Gauge symmetry of the *N*-body problem in the Hamilton–Jacobi approach

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In most books the Delaunay and Lagrange equations for the orbital elements are derived by the Hamilton–Jacobi method: one begins with the two-body Hamilton equations in spherical coordinates, performs a canonical transformation to the orbital elements, and obtains the Delaunay system. A standard trick is then used to generalize the approach to the N-body case. We reexamine this step and demonstrate that it contains an implicit condition which restricts the dynamics to a $9(N-1)$-dimensional submanifold of the $12(N-1)$-dimensional space spanned by the elements and their time derivatives. The tacit condition is equivalent to the constraint that Lagrange imposed ''by hand'' to remove the excessive freedom, when he was deriving his system of equations by variation of parameters. It is the condition of the orbital elements being osculating, i.e., of the instantaneous ellipse (or hyperbola) being always tangential to the physical velocity. Imposure of any supplementary condition different from the Lagrange constraint (but compatible with the equations of motion) is legitimate and will not alter the physical trajectory or velocity (though will alter the mathematical form of the planetary equations). This freedom of nomination of the supplementary constraint reveals a gauge-type internal symmetry instilled into the equations of celestial mechanics. Existence of this internal symmetry has consequences for the stability of numerical integrators. Another important aspect of this freedom is that any gauge different from that of Lagrange makes the Delaunay system noncanonical. In a more general setting, when the disturbance depends not only upon positions but also upon velocities, there is a ''generalized Lagrange gauge'' wherein the Delaunay system is symplectic. This special gauge renders orbital elements that are osculating in the phase space. It coincides with the regular Lagrange gauge when the perturbation is velocity independent.
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I. EULER AND LAGRANGE

A. The history

The planetary equations, which describe the evolution of the orbital elements, constitute the cornerstone of the celestial mechanics. Description of orbits in the language of Keplerian elements (rather than in terms of the Cartesian coordinates) is not only physically illustrative but also provides the sole means for analysis of resonant interactions. These equations exist in a variety of equivalent forms (those of Lagrange, Delaunay, Gauss, Poincare) and can be derived by several different methods.

The earliest sketch of the method dates back to Euler's paper of 1748, which addresses the perturbations exerted upon one another by Saturn and Jupiter. In the publication on the Lunar

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motion, dated by 1753, Euler derived the equations for the longitude of the node, Ω, the inclination, i, and the quantity $p \equiv a(1-e^2)$. Time derivatives of these three elements were expressed through the components of the disturbing force. Sixty years later the method was amended by Gauss who wrote down similar equations for the other three elements and, thus, obtained what we now call the Gauss system of planetary equations. The history of this scientific endeavour was studied by Subbotin (1958), who insists that the Gauss system of planetary equations should rather be called Euler system. A modern but still elementary derivation of this system belongs to Burns (1976).

In his mémoires of 1778, which received an honorable prize from the Académie des Sciences of Paris, Lagrange employed the method of variation of parameters (VOP) to express the time derivatives of the orbital elements through the disturbing functions' partial derivatives with respect to the Cartesian coordinates. In his mémoire of 1783, Lagrange furthered this approach, while in Lagrange (1808, 1809, 1810) these equations acquired their final, closed, shape: they expressed the orbital elements' evolution in terms of the disturbing potentials' derivatives with respect to the elements. Lagrange's derivation rested upon an explicit imposure of the osculation condition, i.e., of a supplementary vector equation (called the Lagrange constraint) which guaranteed that the instantaneous ellipses (in the case of bound motions) or hyperbolae (in the case of flyby ones) are always tangential to the physical trajectory. Though it has been long known (and, very possibly, appreciated by Lagrange himself) that the choice of the supplementary conditions is essentially arbitrary, and though a couple of particular examples of use of nonosculating elements appeared in the literature (Goldreich, 1965; Brumberg *et al.*, 1971; Borderies and Longaretti, 1987), a comprehensive study of the associated freedom has not appeared until very recently (Efroimsky, 2002, 2003).

In the middle of the 19th century Jacobi applied a canonical-transformation-based procedure (presently known as the Hamilton–Jacobi approach) to the orbital dynamics, and offered a method of deriving the Lagrange system. This technique is currently regarded standard and is offered in many books. Though the mathematical correctness of the Hamilton–Jacobi method is beyond doubt, its application to celestial mechanics contains an aspect that has long been overlooked (at least, in the astronomical literature). This overlooked question is as follows: where in the Hamilton–Jacobi derivation of the planetary equations is the Lagrange constraint tacitly imposed, and what happens if we impose a different constraint? This issue will be addressed in our article.

B. The gauge freedom

Mathematically, we shall concentrate on the *N*-body problem of celestial mechanics, a problem that for each body can be set as

$$\ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^2}\frac{\vec{\mathbf{r}}}{r} = \Delta\vec{\mathbf{F}}, \tag{1}$$

$\Delta\vec{\mathbf{F}}$ being the disturbing force that vanishes in the (reduced) two-body case and $\vec{\mathbf{r}}$ being the position relative to the primary, and μ standing for $G(m_{\text{planet}} + m_{\text{sun}})$. A solution to the unperturbed problem is a Keplerian ellipse (or hyperbola)

$$\vec{\mathbf{r}} = \vec{\mathbf{f}}(C_1,\ldots,C_6,t) \tag{2}$$

parametrized by six constants (which may be, for example, the Kepler or Delaunay elements). In the framework of the VOP approach it gives birth to the ansatz

$$\vec{\mathbf{r}} = \vec{\mathbf{f}}(C_1(t),\ldots,C_6(t),t), \tag{3}$$

the "constants" now being time dependent and the functional form of $\vec{\mathbf{f}}$ remaining the same as in (2). Substitution of (3) into (1) results in three scalar equations for six independent functions $C_i(t)$. In order to make the problem defined, Lagrange applied three extra conditions

$$\sum_i \frac{\partial \vec{f}}{\partial C_i} \frac{dC_i}{dt} = 0 \tag{4}$$

that are often referred to as ''the Lagrange constraint.'' This constraint guarantees osculation, i.e., that the functional dependence of the perturbed velocity upon the ''constants'' is the same as that of the unperturbed one. This happens because the physical velocity is

$$\dot{\vec{r}} = \vec{g} + \sum_i \frac{\partial \vec{f}}{\partial C_i} \frac{dC_i}{dt}, \tag{5}$$

where $\vec{g}$ stands for the unperturbed velocity that emerged in the two-body setting. This velocity is, by definition, a partial derivative of $\vec{f}$ with respect to the last variable:

$$\vec{g}(C_1,\ldots,C_6,t) \equiv \frac{\partial \vec{f}(C_1,\ldots,C_6,t)}{\partial t}. \tag{6}$$

This choice of supplementary conditions is convenient, but not at all necessary. A choice of any other three scalar relations (consistent with one another and with the equations of motion) will give the same physical trajectory, even though the appropriate solution for nonosculating variables C_i will differ from the solution for osculating ones.

Efroimsky (2002, 2003) suggested to relax the Lagrange condition and to consider

$$\sum_i \frac{\partial \vec{f}}{\partial C_i} \frac{dC_i}{dt} = \vec{\Phi}(C_1,\ldots,C_6,t), \tag{7}$$

$\vec{\Phi}$ being an arbitrary function of time, ''constants'' C_i and their time derivatives of all orders. For practical reasons it is convenient to restrict $\vec{\Phi}$ to a class of functions that depend upon the time and the ''constants'' only. (The dependence upon derivatives would yield higher-than-first-order time derivatives of the C_i in the subsequent developments, which would require additional initial conditions, beyond those on $\vec{r}$ and $\dot{\vec{r}}$, to be specified in order to close the system.) Different choices of $\vec{\Phi}$ entail different forms of equations for C_i and, therefore, different mathematical solutions in terms of these ''constants.'' A transition from one such solution to another will, though, be a mere reparametrization of the orbit. The physical orbit itself will remain invariant. Such invariance of the physical content of a theory under its mathematical reparametrizations is called gauge symmetry. On the one hand, it is in a close analogy with the gradient invariance of the Maxwell electrodynamics and other field theories. On the other hand, it illustrates some general mathematical structure emerging in the ODE theory (Newman and Efroimsky, 2003).

