed of FeS would be larger than a Fe core since FeS is less dense. Also, FeS is not as good an electrical conductor as Fe, so the small magnetic field is not a consideration. The range of models investigated by Okal & Anderson (1978) included cores with radii from 20% to 60% of Mars' radius and contained 7% to 31% of its total mass. The key to determining the core's composition is determining its mean radius which, along with the planetary inertia ratio, will determine its mean density.

By using the Earth as an example for how the motion of the pole is affected by a planet's structure and keeping in mind the known and possible differences between Mars and the Earth, it is possible to use the motion of Mars' pole as a probe of its interior structure. The information needed to discriminate between the various models of the Martian interior can be obtained from a few, highly accurate planetary orientation sensors as opposed to a large net of seismometers needed for a classic determination of the planetary interior.

Mars has been the subject of most of the attempts to determine the precession and nutation of other planets. One of the first attempts to calculate Mars' precession was by Struve (1898) who obtained a value of $-7\farcs07$ per terrestrial year. More recent attempts to determine the precession of Mars theoretically are those of de Vaucouleurs (1964) with a precession rate of $-7\farcs07$ per year and Reasenberg & King (1979) who determined a precession rate of $-7\farcs575$ per year along with nine nutation components in longitude and seven nutation components in obliquity. These nutation components are the result of the direct solar torque and long-period changes in Mars' orbit only. The nutation components range in amplitude from $1\farcs097$ to $0\farcs0004$ with periods ranging from 114.5 days to 686.9 days. Reasenberg & King also made a first study of the effects of other gravitational sources on the above terms. However, some sources for motion of the pole have not been taken into account, such as (1) tidal distortion, that is, planetary elasticity, (2) diurnal polar motion due to tidal distortion, (3) regional tectonocism which may be responsible for the buildup of the Tharsis region of Mars (Reasenberg 1977) and the formation of the Valles Marineris (Blasius *et al.* 1977), and (4) the torques of the Martian satellites on the precession and nutation of Mars. Although both satellites are small and in nearly equatorial orbits, they are close enough to Mars that preliminary calculations indicate that Phobos has the potential for producing a precession of at least a few tenths of an arcsecond per year. Ward (1974) and Borderies (1980) have done work on the long period nutation of Mars.

A preliminary study of the motion of Mars' pole from Viking data has been carried out by Borderies *et al.* (1980). This paper uses the data from the Martian ranging and Doppler shift information from the first nine months of the Viking landers on Mars. The results from these preliminary measurements show (1) there are definite systematic residuals left from their solution for the planetary motion and (2) the data do not provide a long enough baseline for a definitive measurement of the motion of Mars' pole.

The object of this paper is to determine the theoretical basis for the rigid-body precession and nutation of Mars including the contributions of Phobos, Deimos, and the planets. A future paper will take the rigid-body framework developed here, and apply it to an elastic planet. Models of Mars using either solid or liquid core configurations will show how the radius of the core and the core's state affect the precession, nutation, and other elements of the motion of Mars' pole. This pair of papers will then provide a basis for the first-order determination of Mars' structure based on the change in its spatial orientation with time.

2. THE EQUATION OF MOTION

The gravitational potential of a test particle at $\mathbf{r}$ by an extended mass $\mathscr{M}$ is expressed by

$$\Phi(\mathbf{r}) = \frac{-G}{r}\left(M + \frac{1}{r^2}\int_V P_1(\mathbf{r}\cdot\mathbf{r}')\,dm + \frac{1}{r^4}\int_V P_2(\mathbf{r}\cdot\mathbf{r}')\,dm + \cdots\right), \tag{2}$$

where $\mathbf{r}'$ is the position of an infinitesimal mass element dm within the extended mass and $P_n(\mathbf{r}\cdot\mathbf{r}')$ are the Legendre polynomials of order n. The geometric algebra form of the Legendre polynomial uses the dot product of two vectors rather than the usual cosine of the angle between the vectors for the free parameter.

From Appendix A the first-order equation for the torque of a point mass m on a spheroidal planet is

$$\mathbf{F}_\perp \approx -3\,\frac{G\mathscr{M}m}{r^2}\,J_2\left(\frac{R}{r}\right)^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{e}})\hat{\mathbf{e}}, \tag{3}$$

where m is the mass of the perturbing body, r is the distance from the center of mass of the planet to the perturbing body, $\hat{\mathbf{r}}$ is the unit vector pointing toward the perturbing body from the center of mass of $\mathscr{M}$, $\hat{\mathbf{e}}$ is the unit vector along the axis of the greatest principal moment of inertia, J_2 is the dynamical form factor, and R is the equatorial radius of the planet.

The equation for the forced motion of the pole of a spheroidal body is found to be

$$(-\dot{\lambda}\sin\epsilon, -\dot{\beta}, 0) = \frac{3Gm}{r^3\Omega}\,\frac{J_2}{q}(\sin\delta\cos\delta\sin\alpha, \sin\delta\cos\delta\cos\alpha, 0), \tag{4}$$

where $\dot{\lambda}$ is the time derivative of the motion in longitude with respect to a given inertial coordinate plane, $\dot{\beta}$ is time derivative of the latitude, α is the longitude of the perturbing body with respect to the equator of the planet, and δ is the latitude of the perturbing body with respect to the planet's equator.

3. SOLAR PRECESSION

For the solar precession and nutation two additional simplifications can be made. First, the inclination of the orbital plane of the Sun about Mars and the orbital plane of Mars about the Sun are identical, so $\epsilon = \epsilon_0$. Second, the mean orbital plane at the chosen epoch is identical to the reference plane, that is $\beta = 0$, so $\cos\beta = 1$ and $\sin\beta = 0$.

3.1 Precession and Nutation for a Circular Orbit

In Eq. (4) the angle ϵ varies only a small amount over relatively short periods of time ($\sim 10^4$ yr). Ward (1974) and Borderies (1980) showed that the obliquity of Mars oscillates about 4° on a time scale of 1.2×10^5 yr and about 7° over 1.2×10^6 yr. These changes in inclination are mainly a result of changes in the orbital plane of Mars rather than changes in the orientation of the rotation axis. Thus, for the present purposes, ϵ is considered constant. The effect of the change in the obliquity of Mars on its precession and nutation will be covered in Sec. *3.3* and Appendix D. Therefore, for a circular orbit, the only quantity that changes with time on the right-hand side of Eq. (4) is the longitude λ. The zeroth-order approximation for λ is a monotonically increasing function of time:

$$\lambda(t) = nt + \lambda_0, \tag{5}$$

where λ_0 is the ecliptic longitude at some initial epoch t_0. Using Eq. (5) in Eq. (4) gives

$$\dot{\lambda} \sin \epsilon = \frac{-3G}{\Omega} \frac{J_2}{q} \frac{m}{r^3} \sin \epsilon \cos \epsilon \sin^2(nt + \lambda_0) \tag{6}$$

and

$$\dot{\beta} = \frac{3G}{\Omega} \frac{J_2}{q} \frac{m}{r^3} \sin(nt + \lambda_0) \sin \epsilon \cos(nt + \lambda_0)\ . \tag{7}$$

These two equations for $\dot{\lambda}$ and $\dot{\beta}$ are integrated to determine both the precession in longitude and the first nutation terms in longitude and latitude that arise from a circular orbit of Mars about the Sun. Equation (6) is divided by $\sin \epsilon$ and integrated over the orbital period. The change in longitude of the pole as a function of time is

$$\lambda = \frac{-3G}{\Omega} \frac{J_2}{q} \frac{m}{r^3} \cos \epsilon \left(\frac{1}{2} t - \frac{1}{4n} \sin 2(nt + \lambda_0) \right) + C. \tag{8}$$

Integrating Eq. (27) gives

$$\beta = \frac{-3G}{\Omega} \frac{J_2}{q} \frac{m}{r^3} \sin \epsilon \frac{1}{4n} \cos 2(nt + \lambda_0) + C'. \tag{9}$$

Using the definition of the Euler angles $\Delta\epsilon = -\Delta\beta$ and $\Delta\psi = -\Delta\lambda$. Thus, the negatives of Eqs. (8) and (9) give the precession in ψ and the circular orbit nutation terms in both ϵ and ψ.

For the solar precession and nutation of Mars, the values for the quantities needed to evaluate Eqs. (8) and (9) are

$G = (6.672 \pm 0.001) \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$ (Kaplan 1981),

$m = (1.9891 \pm 0.0001) \times 10^{30}$ kg (Kaplan 1981),

$r = (2.279\,390\,77 \times 10^{11} \pm 1500)$m (Kaplan 1981; Bretagnon 1982),

$J_2 = (1.959\,0468 \pm 0.000\,138\,70) \times 10^{-3}$ (Christensen & Williams 1978),

$q = 0.3654 \pm 0.0010$ (Reasenberg 1979),

$\Omega = (7.088\,2181 \times 10^{-5} \pm 1 \times 10^{-12})\mathrm{s^{-1}}$ (Mayo *et al.* 1977),

$\epsilon = (25\overset{\circ}{.}20 \pm 0\overset{\circ}{.}02)$ (Sinclair, 1972).

The exact value for the inertia ratio q is still unknown as pointed out by Bills (1989). The value of q is derived from models of Mars' gravity field. These gravity field models incorporate an assumed orientation for the nonhydrostatic component of Mars' mass distribution with respect to its equator. Reasenberg's (1977) value finds that the pole of the nonhydrostatic component of Mars' structure is oriented along the pole of Mars' greatest principal moment of inertia. Bills, however, argues based on statistics that the nonhydrostatic component of Mars' mass distribution is more likely to have the pole of Mars' nonhydrostatic component in the midlatitude regions. For this orientation of the nonhydrostatic component the best value for Mars' inertia ratio is $q = 0.345$. However, the two largest visible nonyhydrostatic components of Mars, the Valles Marineris and the Tharsis uplift, are both centered near the equator, and Mars, unlike the Earth, has a definite correlation between its low-order gravitational field components and its terrain features (Lorell *et al.* 1972). Other arguments favoring Reasenberg's inertia ratio are presented by Kaula *et al.* (1989). Therefore the value for the inertia ratio used here is 0.3654 (Reasenberg 1977).

Using the above constants for Mars, the precession of Mars, the first term in Eq. (28) is

$$\dot{\psi} = -(1.1504 \pm 0.0032) \times 10^{-12}\ \mathrm{s^{-1}}$$
$$= -(7\overset{''}{.}488 \pm 0\overset{''}{.}021)\ \mathrm{yr^{-1}}.$$

This value for the mean precession of Mars' pole compares very favorably with previously derived values for the polar precession. By far, the largest source of uncertainty (99.7%) in the precession rate is the uncertainty in the inertia ratio q.

The amplitude of the two circular orbit nutation components driven by the torque of the Sun on Mars are determined using the mean orbital motion of Mars taken from Bretagnon (1982)

$$n = 1.058\,589\,015 \times 10^{-7} \pm 1 \times 10^{-16}\ \mathrm{rad\,s^{-1}}.$$

Substituting this into Eq. (8) gives a nutation amplitude of

$$\frac{3}{4n} \frac{G}{\Omega} \frac{J_2}{q} \frac{m}{r^3} \cos \epsilon = (5.434 \pm 0.015) \times 10^{-6}\ \mathrm{rad}$$
$$= 1\overset{''}{.}121 \pm 0\overset{''}{.}003.$$

Similarly, using the value for $\sin \epsilon$ the amplitude of the main nutation in obliquity determined from Eq. (9):

$$\frac{-3}{4n} \frac{G}{\Omega} \frac{J_2}{q} \frac{m}{r^3} \sin \epsilon = (-2.557 \pm 0.007) \times 10^{-6}\ \mathrm{rad}$$
$$= -0\overset{''}{.}527 \pm 0\overset{''}{.}002.$$

Both of these nutation amplitudes are in good agreement with the primary nutation amplitudes determined by Reasenberg & King (1979) of $1\overset{''}{.}097$ in longitude and $0\overset{''}{.}5157$ in obliquity.

Now only the initial phase λ_0 needs to be determined. The restoring force on Mars is at its greatest when Mars is at its solstices. Therefore, the greatest deviation from the mean will also occur at the solstices. Letting the mean longitude of Mars be represented by $\mathscr{L}$ and the angle between the perihelion of Mars and its vernal equinox be Λ, then

$$nt + \lambda_0 = \mathscr{L} - \Lambda. \tag{10}$$

The circular orbit precession and nutation terms for Mars are

$$\psi = -7''488t + 1''121 \cos 2(\mathscr{L} - \Lambda),$$
$$\delta\epsilon = 0''527 \sin 2(\mathscr{L} - \Lambda). \quad (11)$$

3.2 Components from the Orbital Eccentricity

The next level of complication to add to the determination of the amplitudes of Mars' nutation components is the change in the torque on the planet as a function of distance from the Sun. The nutation will be determined by using a series expansion in sine and cosine functions of angular variables. The angular variables are chosen to be linear functions of time.

