SAMPLE UNT DISSERTATION WITH
A TWO LINE TITLE

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I would like to think my puppy Sparky for not eating the only extant copy of my dissertation when he was gnawing through all my library books.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Homeomorphic Measures

Two measures μ and ν defined on the family of Borel subsets of a topological space X are said to be *homeomorphic* or *topologically equivalent* provided there exists a homeomorphism h of X onto X such that μ is the image measure of ν under h: $\mu = \nu h^{-1}$. This means $\mu(E) = \nu(h^{-1}(E))$ for each Borel subset E of X.

1.1.1. Testing Subsection

One may be interested in the structure of these equivalence classes of measures or in a particular equivalence class. For example, a probability measure μ on $[0,1]$ is topologically equivalent to Lebesgue measure if and only if μ gives every point measure 0 and every non-empty open set positive measure. (The distribution function of μ is a homeomorphism on $[0,1]$ witnessing this equivalence.) This is a special case of a result of Oxtoby and Ulam [5], who characterized those probability measures μ on finite dimensional cubes $[0,1]^n$ which are homeomorphic to Lebesgue measure. For this to be so, μ must give points measure 0, non-empty open sets positive measure, and the boundary of the cube measure 0. Later Oxtoby and Prasad [4] extended this result to the Hilbert cube. These results have been extended and applied to various manifolds. The book of Alpern and Prasad [1] is an excellent source for these developments. Oxtoby [3] also characterized those probability measures on the space of irrational numbers in $[0,1]$ which are homeomorphic to Lebesgue measure as those which give points measure zero and open sets positive measure.

It turns out that the Cantor space is more rigid than the above spaces for measure homeomorphisms – it is not true that two probability measures on $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ which give points measure 0 and non-empty open sets positive measure are homeomorphic. Since $\mathcal{C}$ has countably many clopen sets, the set of values taken on clopen sets by such a measure will be a countable dense subset of $[0,1]$. I will refer to this set as the *clopen values set* of such a measure. Even two well behaved measures on $\mathcal{C}$ will typically have different clopen values sets, and so cannot be homeomorphic. A first conjecture at getting around this may be to ask whether any two measures on $\mathcal{C}$ with the same clopen values sets are homeomorphic. This turns out to fail, and it appears unlikely that adding additional conditions will provide a satisfactory theorem, as in some sense there are just too many measures possible. I therefore restrict attention to a particular class of measures which arise frequently.

1.2. Bernoulli Trial Measure

Regard $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ as the set of all infinite words on the alphabet $\{0,1\}$, and for $e = e_1 e_2 \dots e_n$ a finite word, let $[e]$ denote the set of all infinite words beginning with e. Such sets are called *cylinder sets*. Notice that they form a basis for $\mathcal{C}$. Define the *length* of such a set to be the length of the word e.

If $0 \leq r \leq 1$, let μ_r denote Bernoulli trial measure with probability r of success, sometimes called coin tossing measure. To be specific, μ_r is the unique measure for which the sets $\{\pi_n^{-1}(1)\}_{n \geq 1}$ are independent, and each has measure r. Note that if e is a word of length n having i occurrences of the letter 1, then $\mu_r([e]) = r^i(1-r)^{n-i}$.

When the measures μ_r and μ_s are homeomorphic, write $r \sim_{top} s$. In 1979, Oxtoby began to publish papers investigating this equivalence relation on $[0,1]$. In this paper we give a complete characterization of when two such measures are homeomorphic, answering Oxtoby's question.

In Chapter 2 I define terminology, review some previous results, and prove a few preliminary lemmas, finally stating the main result, that four statements are equivalent. Chapters 3 through 5 prove this result, each addressing one of the three non-trivial implications. Chapter 6 provides some examples and additional results, and raises some questions for further research.

CHAPTER 2

THE FAMILY $\mathcal{H}$

2.1. Definition of $\mathcal{H}$ and the Uniformly Expanding Property

In this section we define the family $\mathcal{H}$ and we establish basic dynamical properties of a map $f_a \in \mathcal{H}$. Then we we prove the important Lemma 2.1.