If the Lagrange gauge (4) is fixed, the parameters obey the equation

$$\sum_j [C_n \ C_j] \frac{dC_j}{dt} = \frac{\partial \vec{f}}{\partial C_n} \Delta \vec{F}, \tag{8}$$

$[C_n \ C_j]$ standing for the unperturbed (i.e., defined as in the two-body case) Lagrange brackets:

$$[C_n \ C_j] \equiv \frac{\partial \vec{f}}{\partial C_n} \frac{\partial \vec{g}}{\partial C_j} - \frac{\partial \vec{f}}{\partial C_j} \frac{\partial \vec{g}}{\partial C_n}. \tag{9}$$

To arrive at formula (8), one should, according to Lagrange (1778, 1783, 1808, 1809), differentiate (5), insert the outcome into (1), and then combine the result with the Lagrange constraint (4). (In the modern literature, this derivation can be found, for example, in Brouwer and Clemence (1961), Efroimsky (2002, 2003), Newman and Efroimsky (2003), Efroimsky and Goldreich (2003).) In the

simplest case the perturbing force depends only upon positions and is conservative: $\Delta\vec{\mathbf{F}} = \partial R(\vec{\mathbf{r}})/\partial\vec{\mathbf{r}}$. Then the right-hand side of (8) will reduce to the partial derivative of the disturbing function $R(\vec{\mathbf{r}})$ with respect to C_n, whereafter inversion of the Lagrange-bracket matrix will entail' the Lagrange system of planetary equations (for C_i being the Kepler elements) or the Delaunay system (for the parameters chosen as the Delaunay elements).

As explained in Efroimsky (2003), in an arbitrary gauge $\vec{\Phi}$ Eq. (8) will generalize to its gauge-invariant form

$$\sum_j \left[C_n \ C_j \right] \frac{dC_j}{dt} = \frac{\partial\vec{\mathbf{f}}}{\partial C_n} \Delta\vec{\mathbf{F}} - \frac{\partial\vec{\mathbf{f}}}{\partial C_n} \frac{d\vec{\Phi}}{dt} - \frac{\partial\vec{\mathbf{g}}}{\partial C_n} \vec{\Phi}, \tag{10}$$

the Lagrange brackets $\left[C_n \ C_j \right]$ being still defined through (9). If we agree that $\vec{\Phi}$ is a function of both time and the parameters C_i, but not of their derivatives, then the right-hand side of (10) will implicitly contain the first time derivatives of C_i. It will then be reasonable to move them into the left-hand side. Hence, (10) will be reshaped into

$$\sum_j \left(\left[C_n \ C_j \right] + \frac{\partial\vec{\mathbf{f}}}{\partial C_n} \frac{\partial\vec{\Phi}}{\partial C_j} \right) \frac{dC_j}{dt} = \frac{\partial\vec{\mathbf{f}}}{\partial C_n} \Delta\vec{\mathbf{F}} - \frac{\partial\vec{\mathbf{f}}}{\partial C_n} \frac{\partial\vec{\Phi}}{\partial t} - \frac{\partial\vec{\mathbf{g}}}{\partial C_n} \vec{\Phi}. \tag{11}$$

This is the general form of the gauge-invariant perturbation equations of celestial mechanics, which follows from the VOP method, for an arbitrary disturbing force $\Delta\vec{\mathbf{F}}(\vec{\mathbf{r}},\dot{\vec{\mathbf{r}}},t)$ and under the simplifying assumption that the arbitrary gauge function $\vec{\Phi}$ is chosen to depend on the time and the parameters C_i, but not on their derivatives.

For performing further algebraic developments of (10) and (11), let us decide what sort of interactions will fall within the realm of our interest. On general grounds, it would be desirable to deal with forces that permit description in the language of Lagrangians and Hamiltonians.

II. DELAUNAY

A. Perturbations of Lagrangians and Hamiltonians

Contributions to the disturbing force $\Delta\vec{\mathbf{F}}$ generally consist of two types, physical and inertial. Inputs can depend upon velocity as well as upon positions. As motivation for this generalization we consider two practical examples. One is the problem of orbital motion around a precessing planet: the orbital elements are defined in the precessing frame, while the velocity-dependent fictitious forces play the role of the perturbation (Goldreich, 1965; Brumberg et al., 1971; Efroimsky and Goldreich, 2003). Another example is the relativistic two-body problem where the relativistic correction to the force is a function of both velocity and position, as explained, for example, in Brumberg (1992) and Klioner and Kopeikin (1994). (It turns out that in relativistic dynamics even the two-body problem is disturbed, the relativistic correction acting as disturbance. This yields the gauge symmetry that will cause ambiguity in defining the orbital elements of a binary.) Finally, we shall permit the disturbances to bear an explicit time dependence. Such a level of generality will enable us to employ our formalism in noninertial coordinate systems.

Let the unperturbed Lagrangian be $\dot{\vec{\mathbf{r}}}^2/2 - U(\vec{\mathbf{r}})$. The disturbed motion will be described by the new, perturbed, Lagrangian

$$\mathcal{L} = \frac{\dot{\vec{\mathbf{r}}}^2}{2} - U(\vec{\mathbf{r}}) + \Delta\mathcal{L}(\vec{\mathbf{r}},\dot{\vec{\mathbf{r}}},t), \tag{12}$$

and the appropriately perturbed canonical momentum and Hamiltonian,

$$\vec{\mathbf{p}} = \dot{\vec{\mathbf{r}}} + \frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}}, \tag{13}$$

$$\mathcal{H} = \vec{\mathbf{p}} \dot{\vec{\mathbf{r}}} - \mathcal{L} = \frac{\vec{\mathbf{p}}^2}{2} + U + \Delta \mathcal{H}, \quad \Delta \mathcal{H} \equiv -\Delta \mathcal{L} - \frac{1}{2} \left(\frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \right)^2. \tag{14}$$

The Euler–Lagrange equations written for the perturbed Lagrangian (12) are

$$\ddot{\vec{\mathbf{r}}} = -\frac{\partial U}{\partial \vec{\mathbf{r}}} + \Delta \vec{\mathbf{F}}, \tag{15}$$

where the disturbing force is given by

$$\Delta \vec{\mathbf{F}} \equiv \frac{\partial \Delta \mathcal{L}}{\partial \vec{\mathbf{r}}} - \frac{d}{dt} \left(\frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \right). \tag{16}$$

We see that in the absence of velocity dependence the perturbation of the Lagrangian plays the role of disturbing function. Generally, though, the disturbing force is not equal to the gradient of $\Delta \mathcal{L}$, but has an extra term called into being by the velocity dependence.

As we already mentioned, this setup is sufficiently generic. For example, it is convenient for description of a satellite orbiting a wobbling planet: the inertial forces, which emerge in the planet-related noninertial frame, will nicely fit in the above formalism.

It is worth emphasizing once again that, in the case of velocity-dependent disturbances, the disturbing force is equal neither to the gradient of the Lagrangian's perturbation nor to the gradient of negative Hamiltonian's perturbation. This is an important thing to remember when comparing results obtained by different techniques. For example, in Goldreich (1965) the term "disturbing function" was used for the negative perturbation of the Hamiltonian. For this reason, the gradient of a so defined disturbing function was not equal to the disturbing force. A comprehensive comparison of the currently developed theory with that offered in Goldreich (1965) will be presented in a separate publication (Efroimsky and Goldreich, 2003), where we shall demonstrate that the method used there was equivalent to fixing a special gauge (one described in Sec. II C of this article).