For the Sun's torque on Mars, the angular variables used are $\mathscr{L}$ and Λ. The result to expect from such an expansion is that the two circular orbit nutation terms in Eqs. (10) and (11) above will be split into two series of nutation components for the planet. From Appendix B the precession and nutation are

$$\lambda = \frac{-Q}{a^3} \cos \epsilon \left(A_0 t + \sum_{n=1}^{\infty} \frac{1}{n\mathscr{L}} [A_n \sin (n\mathscr{L}) - C_n \cos (2\Lambda + n\mathscr{L})] + \frac{B_n}{n\mathscr{L}} \sin (2\Lambda + n\mathscr{L}) \right),$$
$$\beta = \frac{-Q}{a^3} \sin \epsilon \left(\sum_{n=0}^{\infty} \frac{B_n}{n\mathscr{L}} \cos (2\Lambda + n\mathscr{L}) + \frac{C_n}{n\mathscr{L}} \sin (2\Lambda + n\mathscr{L}) \right), \quad (12)$$

where A_0, A_n, B_n, and C_n are coefficients determined in Appendix B. Inserting the appropriate values for the constants for Mars the coefficient for the precession becomes

$$\psi = -7''587 \pm 0''021 \ \mathrm{yr}^{-1}.$$

The coefficients for the nutation terms larger than 0"001 are given in Table 1. All of the nutation amplitudes from C_n coefficients are smaller than 0"001 so there are no [cos $(2\Lambda + n\mathscr{L})$] terms in longitude or (sin 2Λ sin $n\mathscr{L}$) terms in obliquity in Table 1. As before, the largest source of uncertainty is the inertia ratio of Mars.

3.2.1 Changes of eccentricity with time

Mars is subject to planetary gravitational torques as well as the torque provided by the Sun. The planetary perturbations cause the shape and orientation of Mars' orbit to vary with time. The two changes in the orbit and orientation which are of concern are the eccentricity of Mars' orbit and the obliquity of Mars to its orbit. The change in the obliquity does cause small but significant changes in the nutation in obliquity and will be treated in the next section. In this section the effect of change in the eccentricity of the orbit will be explored.

The treatment of the eccentricity uses the Bretagnon (1982) semianalytic theory for Mars and is divided into periodic and secular terms. Appendix C shows that all of the periodic components caused by the change in eccentricity are insignificant. Substituting the nonperiodic, zeroth-order corrections into Appendix C the total precession becomes

$$\psi = -7''588 \pm 0''021 \ \mathrm{yr}^{-1}.$$

The change of 0"001 yr^{-1} is smaller than the uncertainty in the precession.

Including the secular zeroth-order part for eccentricity from Bretagnon's theory does cause a change that is larger than the uncertainty of the individual terms for most of the nutation terms (Table 2). For most of the components, however, the change in amplitude is too small to avoid being lost in the uncertainty of the larger nutation components. The effect of the change of eccentricity with time is seen as a small change in the amplitudes of the nutation components.

TABLE 1. Amplitudes for the rigid body solar nutation components of Mars.

Nutation Components in Longitude			
Term	Amplitude (")	Error (")	Period (day)
sin L	-0.6343	0.0018	686.93
sin $2L$	-0.0443	0.0001	343.46
sin $3L$	-0.00405	0.00001	228.98
sin $(2\Lambda + L)$	-0.1046	0.0003	686.72
sin $(2\Lambda + 2L)$	1.0963	0.0031	343.41
sin $(2\Lambda + 3L)$	0.2396	0.0007	228.96
sin $(2\Lambda + 4L)$	0.0407	0.0001	171.72
sin $(2\Lambda + 5L)$	0.00630	0.00002	137.38
sin $(2\Lambda + 6L)$	0.000926	0.000003	114.48

Nutation Components in Latitude			
Term	Amplitude (")	Error (")	Period (day)
cos $(2\Lambda + L)$	-0.0492	0.0001	686.72
cos $(2\Lambda + 2L)$	0.5159	0.0014	343.41
cos $(2\Lambda + 3L)$	0.1127	0.0003	228.96
cos $(2\Lambda + 4L)$	0.01917	0.00005	171.72
cos $(2\Lambda + 5L)$	0.002963	0.000008	137.38

L = Mars' mean longitude
Λ = angle between the perihelion and the vernal equinox of Mars

3.3 Changes in Orbital Inclination and the Ascending Node

A final source for solar nutation components is long term change in the obliquity of Mars' equator to its orbit. The source of the changes in obliquity is not the nutation in obliquity, terms which have periods of a Martian sidereal year (686.9297 days) or less. The change in obliquity is the result of changes in the orientation of Mars' orbital plane with time in response to the perturbations of the other planets in the solar system. Ward (1974) and Borderies (1980) show that the variation in the obliquity of Mars resulting from changes in the orbital plane can be represented by a pair of approximately equal amplitude oscillations; the first oscillation has a period of 1.14×10^5 yr, the second oscillation has a period of 1.26×10^5 yr, and the combined amplitude is about 10°3.

The equations of motion for Mars developed by Ward (1974) are

$$\frac{d\epsilon}{dt} = -\cos \psi \sin i \frac{d\Omega}{dt} + \sin \psi \frac{di}{dt} \quad (13)$$

and

$$\frac{d\psi}{dt} = (\sin i \sin \psi \cot \epsilon - \cos i) \frac{d\Omega}{dt} - \alpha \cos \epsilon + \cos \psi \cot \epsilon \frac{di}{dt}. \quad (14)$$

This is a pair of coupled first-order equations for ϵ and ψ as a function of time. These equations are solved by making

TABLE 2. Amplitudes for the rigid body solar nutation components of Mars including the change in eccentricity with time.

Nutation Components in Longitude

Term	Amplitude (")	Error (")	Period (day)
sin L	-0.6357	0.0018	686.93
sin $2L$	-0.0445	0.0001	343.46
sin $3L$	-0.00408	0.00001	228.98
sin $(2\Lambda + L)$	-0.1047	0.0003	686.72
sin $(2\Lambda + 2L)$	1.0962	0.0031	343.41
sin $(2\Lambda + 3L)$	0.2401	0.0007	228.96
sin $(2\Lambda + 4L)$	0.0409	0.0001	171.72
sin $(2\Lambda + 5L)$	0.00634	0.00002	137.38
sin $(2\Lambda + 6L)$	0.000934	0.000003	114.48

Nutation Components in Latitude

Term	Amplitude (")	Error (")	Period (day)
cos $(2\Lambda + L)$	-0.0493	0.0001	686.72
cos $(2\Lambda + 2L)$	0.5158	0.0014	343.41
cos $(2\Lambda + 3L)$	0.1130	0.0003	228.96
cos $(2\Lambda + 4L)$	0.01925	0.00005	171.72
cos $(2\Lambda + 5L)$	0.002982	0.000008	137.38

L = Mars' mean longitude
Λ = angle between the perihelion and the vernal equinox of Mars

repeated approximations of their actual values. The first-order solutions to the equations are a very good approximation to the motion. Ward (1974) solved the first-order equation for ϵ and gives a first-order estimate to the first-order solution for the motion in longitude ψ. The first-order solution to the equation of motion in latitude and the third-order estimate to the first-order solution to the equation of motion in longitude are developed in Appendix *D*.

Over short periods of time near the present era, $\pm$ 10 000 yr (Fig. 1), the motion in obliquity of the Martian pole as a result of the change in orientation of Mars' orbit can be represented by the linear function

$$\epsilon = (25^\circ\!.2 \pm 0^\circ\!.01) + 0''\!.4255t, \tag{15}$$

where t is the time, in years, from J2000.0. Also, the zeroth- and first-order parts of the first-order solution for precession and nutation in longitude appear as a contribution to the precession of

$$\psi_{11} = 0''\!.2127 \pm 0''\!.0006 \ \text{yr}^{-1}$$

shown in Fig. 2.

The secular portion of the second-order part to the first-order solution gives an additional correction in precession of

$$\delta\psi_{12} = 0''\!.0797 \pm 0''\!.0002 \ \text{yr}^{-1}.$$

Overall, the change of Mars' orbit with time causes two measurable effects in longitude. (1) A set of very long-term nutation components from the first-order term. These nutation components are seen over short periods of time as a "precession" of $0''\!.2137$ yr^{-1}. (2) There is an addition to the precession from the second-order term of $0''\!.0797$ yr^{-1}. Adding these terms to the solar precession the total precession in longitude of Mars is

$$\psi = -7''\!.296 \pm 0''\!.021 \ \text{yr}^{-1}.$$

In addition, there is a "precession" in latitude of $0''\!.4255$ yr^{-1}.

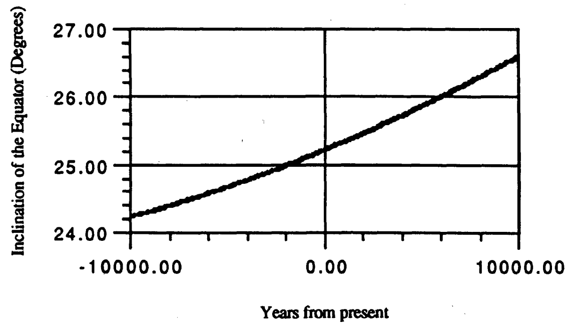

FIG. 1. Very long-period nutation in obliquity over historical periods of time. Over short time spans $\pm$ 10 000 yr the change in the inclination of Mars' pole is nearly linear with time.

4. PRECESSION AND NUTATION FROM NATURAL SATELLITES

Mars has two satellites, Phobos and Deimos, to contribute to Mars' precession and nutation, just as the Moon contributes to the Earth's precession and nutation. Although both of these satellites are small, their contributions to Mars' nutation may be fairly large because their mean distances from Mars are very small and forced nutation is a $1/r^3$ effect. The orbital elements of the satellites taken from Sinclair (1972) and the masses taken from Morley (1990) are

Phobos:

$$m = (1.05 \pm 0.1) \times 10^{16} \ \text{kg},$$
$$r = (9.378 \pm 0.001) \times 10^{6} \ \text{m},$$
$$i = 1^\circ\!.01 \pm 0^\circ\!.07,$$
$$n = 7196.1730 \pm 0.0064 \ \text{rad/yr},$$
$$\lambda_0 = 2.817 \pm 0.003 \ \text{rad},$$
$$\dot{N} = -2.776 \pm 0.003 \ \text{rad/yr},$$
$$N = 2.65 \pm 0.07 \ \text{rad},$$

where i is the inclination of Phobos' orbit to the equator of Mars, n is the mean motion of Phobos, λ_0 is the longitude of Phobos at J2000.0, $\dot{N}$ is the motion of the node of Phobos' orbit, and N is the position of the node of Phobos at J2000.0. These values for the physical quantities for Phobos give a precession constant of

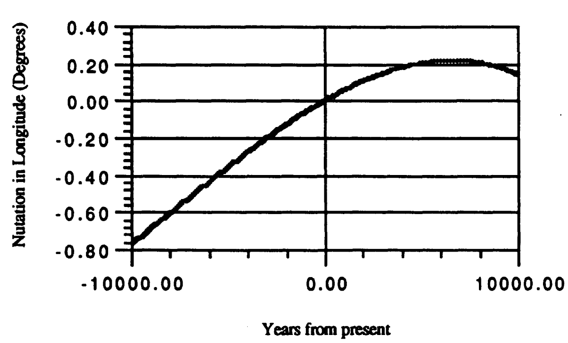

FIG. 2. Very long-period nutation in longitude over historical periods of time. Over short time spans $\pm$ 10 000 yr the change in the longitude of Mars' pole is nearly quadratic with time. The coefficient for the quadratic term is only $0''\!.0001$ yr^{-2}.

$$Q = \frac{3Gm}{\Omega r^3}\frac{J_2}{q} = (1.95 \pm 0.15) \times 10^{-13} \text{ rad/s}$$
$$= 1''.26 \pm 0''.10 \text{ yr}^{-1}.$$

Deimos:

$m = (1.8 \pm 0.15) \times 10^{15}$ kg,
$r = (2.3459 \pm 0.0001) \times 10^{7}$ m,
$i = 2^\circ.695 \pm 0^\circ.030$,
$n = 1817.8545 \pm 0.0003$ rad/yr,
$\lambda_0 = 3.768 \pm 0.001$ rad,
$\dot{N} = -0.1156 \pm 0.0001$ rad/yr,
$N = 0.16 \pm 0.01$ rad.