2.1.1. Definition of $\mathcal{H}$

We define the family $\mathcal{H}$ as a family of maps in the Speiser class of transcendental entire functions of finite singular type.

Let $a = (a_0, a_1, \cdots, a_n) \in \mathbb{C}^{n+1}$ be a vector such that $a_0 \neq 0$, $a_n \neq 0$,

$$P_a(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$$

and

$$g_a(z) = \frac{P_a(z)}{z^k}$$

where k is a positive integer strictly less than $n = \deg(P_a) \geq 2$. Define

$$f_a(z) = g_a \circ \exp(z) = \tfrac{a_n e^{nz} + a_{n-1} e^{(n-1)z} + \cdots + a_1 e^z + a_0}{e^{zk}} = \textstyle\sum_{j=0}^{n} a_j e^{(j-k)z}$$

Observe that maps of this form do not have any finite asymptotic values. This is the reason why we restricted ourselves to integers k satisfying condition $0 < k < n$. As it was mentioned in Chapter 1, the most well known examples of this type of maps are maps from the *cosine* family.

We denote by $\mathrm{Crit}(f_a)$ the set $\{z : f_a'(z) = 0\}$. Observe that

$$f_a'(z) = \sum_{j=0}^{n} a_j (j-k) e^{(j-k)z}$$

and that $g_a'(z) = 0$ if and only if $zP_a'(z) - kP_a(z) = 0$, which is equivalent to

$$\sum_{j=0}^{n} a_j (j-k) z^j = 0.$$

Therefore, there exist n non-zero complex numbers (counting multiplicities) $s_1, s_2, \cdots, s_n$ such that $z \in \text{Crit}(f_a)$ if and only if $e^z = s_k$ for some $k = 1, 2, \cdots, n$ i.e.

$$\{z_k = \log s_k + 2\pi i m : m \in \mathbb{Z}, k = 1, \cdots, n\}$$

is the set of critical points and observe that the set of critical values of a map f_a is finite.

Denote by $\mathcal{H}$ the family of functions

$$\mathcal{H} = \left\{ f_a(z) = \frac{P_a(e^z)}{e^{kz}} : \deg P_a > k > 0 \text{ and } \delta_a > 0 \right\},$$

where by $\mathcal{P}_{f_a}$ we denote the post-critical set of f_a i.e. the set

$$\mathcal{P}_{f_a} = \overline{\bigcup_{n \geq 0} f_a^n(Crit(f_a))}$$

and

$$\delta_a = \frac{1}{2} \min \left\{ \frac{1}{2}, \text{dist}(J_{f_a}, \mathcal{P}_{f_a}) \right\},$$

where

$$\text{dist}(J_{f_a}, \mathcal{P}_{f_a}) = \inf\{|z_1 - z_2| : z_1 \in J_{f_a}, z_2 \in \mathcal{P}_{f_a}\}$$

is the Euclidean distance between the Julia set of f_a, J_{f_a}, and the post-critical set of f_a, $\mathcal{P}_{f_a}$.

The reason we define δ_a in such a way will be more visible later on, starting with Chapter 3, and is due to the application (we shall need) of the Koebe Distortion Theorem since one can observe that, for every $y \in J_{f_a}$ and for every $n \geq 1$, there exists a unique holomorphic inverse branch

$$(f_a^n)_y^{-1} : B(f_a^n(y), 2\delta_a) \to \mathbb{C}$$

such that $(f_a^n)_y^{-1} \circ (f_a^n)(y) = y$.

Then there exists a numerical constant K such that, for $z_1, z_2 \in J_{f_a}$ with $|z_1 - z_2| < \delta_a$ and for $y \in f_a^{-n}(z_1)$,

$$\frac{1}{K} \leq \frac{|((f_a^n)_y^{-1})'(z_1)|}{|((f_a^n)_y^{-1})'(z_2)|} \leq K. \tag{1}$$

Observe that $Crit(f_a) \subset F_{f_a}$, where F_{f_a} is the Fatou set of f_a. Consequently, maps in the family $\mathcal{H}$ do not have neither parabolic domains nor Herman rings nor Siegel disks. Moreover, as was written in Chapter 1 they do not have neither wandering nor Baker domains. Also for every point z in the Fatou set there exists (super)attracting cycle such that the trajectory of z converges to this cycle.