B. Gauge-invariant planetary equations

Insertion of the generic force (16) into (10) will bring us

$$\sum_j \left[C_n \ C_j \right] \frac{dC_j}{dt} = \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \frac{\partial \Delta \mathcal{L}}{\partial \vec{\mathbf{r}}} - \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \frac{d}{dt} \left(\vec{\boldsymbol{\Phi}} + \frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \right) - \frac{\partial \vec{\mathbf{g}}}{\partial C_n} \vec{\boldsymbol{\Phi}}. \tag{17}$$

If we recall that, for a velocity-dependent disturbance,

$$\frac{\partial \Delta \mathcal{L}}{\partial C_n} = \frac{\partial \Delta \mathcal{L}}{\partial \vec{\mathbf{r}}} \frac{\partial \vec{\mathbf{f}}}{\partial C_n} + \frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \frac{\partial \dot{\vec{\mathbf{r}}}}{\partial C_n} = \frac{\partial \Delta \mathcal{L}}{\partial \vec{\mathbf{r}}} \frac{\partial \vec{\mathbf{f}}}{\partial C_n} + \frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \frac{\partial (\vec{\mathbf{g}} + \vec{\boldsymbol{\Phi}})}{\partial C_n}, \tag{18}$$

then equality (17) will look like this:

$$\sum_j \left[C_n \ C_j \right] \frac{dC_j}{dt} = \frac{\partial \Delta \mathcal{L}}{\partial C_n} - \frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \frac{\partial \vec{\boldsymbol{\Phi}}}{\partial C_n} - \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \frac{d}{dt} \left(\vec{\boldsymbol{\Phi}} + \frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \right) - \frac{\partial \vec{\mathbf{g}}}{\partial C_n} \left(\vec{\boldsymbol{\Phi}} + \frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \right). \tag{19}$$

After subsequent addition and subtraction of $\frac{1}{2} \partial ((\partial \Delta \mathcal{L} / \partial \dot{\vec{\mathbf{r}}})^2) / \partial C_n$ on the right-hand side, the gauge function $\vec{\boldsymbol{\Phi}}$ will everywhere appear in the company of $+ \partial (\Delta \mathcal{L}) / \partial \dot{\vec{\mathbf{r}}}$:

$$\sum_j \left([C_n \ C_j] + \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \frac{\partial}{\partial C_j} \left(\frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} + \vec{\boldsymbol{\Phi}} \right) \right) \frac{dC_j}{dt}$$

$$= \frac{\partial}{\partial C_n} \left[\Delta \mathcal{L} + \frac{1}{2} \left(\frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \right)^2 \right] - \left(\frac{\partial \vec{\mathbf{g}}}{\partial C_n} + \frac{\partial \vec{\mathbf{f}}}{\partial C_n} \frac{\partial}{\partial t} + \frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \frac{\partial}{\partial C_n} \right) \left(\vec{\boldsymbol{\Phi}} + \frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} \right), \quad (20)$$

the sum in square brackets being equal to $-\Delta \mathcal{H}$. While (11) expressed the VOP method in the most generic form it can have in terms of disturbing forces $\Delta \vec{\mathbf{F}}(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t)$, Eq. (20) furnishes the most general form in terms of the Lagrangian perturbation $\Delta \mathcal{L}(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t)$ (under the simplifying assumption that the arbitrary gauge function $\vec{\boldsymbol{\Phi}}$ is set to depend only upon the time and the parameters C_i, but not upon their derivatives).

The Lagrange brackets in (19) are gauge-invariant; they contain only functions $\vec{\mathbf{f}}$ and $\vec{\mathbf{g}}$ that were defined in the unperturbed, two-body, setting. This enables us to exploit the well-known expressions for the inverse of this matrix. These look most simple (and are either zero or unity) in the case when one chooses as the ''constants'' the Delaunay set of orbital variables. As is well known, this simplicity of the Lagrange and their inverse, Poisson, brackets of the Delaunay elements is the proof of these elements' canonicity in the unperturbed, two-body, problem. When only a position-dependent disturbing function $R(\vec{\mathbf{r}}) = \Delta \mathcal{L}(\vec{\mathbf{r}})$ is ''turned on,'' the Delaunay elements still remain canonical, provided the Lagrange gauge is imposed. This happens because, as is well known (Brouwer and Clemence, 1961), the equations of motion together with the Lagrange constraint yield, in that case, the following equation,

$$\sum_j [C_n \ C_j] \frac{dC_j}{dt} = \frac{\partial \Delta \mathcal{L}}{\partial C_n}, \quad \Delta \mathcal{L} = \Delta \mathcal{L}(\vec{\mathbf{f}}(C_1, \ldots, C_6, t)) = R(\vec{\mathbf{f}}(C_1, \ldots, C_6, t)), \quad (21)$$

which, is its turn, results in the standard Delaunay system.

In our case, though, the perturbation depends also upon velocities; beside this, the gauge $\vec{\boldsymbol{\Phi}}$ is set arbitrary. Then our Eq. (20) will entail the gauge-invariant Lagrange-type and Delaunay-type systems of equations that are presented in Appendix A. Interestingly, the gauge-invariant Delaunay-type system is, generally, nonsymplectic. It regains the canonical form only in one special gauge considered below (a gauge which coincides with the Lagrange gauge when the perturbation bears no velocity dependence). This can be proven by a direct substitution of that special gauge condition into the gauge-invariant Delaunay-type system given in Appendix A. An easier way would be to fix the gauge already in (20), and this is what we shall do in the next subsection.

C. The generalized Lagrange gauge: Gauge wherein the Delaunay-type system becomes canonical

We transformed (17) into (20) for two reasons: to single out the negative perturbation of the Hamiltonian and to reveal the advantages of the gauge

$$\vec{\boldsymbol{\Phi}} = -\frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}}, \quad (22)$$

which reduces to $\vec{\boldsymbol{\Phi}} = 0$ for velocity-independent perturbations. The first remarkable peculiarity of (22) is that in this gauge the canonical momentum $\vec{\mathbf{p}}$ is equal to $\vec{\mathbf{g}}$ [as can be seen from (5) and (13)]:

$$\vec{\mathbf{g}} = \dot{\vec{\mathbf{r}}} - \vec{\boldsymbol{\Phi}} = \dot{\vec{\mathbf{r}}} + \frac{\partial \Delta \mathcal{L}}{\partial \dot{\vec{\mathbf{r}}}} = \vec{\mathbf{p}}. \quad (23)$$

We see that in this gauge not the velocity but the momentum in the disturbed setting is the same function of time and "constants" as it used to be in the unperturbed, two-body case. Stated differently, the instantaneous ellipses (or hyperbolae) defined in this gauge will osculate the orbit **in the phase space**. For this reason our special gauge (22) will be called the "generalized Lagrange gauge."

Another good feature of (22) is that in this gauge Eq. (20) acquires an especially simple form:

$$\sum_j [C_n \ C_j] \frac{dC_j}{dt} = -\frac{\partial \Delta \mathcal{H}}{\partial C_n}, \tag{24}$$

whose advantage lies not only in its brevity, but also in the invertibility of the matrix emerging on its left-hand side.

As already mentioned above, the gauge invariance of definition (9) enables us to employ the standard (Lagrange-gauge) expressions for $[C_n \ C_j]^{-1}$ and, thus, to get the planetary equations by inverting matrix $[C_n \ C_j]$ in (19). The resulting gauge-invariant Lagrange- and Delaunay-type systems are presented in Appendix A. In the special gauge (22), however, the situation is much better. Comparing (21) with (24), we see that in the general case of an arbitrary $R = \Delta \mathcal{L}(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t)$ one arrives from (24) to the same equations as from (21), except that now they will contain $-\Delta \mathcal{H}$ instead of $R = \Delta \mathcal{L}$. These will be the Delaunay-type equation in the generalized Lagrange gauge:

$$\frac{dL}{dt} = \frac{\partial \Delta \mathcal{H}}{\partial (-M_o)}, \quad \frac{d(-M_o)}{dt} = -\frac{\partial \Delta \mathcal{H}}{\partial L}, \tag{25}$$

$$\frac{dG}{dt} = \frac{\partial \Delta \mathcal{H}}{\partial (-\omega)}, \quad \frac{d(-\omega)}{dt} = -\frac{\partial \Delta \mathcal{H}}{\partial G}, \tag{26}$$

$$\frac{dH}{dt} = \frac{\partial \Delta \mathcal{H}}{\partial (-\Omega)}, \quad \frac{d(-\Omega)}{dt} = -\frac{\partial \Delta \mathcal{H}}{\partial H}, \tag{27}$$

where

$$L \equiv \mu^{1/2} a^{1/2}, \quad G \equiv \mu^{1/2} a^{1/2} (1 - e^2)^{1/2}, \quad H \equiv \mu^{1/2} a^{1/2} (1 - e^2)^{1/2} \cos i. \tag{28}$$

We see that in this special gauge the Delaunay-type equations indeed become canonical, and the role of the effective new Hamiltonian is played exactly by the Hamiltonian perturbation which emerged earlier in (14).

Thus we have proven an interesting **THEOREM: Even though the gauge-invariant Delaunay-type system (A7)−(A12) is not generally canonical, it becomes canonical in one special gauge (22) which we call the "generalized Lagrange gauge."** This theorem can be proved in a purely Hamiltonian language, as is done in Sec. III C.