These values give the precession constant due to Deimos of

$$Q = \frac{3Gm}{\Omega r^3}\frac{J_2}{q} = (2.1 \pm 0.2) \times 10^{-15} \text{ rad/s}$$
$$= 0''.014 \pm 0''.001 \text{ yr}^{-1}.$$

Thus, both of the satellites of Mars have the potential of contributing significantly to the precession and nutation of Mars.

Since both of the satellites orbit Mars in planes near that of the planet's equator, the easiest method to deal with them is to return to the equations of motion (4) and rewrite them

$$-\dot{\beta} = Q \sin\delta \cos\delta \cos\alpha = \frac{Q}{2}\sin 2\delta \cos\alpha,$$
$$-\dot{\lambda} \sin\epsilon = Q \sin\delta \cos\delta \sin\alpha = \frac{Q}{2}\sin 2\delta \sin\alpha, \quad (16)$$

where $Q = (3GmJ_2)/(r^3\Omega q)$. Since the orbital planes of Mars' moons are nearly the same as the equatorial plane of Mars, the values for α and δ as functions of time are

$$\alpha(t) \approx nt + \lambda_0,$$
$$\delta(t) \approx i \sin(nt + \lambda_0 - \dot{N}t - N), \quad (17)$$

where i is the inclination of the satellite's orbit to Mars' equator. Substituting these relations for α and δ into the first of Eqs. (16) and assuming i is small gives to first order:

$$-\dot{\beta} \approx \frac{Q}{2}\sin 2[i \sin(nt + \lambda_0 - \dot{N}t - N)]\cos(nt + \lambda_0)$$
$$= \frac{Qi}{2}[\sin(2nt + 2\lambda_0 - \dot{N}t - N) + \cos(\dot{N}t + N)]. \quad (18)$$

Integrating Eq. (18) gives the nutation in latitude as a function of time as

$$\epsilon = -\beta = \frac{Qi}{2}\left(\frac{1}{\dot{N}}\sin(\dot{N}t + N) - \frac{1}{2n - \dot{N}}\cos(2nt + 2\lambda_0 - \dot{N}t - N)\right) + C. \quad (19)$$

Performing the same set of operations on the second of Eqs. (16) and using $\psi = -\lambda$ gives the nutation in longitude as a function of time as

$$\psi = \frac{Qi}{2\sin\epsilon}\left(\frac{1}{2n - \dot{N}}\sin(2nt + 2\lambda_0 - \dot{N}t - N) - \frac{1}{\dot{N}}\sin(\dot{N}t + N)\right) + C. \quad (20)$$

There is no precession term contributed by the Martian satellites in Eq. (20) because the invariable planes of the satellites' orbits are both essentially the plane of Mars' equator. The similarity of the two planes is implicitly included in Eq. (17) for $\delta(t)$. Thus, the precession contributions of the satellites are cancelled out by the motions of the nodes of the orbits.

Evaluating Eqs. (19) and (20) and setting the constants of integration to $\psi(0) = 0$ and $\epsilon(0) = \epsilon_0$, using the orbital parameters for Phobos and Deimos gives

Phobos:

$$\epsilon = (0''.0040 \pm 0''.0003)\sin[(-2^\circ.776 \pm 0^\circ.003)t + (2^\circ.65 \pm 0^\circ.07)] + (7''.7 \pm 0''.7 \times 10^{-7}) \times \cos[(14\,395^\circ.12 \pm 0.01)t + (2^\circ.98 \pm 0.07)] + \epsilon_0,$$
$$\psi = (4''.4 \pm 0''.2) \times 10^{-5}\sin[(14\,395^\circ.12 \pm 0^\circ.01)t + (2^\circ.98 \pm 0''.07)] + (0''.23 \pm 0''.01) \times \cos[(-2^\circ.776 \pm 0.003)t + (2^\circ.65 \pm 0.07)].$$

Deimos:

$$\epsilon = (0''.0028 \pm 0''.0002)\sin[(-0^\circ.1156 \pm 0^\circ.0001)t + (0^\circ.16 \pm 0^\circ.01)] - (9''.0 \pm 0''.7 \times 10^{-8}) \times \cos[(3635^\circ.8264 \pm 0.0004)t + (7^\circ.38 \pm 0.01)] + \epsilon_0,$$
$$\psi = (1''.9 \pm 0''.2 \times 10^{-6})\sin[(3635^\circ.8264 \pm 0.0004)t + (7^\circ.38 \pm 0.01)] + (0''.060 \pm 0.004)\cos[(-0^\circ.1156 \pm 0.0001)t + (0^\circ.16 \pm 0.01)],$$

where the time t is in years. Only one term in each of these expressions for the nutation of Mars is large enough to be significant. The nutation in longitude from the torque of the Martian moons is deceptively large. The reason for the large nutation in longitude is because it is proportional to $(\sin\epsilon)^{-1}$. That is, the motion of the pole in longitude moves a large distance, but along a very small circle on the sky. The main source of uncertainty for the nutation from the satellites is the inclination of Phobos' orbital plane and mass for the nutation driven by Phobos and Deimos' mass for the nutation driven by Deimos. Deimos, despite its small mass and greater distance from Mars than Phobos, contributes to the nutation because of the slow motion of its node.

The third-order term for the expansion of $\sin[2i \sin(nt + \lambda_0 - \dot{N}t - N)]$ in Eqs. (17) is at least 3.7 orders of magnitude smaller than the first-order term for an inclination of $\geqslant 2^\circ$. Thus, the higher-order terms in the expansion are insignificant.

The nutation that arises from the eccentricity of the satellites' orbits is all insignificant because the periods for the largest nutation components are on the order of $1/n$, where n is the mean orbital motion. Since the amplitude of a nutation is inversely proportional to its period, a long-period nutation has a larger amplitude than a short-period one. The nutation components arising from the motion of the nodes are significant because the periods are long (828 days for Phobos and 19 850 days for Deimos). The main nutation components arising from the eccentricity of the orbits will have periods of 0.319 days for Phobos and 1.26 days for Deimos. Hence these nutation components will be about the same size as the two small nutation components that were found in the first order expansion in Eqs. (19) and (20), and, therefore, insignificant.

5. PRECESSION CONTRIBUTIONS FROM THE PLANETS

Appendix D shows that the perturbations of the planets on Mars' orbit cause change in the orbit and hence a change in the precession rate and nutation component amplitudes from the change in the gravitational pull of the Sun. In this section the direct effect of the gravitational torques of the planets on Mars will be assessed.

Again returning to Eq. (4) and rewriting its first two terms gives the equation of motion as

$$-\dot{\beta} = \frac{3G}{\Omega}\frac{J_2}{q}\frac{m}{r^3}\sin\delta\cos\delta\cos\alpha$$
$$= Q\frac{m}{4r^3}(2\sin 2\beta\cos\epsilon + \sin 2\lambda\sin\epsilon$$
$$+ \sin 2\lambda\cos 2\lambda.\cos\epsilon) \quad (21)$$

and

$$-\dot{\lambda}\sin\epsilon = \frac{3G}{\Omega}\frac{J_2}{q}\frac{m}{r^3}\sin\delta\cos\delta\sin\alpha$$
$$= Q\frac{m}{8r^3}(3\cos 2\beta\sin 2\epsilon - \sin 2\epsilon$$
$$- \cos 2\lambda\sin 2\epsilon - \cos 2\beta\cos 2\lambda\sin 2\epsilon$$
$$+ 4\sin 2\beta\sin\lambda\cos 2\epsilon), \quad (22)$$

where m is the mass of the perturbing planet and r is the distance between the planet and Mars. Because both Mars and the perturbing planet are circling the Sun the distance between the planet and Mars varies greatly as a function of time. Thus, the distance as a function of time will be estimated using a series approximation in Appendix B.

All of the functions on the right-hand side of Eq. (96) in Appendix E have the common factor

$$Q_1 = Q\frac{m}{2r\alpha^3}\frac{\sin 2\epsilon}{\sin\epsilon}. \quad (23)$$

Substituting in the appropriate values for the effect of the Earth on Mars gives

$$Q_{1E} = 0\rlap{.}''000\,1044\frac{\text{Ea}}{\text{AU}^3\ \text{yr}}\frac{1.000\text{Ea}}{2(1.524\ \text{AU})^3}\frac{\sin 50\rlap{.}^\circ 40}{\sin 25\rlap{.}^\circ 20}$$
$$= 0\rlap{.}''000\,026\,69\ \text{yr}^{-1},$$

where Ea is the mass in Earth masses. Substituting in the appropriate constant values for Jupiter gives

$$Q_{1J} = 0\rlap{.}''000\,2132\ \text{yr}^{-1}.$$

The precession in longitude is determined from the first full term on the right-hand side of Eq. (96):

$$\Psi = Q_1\frac{1}{4}B_0^{5/2}(1+\alpha^2)t. \quad (24)$$

Substituting in the values Q_{1E}, α, and $B_0^{5/2}$ for the torque of the Earth on Mars gives

$$\Psi_E = (0\rlap{.}''000\,026\,69\ \text{yr}^{-1})(38.006)$$
$$\times\left[1+\left(\frac{1.000\ \text{AU}}{1.524\ \text{AU}}\right)^2\right]t = 0\rlap{.}''000\,363\ \text{yr}^{-1}t.$$

This precession is almost three orders of magnitude smaller than the uncertainty in the solar precession alone and, hence, insignificant. Substituting in the similar set of constants for Jupiter gives a precession of

$$\Psi_J = 0\rlap{.}''000\,198\ \text{yr}^{-1}t.$$

Again the precession contributed by Jupiter is insignificant. Summing the contributions from all of the planets in the solar system, the total precession contributed by the planets is calculated to be

$$\Psi_T = 0\rlap{.}''000\,624\ \text{yr}^{-1}t.$$

Thus, the Earth and Jupiter contribute 89.9% of the total planet driven component of Mars' precession.

Similarly, the nutation in longitude components all have the general form of

$$Q_1\frac{1}{4}B_k^{5/2}\frac{\alpha^j}{n\sin nt}, \quad (25)$$

where $j = 0$, 1, or 2, and n is the sum of the mean motion for each of the perturbing planets and Mars times some integer coefficient. Therefore, based on the size of the coefficient Q_{1E} the amplitude of a nutation term contributed by the Earth will be significant only if

$$B_k^{5/2}\frac{\alpha^j}{n} > 149.9.$$

With the correspondingly larger coefficient Q_{1J}, the amplitude of a nutation term contributed by Jupiter will be significant only if

$$B_k^{5/2}\frac{\alpha^j}{n} > 18.76.$$

The Laplace coefficients for both planets steadily decrease with increasing k. The largest coefficient value for the Earth is $B_0^{5/2} = 38.006$ and the largest Laplace coefficient value for Jupiter is $B_0^{5/2} = 3.418$. The value of α^n will always be less than or equal to 1. Therefore, if a nutation term contributed by the Earth is large enough to be significant it must have a mean angular motion of

$$n < 38.006/149.9 = 0.2535\ \text{rad/yr}.$$

And a significant nutation term for Jupiter must have

$$n < 0.1822\ \text{rad/yr}.$$

For the Earth, the smallest n for $k<9$ is for $n_E - 3n_{\mathscr{M}} = 1.7511$ rad/yr. Therefore, a significant nutation where the Earth as the driving body does not exist. However, for Jupiter, the smallest possible n for $k<9$ is $3n_J - n_{\mathscr{M}} = 0.1107$. Picking out the related terms for this value of n gives the term

$$\Psi_J = \frac{Qm}{2r^3}\frac{\sin 2\epsilon}{\sin\epsilon}\frac{1}{4}[(1+\alpha^2)B_1^{5/2} + \alpha B_2^{5/2}]$$
$$\times\frac{1}{3n_J - n_{\mathscr{M}}}\sin(3n_J - n_{\mathscr{M}})t. \quad (26)$$

Evaluating this equation gives a nutation term of

$$\Psi_J = (0\rlap{.}''000\,714 \pm 0\rlap{.}''000\,002)\sin(6\rlap{.}^\circ 342t + 315\rlap{.}^\circ 111),$$

where t is in years. This is the largest nutation in longitude term for Mars being driven by Jupiter. Although it is a factor of 3.2 larger than any planetary nutation component of on the Earth (Vondrak 1982), it is just barely significant.