2.1.2. The Cylinder and the Definition of $J_{F_a}^r$

Since the map $f_a \in \mathcal{H}$ is periodic with period $2\pi i$, we consider it on the quotient space $P = \mathbb{C}/\sim$ (the cylinder) where

$$z_1 \sim z_2 \text{ iff } z_1 - z_2 = 2k\pi i \text{ for some } k \in \mathbb{Z}.$$

If $\pi : \mathbb{C} \to P$ is the natural projection, then, since the map $\pi \circ f_a : \mathbb{C} \to P$ is constant on equivalence classes of relation $\sim$, it induces a holomorphic map

$$F_a : P \to P.$$

The cylinder P is endowed with Euclidean metric which will be denoted in what follows by the same symbol $|w - z|$ for all $z, w \in P$. The Julia set of F_a is defined to be

$$J_{F_a} = \pi(J_{f_a})$$

and observe that

$$F_a(J_{F_a}) = J_{F_a} = F_a^{-1}(J_{F_a}).$$

We shall study the set $J_{f_a}^r$ consisting of those points of J_{f_a} that do not escape to infinity under positive iterates of f_a. In other words, if

$$I_\infty(f_a) = \{z \in \mathbb{C} : \lim_{n\to\infty} f_a^n(z) = \infty\},$$

then

$$J_{f_a}^r = J_{f_a} \backslash I_\infty(f_a)$$

and, if

$$I_\infty(F_a) = \{z \in P : \lim_{n\to\infty} F^n(z) = \infty\},$$

then

$$J_{F_a}^r = J_{F_a} \backslash I_\infty(F_a).$$

In what follows we fix $a \in \mathbb{C}^{n+1}$ and we denote for simplicity $f_a \in \mathcal{H}$ by f. The following Lemma reveals some background information for a better understanding of the dynamical behavior of maps in our family $\mathcal{H}$. This lemma will be used several times and it will be a key technical ingredient for many proofs.

Observe first that, if we consider $a = (a_0, \cdots, a_n) \in \mathbb{C}^{n+1}$, since

$$f_a(z) = \sum_{j=0}^{n} a_j e^{(j-k)z} \tag{2}$$

we have

$$f'_a(z) = \sum_{j=0}^{n} a_j (j-k) e^{(j-k)z}. \tag{3}$$

LEMMA 2.1. *Let f_a be a function of form (2). Then there exist $M_1, M_2, M_3 > 0$ such that, for every z with $|Re\ z| \geq M_3$, the following inequalities hold.*

(1) $M_1 e^{q|Re\ z|} \leq |f_a(z)| \leq M_2 e^{q|Re\ z|}$

(2) $M_1 e^{q|Re\ z|} \leq |f'_a(z)| \leq M_2 e^{q|Re\ z|}$

(3) $\frac{M_1}{M_2}|f'_a(z)| \leq |f_a(z)| \leq \frac{M_2}{M_1}|f'_a(z)|$

where $q = \begin{cases} k & \textit{if } Re\ z < 0 \\ n-k & \textit{if } Re\ z > 0. \end{cases}$

PROOF. Note that (iii) follows from (i) and (ii). The proof of (i) and (ii) follows from the fact that

$$|f_a(z)| = |a_n| e^{(n-k)\mathrm{Re}\ z} + o(e^{(n-k)\mathrm{Re}\ z}) \text{ as Re } z \to \infty$$

$$|f_a(z)| = |a_0| e^{-k\mathrm{Re}\ z} + o(e^{-k\mathrm{Re}\ z}) \text{ as Re } z \to -\infty$$

and from the observation that f'_a is a function of the same (algebraic) type as f_a (see (3)). $\square$

2.1.3. The Uniformly Expanding Property

In this section we shall prove, mainly, the very important result, Proposition 2.2, using McMullen's result from [2], that any map $f_a \in \mathcal{H}$ is uniformly expanding on its Julia set.