III. HAMILTON AND JACOBI

A. The concept

A totally different approach to derivation of the planetary equations is furnished by the technique worked out in 1834–1835 by Hamilton and refined several years later by Jacobi. In the lecture course shaped by 1842 and published as a book in 1866, Jacobi formulated his famous theorem and applied it to the celestial motions. Jacobi chose orbital elements that were some combinations of the Keplerian ones. His planetary equations can be easily transformed into the Lagrange system by the differentiation chain rule (Subbotin, 1968). Later authors used this method for a direct derivation of the Lagrange and Delaunay systems, and thus the Hamilton–Jacobi approach became a part and parcel of almost any course in celestial mechanics. To some of these
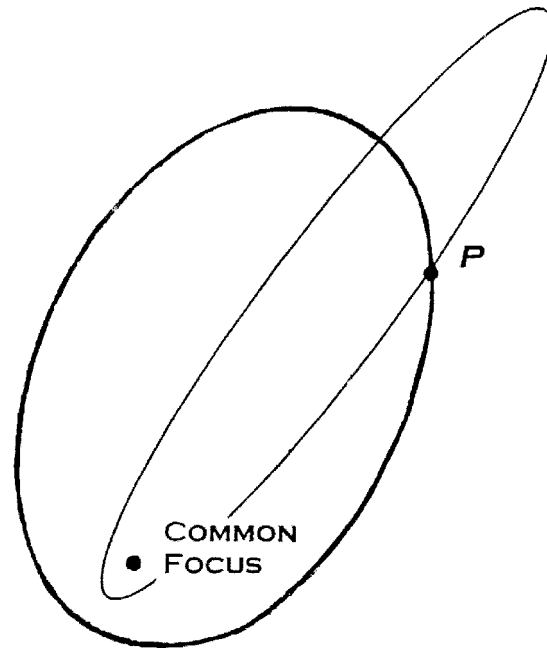
FIG. 1. These two coplanar ellipses share one of their foci and are assumed to rotate about this common focus in the same direction, always remaining within their plane. Suppose that the rotation of one ellipse is faster than that of the other, and that a planet is located at one of the points of these ellipses' intersection, *P*, and that the rotation of the ellipses is such that the planet is always at the instantaneous point of their intersection. We may say that the planet is swiftly moving along the slower rotating ellipse, while it would be equally legitimate to state that the planet is slowly moving along the fast-rotating ellipse. Both interpretations are valid, because one can divide, in an infinite number of ways, the actual motion of the planet into a motion along some ellipse and a simultaneous evolution of that ellipse. The Lagrange constraint (4) singles out, of all the multitude of evolving ellipses, that unique ellipse which is always tangential to the total, physical, velocity of the planet.

sources we shall refer below. The full list of pertinent references would be endless, so it is easier to single out a couple of books that break the code by offering alternative proofs: these exceptions are Kaula (1968) and Brouwer and Clemence (1961).

Brouwer and Clemence (1961) use the VOP method [like in Lagrange (1808, 1809, 1810)], which makes the imposition of the Lagrange constraint explicit. Kaula (1968) undertakes, by means of the differentiation chain rule, a direct transition from the Hamilton equations in a Cartesian frame to those in terms of orbital elements. As explained in Efroimsky (2002, 2003), in Kaula's treatment the Lagrange constraint was imposed tacitly.

It is far less easy to understand where the implicit gauge fixing is used in the Hamilton–Jacobi technique. This subtlety of the Hamilton–Jacobi method is so well camouflaged that through the century and a half of the method's life this detail has never been brought to light (at least, in the astronomical literature). The necessity of such a constraint is evident: one has to choose one out of infinitely many sets of orbital elements describing the physical trajectory. Typically, one prefers the set of orbital elements that osculates with the trajectory, so that the physical orbit be always tangential to the instantaneous ellipse, in the case of bound orbits, or to the instantaneous hyperbola, in the case of flybys. This point is most easily illustrated by the following simple example depicted on Fig. 1. Consider two coplanar ellipses with one common focus. Let both ellipses rotate, in the same direction within their plane, about the shared focus; and let us assume that the rotation of one ellipse is faster than that of the other. Now imagine that a planet is located at one of the points of these ellipses' intersection, and that the rotation of the ellipses is such that the planet is always at the instantaneous point of their intersection. One observer will say that the planet is swiftly moving along the slower rotating ellipse, while another observer will argue that the planet is slowly moving along the fast-rotating ellipse. Both viewpoints are acceptable, be-

cause one can divide, in an infinite number of ways, the actual motion of the planet into a motion along some ellipse and a simultaneous evolution of that ellipse. The Lagrange constraint (4) singles out, of all the multitude of evolving ellipses, that unique ellipse which is always tangential to the total (physical) velocity of the body. This way of gauge fixing is natural but not necessary. Besides, as we already mentioned, the chosen gauge (4) will not be preserved in the course of numerical computations. Sometimes osculating elements do not render an intuitive picture of the motion. In such situations other elements are preferred. One such example is a circular orbit about an oblate planet. The osculating ellipse precesses with the angular velocity of the satellite, and its eccentricity is proportional to the oblateness factor J_2. Under these circumstances the so-called geometric elements are more convenient than the osculating ones (Borderies and Longaretti 1987).

We remind the reader that the Hamilton–Jacobi treatment is based on the simple facts that the same motion can be described by different mutually interconnected canonical sets $(q,p,\mathcal{H}(q,p))$ and $(Q,P,\mathcal{H}^*(Q,P))$, and that fulfillment of the Hamilton equations along the trajectory makes the infinitesimally small quantities

$$d\theta = pdq - \mathcal{H}dt \tag{29}$$

and

$$d\tilde{\theta} = PdQ - \mathcal{H}^*dt \tag{30}$$

perfect differentials. Subtraction of the former from the latter shows that their difference,

$$-dW \equiv d\tilde{\theta} - d\theta = PdQ - pdq - (\mathcal{H}^* - \mathcal{H})dt, \tag{31}$$

is a perfect differential, too. Here q, p, Q, and P contain N components each. If we start with a system described by $(q,p,\mathcal{H}(q,p))$, it is worth looking for such a reparametrization $(Q,P,\mathcal{H}^*(Q,P))$ that the new Hamiltonian H^* is constant in time, because in these variables the canonical equations simplify. Especially convenient is to find a transformation that nullifies the new Hamiltonian $\mathcal{H}^*$, for in this case the new canonical equations will render the variables (Q, P) constant. One way of seeking such transformations is to consider W as a function of only q, Q, and t. Under this assertion, the above equation will entail

$$-\frac{\partial W}{\partial t}dt - \frac{\partial W}{\partial Q}dQ - \frac{\partial W}{\partial q}dq = PdQ - pdq + (\mathcal{H} - \mathcal{H}^*)dt, \tag{32}$$

whence

$$P = -\frac{\partial W}{\partial Q}, \quad p = \frac{\partial W}{\partial q}, \quad \mathcal{H} + \frac{\partial W}{\partial t} = \mathcal{H}^*. \tag{33}$$

The function W can be then found by solving the Jacobi equation

$$\mathcal{H}\left(q, \frac{\partial W}{\partial q}, t\right) + \frac{\partial W}{\partial t} = \mathcal{H}^*, \tag{34}$$

where $\mathcal{H}^*$ is a constant. It is very convenient to make it equal to zero. Then the above equation can be easily solved in the unperturbed (reduced) two-body setting. This solution, which has long been known, is presented, in a very compact form, in Appendix B. It turns out that, if the spherical coordinates and their conjugate momenta are taken as a starting point, then the eventual canonical variables Q, P corresponding to $\mathcal{H}^*(Q,P) = 0$ are the Delaunay elements:

$$Q_1 \equiv L = \sqrt{\mu a}, \quad P_1 = -M_o,$$

$$Q_2 \equiv G = \sqrt{\mu a(1-e^2)}, \quad P_2 = -\omega, \tag{35}$$

$$Q_3 \equiv H = \sqrt{\mu a(1-e^2)} \cos i, \quad P_3 = -\Omega. $$

B. Where can free cheese be found?

The transition from two-body to *N*-body celestial mechanics is presented in numerous books. However, none of them explain how the Lagrange constraint is implicitly involved in the formalism.