Like Eq. (96), Eq. (98) in Appendix E has a very simple general form that makes the search for nutation in latitude components that are worth evaluating simple. The general form of the motion in latitude terms contains

$$Q_2 B_0^{5/2} \frac{\alpha^k}{n} \cos nt, \qquad (27)$$

where $Q_2 = Q\mathcal{m}/2r_a^3 \sin \epsilon$. Comparing Q_2 with Q_1 for the nutation in longitude equation gives

$$Q_1/Q_2 = \sin 2\epsilon / \sin^2 \epsilon. \qquad (28)$$

Substituting in the present obliquity of Mars to its orbit gives

$$Q_1/Q_2 = 0.2658.$$

Thus, a given nutation in obliquity term is only about 27% the size of the equivalent nutation in longitude term. Therefore, the one nutation in obliquity term that corresponds to $3n_J - n_{\mathcal{M}}$ is insignificant. Its amplitude is

$$\delta\epsilon = (0''.000\,1897 \pm 0''.000\,0005)\cos(6^\circ.342t + 315^\circ.111).$$

The direct gravitational attraction of the planets on Mars does not contribute significantly to the precession and nutation of Mars. However, as shown in Sec. *3.3*, the change in the orbit of Mars arising from perturbations by the planets does cause significant changes in the solar precession and nutation of Mars over very long periods of time.

6. CONCLUSIONS

The total precession in longitude for Mars is

$$\Delta\Psi = -7''.296 \pm 0''.021 \text{ yr}^{-1}.$$

This precession includes two components resulting from extremely long-period (> 210 000 yr) nutation components from long period changes in Mars' orbit. Aside from the two contributions from the change in Mars' orbital plane with time, the precession is in excellent agreement with the precession found by Reasenberg & King (1979). These same very long-period nutation components also contribute a "precession" in latitude of

$$\Delta\epsilon = 0''.4255 \pm 0''.0012 \text{ yr}^{-1}.$$

The nutation components in longitude with amplitudes larger than 0″.001 consist of eight solar terms in Table 2 and one nutation each contributed by the motions of the nodes of Phobos and Deimos. The nutation components in obliquity consist of the six solar terms in Table 2 and one nutation term each from the motions of the nodes of Phobos and Deimos. Again, the solar nutation in both longitude and latitude are in excellent agreement with those found by Reasenberg & King (1979).

Although the torques of the Earth and Jupiter on Mars are several times larger than the torques of the planets in the solar system on the Earth, only a single nutation in longitude driven by Jupiter is significant at the milliarcsecond level.

Looking at the nutation as a whole, Fig. 3 shows the nutation of the Martian pole with respect to the mean pole over a period of 686.9 days (1 Martian sidereal year). This figure is very different from the same one for the Earth (Fig. 4). The difference between the two figures is the result of the difference in the main driving mechanisms for the dominant terms in the nutation series for the two planets. For the Earth, each of the dominant terms is a factor of a few smaller than the next larger term and the periods are different by a factor of several. The periods and amplitudes are the result of the Earth having two major bodies, the Sun and the Moon, driving its significant nutation components. For example, the three largest nutation components in longitude for the Earth have amplitudes of −17″.20, −1″.32, and −0″.23 with periods of 6798, 183, and 14 days, respectively. The irrational ratios of the periods and amplitudes give rise to the ellipse on ellipse structure for the Earth's nutation in Fig. 4. For Mars, however, the three largest nutation components in longitude have amplitudes of 1″.10, −0″.64, and −0″.24 with periods of 344, 687, and 229 days, respectively. The ratio of the periods is nearly 2:1:3. The ratios of the periods are nearly in resonance because the only motions involved in driving the nutation are Mars' orbital motion and the motion of the perihelion with respect to the ascending node of the equator of Mars. The motion of the perihelion with respect to the

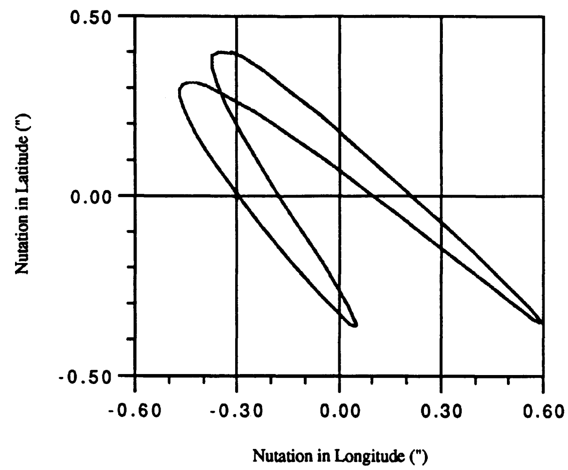

FIG. 3. Path of Mars' true celestial pole with respect to its mean celestial pole. The path of Mars' true celestial pole on the sky over the period of 1 Martian sidereal year (686.93 days). The mean pole moves at a rate of −7″.3 along the small circle of precession. The motion of the mean pole on the sky is about 3.1 arcsec per year. The "boomerang" motion of the true pole about the mean pole is the nutation.

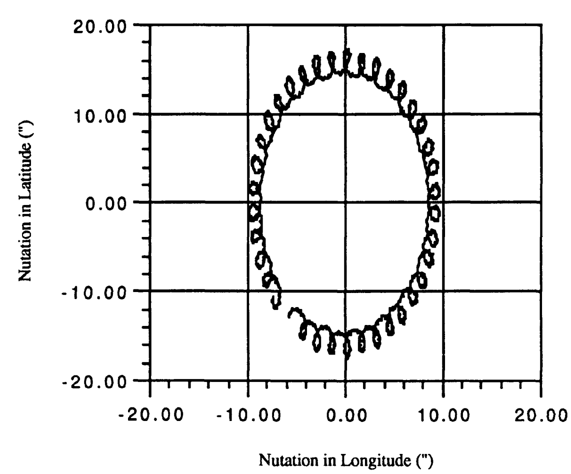

FIG. 4. Path of the Earth's true celestial pole with respect to its mean celestial pole. The path of the Earth's true celestial pole in the sky, over an 18 yr period beginning at J2000.0, with respect to the mean pole. The mean pole moves along a smooth arc at a rate of about 50″ per year along the small circle described by precession. The motion along the small circle is seen as a motion of 20 arcsec per year on the sky. The complex epicyclic motion of the true pole is nutation.

TABLE 3. The apparent amplitude of the combined mean motion and perihelion terms for the three largest nutation components in longitude.

Period (day)	Amplitude (")	Phase (at J2000.0) (°)
686.82	-0.5577	263.27
343.44	1.131	127.20
228.97	0.2433	110.19

equinox, 0°03 yr^{-1}, is very slow in comparison to the orbital motion, 191°417 yr^{-1} (Bretagnon 1982). The very slow motion of the perihelion causes the periods of the dominant nutation components in longitude and latitude to have periods that are nearly simple ratios. This simple structure is seen on the sky as Fig. 3.

The slow motion of the perihelion also causes the nutation components in longitude with the forms of sin $(2\Lambda + n\mathscr{L})$ to appear over short periods of time (decades) as a single nutation with periods of $n\mathscr{L}$ and amplitudes and phases of

$$A \sin(2\Lambda + n\mathscr{L}) + B \sin n\mathscr{L} = \sqrt{A^2 + B^2 + 2AB \sin 2\Lambda} \cos \times \left(n\mathscr{L} + \tan^{-1} \frac{A \cos 2\Lambda + B}{B}\right). \tag{29}$$

At J2000.0

$$\Lambda = 250°70.$$

Table 3 gives the apparent amplitudes and phases for the three largest nutation components in longitude.

APPENDIX A: THE EQUATION OF MOTION

The first four Legendre polynomials used in the equation of motion (2) in geometric algebra form are

$$P_0(\mathbf{r}\cdot\mathbf{r}') = 1, \tag{30}$$

$$P_1(\mathbf{r}\cdot\mathbf{r}') = \mathbf{r}\cdot\mathbf{r}', \tag{31}$$

$$P_2(\mathbf{r}\cdot\mathbf{r}') = \tfrac{1}{2}[3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2r'^2], \tag{32}$$

$$P_3(\mathbf{r}\cdot\mathbf{r}') = \tfrac{1}{2}[5(\mathbf{r}\cdot\mathbf{r}')^3 - 3r^2r'^2(\mathbf{r}'\cdot\mathbf{r}')]. \tag{33}$$

Integrating Eq. (2) for the potential, the $P_1(\mathbf{r}\cdot\mathbf{r}')$ term disappears. The gravitational potential for a planet is then

$$\Phi = \frac{-G}{r}\left(\mathscr{M} + \frac{1}{2}\mathbf{r}\cdot\frac{S(\mathbf{r})}{r^4} + \cdots\right), \tag{34}$$

where

$$S(\mathbf{r}) = r^2\,\mathrm{Tr}(I) - 3\mathbf{r}\cdot I(\mathbf{r}), \tag{35}$$

where I is the inertia tensor and Tr(I) is the trace of the inertia tensor.

The gravitational field of an extended body is found by taking the gradient of Eq. (34)

$$\begin{aligned}\mathbf{g}(\mathbf{r}) &= -\nabla\Phi(\mathbf{r}) \\ &= -\nabla\left[\frac{-G}{r}\left(\mathscr{M} + \tfrac{1}{2}\mathbf{r}\cdot\frac{S(\mathbf{r})}{r^4} + \cdots\right)\right], \\ &= \frac{-G}{r^2}\left(\mathscr{M}\hat{\mathbf{r}} - \frac{1}{r^2}\left[S(\hat{\mathbf{r}}) - \tfrac{5}{2}\hat{\mathbf{r}}\cdot S(\hat{\mathbf{r}})\hat{\mathbf{r}}\right] + \cdots\right).\end{aligned} \tag{36}$$

Substituting for $S(\mathbf{r})$ the gravitational field becomes

$$\mathbf{g}(\mathbf{r}) = \frac{-G\mathscr{M}}{r^2}\left[\hat{\mathbf{r}} - 3J_2\left(\frac{R}{r}\right)^2\{[5(\hat{\mathbf{r}}\cdot\hat{\mathbf{e}})^2 - 1]\hat{\mathbf{r}} - 2[\hat{\mathbf{r}}\cdot\hat{\mathbf{e}}]\hat{\mathbf{e}}\} + \cdots\right], \tag{37}$$

where R is the equatorial radius for the body, $\hat{\mathbf{e}}$ is the vector for the axis of the principal moment of inertia about the pole of the assumed hydrostatic spheroid for the planet, $\hat{\mathbf{r}}$ is the unit vector from the planet to the torquing body, and J_2 is the dynamical form factor defined by

$$J_2 \equiv \frac{I_3 - I}{\mathscr{M}R^2}, \tag{38}$$

where I_3 is the principal moment of inertia about the axis of figure and I is the equatorial principal moment of inertia for a spheroidal planet.

Thus, the force of a point mass m on an extended mass $\mathscr{M}$ is

$$\mathbf{F} = m\mathbf{g}(\mathbf{r}) = \frac{-G\mathscr{M}m}{r^2}\left[\hat{\mathbf{r}} - 3J_2\left(\frac{R}{r}\right)^2\{[5(\hat{\mathbf{r}}\cdot\hat{\mathbf{e}})^2 - 1]\hat{\mathbf{r}} - 2[\hat{\mathbf{r}}\cdot\hat{\mathbf{e}}]\hat{\mathbf{e}}\} + \cdots\right]. \tag{39}$$

The torque of m on $\mathscr{M}$ is perpendicular to the vector from $\mathscr{M}$ to m, that is $\Gamma = \mathbf{r}\times\mathbf{F}$, so all of the terms in the direction of $\hat{\mathbf{r}}$ can be ignored. The first-order perpendicular component of $\mathbf{F}$ is simply

$$\mathbf{F}_\perp \approx -3\frac{G\mathscr{M}m}{r^2}J_2\left(\frac{R}{r}\right)^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{e}})\hat{\mathbf{e}}. \tag{40}$$

The vector for the torque per unit moment of inertia $\mathbf{H}$ is

$$\mathbf{H} = \frac{r\mathbf{F}_\perp}{I} \approx -3\frac{Gm}{r^3}\frac{J_2}{q}(\hat{\mathbf{r}}\cdot\hat{\mathbf{e}})\hat{\mathbf{e}}. \tag{41}$$

The steady precession for a torquing body whose position is fixed with respect to the planet is

$$\boldsymbol{\Omega}_1 = \frac{-H}{\boldsymbol{\Omega}\cdot\hat{\mathbf{e}}}. \tag{42}$$

Substituting in Eq. (41) for the torque per unit moment of inertia gives

$$\boldsymbol{\Omega}_1 \approx \frac{3Gm}{\Omega r^3}\frac{J_2}{q}\frac{\hat{\mathbf{r}}\cdot\hat{\mathbf{e}}}{r}\hat{\mathbf{e}}. \tag{43}$$