Before we move on, let us cast a look back at what has been accomplished in the two-body case. We started out with a Hamiltonian problem $(q, p, \mathcal{H})$ and reformulated its equations of motion

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \tag{36}$$

in terms of another set $(Q, P, \mathcal{H}^*)$:

$$q = \phi(Q, P, t), $$
$$p = \psi(Q, P, t), \tag{37}$$

so that the above equations are mathematically equivalent to the new system

$$\dot{Q} = \frac{\partial \mathcal{H}^*}{\partial P}, \quad \dot{P} = -\frac{\partial \mathcal{H}^*}{\partial Q}. \tag{38}$$

The simple nature of the two-body setting enabled us to carry out this transition so that our new Hamiltonian $\mathcal{H}^*$ vanishes and the variables Q and P are, therefore, constants. This was achieved by means of a transformation-generating function $W(q, Q, t)$ obeying the Jacobi equation (34). After formula (B12) for this function is written down, the explicit form of dependence (37) can be found through the relations $P = -\partial W / \partial Q$. This is given by (B15).

To make this machinery function in an *N*-body setting, let us first consider a disturbed two-body case. The number of degrees of freedom is still the same (three coordinates q and three conjugate momenta p), but the initial Hamiltonian is perturbed:

$$\dot{q} = \frac{\partial(\mathcal{H} + \Delta\mathcal{H})}{\partial p}, \quad \dot{p} = -\frac{\partial(\mathcal{H} + \Delta\mathcal{H})}{\partial q}. \tag{39}$$

Trying to implement the Hamilton-Jacobi method (32)–(34), for the new Hamiltonians $(\mathcal{H} + \Delta\mathcal{H})$, $(\mathcal{H}^* + \Delta\mathcal{H})$ and for some generating function $V(q, Q, t)$, we shall arrive at

$$-\frac{\partial V}{\partial t} dt - \frac{\partial V}{\partial Q} dQ - \frac{\partial V}{\partial q} dq = P\,dQ - p\,dq + [(\mathcal{H} + \Delta\mathcal{H}) - (\mathcal{H}^* + \Delta\mathcal{H})]\,dt, \tag{40}$$

$$P = -\frac{\partial V}{\partial Q}, \quad p = \frac{\partial V}{\partial q}, \quad \mathcal{H} + \Delta\mathcal{H} + \frac{\partial V}{\partial t} = \mathcal{H}^* + \Delta\mathcal{H}, \tag{41}$$

$$\mathcal{H}\left(q, \frac{\partial V}{\partial q}, t\right) + \frac{\partial V}{\partial t} + \mathcal{H}^*. \tag{42}$$

We see that V obeys the same equation as W and, therefore, may be chosen to coincide with it. Hence, the new, perturbed, solution (q,p) will be expressed through the perturbed "constants" $Q(t)$ and $P(t)$ in the same fashion as the old, undisturbed, q and p were expressed through the old constants Q and P:

$$q = \phi(Q(t), P(t), t),$$

$$p = \psi(Q(t), P(t), t),$$

(43)

ϕ and ψ being the same functions as those in (37). Benefitting from this form-invariance, one can now master the N-particle problem. To this end, one should choose the transformation-generating function V to be additive over the particles, whereafter the content of Sec. III A shall be repeated verbatim for each of the bodies, separately. In the end of this endeavour one will arrive to $N-1$ Delaunay sets similar to (B15), except that now these sets will be constituted by **instantaneous** orbital elements. The extension of the preceding subsection's content to the N-body case seems to be most straightforward and to involve no additional assumptions. To dispel this illusion, two things should be emphasized. One, self-evident, fact is that the quantities Q and P are no longer conserved after the disturbance $\Delta\mathcal{H}$ is added to the zero Hamiltonian $\mathcal{H}^*$. The second circumstance is that a change in a Hamiltonian implies an appropriate alteration of the Lagrangian. In the simple case of $\Delta\mathcal{H}$ being a function of the coordinates and time only (not of velocities or momenta), addition of $\Delta\mathcal{H}$ to the Hamiltonian implies addition of its opposite to the Lagrangian. Since this extra term bears no dependence upon velocities, the expressions for momenta through the coordinates and time will stay form-invariant. Hence (if the Lagrangian is not singular), the functional dependence of the velocities upon the coordinates and momenta will, also, preserve their functional form $v(q,p,t)$:

$$\text{without perturbation:} \quad p \equiv \frac{\partial\mathcal{L}(q,\dot{q},t)}{\partial\dot{q}} \Rightarrow \dot{q} = v(q,p,t),$$

(44)

$$\text{with perturbation:} \quad p \equiv \frac{\partial(\mathcal{L}(q,\dot{q},t) + \Delta\mathcal{L}(q,t))}{\partial\dot{q}} = \frac{\partial\mathcal{L}(q,\dot{q},t)}{\partial\dot{q}} \Rightarrow \dot{q} = v(q,p,t),$$

where the new, perturbed dependence $\dot{q} = v[q(Q(t), P(t), t), p(Q(t), P(t), t), t]$ has the same functional form as the old one, $\dot{q} = v[q(Q, P, t), p(Q, P, t), t]$. Together with (43), this means that the dependence of the new $\dot{q}$ upon the new $P(t)$ and $Q(t)$ will have the same functional form as the dependence of the old $\dot{q}$ upon the constants Q and P:

$$\frac{d}{dt}q(Q(t), P(t), t) = \frac{\partial}{\partial t}q(Q(t), P(t), t).$$

(45)

In other words,

$$\sum_{i=1}^{6} \frac{\partial q}{\partial D_i}\dot{D}_i = 0,$$

(46)

where D_i denotes the set of perturbed variables $(Q(t), P(t))$. In the astronomical applications, D_i may stand for the Delaunay set.

This is the implicit condition under which the Hamilton–Jacobi method works (in the above case of velocity-independent disturbance). Violation of (46) would invalidate our cornerstone assumption (38). This circumstance imposes a severe restriction on the applicability of the Hamilton–Jacobi theory. In the astronomical context, this means that the Delaunay elements (B15) must be osculating. Indeed, if D_i denote a set of orbital elements, then expression (46) is equiva-

lent to the Lagrange constraint (4) discussed in Sec. I. There the constraint was imposed upon the Keplerian elements; however, its equivalence to (46), which is written in terms of the Delaunay variables, can be easily proven by the differentiation chain rule.

C. The case of momentum-dependent disturbances

When the perturbation of the Lagrangian depends also upon velocities (and, therefore, the Hamiltonian perturbation carries dependence upon the canonical momenta), the special gauge (22) wherein the Delaunay-type system preserves its canonicity differs from the Lagrange gauge. This was proven in Sec. II C in the Lagrangian language. Now we shall study this in Hamiltonian terms. Our explanation will be sufficiently general and will surpass the celestial-mechanics setting. For this reason we shall use notations q, p, not $\vec{\mathbf{r}}, \vec{\mathbf{p}}$. The development will, as ever, begin with an unperturbed system described by canonical variables obeying

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}. \tag{47}$$

This dynamics may be reformulated in terms of the new quantities (Q, P):

$$
\begin{aligned}
q &= \phi(Q, P, t), \\
p &= \psi(Q, P, t),
\end{aligned}
\tag{48}
$$

so that the Hamiltonian equations (47) are equivalent to

$$\dot{Q} = \frac{\partial \mathcal{H}^*}{\partial P}, \quad \dot{P} = -\frac{\partial \mathcal{H}^*}{\partial Q}. \tag{49}$$

For simplicity, we shall assume that $\mathcal{H}^*$ is zero. Then the new canonical variables will play the role of adjustable constants upon which the solution (48) of (47) depends.

We now wish to know under what circumstances a modified canonical system

$$\dot{q} = \frac{\partial(\mathcal{H} + \Delta\mathcal{H})}{\partial p}, \quad \dot{p} = -\frac{\partial(\mathcal{H} + \Delta\mathcal{H})}{\partial q}, \quad \Delta\mathcal{H} = \Delta\mathcal{H}(q, p, t) \tag{50}$$

will be satisfied by the solution

$$
\begin{aligned}
q &= \phi(Q(t), P(t), t), \\
p &= \psi(Q(t), P(t), t)
\end{aligned}
\tag{51}
$$

of the same functional form as (48) but with time-dependent parameters obeying

$$\dot{Q} = \frac{\partial \Delta\mathcal{H}}{\partial P}, \quad \dot{P} = -\frac{\partial \Delta\mathcal{H}}{\partial Q}. \tag{52}$$

This situation is of a more general sort than that addressed in Sec. III B, in that the perturbation $\Delta\mathcal{H}$ now depends also upon the momentum.