This motion is the mean precession of Mars assuming that the position of the body providing the torque is fixed with respect to the planet. Mars, however, is moving with respect to the other masses in the solar system, so the time averaged value of $(\mathbf{r}\cdot\hat{\mathbf{e}})/r$ is needed. The vector $\mathbf{r}$ is a function of time so it is now convenient to introduce coordinates that will represent $\mathbf{r}(t)$ and can be used to produce measurable quantities for the precession and nutation of Mars. The coordinate system most useful for determining $[\mathbf{r}(t)\cdot\hat{\mathbf{e}}]$ and the coordinate system most suited for measuring precession and nutation are rotated with respect to one another. First, the celestial coordinate system for the position of a perturbing body with respect to Mars is the equatorial coordinate system, the Martian Right Ascension α, and Martian declination δ of the perturbing body. These coordinates are defined to be exactly analogous to the Earth's equatorial celestial coordinate system. The epoch used for the coordinate system is J2000.0. In this system of coordinates the dot product $(\mathbf{r}\cdot\hat{\mathbf{e}})/r$ is simply

$$\mathbf{r}(t)\cdot\hat{\mathbf{e}}/r = \sin\delta(t). \quad (44)$$

The direction vector $\hat{\mathbf{e}}$ becomes

$$\hat{\mathbf{e}} = \cos\delta(t)\hat{\alpha}, \quad (45)$$

where $\hat{\alpha}$ is the direction vector in curvilinear coordinates of increasing α. Finally, since the torque is the cross product of **F** and **r**, the motion is in the $\hat{\alpha}$ direction. Substituting these values for $(\mathbf{r}\cdot\hat{\mathbf{e}})/r$ and $\hat{\mathbf{e}}$ into Eq. (43) gives

$$\mathbf{H} = \frac{-3G\mathscr{m}}{r^3}\frac{J_2}{q}\sin\delta(t)\cos\delta(t)\hat{\alpha}. \quad (46)$$

The second coordinate system used is for the measurement of precession and nutation. Precession and nutation involve a change in the orientation of the rotation axis with time, so it is preferable to describe their components in a coordinate system that is independent of the position of the planetary pole. The coordinate system usually used is the Euler angles, ψ, ϕ, and ϵ. The angle ψ is the angle from the equinox of a chosen epoch to the intersection of the ascending node of Mars' equator of date along the plane of the orbit. The angle ϕ is the rotation of a point on the surface of Mars about the axis of rotation. The angle ϵ is the inclination of the equator of Mars to the plane of its orbit about the Sun.

Using the Euler angles and the ecliptic coordinates the angular momentum vector of the planet is

$$\mathbf{l} = I_3\boldsymbol{\Omega}[(\cos\beta\cos\lambda),(\cos\beta\sin\lambda\cos\epsilon - \sin\beta\sin\epsilon), (\cos\beta\sin\lambda\sin\epsilon + \sin\beta\cos\epsilon)], \quad (47)$$

where λ is the ecliptic longitude of the pole of rotation and β is the ecliptic latitude of the pole. Thus, the time derivative of the angular momentum in the ecliptic coordinate system is

$$\begin{aligned}\dot{\mathbf{l}} = I_3\boldsymbol{\Omega}\{&[-\dot\beta\sin\beta\cos\lambda - \dot\lambda\cos\beta\sin\lambda],\\ &[\dot\beta(\sin\beta\sin\lambda\cos\epsilon + \cos\beta\sin\epsilon) + \dot\lambda(\cos\beta\cos\lambda\cos\epsilon) - \dot\epsilon(\cos\beta\sin\lambda\sin\epsilon + \sin\beta\cos\epsilon)],\\ &[\dot\beta(\cos\beta\cos\epsilon - \sin\beta\sin\lambda\sin\epsilon) + \dot\lambda(\cos\beta\cos\lambda\sin\epsilon) + \dot\epsilon(\cos\beta\sin\lambda\cos\epsilon - \sin\beta\sin\epsilon)]\}.\end{aligned} \quad (48)$$

If the epoch of the ecliptic coordinate system is chosen to be near the time of observation, then the angular momentum vector will point near $\lambda = 90°$, $\beta = 90° - \epsilon$, and the obliquity ϵ is approximately constant so

$$\cos\lambda \approx 0, \quad \sin\lambda \approx 1,$$
$$\cos\beta \approx \sin\epsilon, \quad \sin\beta \approx \cos\epsilon,$$

and

$$\dot\epsilon \approx 0. \quad (49)$$

Using approximations (49) in Eq. (48) the time derivative of the angular momentum vector becomes

$$\begin{aligned}\dot{\mathbf{l}} &\approx I_3\boldsymbol{\Omega}[-\dot\lambda\sin\epsilon, -\dot\beta(\cos^2\epsilon + \sin^2\epsilon), \dot\beta(\sin\epsilon\cos\epsilon - \cos\epsilon\sin\epsilon)]\\ &= I_3\boldsymbol{\Omega}(-\dot\lambda\sin\epsilon, -\dot\beta, 0).\end{aligned} \quad (50)$$

It is now possible to resolve **H**, Eq. (43), into its component parts and equate it with $\dot{\mathbf{l}}$. Since $\dot{\mathbf{l}} = \boldsymbol{\Gamma} = \mathscr{M}\mathbf{H}$

$$(-\dot\lambda\sin\epsilon, -\dot\beta, 0) = \frac{3G\mathscr{m}}{r^3\Omega}\frac{J_2}{q}(\sin\delta\cos\delta\sin\alpha, \sin\delta\cos\delta\cos\alpha, 0). \quad (51)$$

The equatorial coordinates used for the position of the body supplying the torque needs to be transformed to the ecliptic coordinate system. The transformation between the two coordinate systems is given in *The Explanatory Supplement to The American Ephemeris and Nautical Almanac* (1974).

APPENDIX B: EXPANSION OF THE EQUATION OF MOTION FOR AN ELLIPTICAL ORBIT

The process of expansion for an elliptical orbit is begun by dropping the assumption of uniform motion and distance from the attracting body. Equations (6) and (7) are rewritten

$$\dot\lambda\sin\epsilon = \frac{-Q}{r^3}\sin\epsilon\cos\epsilon\sin^2(f-\Lambda),$$
$$\dot\beta = \frac{Q}{r^3}\sin\epsilon\sin(f-\Lambda)\cos(f-\Lambda), \quad (52)$$

where $Q = (3\mathscr{m}GJ_2)/(\Omega q)$ and f is the true anomaly of Mars. Using standard trigonometric identities the sine and cosine functions in Eq. (52) are rewritten

$$\sin^2(f-\Lambda) = \tfrac{1}{2} - \tfrac{1}{2}\cos 2f\cos 2\Lambda - \tfrac{1}{2}\sin 2f\sin 2\Lambda$$

and (53)

$$\sin(f-\Lambda)\cos(f-\Lambda) = \tfrac{1}{2}\sin 2f\cos 2\Lambda - \tfrac{1}{2}\cos 2f\sin 2\Lambda.$$

Neither the true anomaly nor the distance from the Sun, r, are linear functions of time. However, both the true anomaly and the distance r can be represented by series expansions, Smart (1953) Chap. 3:

$$\frac{a^3}{r^3} = \sum_{n=0}^{\infty} D_n\cos(n\mathscr{L}),$$
$$\frac{a^3}{r^3}\sin 2f = \sum_{n=0}^{\infty} E_n\sin(n\mathscr{L}), \quad (54)$$
$$\frac{a^3}{r^3}\cos 2f = \sum_{n=0}^{\infty} F_n\cos(n\mathscr{L}),$$

where a is the mean distance of the planet from the Sun. The coefficients for these series have been tabulated by Jarnagin (1965) and the series for Mars, truncated to six significant figures in the eccentricity e, are given in Table 4. The value used for the mean eccentricity e is

$e = 0.093\,400\,619\,9474$ (Bretagnon 1982).

Substituting Eqs. (54) into Eqs. (53) and then substituting that result into equations of motion (52) and simplifying gives the motion in longitude and latitude as

$$\dot\lambda = \frac{-Q}{a^3}\cos\epsilon\left[\frac{D_0}{2} + \sum_{n=1}^{\infty}\left(\frac{D_n}{2} - \frac{F_n}{2}\cos 2\Lambda\right)\right. \left.\times\cos(n\mathscr{L}) - \frac{B_n}{2}\sin 2\Lambda\sin(n\mathscr{L})\right),$$

TABLE 4. Coefficient series in eccentricity, e, for polynomial expansions in the mean longitude, $\mathscr{L}$.

Series		Trig. Func. of $\mathscr{L}$	Coefficient Series	Value
$(a/r)^3$	cos 2f	cos $1\mathscr{L}$	$-e + 3/48\, e^3 - 5/384\, e^5$	-0.0466495
		cos $2\mathscr{L}$	$1 - 5/2\, e^2 + 13/16\, e^4 - 35/288\, e^6$	0.9782526
		cos $3\mathscr{L}$	$7/2\, e - 123/16\, e^3 + 489/128\, e^5$	0.3206656
		cos $4\mathscr{L}$	$17/2\, e^2 - 115/6\, e^4 + 601/48\, e^6 - 1423/360\, e^8$	0.0727009
		cos $5\mathscr{L}$	$845/48\, e^3 - 32525/768\, e^5 + 1457575/43008\, e^7$	0.0140167
		cos $6\mathscr{L}$	$533/16\, e^4 - 13827/160\, e^6 + 728889/8760\, e^8$	0.0024783
		cos $7\mathscr{L}$	$228347/3840\, e^5 - 3071075/18432\, e^7$	0.0004123
		cos $8\mathscr{L}$	$73369/720\, e^6 - 775727/2520\, e^8$	0.0000659
		cos $9\mathscr{L}$	$12144273/71680\, e^7$	0.0000105
		cos $10\mathscr{L}$	$634085\, e^8$	0.0000016
$(a/r)^3$	sin 2f	sin $1\mathscr{L}$	$1/48\, e^3 + 11/768\, e^5$	0.0000171
		sin $2\mathscr{L}$	$2/48\, e^4 + 42/1440\, e^6$	0.0000032
		sin $3\mathscr{L}$	$81/1280\, e^5$	0.0000004
		sin $4\mathscr{L}$	$64/720\, e^6$	0.0000001
$(a/r)^3$		cos $0\mathscr{L}$	$1 + 3/2\, e^2 + 15/8\, e^4 + 35/16\, e^6$	1.0132297
		cos $1\mathscr{L}$	$3\, e + 27/8\, e^3 + 261/64\, e^5$	0.2829808
		cos $2\mathscr{L}$	$9/2\, e^2 + 7/2\, e^4 + 141/32\, e^6$	0.0395258
		cos $3\mathscr{L}$	$53/8\, e^3 + 393/128\, e^5 + 24753/5120\, e^7$	0.0054202
		cos $4\mathscr{L}$	$77/8\, e^4 + 129/80\, e^6$	0.0007336
		cos $5\mathscr{L}$	$1773/128\, e^5$	0.0000985
		cos $6\mathscr{L}$	$3167/160\, e^6$	0.0000131
		cos $7\mathscr{L}$	$432091/15360\, e^7$	0.0000017

$$\dot{\beta} = \frac{Q}{a^3} \sin \epsilon \left(\sum_{n=0}^{\infty} \frac{E_n}{2} \cos 2\Lambda \sin(n\mathscr{L}) - \frac{F_n}{2} \sin 2\Lambda \cos(n\mathscr{L}) \right). \quad (55)$$

The position of the perihelion with respect to the Martian vernal equinox is the difference between the advance of the perihelion and the precession of Mars. Using the Bretagnon (1982) mean orbit for Mars and the circular orbit precession previously determined, the position of the perihelion is

$$\Lambda = 94''.050 - (-7''.488) = 101''.538 \text{ yr}^{-1}.$$

The motion of precession also folds in the effect of the motion of Mars' perihelion with respect to its vernal equinox. However, the change in the precession rate from the circular orbit value is too small to affect the rate of precession from its inclusion in Λ or the amplitudes or periods of the nutation components.

Collecting like terms together, constant Eqs. (55) integrate to give the nutation in longitude and latitude as

$$\lambda = \frac{-Q}{a^3} \cos \epsilon \left(A_0 t + \sum_{n=1}^{\infty} \frac{1}{n\mathscr{L}} [A_n \sin (n\mathscr{L})] - C_n \cos (2\Lambda + n\mathscr{L}) + \frac{B_n}{n\mathscr{L}} \sin(2\Lambda + n\mathscr{L}) \right),$$

$$\beta = \frac{-Q}{a^3} \sin \epsilon \left(\sum_{n=0}^{\infty} \frac{B_n}{n\mathscr{L}} \cos (2\Lambda + n\mathscr{L}) + \frac{C_n}{n\mathscr{L}} \sin (2\Lambda + n\mathscr{L}) \right). \quad (56)$$

The coefficients A_0, A_n, B_n, and C_n are linear combinations of the coefficients in Eq. (54).

APPENDIX C: PRECESSION AND NUTATION FROM CHANGES IN ECCENTRICITY

The Bretagnon (1982) theory for Mars uses the canonical elements $k = e \cos \Pi$ and $h = e \sin \Pi$, where e is the eccentricity and Π is the longitude of perihelion. Both of the elements k and h are given in the form

$$k = \sum_n T^{A_n} (B_n \sin \gamma_n + C_n \cos \gamma_n),$$

$$h = \sum_n T^{D_n} (E_n \sin \gamma_n + F_n \cos \gamma_n), \quad (57)$$

where T is the time in thousands of Julian years from J2000.0, A_n and D_n are the integer exponential coefficients of the time dependence, B_n, C_n, E_n, and F_n are the amplitude coefficients for the sine and cosine functions, and

$$\gamma_n = K_{n1} Z + K_{n2} V + K_{n3} E + K_{n4} M + K_{n5} J + K_{n6} S + K_{n7} U + K_{n8} N, \quad (58)$$

where the variables Z, V, E, M, J, S, U, and N are the mean ecliptic longitudes of Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune, respectively, and the K_{nm} are integer coefficients of the mean longitudes. Only the mean ecliptic longitudes of the Earth, Mars, and Jupiter are involved in those terms which may be large enough to significantly affect the amplitudes of Mars' nutation components.

For a given term n in the series making up the periodic terms in eccentricity, the conversion from h_n and k_n to e_n^2 is

$$e_n^2 = h_n^2 + k_n^2 = G_n + H_n \cos(2\gamma_n + \chi_n), \quad (59)$$

where

$$G_n = \tfrac{1}{2}(B_n^2 + E_n^2 - C_n^2 - F_n^2),$$

$$H_n = \tfrac{1}{4}(C_n^2 + F_n^2 - B_n^2 - E_n^2)^2 + (B_n C_n + E_n F_n)^2,$$

and

$$\chi_n = \arctan \frac{2(B_n C_n + E_n F_n)}{B_n^2 + E_n^2 - C_n^2 - F_n^2}. \quad (60)$$

The conversion from h and k to e^2 has a constant offset in eccentricity and a cosine function with an angular motion that is twice that of the original angular motion in h and k. The eccentricity is determined from e^2 by applying a Taylor series expansion to Eq. (60). To first order the expansion is

$$e = (G_n + H_n \cos \chi_n)^{1/2} - \frac{H_n \sin \chi_n}{2(G_n + H_n \cos \chi_n)^{1/2} \gamma_n}. \quad (61)$$

In Bretagnon's theory for Mars there are seven terms in h and k with $A_n = D_n = 0$ for which the zeroth-order Taylor series term for eccentricity is larger than 0.000 01 and there is one term, with $A_n = D_n = 1$, for which the zeroth-order term is larger than 0.000 001. Inserting the values of these terms into Eq. (41) shows that the largest first-order term in Eq. (41) is a factor of 11.4 smaller than the smallest zeroth-order term. Hence only the zeroth-order terms are significant.

APPENDIX D: PRECESSION AND NUTATION FROM CHANGES IN THE ORBITAL PLANE

Because the change in obliquity is caused by changes in Mars' orbital plane, a constant plane needs to be chosen to describe its motion. The inertial reference plane chosen is the ecliptic plane of the Earth at J2000.0 and the variables determining the orientation of Mars' orbital plane are the inclination of the orbit to the ecliptic i and the position of the ascending node of the orbit on the ecliptic Ω. The general development of the long-period motion of the pole will follow that of Ward (1974) and Reasenberg & King (1979). The main results of this section are (1) a "linear" term added to the precession as a result of the changes in the orbital plane and (2) a "linear" term for the change of the obliquity in time. Both of these terms are the first-order approximations of sinusoidal motions which are approximately linear over periods on the order of 10 000 yr.

The first-order equations of motion are

$$\frac{d\epsilon_1}{dt} = \sin i \cos \psi_0\left(\alpha \cos \epsilon_0 + \frac{d\psi_0}{dt}\right) + \sin \psi_0 \frac{di}{dt},$$
$$\frac{d\psi_1}{dt} = -\alpha t \cos \epsilon_1 - \frac{d\Omega}{dt} - \sin i \cot \epsilon_0 \times \sin \psi_0\left(\alpha \cos \epsilon_0 + \frac{d\psi_0}{dt}\right) + \cot \epsilon_0 \cos \psi_0 \frac{di}{dt}, \tag{62}$$

where ϵ_0 and ψ_0 are the zeroth-order solutions for the motion

$$\epsilon = \epsilon_0$$
and
$$\psi_0 = \Psi - \alpha t \cos \epsilon_0 + \Omega_0 - \Omega(t), \tag{63}$$

where Ψ and Ω_0 are the values of ψ and Ω at some given initial time t_0 and $\Omega(t)$ is the value of Ω as a function of time.

The motion in latitude is rewritten

$$\frac{d\epsilon_1}{dt} = \alpha \cos \epsilon_0 [p \cos(\alpha t \cos \epsilon_0 - \Omega_0 - \Psi) - q \sin(\alpha t \cos \epsilon_0 - \Omega_0 - \Psi)] + \frac{d}{dt} \sin i \sin \psi_0, \tag{64}$$

where $p = \sin i \cos \Omega(t)$ and $q = \sin i \sin \Omega(t)$, the canonical elements (Bretagnon 1982) for the orientation of the Mars' orbit. This allows the motion in latitude to be solved for as a function of the change in the orientation of Mars' orbit with time.

The canonical elements q and p are also functions of time. The first-order solutions for p and q are represented by

$$q_f = \sum_j A_{jf} \sin(\varpi_j t + \delta_j),$$
$$p_f = \sum_j A_{jf} \cos(\varpi_j t + \delta_j). \tag{65}$$

The constants A_{jf}, ϖ_j, and δ_j have been solved for the major solar system bodies, excluding Pluto, by Brouwer & van Woerkom (1950). These constants for Mars, brought forward from the ecliptic and equinox of B1950.0 to the ecliptic and equinox of J2000.0 and modified for the IAU 1984 values for the reciprocal masses of the planets (Kaplan 1981), are given in Table 5.

Substituting Eqs. (65) for p and q into Eqs. (64) gives

$$\frac{d\epsilon_1}{dt} = \alpha \cos \epsilon_0 \sum_j A_{4j} \cos(\varpi_j t + \alpha t \cos \epsilon_0 + \delta_j - \Omega_0 - \Psi) + \frac{d}{dt} \sin i \sin \psi_0. \tag{66}$$

Equation (66) is integrated to obtain the first-order equation for the nutation in latitude of Mars as a function of time resulting from the change in the orientation of Mars' orbit with time.

$$\epsilon_1 = \alpha \cos \epsilon_0 \sum_j \frac{A_{4j}}{\varpi + \alpha \cos \epsilon_0} \times \sin(\varpi_j t + \alpha t \cos \epsilon_0 + \delta_j - \Omega_0 - \Psi) + \sin i \sin \psi_0 + C, \tag{67}$$

where C is the constant of integration. Since the obliquity to the ecliptic at $t = 0$ is ϵ_0, the value of the constant of integration is

$$C = \epsilon_0 - \sum_j \frac{\varpi_j \mathcal{M}_{4j}}{\varpi_j + \alpha \cos \epsilon_0} \sin(\delta_j - \Psi - \Omega_0) \tag{68}$$

and

$$\epsilon_1 = \epsilon_0 - \sum_j \frac{\varpi_j \mathcal{M}_{4j}}{\varpi_j + \alpha \cos \epsilon_0} \times [\sin(\varpi_j t + \alpha t \cos \epsilon_0 + \delta_j - \Psi - \Omega_0) + \sin(\delta_j - \Psi - \Omega_0)]. \tag{69}$$

Plotting this equation for ϵ_1 in Fig. 5 shows a change in obliquity with time that appears to be a beat frequency of two sine waves with periods of 1.16×10^5 yr and 1.29×10^5 yr. These are the frequencies of the perturbations of Jupiter and the Earth which supply 83.1% of the amplitude to the perturbation in latitude. Over short periods of time near the present era, $\pm$ 10 000 yr (Fig. 1), the motion in obliquity of the Martian pole as a result of the change in orientation of Mars' orbit can be represented by the linear function

TABLE 5. Constants for the canonical elements for Mars.

Planet (i)	ϖ_i ("/yr.)	δ_i (°)	A_{4i} (rad.)	A_{4i} (")
1	-5.202	271.99	0.0017940	370.04
2	-6.571	209.97	0.0017989	371.05
3	-18.744	147.13	-0.0359444	-7414.06
5	-17.633	188.68	0.0502514	10365.1
6	-25.734	19.22	0.0096568	1991.86
7	-2.903	207.44	-0.0012561	-259.09
8	-0.678	95.00	-0.0012286	-253.42

Planets are numbered by their increasing mean distance from the Sun

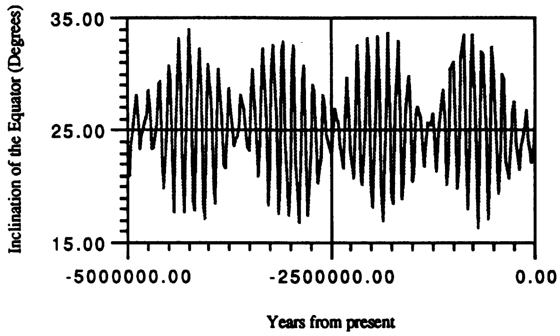

FIG. 5. Very long-period nutation in obliquity. The change in the inclination of Mars' pole with respect to its orbital plane resulting from the change in Mars' orbital plane with time.

$$\epsilon = (25.^\circ 2 \pm 0.^\circ 01) + 0.''4255t, \qquad (70)$$

where t is the time, in years, from J2000.0. This linear equation is obtained by doing a quadratic fit to function (69) for ϵ over the period J2000.0 $\pm$ 10 000 yr. The quadratic term of $7.''157 \times 10^{-6}$ yr^{-2} is insignificant. The uncertainty in the linear term which appears as a "precession" in inclination in the final solar nutation series is $\pm 0.''0012$. The main source of uncertainty is the value of the inertia ratio.

Substituting Eq. (69) for ϵ_1 into Eqs. (62) the first-order equation of motion in longitude becomes

$$\frac{d\psi_1}{dt} = -\alpha \cos\Big(\epsilon_0 - \sum_j \frac{\varpi_j N_j}{\varpi_j + \alpha \cos \epsilon_0} \times [\sin(\varpi_j t + \alpha t \cos \epsilon_0 + \delta_j - \Psi - \Omega_0) + \sin(\delta_j - \Psi - \Omega_0)]\Big) - \frac{d\Omega}{dt} - \sin i \cot \epsilon_0 \times \sin \psi_0 \Big(\alpha \cos \epsilon + \frac{d\Psi_0}{dt}\Big) + \cot \epsilon_0 \cos \psi_0 \frac{di}{dt}. \qquad (71)$$

Approximating $\cos i \approx 1$, putting all of the time derivatives on the left side of the equation, and substituting Eq. (63) for ψ_0 into Eq. (71) gives

$$\frac{d}{dt}(\psi_1 + \Omega - \sin i \cot \epsilon_0 \cos \psi_0) = -\alpha \cos\Big(\epsilon_0 - \sum_j \frac{\varpi_j N_j}{\varpi_j + \alpha \cos \epsilon_0} [\sin(\varpi_j t + \alpha t \cos \epsilon_0 + \delta_j - \Psi - \Omega_0) + \sin(\delta_j - \Psi - \Omega_0)]\Big) + \alpha \sin i \cot \epsilon_0 \cos \epsilon_0 \sin[\Omega(t) + \alpha t \cos \epsilon_0 - \Psi - \Omega_0]. \qquad (72)$$

The above equation, like Eq. (64) for ϵ_1, depends on $(\sin i \sin \Omega)$, so the same set of identities for p and q, Eqs. (65), are substituted to give

$$\frac{d}{dt}(\psi_1 + \Omega - \sin i \cot \epsilon_0 \cos \psi_0) = -\alpha \cos \epsilon_0^* \cos\Big(\sum_j \frac{\varpi_j N_j}{\kappa_j} \sin(\kappa_j + \mu_j)\Big) - \alpha \sin \epsilon_0^* \sin\Big(\sum_j \frac{\varpi_j N_j}{\kappa_j} \sin(\kappa_j + \mu_j)\Big) + \alpha \sin i \cot \epsilon_0 \cos \epsilon_0 \sum_j N_j \sin(\varpi_j t + \alpha t \cos \epsilon_0 + \delta_j - \Psi - \Omega_0), \qquad (73)$$

where

$$\epsilon_0^* = \epsilon_0 + \sum_j \frac{\varpi_j N_j}{\kappa_j} \sin \mu_j,$$

$$\kappa_j = \varpi_j + \alpha \cos \epsilon_0,$$

and

$$\mu_j = \delta_j - \Psi - \Omega_0.$$

The sine and cosine functions in Eq. (73) are approximated in Ward (1974) by

$$\cos\Big(\sum_j \frac{\varpi_j N_j}{\kappa_j} \sin(\kappa_j t + \mu_j)\Big) = 1 \qquad (75)$$

and

$$\sin\Big(\sum_j \frac{\varpi_j N_j}{\kappa_j} \sin(\kappa_j t + \mu_j)\Big) = \sum_j \frac{\varpi_j N_j}{\kappa_j} \sin(\kappa_j t + \mu_j). \qquad (76)$$

However, the total amplitude for the second-order terms for the cosine function approximation is

$$\frac{1}{2} \sum_j \left| \frac{\varpi_j N_j}{\kappa_j} \right|^2 = 0.015\,958\,143$$

which when multiplied by $\alpha \cos \epsilon^*$ gives a total amplitude of $0.''2422 \pm 0.''0007$.