Under the assumption of (48) being (at least, locally) invertible, substitution of the equalities

$$\dot{Q} = \frac{\partial \Delta\mathcal{H}}{\partial P} = \frac{\partial \Delta\mathcal{H}}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial \Delta\mathcal{H}}{\partial p}\frac{\partial p}{\partial P} \tag{53}$$

and

$$\dot{P} = -\frac{\partial \Delta \mathcal{H}}{\partial Q} = -\frac{\partial \Delta \mathcal{H}}{\partial q}\frac{\partial q}{\partial Q} - \frac{\partial \Delta \mathcal{H}}{\partial p}\frac{\partial p}{\partial Q} \tag{54}$$

into the expression for velocity

$$\dot{q} = \frac{\partial q}{\partial t} + \frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial P}\dot{P} \tag{55}$$

leads to

$$\dot{q} = \frac{\partial q}{\partial t} + \left(\frac{\partial q}{\partial Q}\frac{\partial q}{\partial P} - \frac{\partial q}{\partial P}\frac{\partial q}{\partial Q}\right)\frac{\partial \Delta \mathcal{H}}{\partial q} + \left(\frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial q}{\partial P}\frac{\partial p}{\partial Q}\right)\frac{\partial \Delta \mathcal{H}}{\partial p}. \tag{56}$$

Here the coefficient accompanying $\partial \Delta \mathcal{H}/\partial q$ identically vanishes, while that accompanying $\partial \Delta \mathcal{H}/\partial p$ coincides with the Jacobian of the canonical transformation and is, therefore, unity:

$$\frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial q}{\partial P}\frac{\partial p}{\partial Q} = 1. \tag{57}$$

So if we introduce, in the spirit of (6), notation

$$g \equiv \frac{\partial q}{\partial t}, \tag{58}$$

then (56) will lead to

$$\dot{q} = g + \left(\frac{\partial \Delta \mathcal{H}}{\partial p}\right)_{q,t}. \tag{59}$$

Expression (59) establishes the link between the regular VOP method and the canonical treatment. It shows that, to preserve the symplectic description, one must always choose a particular gauge $\Phi = \partial \Delta \mathcal{H}/\partial p$. Needless to say, this is exactly the generalized Lagrange gauge (22) discussed in Sec. II C. A direct, though very short, proof is as follows.

   On the one hand, the Hamilton equation for the perturbed Hamiltonian (14) is

$$\dot{q} = \frac{\partial(\mathcal{H} + \Delta \mathcal{H})}{\partial p} = p + \frac{\partial \Delta \mathcal{H}}{\partial p}, \tag{60}$$

while, on the other hand, the definition of momentum entails, for the Lagrangian (12),

$$p \equiv \frac{\partial(\mathcal{L}(q,\dot{q},t) + \Delta \mathcal{L}(q,\dot{q},t))}{\partial \dot{q}} = \dot{q} + \frac{\partial \Delta \mathcal{L}}{\partial \dot{q}}. \tag{61}$$

By comparing the latter with the former we arrive at

$$\Phi \equiv \left(\frac{\partial \Delta \mathcal{H}}{\partial p}\right)_{q,t} = -\left(\frac{\partial \Delta \mathcal{L}}{\partial \dot{q}}\right)_{q,t}, \tag{62}$$

which coincides with (22). Thus we see that transformation (48) being canonical is equivalent to the partition of the physical velocity $\dot{q}$ in a manner prescribed by (59), where $\Phi = \partial \Delta \mathcal{H}/\partial p$. This is equivalent to our theorem from Sec. II C. Evidently, for disturbances dependent solely upon the coordinates, we return to the case explained in Sec. III B [Eqs. (45) and (46)]: in that case, the Hamiltonian formulation of the problem demanded imposition of the Lagrange constraint (46).

To draw to a close, the generalized Lagrange constraint, $\vec{\Phi} = -\partial\Delta\mathcal{L}/\partial\dot{q}$, is stiffly embedded in the Hamilton–Jacobi technique. Hence this technique breaks the gauge invariance and is unfit (at least, in its straightforward form) to describe the gauge symmetry of the planetary equations. It is necessary to sacrifice gauge freedom by imposing the generalized Lagrange constraint to make use of the Hamilton–Jacobi development.

In this special gauge, the perturbed momentum coincides with the unperturbed one (which was equal to $\vec{\mathbf{g}}$). Indeed, we can rewrite (61) as

$$p \equiv \frac{\partial(\mathcal{L}(q,\dot{q},t) + \Delta\mathcal{L}(q,\dot{q},t))}{\partial\dot{q}} = \dot{q} - \Phi = g, \tag{63}$$

which means that, in the astronomical implementation of this theory, the Hamilton–Jacobi treatment necessarily demands the orbital elements to osculate in the phase space. Naturally, this demand reduces to that of regular osculation in the simple case of velocity-independent $\Delta\mathcal{L}$ that was explored in Sec. III B.

IV. CONCLUSIONS

We have studied, in an arbitrary gauge, the VOP method in celestial mechanics in the case when the perturbation depends on both positions and velocities. Such situations emerge when relativistic corrections to the Newton law are taken into account or when the VOP method is employed in noninertial frames of reference (a satellite orbiting a precessing planet being one such example). The gauge-invariant (and generalized to the case of velocity-dependent disturbances) Delaunay-type system of equations is not canonical. We, though, have proven a theorem establishing a particular gauge (which coincides with the Lagrange gauge in the absence of velocity dependence of the perturbation) that renders this system canonical. We called that gauge the ''generalized Lagrange gauge.''

We have explained where the Lagrange constraint tacitly enters the Hamilton–Jacobi derivation of the Delaunay equations. This constraint turns out to be an inseparable (though not easily visible) part of the method: in the case of momentum-independent disturbances, the *N*-body generalization of the two-body Hamilton–Jacobi technique is legitimate only if we use orbital elements that are osculating, i.e., if we exploit only the instantaneous ellipses (or hyperbolae, in the flyby case) that are always tangential to the velocity vector. Oddly enough, an explicit mention of this circumstance has not appeared in the astronomical literature (at least to the best of our knowledge).

In the case of momentum-dependent disturbances, the above restriction generalizes, in that the instantaneous ellipses (hyperbolae) must be osculating in the phase space. This is equivalent to the imposition of the generalized Lagrange gauge.

Comparing the good old VOP method with that based on the Jacobi theorem, we have to acknowledge that the elegance of the latter does not outweigh the power of the former. If we decide to explore the infinite multitude of gauges or to study the numerical-error-invoked gauge drift, we shall not be able to employ the Hamilton–Jacobi theory without additional structure. However, the direct VOP method unencumbered with the canonicity demand will immediately yield gauge-invariant equations for the Delaunay elements obeying an arbitrary gauge condition

$$\sum_i \frac{\partial\vec{\mathbf{f}}}{\partial D_i}\frac{dD_i}{dt} = \vec{\Phi}(D_i, t), \tag{64}$$

$\vec{\Phi}$ being some function of time and elements D_i. In Efroimsky (2002) these equations were written down for the case of velocity-independent perturbation. If the disturbing force depends also upon velocities, the Delaunay-type equations will acquire even more terms and will read as (A7)–(A12). In the simple case of a velocity-independent disturbance, any supplementary condi-

tion different from that of Lagrange will drive the Delaunay system away from its canonical form. If we permit the disturbing force to depend also upon velocities, the Delaunay equations will retain their canonicity only in the generalized Lagrange gauge.

In the language of modern physics, this may be put in the following wording. *N*-body celestial mechanics is a gauge theory but is not genuinely symplectic insofar as the language of orbital elements is used. It, though, becomes canonical in the generalized Lagrange gauge.

The applications of this formalism to motions in noninertial frames of reference will be studied in Efroimsky and Goldreich (2003). Some other applications were addressed in Slabinski (2003).