Similarly, the total amplitude for the third-order term in the approximation for the sine function is

$$\frac{1}{6} \sum_j \left| \frac{\varpi_j N_j}{\kappa_j} \right|^3 = 0.000\,950\,3144.$$

Multiplying this by $\alpha \sin \epsilon^*$, the total amplitude of the third-order term is

$$0.''003\,393 \pm 0.''000\,009.$$

Thus, both the second- and third-order terms of the sine and cosine functions are potentially large enough to make a significant contribution to the precession of Mars. Therefore, the equation of motion in longitude resulting from perturbations in Mars' orbit will be approximated to third order by

$$\frac{d}{dt}(\psi_1+\Omega-\sin i\cot\epsilon_0\cos\psi_0)=-\alpha\cos\epsilon_0^*\left[1-\frac{1}{2}\left(\sum_j\frac{\varpi_jN_j}{\kappa_j}\sin(\kappa_jt+\mu_t)\right)^2\right]$$
$$-\alpha\sin\epsilon_0^*\sum_j\left[\frac{\varpi_jN_j}{\kappa_j}\sin(\kappa_jt+\mu_j)+\frac{1}{6}\left(\frac{\varpi_jN_j}{\kappa_j}\sin(\kappa_jt+\mu_j)\right)^3\right]$$
$$+\alpha\cos\epsilon_0\cot\epsilon_0\sum_jN_j\sin(\kappa_jt+\mu_j). \quad (77)$$

D.1 Zeroth and First-Order Longitude Terms

Equation (77) is the same as that derived by Ward (1974) with the addition of the second- and third-order terms in the approximation for the sine and cosine terms. The second- and third-order terms in Eq. (77) are related to the zeroth and first-order terms by addition, so they will be integrated separately and the results added to the first-order terms.

Keeping the zeroth- and first-order terms only, Eq. (77) becomes

$$\frac{d}{dt}(\psi_{11}+\Omega-\sin i\cot\epsilon_0\cos\psi_0)$$
$$=-\alpha\cos\epsilon_0^*-\alpha\sin\epsilon_0^*\sum_j\frac{\varpi_jN_j}{\kappa_j}\sin(\kappa_jt+\mu_j)$$
$$+\alpha\cos\epsilon_0\cot\theta_0\sum_jN_j\sin(\kappa_jt+\mu_j). \quad (78)$$

Integrating Eq. (78) gives

$$\psi_{11}+\Omega(t)-\sin i\cot\epsilon_0\cos\psi_0$$
$$=-\alpha t\cos\epsilon_0^*+\alpha\sin\epsilon_0^*\sum_j\frac{\varpi_jN_j}{\kappa_j^2}\cos(\kappa_jt+\mu_j)$$
$$-\alpha\cos\epsilon_0\cot\epsilon_0\sum_j\frac{N_j}{\kappa_j}\cos(\kappa_jt+\mu_j)+C. \quad (79)$$

Evaluating Eq. (64) for ϵ^* gives

$\epsilon^*=25\overset{\circ}{.}199\pm0\overset{\circ}{.}02.$

This value is indistinguishable from ϵ_0. The values of the other angles at $t(0)$ are $\Omega(0)=\Omega_0$ and $\psi_{11}(0)=\Psi$. Thus, the constant of integration is

$$C'=-\sum_j\frac{\varpi_jN_j}{\sin\epsilon_0\kappa_j^2}(\alpha+\varpi_j\cos\epsilon_0)\cos\mu_j. \quad (80)$$

The first-order approximation for $\psi_1(t)$ is

$$\psi_{11}=\Psi-\alpha t\cos\epsilon_0^*+\Omega_0-\Omega(t)$$
$$+\sum_j\frac{\varpi_jN_j}{\sin\epsilon_0\kappa_j^2}[(\alpha+\varpi_j\cos\epsilon_0)$$
$$\times\cos(\kappa_jt+\mu_j)+\cos\mu_j]. \quad (81)$$

The second term on the right-hand side of Eq. (81) is the already determined precession of Mars. The fourth term is the motion of the node of Mars' orbit along the ecliptic of J2000.0. The first and third terms in Eq. (81) are the positions of the ascending node of Mars' orbit on its equator and the node of the orbit of Mars at J2000.0 with respect to the ecliptic and equinox of the Earth at J2000.0.

The last term on the right side of the equation is the first-order motion of the longitude caused by the changes in the orientation of Mars' orbit with time. As with the motion in latitude, the motion in longitude is represented as a series of nutation components with periods varying from 71 421 yr to 1270 000 yr (Fig. 6). The amplitudes of these changes in longitude are evaluated to range from 32 200″ from the perturbations by Jupiter to 65″.7 from the perturbations by Neptune; however, 88.5% of the total amplitude of these perturbations are the result of the influences of Jupiter, the Earth, and Venus. Over short periods of time (< 10 000 yr) the zeroth- and first-order parts of the first-order solution for precession and nutation in longitude appear as a contribution to the precession of

$\psi_{11}=0\overset{''}{.}2127\pm0\overset{''}{.}0006\ \mathrm{yr}^{-1}$

shown in Fig. 2.

D.2 Second and Third-Order Longitude Terms

Carrying out the squaring for the second-order cosine term in Eq. (77) and simplifying it gives

$$\alpha\cos\epsilon^*\frac{1}{2}\left(\sum_j\frac{\varpi_jN_j}{\kappa_j}\sin(\kappa_jt+\mu_j)\right)^2$$
$$=\alpha\cos\epsilon^*\frac{1}{2}\left\{\sum_j\left[\frac{1}{2}\left(\frac{\varpi_jN_j}{\kappa_j}\right)^2[1-\cos2(\kappa_jt+2\mu_j)]\right]\right.$$
$$\left.+\sum_j\sum_k\left(\frac{\varpi_j\varpi_kN_jN_k}{\kappa_j\kappa_k}\{\cos[(\kappa_j+\kappa_k)t+\mu_j+\mu_k]-\cos[(\kappa_j-\kappa_k)t+\mu_j-\mu_k]\}\right)\right\}, \quad (82)$$

where k runs from $j+1$ to 8. This is then integrated to obtain

$$\alpha\cos\epsilon^*\frac{1}{2}\left[\sum_j\left\{\frac{1}{2}\left[\left(\frac{\varpi_jN_j^2}{\kappa_j}\right)t-\frac{(\varpi_jN_j)^2}{2\kappa_j^3}\sin2(\kappa_jt+2\mu_j)\right]\right\}\right.$$
$$\left.+\sum_j\sum_k\left(\frac{\varpi_j\varpi_kN_jN_k}{\kappa_j\kappa_k}\left\{\frac{1}{\kappa_j+\kappa_k}\sin[(\kappa_j+\kappa_k)t+\mu_j+\mu_k]-\frac{1}{\kappa_j-\kappa_k}\sin[(\kappa_j-\kappa_k)t+\mu_j-\mu_k]\right\}\right)\right]+C. \quad (83)$$

The first term in Eq. (83) evaluates to be a secular precession of

$$\delta\psi_{12} = 0\farcs0797 \pm 0\farcs0002 \ \text{yr}^{-1}.$$

The other three terms in Eq. (83) are 63 periodic components with amplitudes as large as 0″.0650 with periods ranging from 44 229 to 946 700 yr. The evaluation of the periodic terms was carried out using MathCAD 2.0 on a Zenith 248PC. The motion over 5×10^6 yr shows that the main terms combine to form a quasiperiodic function with a period of about 1.17×10^6 yr and an amplitude of 0″.15. However, over the short term the combined effect of these 63 terms is an additional "precession" of only $1\farcs00\times 10^{-6}$ yr^{-1} which is far too small to be significant.

Finally, cubing the third-order sine term in Eq. (77) and simplifying using μMath a simple algebraic manipulation program on a Zenith 248 PC gives

$$\begin{aligned}
-\alpha\sin\epsilon^*\frac{1}{6}\left(\sum_j \frac{\varpi_j N_j}{\kappa_j}\sin(\kappa_j t+\mu_j)\right)^3 = &-\alpha\sin\epsilon^*\frac{1}{6}\left\{\sum_j\left[\left(\frac{\varpi_j N_j}{\kappa_j}\right)^3[3\sin(\kappa_j t+\mu_j)-\sin(3\kappa_j t+3\mu_j)]\right.\right.\\
&-\frac{3}{4}\sum_k \frac{\varpi_j N_j}{\kappa_j}\left(\frac{\varpi_k N_k}{\kappa_k}\right)^2(\sin[(\kappa_j+2\kappa_k)t+\mu_j+2\mu_k]\\
&+\sin[(\kappa_j-2\kappa_k)t+\mu_j-2\mu_k]-2\sin(\kappa_j t+\mu_j))(1-\delta_{jk})\\
&+\frac{3}{2}\sum_l\sum_m \frac{\varpi_j\varpi_l\varpi_m N_j N_l N_m}{\kappa_j\kappa_l\kappa_m}(\sin[(\kappa_j+\kappa_l+\kappa_m)t+\mu_j+\mu_l+\mu_m]\\
&+\sin[(\kappa_j-\kappa_l+\kappa_m)t+\mu_j-\mu_l+\mu_m]+\sin[(\kappa_j+\kappa_l-\kappa_m)t\\
&\left.\left.+\mu_j+\mu_l-\mu_m]+\sin[(\kappa_j-\kappa_l-\kappa_m)t+\mu_j-\mu_l-\mu_m])\right]\right\}, \qquad (84)
\end{aligned}$$

where k is summed from 1 to 8, l is summed from $j+1$ to 8, m is summed from $k+1$ to 8, and δ_{jk} is the Kronecker delta. Equation (84) consists of 980 terms. Integrating this equation with respect to time gives

$$\begin{aligned}
\alpha\sin\epsilon^*\frac{1}{6}&\left\{\sum_j\left[\frac{1}{4}\left(\frac{\varpi_j N_j}{\kappa_j}\right)^3\left(\frac{3}{\kappa_j}\cos(\kappa_j t+\mu_j)-\frac{1}{3\kappa_j}\cos(3\kappa_j t+3\mu_j)\right)\right.\right.\\
&-\frac{3}{4}\sum_k \frac{\varpi_j N_j}{\kappa_j}\left(\frac{\varpi_k N_k}{\kappa_k}\right)^2\left(\frac{1}{\kappa_j+2\kappa_k}\cos[(\kappa_j+2\kappa_k)t+\mu_j+2\mu_k]\right.\\
&\left.+\frac{1}{\kappa_j-2\kappa_k}\cos[(\kappa_j-2\kappa_k)t+\mu_j-2\mu_k]-\frac{2}{\kappa_j}\cos(\kappa_j t+\mu_j)\right)(1-\delta_{jk})\\
&+\frac{3}{2}\sum_l\sum_m \frac{\varpi_j\varpi_l\varpi_m N_j N_l N_m}{\kappa_j\kappa_l\kappa_m}\left(\frac{1}{\kappa_j+\kappa_l+\kappa_m}\cos[(\kappa_j+\kappa_l+\kappa_m)t+\mu_j+\mu_l+\mu_m]+\frac{1}{\kappa_j-\kappa_l+\kappa_m}\right.\\
&\times\cos[(\kappa_j-\kappa_l+\kappa_m)t+\mu_j-\mu_l+\mu_m]+\frac{1}{\kappa_j+\kappa_l-\kappa_m}\cos[(\kappa_j+\kappa_l-\kappa_m)t+\mu_j+\mu_l-\mu_m]\\
&\left.\left.\left.+\frac{1}{\kappa_j-\kappa_l-\kappa_m}\cos[(\kappa_j-\kappa_l-\kappa_m)t+\mu_j-\mu_l-\mu_m]\right)\right]\right\}+C. \qquad (85)
\end{aligned}$$

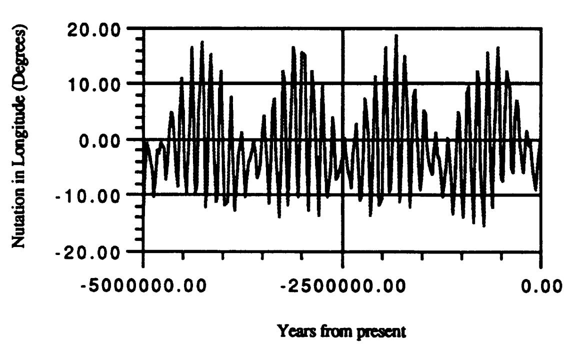

FIG. 6. Very long-period nutation in longitude. The first-order change in the longitude of Mars' pole with respect to its orbital plane resulting from the change in Mars' orbital plane with time.

Evaluating Eq. (85), also using MathCAD, shows that the amplitudes of the individual terms are as large as 0″.002 097 ± 0″.000 006 and periods ranging from 23 807 yr to 1.6×10^7 yr. Evaluation over 5000 yr shows a change in orientation of about 0″.003 with no discernible period. There does, however, appear to be a "short-period" term with a period of about 130 000 yr and an amplitude of about 0″.0003. The short-period (< 10 000 yr) change in orientation with time is negligible. Therefore, the third-order approximation of the sine function does not make a significant addition to the precession of Mars.

APPENDIX E: PLANETARY TORQUES ON MARS

In the zeroth-order estimate for the effect of the planets upon Mars it will be assumed that all the planets in the solar

TABLE 6. Coefficients for the $B_0^{5/2}$ term for the precession caused by the planets on Mars.

Planet	Coefficient
Mercury	2.992
Venus	8.411
Earth	38.006
Jupiter	3.418
Saturn	2.344
Uranus	2.080
Neptune	2.032

TABLE 7. The $B_n^{5/2}$ coefficients for the nutation caused by the Earth and Jupiter on Mars.

n	Earth	Jupiter
0	38.006	3.418
1	36.203	2.163
2	32.188	1.051
3	27.254	0.449
4	22.265	0.178
5	17.701	0.067
6	13.775	0.024
7	10.538	0.009
8	7.949	0.003

system travel in circular orbits in a common plane. These two assumptions reduce Eqs. (21) and (22) to

$$-\dot{\beta} = Q\frac{m}{2r^3}\sin 2\lambda \sin\epsilon,$$

$$-\dot{\lambda}\sin\epsilon = Q\frac{m}{2r^3}\sin 2\epsilon\cos^2\lambda. \qquad (86)$$

Expressions for the "ecliptic" longitude and distance r as a function of time need to be derived before Eqs. (86) can be solved. Basic vector addition represents the position of a planet with respect to Mars by

$$r\sin\lambda = r_p\sin L_p - r_M\sin\mathscr{L},$$

$$r\cos\lambda = r_p\cos L_p - r_M\cos\mathscr{L}, \qquad (87)$$

where r_p is the distance of the planet from the Sun, r_M is the distance of Mars from the Sun, L_p is the heliocentric longitude of the planet, and $\mathscr{L}$ is the heliocentric longitude of Mars. Under the assumption of perturbation free, circular orbits, r_p and r_M are constants and

$$L_p = n_p t + L_{p0},$$

$$\mathscr{L} = n_M t + \mathscr{L}_0, \qquad (88)$$

where n_p and n_M are the mean motion of the planet and Mars, respectively, and L_{p0} and $\mathscr{L}_0$ are the heliocentric longitudes of the planet and Mars at some epoch t_0. For ease of notation, the heliocentric longitudes of epoch, L_{p0} and $\mathscr{L}_0$, will be dropped until they are needed for evaluation of the precession and nutation.

Using Eqs. (87) for $\sin\lambda$ and $\cos\lambda$ in Eqs. (86) gives

$$-\dot{\beta} = Q\frac{m}{r}\sin\epsilon\frac{1}{r^2}\left(\frac{r_p^2}{2}\sin 2n_p t + \frac{r_M^2}{2}\cos 2n_M t - r_p r_M\sin(n_p + n_M)\right) \qquad (89)$$

and

$$-\dot{\lambda}\sin\epsilon = Q\frac{m}{2r^3}\sin 2\epsilon\frac{1}{r^2}\left(\frac{r_p^2}{2} + \frac{r_M^2}{2} + \frac{r_p^2}{2}\cos 2n_p t + \frac{r_M^2}{2}\cos 2n_M t + r_p r_M\cos(n_p + n_M)t + r_p r_M\cos(n_p - n_M)t\right). \qquad (90)$$

The determination of the value of r as a function of time begins with the cosine law

$$r^2 = r_p^2 + r_M^2 - 2r_p r_M\cos(n_p - n_M)t. \qquad (91)$$

In both (89) and (90) r appears as some power r^{-s}, where s is a positive integer. Using Eq. (91) r^{-s} is rewritten

$$r^{-s} = [r_p^2 + r_M^2 - 2r_p r_M\cos(n_p - n_M)t]^{-s/2}$$

$$= r_p^{-s}\left[1 + \left(\frac{r_M}{r_p}\right)^2 - 2\frac{r_M}{r_p}\cos(n_p - n_M)t\right]^{-s/2} \qquad (92)$$

or

$$= r_M^{-s}\left[1 + \left(\frac{r_p}{r_M}\right)^2 - 2\frac{r_p}{r_M}\cos(n_p - n_M)t\right]^{-s/2}. \qquad (93)$$

Both of Eqs. (92) and (93) have the general form $(1 + x)^{-s/2}$, where $x = b^2 - 2b\cos(n_p - n_M)t$ and b is either the ratio r_p/r_M or r_M/r_p. The solution to these equations, provided the smaller of the ratios r_p/r_M or r_M/r_p is always used, is

$$r^{-s} = r_\alpha^{-s}\left\{\frac{1}{2}B_0^{s/2} + \sum_k B_k^{s/2}\cos[k(n_p - n_M)t]\right\}, \qquad (94)$$

where r_α is the smaller of r_p or r_M and $B_k^{s/2}$ are the Laplace coefficients. The Laplace coefficients frequently arise in classical celestial mechanics multibody problems such as the perturbation of the orbit of one planet by others. The method of solving for the value of any Laplace coefficients can be found in several places including Smart (1953). The values for $B_0^{5/2}$ for each of the planets torque on Mars are given in Table 6 and the values of $B_0^{5/2}$ through $B_8^{5/2}$ for the Earth and Jupiter with Mars are given in Table 7.

Substituting Eq. (94) into the equation for the motion in longitude of the pole of Mars, Eq. (90), gives

$$-\dot{\lambda}\sin\epsilon=\frac{Qm}{r_\alpha^3}\sin 2\epsilon\Big\{\frac{1}{2}B_0^{5/2}\Big(\frac{1}{4}(1+\alpha^2)+\frac{1}{4}\cos 2n_Mt+\frac{1}{4}\alpha^2\cos 2n_pt$$
$$+\frac{\alpha}{2}\cos(n_p+n_M)t+\frac{\alpha}{2}\cos(n_p-n_M)t\Big)+\sum_k B_k^{5/2}\Big(\frac{1}{2}(1+\alpha^2)\cos k(n_p-n_M)t$$
$$+\frac{1}{4}\cos[kn_p-(k-2)n_M]t+\frac{1}{4}\cos[kn_p-(k+2)n_M]t+\frac{r_p^2}{4}\{\cos[(k-2)n_p-kn_M]t$$
$$+\cos[(k+2)n_p-kn_M]t\}+\tfrac{1}{2}\alpha\{\cos[(k+1)n_p-(k-1)n_M]t$$
$$+\cos[(k-1)n_p-(k+1)n_M]t+\cos[(k+1)(n_p-n_M)]t+\cos[(k+1)(n_p-n_M)]t\}\Big)\Big\}. \quad (95)$$

This equation is integrated to yield

$$-\lambda\sin\epsilon=\frac{Qm}{r_\alpha^3}\sin 2\epsilon\Big\{\frac{1}{2}B_0^{5/2}\Big[\frac{1}{4}(1+\alpha^2)t+\frac{1}{8n_M}\sin 2n_Mt+\frac{\alpha^2}{8n_p}\sin 2n_pt$$
$$+\frac{\alpha}{2}\Big\{\frac{1}{n_p+n_M}\sin(n_p+n_M)t+\frac{1}{n_p-n_M}\sin(n_p-n_M)t\Big\}\Big]$$
$$+\sum_k\frac{1}{2}B_k^{5/2}\Big[\frac{(1+\alpha^2)}{k(n_p-n_M)}\sin k(n_p-n_M)t+\frac{1}{2[kn_p-(k-2)n_M]}\sin[kn_p-(k-2)n_M]t$$
$$+\frac{1}{8[kn_p-(k+2)n_M]}\sin[kn_p-(k+2)n_M]t$$
$$+\frac{\alpha^2}{2}\Big\{\frac{1}{[(k-2)n_p-kn_M]}\sin[(k-2)n_p-kn_M]t+\frac{1}{[(k+2)n_p-kn_M]}\sin[(k+2)n_p-kn_M]t\Big\}$$
$$+\alpha\Big\{\frac{1}{[(k+1)n_p-(k-1)n_M]}\cos[(k+1)n_p-(k-1)n_M]t$$
$$+\frac{1}{[(k-1)n_p-(k+1)n_M]}\sin[(k-1)n_p-(k+1)n_M]t$$
$$+\frac{1}{[(k+1)(n_p-n_M)]}\sin[(k+1)(n_p-n_M)]t$$
$$+\frac{1}{[(k+1)(n_p-n_M)]}\cos[(k+1)(n_p-n_M)]t\Big\}\Big]\Big\}. \quad (96)$$

Like any equation involving numerous sine functions Eq. (96) looks daunting but it is actually quite simple.
Similarly, substituting the equation for the expression $r^{-s/2}$ into the equation for the motion in latitude gives

$$-\dot{\beta}=\frac{Qm}{2r_\alpha^3}\sin\epsilon\Big[B_0^{5/2}[\alpha^2\sin 2n_pt+\sin 2n_Mt-2\alpha\sin(n_p-n_M)t]$$
$$+\sum_k B_k^{5/2}\Big(\frac{\alpha^2}{2}\{\sin[(k+2)n_p-kn_M]t+\sin[(k-2)n_p-kn_M]t\}$$
$$+\tfrac{1}{2}\{\sin[kn_p-(k-2)n_M]t+\sin[kn_p-(k+2)n_M]t\}$$
$$-\alpha\{\sin[(k+1)n_p-(k-1)n_M]t+\sin[(k-1)n_p-(k+1)n_M]t\}\Big)\Big]. \quad (97)$$

The integral of Eq. (97) is

$$-\beta=\frac{Qm}{2r_\alpha^3}\sin\epsilon\Big\{B_0^{5/2}\Big(\frac{\alpha^2}{2n_p}\cos 2n_pt+\frac{1}{2n_M}\cos 2n_Mt-\frac{2\alpha}{n_p-n_M}\cos(n_p-n_M)t\Big)$$
$$+\sum_k B_k^{5/2}\Big[\frac{\alpha^2}{2}\Big\{\frac{1}{(k+2)n_p-kn_M}\cos[(k+2)n_p-kn_M]t+\frac{1}{(k-2)n_p-kn_M}\cos[(k-2)n_p-kn_M]t\Big\}$$

$$+\frac{1}{2}\left(\frac{1}{kn_p-(k-2)n_M}\cos[kn_p-(k-2)n_M]t+\frac{1}{kn_p-(k+2)n_M}\cos[kn_p-(k+2)n_M]t\right)$$
$$-\alpha\left(\frac{1}{(k+1)n_p-(k-1)n_M}\cos[(k+1)n_p-(k-1)n_M]t\right.$$
$$\left.+\frac{1}{(k-1)n_p-(k+1)n_M}\cos[(k-1)n_p-(k+1)n_M]t\right)\Bigg]\Bigg\}. \quad (98)$$

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