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APPENDIX A: GAUGE-INVARIANT EQUATIONS OF LAGRANGE AND DELAUNAY TYPES

We present the gauge-invariant Lagrange-type equations. They follow from (19) if we take into account the gauge-invariance of matrix $[C_i \ C_j]$ defined by (9). We denote by $\Delta\mathcal{H}$ the perturbation of the Hamiltonian, connected through (14) with that of the Lagrangian. The latter, in its turn, is connected through (16) with the disturbing force (and acts as the customary disturbing function when the perturbations are devoid of velocity dependence):

$$\frac{da}{dt} = \frac{2}{na}\left[\frac{\partial(-\Delta\mathcal{H})}{\partial M_o} - \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial M_o}\left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) - \left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial M_o} - \frac{\partial\vec{\mathbf{f}}}{\partial M_o}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right],$$

(A1)

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e}\left[\frac{\partial(-\Delta\mathcal{H})}{\partial M_o} - \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial a}\left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) - \left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial M_o} - \frac{\partial\vec{\mathbf{f}}}{\partial M_o}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right]$$

$$- \frac{(1-e^2)^{1/2}}{na^2 e}\left[\frac{\partial(-\Delta\mathcal{H})}{\partial\omega} - \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial\omega}\left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) - \left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial\omega}\right.$$

$$\left. - \frac{\partial\vec{\mathbf{f}}}{\partial\omega}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right],$$

(A2)

$$\frac{d\omega}{dt} = \frac{-\cos i}{na^2(1-e^2)^{1/2}\sin i}\left[\frac{\partial(-\Delta\mathcal{H})}{\partial i} - \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial i}\left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right.$$

$$\left. - \left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial i} - \frac{\partial\vec{\mathbf{f}}}{\partial i}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right] + \frac{(1-e^2)^{1/2}}{na^2 e}\left[\frac{\partial(-\Delta\mathcal{H})}{\partial e} - \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial e}\left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right.$$

$$\left. - \left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial e} - \frac{\partial\vec{\mathbf{f}}}{\partial e}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}} + \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right],$$

(A3)

$$\frac{di}{dt} = \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\left[\frac{\partial(-\Delta\mathcal{H})}{\partial\omega} - \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial\omega}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) - \left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial\omega}\right.$$

$$\left. - \frac{\partial\vec{\mathbf{f}}}{\partial\omega}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right] - \frac{1}{na^2(1-e^2)^{1/2}\sin i}\left[\frac{\partial(-\Delta\mathcal{H})}{\partial\Omega} - \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial\Omega}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right.$$

$$\left. - \left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial\Omega} - \frac{\partial\vec{\mathbf{f}}}{\partial\Omega}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right], \tag{A4}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2(1-e^2)^{1/2}\sin i}\left[\frac{\partial(-\Delta\mathcal{H})}{\partial i} - \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial i}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right.$$

$$\left. - \left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial i} - \frac{\partial\vec{\mathbf{f}}}{\partial i}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right], \tag{A5}$$

$$\frac{dM_o}{dt} = -\frac{1-e^2}{na^2e}\left[\frac{\partial(-\Delta\mathcal{H})}{\partial e} - \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial e}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) - \left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial e} - \frac{\partial\vec{\mathbf{f}}}{\partial e}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right]$$

$$- \frac{2}{na}\left[\frac{\partial(-\Delta\mathcal{H})}{\partial a} - \frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial a}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) - \left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial a} - \frac{\partial\vec{\mathbf{f}}}{\partial a}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\right]. \tag{A6}$$

Similarly, the gauge-invariant Delaunay-type system can be written down as

$$\frac{dL}{dt} = \frac{\partial(-\Delta\mathcal{H})}{\partial M_o} - \frac{\partial\Delta L}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial M_o}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) - \left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial M_o} - \frac{\partial\vec{\mathbf{f}}}{\partial M_o}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right), \tag{A7}$$

$$\frac{dM_o}{dt} = -\frac{\partial(-\Delta\mathcal{H})}{\partial L} + \frac{\partial\Delta L}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial L}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) + \left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial L} + \frac{\partial\vec{\mathbf{f}}}{\partial L}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right), \tag{A8}$$

$$\frac{dG}{dt} = \frac{\partial(-\Delta\mathcal{H})}{\partial\omega} - \frac{\partial\Delta L}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial\omega}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) - \left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial\omega} - \frac{\partial\vec{\mathbf{f}}}{\partial\omega}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right), \tag{A9}$$

$$\frac{d\omega}{dt} = -\frac{\partial(-\Delta\mathcal{H})}{\partial G} + \frac{\partial\Delta L}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial G}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) + \left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial G} + \frac{\partial\vec{\mathbf{f}}}{\partial G}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right), \tag{A10}$$

$$\frac{dH}{dt} = \frac{\partial(-\Delta\mathcal{H})}{\partial\Omega} - \frac{\partial\Delta L}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial\Omega}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) - \left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial\Omega} - \frac{\partial\vec{\mathbf{f}}}{\partial\Omega}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right), \tag{A11}$$

$$\frac{d\Omega}{dt} = -\frac{\partial(-\Delta\mathcal{H})}{\partial H} + \frac{\partial\Delta L}{\partial\dot{\mathbf{r}}}\frac{\partial}{\partial H}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right) + \left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\frac{\partial\vec{\mathbf{g}}}{\partial H} + \frac{\partial\vec{\mathbf{f}}}{\partial H}\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}+\frac{\partial\Delta\mathcal{L}}{\partial\dot{\mathbf{r}}}\right), \tag{A12}$$

where

$$L \equiv \mu^{1/2}a^{1/2}, \quad G \equiv \mu^{1/2}a^{1/2}(1-e^2)^{1/2}, \quad H \equiv \mu^{1/2}a^{1/2}(1-e^2)^{1/2}\cos i, \tag{A13}$$

and the symbols $\vec{\Phi}, \vec{f}, \vec{g}$ denote the functional dependencies of the gauge, position and velocity upon the Delaunay, not Keplerian, elements, and therefore these are functions different from $\vec{\Phi}, \vec{f}, \vec{g}$ used in (A1)–(A6) where they stood for the dependencies upon the Kepler elements. [In Efroimsky (2002) the dependencies $\vec{\Phi}, \vec{f}, \vec{g}$ upon the Delaunay variables were equipped with tilde, to distinguish them from the dependencies upon the Kepler coordinates.]

The above equations do not merely repeat those derived earlier in Efroimsky (2002, 2003), but generalize them to the case of a perturbation $\Delta\mathcal{L}$ which is both position and velocity dependent. For this reason, our gauge-invariant equations can be employed not only in an inertial frame but also in a wobbling one.

To employ the gauge-invariant equations in analytical calculations is a delicate task: one should always keep in mind that, in case $\vec{\Phi}$ is chosen to depend not only upon time but also upon the "constants" (but not upon their derivatives), the right-hand sides of these equation will implicitly contain the first derivatives dC_i/dt, and one will have to move them to the left-hand sides [much like in the transition from (10) to (11)].

APPENDIX B: THE HAMILTON–JACOBI METHOD IN CELESTIAL MECHANICS

The Jacobi equation (34) is a PDE of the first order, in $(N+1)$ variables (q_n, t), and its complete integral $W(q, Q, t)$ will depend upon $N+1$ constants a_n (Jeffreys and Jeffreys, 1972; Courant and Hilbert 1989). One of these constants, a_{N+1}, will be additive, because W enters the above equation only through its derivatives. Since both Hamiltonians are, too, defined up to some constant f, then the solution to (34) must contain that constant multiplied by the time:

$$W(q, \ a_1,...,a_N, \ a_{N+1}, t) = \tilde{W}(q, \ a_1,...,a_N, \ t) - (t - t_o)f(a_1,...,a_N)$$

$$= \tilde{W}(q, \ a_1,...,a_N, \ t) - tf(a_1,...,a_N) - a_{N+1}, \qquad (B1)$$

where the fiducial epoch is connected to the constants through $t_o = -a_{N+1}/f$, and the function $\tilde{W}$ depends upon N constants only. As the total number of independent adjustable parameters is $N+1$, the constant f cannot be independent but must rather be a function of $a_1,...,a_N, a_{N+1}$. Since we agreed that the constant a_{N+1} is additive and shows itself nowhere else, it will be sufficient to consider f as a function of the rest N parameters only. (In principle, it is technically possible to involve the constant a_{N+1}, i.e., the reference epoch, into the mutual transformations between the other constants. However, in the applications that we shall consider, we shall encounter only equations autonomous in time, and so there will be no need to treat a_{N+1} as a parameter to vary. Hence, in what follows we shall simply ignore its existence.) The new function $\tilde{W}$ obeys the simplified Jacobi equation

$$\mathcal{H}\left(q, \frac{\partial \tilde{W}(q,a_1,...,a_N,t)}{\partial q}, t \right) + \frac{\partial \tilde{W}(q,a_1,...,a_N,t)}{\partial t} = f(a_1,...,a_N) + \mathcal{H}^*. \qquad (B2)$$

As agreed above, $\mathcal{H}^*$ is a constant. Hence, we can state about this constant all the same as about the constant f: since the integral W can contain no more than $N+1$ adjustable parameters $a_1,...,a_N, a_{N+1}$, and since we ignore the existence of a_{N+1}, the constant $\mathcal{H}^*$ must be a function of the remaining N parameters: $\mathcal{H}^* = \mathcal{H}^*(a_1,...,a_N)$.

Now, in case $\mathcal{H}$ depends only upon (q, p) and lacks an explicit time dependence, then so will $\tilde{W}$; and the above equation will very considerably simplify:

$$\mathcal{H}\left(q, \frac{\partial \tilde{W}(q,a_1,...,a_N)}{\partial q} \right) = f(a_1,...,a_N) + \mathcal{H}^*(a_1,...,a_N), \qquad (B3)$$

where we deliberately avoided absorbing the constant Hamiltonian $\mathcal{H}^*$ into the function f.

Whenever the integral W can be found explicitly, the constants $(a_1,\ldots,a_N)$ can be identified with the new coordinates Q, whereafter the new momenta will be calculated through $P = -\partial W/\partial Q$. In the special case of zero $\mathcal{H}^*$, the new momenta become constants, because they obey the canonical equations with a vanishing Hamiltonian. In the case where $\mathcal{H}^*$ is a nonzero constant, it must, as explained above, be a function of all or some of the independent parameters $(a_1,\ldots,a_N)$, and, therefore, all or some of the new momenta P will be evolving in time.

Since it is sufficient to find only one solution to the Jacobi equation, one can seek it by means of the variable-separation method: Eq. (B3) will solve in the special case when the generating function (B1) is separable:

$$\widetilde{W}(q_1,\ldots,q_N,a_1,\ldots,a_N) = \sum_{i=1}^{N} \widetilde{W}_i(q_i,a_1,\ldots,a_N). \tag{B4}$$

This theory works very well in application to the unperturbed (two-body) problem (1) of celestial mechanics, a problem that is simple due to its mathematical equivalence to the gravitationally bound motion of a reduced mass $m_{\text{planet}}m_{\text{sun}}/(m_{\text{planet}}+m_{\text{sun}})$ about a fixed center of mass $m_{\text{planet}}+m_{\text{sun}}$. If one begins with the (reduced) two-body Hamiltonian in the spherical coordinates

$$q_1 = r, \quad q_2 = \phi, \quad q_3 = \theta \tag{B5}$$

(where $x = r\cos\phi\cos\theta$, $y = r\cos\phi\sin\theta$, $z = r\sin\phi$), then the expression for Lagrangian,

$$L = T - \Pi = \frac{1}{2}(\dot{q}_1)^2 + \frac{1}{2}(q_1)^2(\dot{q}_2)^2 + \frac{1}{2}(q_1)^2(\dot{q}_3)^2\cos^2 q_2 + \frac{\mu}{q_1}, \tag{B6}$$

will yield the following formulas for the momenta:

$$p_1 \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \dot{q}_1, \quad p_2 \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = q_1^2 \dot{q}_2, \quad p_3 \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_3} = q_1^2 \dot{q}_3 \cos^2 q_2, \tag{B7}$$

whence the initial Hamiltonian will read

$$\mathcal{H} = \sum p\dot{q} - \mathcal{L} = \frac{1}{2}p_1^2 + \frac{1}{2q_1^2}p_2^2 + \frac{1}{2q_1^2\cos^2 q_2}p_3^2 - \frac{\mu}{q_1}. \tag{B8}$$

Then the Hamilton–Jacobi equation (30) will look like this:

$$\frac{1}{2}\left(\frac{\partial W}{\partial q_1}\right)^2 + \frac{1}{2q_1^2}\left(\frac{\partial W}{\partial q_2}\right)^2 + \frac{1}{2q_1^2\cos^2 q_2}\left(\frac{\partial W}{\partial q_3}\right)^2 - \frac{\mu}{q_1} - \frac{\partial W}{\partial t} - \mathcal{H}^* = 0, \tag{B9}$$

while the auxiliary function $\widetilde{W}$ defined through (B1) will obey

$$\frac{1}{2}\left(\frac{\partial \widetilde{W}}{\partial q_1}\right)^2 + \frac{1}{2q_1^2}\left(\frac{\partial \widetilde{W}}{\partial q_2}\right)^2 + \frac{1}{2q_1^2\cos^2 q_2}\left(\frac{\partial \widetilde{W}}{\partial q_3}\right)^2 - \frac{\mu}{q_1} - f - \mathcal{H}^* = 0. \tag{B10}$$

A lengthy but elementary calculation [presented, with some inessential variations, in Plummer (1918), Smart (1953), Pollard (1966), Kovalevsky (1967), Stiefel and Scheifele (1971), and many other books] shows that, for a constant $\mathcal{H}^*$ and in the ansatz (B4), the integral of (B3) takes the form

$$\widetilde{W} = \widetilde{W}_1(q_1,a_1,a_2,a_3) + \widetilde{W}_2(q_2,a_1,a_2,a_3) + \widetilde{W}_3(q_3,a_1,a_2,a_3)$$

$$= \int_{q_1(t_o)}^{q_1(t)} \epsilon_1 \left(2(f+\mathcal{H}^*) + \frac{2\mu}{q_1} - \frac{a_2^2}{q_1^2} \right)^{1/2} dq_1 + \int_0^\phi \epsilon_2 \left(a_2^2 - \frac{a_3^2}{\cos^2 q_2} \right)^{1/2} dq_2 + \int_0^\theta a_3 dq_3,$$

(B11)

where the epoch and factors ϵ_1, ϵ_2 may be taken as in Kovalevsky (1967): time t_o is the instant of perigee passage; factor ϵ_1 is chosen to be $+1$ when $q_1 \equiv r$ is increasing, and is -1 when r is decreasing; factor ϵ_2 is $+1$ when $q_2 \equiv \phi$ is increasing, and is -1 otherwise. This way the quantities under the first and second integration signs have continuous derivatives. To draw conclusions, in the two-body case we have a transformation-generating function

$$W \equiv \widetilde{W} + tf(a_1,\ldots,a_N) = \int_{q_1(t_o)}^{q_1(t)} \epsilon_1 \left(2(f+\mathcal{H}^*) + \frac{2\mu}{q_1} - \frac{a_2^2}{q_1^2} \right)^{1/2} dq_1$$

$$+ \int_0^\phi \epsilon_2 \left(a_2^2 - \frac{a_3^2}{\cos^2 q_2} \right)^{1/2} dq_2 + \int_0^\theta a_3 dq_3 + tf,$$

(B12)

whose time-independent component $\widetilde{W}$ enters Eq. (B3). The first integration in (B12) contains the functions $f(a_1,\ldots,a_N)$ and $\mathcal{H}^*(a_1,\ldots,a_N)$, so that in the end of the day W depends on the N constants $a_1,\ldots,a_N$ (not to mention the neglected t_o, i.e., the a_{N+1}).

Different authors deal differently with the sum $(f+\mathcal{H}^*)$ emerging in (B12). Smart (1953) and Kovalevsky (1967) prefer to put

$$f=0, \quad \mathcal{H}^* = a_1, \quad a_1 = -\mu/(2a),$$

(B13)

whereupon the new momentum $P_1 = -\partial W/\partial Q_1 = -\partial W/\partial a_1$ becomes time dependent (and turns out to equal $-t+t_o$). An alternative choice, which, in our opinion, better reflects the advantages of the Hamilton-Jacobi theory, is furnished by Plummer (1918):

$$f=a_1, \quad \mathcal{H}^* = 0, \quad a_1 = \sqrt{\mu a}.$$

(B14)

This entails the following correspondence between the new canonical variables (the Delaunay elements) and the Keplerian orbital coordinates:

$$Q_1 \equiv a_1 = \sqrt{\mu a}, \quad P_1 = -M_o,$$

$$Q_2 \equiv a_2 = \sqrt{\mu a(1-e^2)}, \quad P_2 = -\omega,$$

(B15)

$$Q_3 \equiv a_3 = \sqrt{\mu a(1-e^2)} \cos i, \quad P_3 = -\Omega.$$

Everywhere in this article we follow the convention (B14) and denote the above variables Q_1, Q_2, Q_3 by *L, G, H*, correspondingly (as is normally done in the astronomical literature).

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