Proposition 2.2. *For every $f \in \mathcal{H}$ there exist $c > 0$ and $\gamma > 1$ such that*

$$|(f^n)'(z)| > c\gamma^n$$

for every $z \in J_f$.

Proof. By [2, Proposition 6.1], for all $z \in J_f$,

$$\lim_{n\to\infty} |(f^n)'(z)| = \infty. \tag{4}$$

Since f is periodic with period $2\pi i$ we consider

$$A = J_f \cap \{z : \text{Im } z \in [0, 2\pi]\}$$

and we let A_m denotes the open set

$$\{z \in A : |(f^m)'(z)| > 2\}.$$

Then by (4) $\{A_m\}_{m\geq 1}$ is an open covering of A. Moreover, it follows from Lemma 2.1 that there exists M such that, if $|\text{Re } z| > M$, then $|f'(z)| > 2$. Therefore

$$\{z \in A : |\text{Re } z| > M\} \subset A_1.$$

Since $A \cap \{z : |\text{Re } z| \leq M\}$ is a compact subset of A, it follows that there exists $k \geq 1$ such that the family $\{A_1, A_2, \dots, A_k\}$ covers A. It implies that, for every $z \in A$, there exists $k(z) \leq k$ for which $|(f^{k(z)})'(z)| > 2$. Therefore, for every $n > 0$ and every $z \in A$ we can split the trajectory $z, f(z), \dots, f^n(z)$ into $l \leq \lfloor \frac{n}{k} \rfloor + 1$ pieces of the form

$$z_i, f(z_i), \dots, f^{k(z_i)-1}(z_i)$$

for $i = 1, \dots, l-1$, and, for $i = l$,

$$z_l, f(z_l), \dots f^j(z_l) = f^n(z),$$

where $z_1 = z$, $z_i = f^{k(z_{i-1})}(z_{i-1})$ and j is some integer smaller than k. Then

$$|(f^n)'(z)| \geq 2^{\lfloor \frac{n}{k} \rfloor} \Delta^{k-1},$$

where

$$\Delta = \inf_{z \in J_f} |f'(z)| \neq 0,$$

since J_f contains no critical points and because of Lemma 2.1 (ii). It follows that

$$|(f^n)'(z)| \geq 2^{\frac{n}{k}-1}\Delta^{k-1} = \frac{\Delta^{k-1}}{2}(2^{\frac{1}{k}})^n.$$

□

2.2. Bounded Orbits and Classical Conformal Repellers.

We fix again $a \in \mathbb{C}^{n+1}$ and we denote f_a by f, F_a by F and the Julia set of F by J_F. Our goal in this section is to prove Proposition 2.5. In order to prove this proposition we apply the thermodynamic formalism for compact repellers.

Definition 2.3. Let f be a holomorphic function from an open subset V of $\mathbb{C}$ into $\mathbb{C}$ and J a compact subset of V. The triplet (J, V, f) is a conformal repeller if

(1) there are $C > 0$ and $\alpha > 1$ such that $|(f^n)'(z)| \geq C\alpha^n$ for every $z \in J$ and $n \geq 1$.
(2) $f^{-1}(V)$ is relatively compact in V with

$$J = \bigcap_{n \geq 1} f^{-n}(V).$$

(3) for any open set U with $U \cap J$ not empty, there is $n > 0$ such that

$$J \subset f^n(U \cap J).$$

It is worth noting that there are no critical points of f in J.

2.2.1. Conformal Repellers

Let (J, V, g) be a (mixing) conformal expanding repeller(see for example [7] for more properties). In the proof of Proposition 2.5, $J = J_1(M)$ is a compact subset of $\mathbb{C}$, limit of a finite conformal iterated function system, $g = F$, is a holomorphic function for which J is invariant and for which there exist $\gamma > 1$ and $c > 0$ such that, for all $n \in \mathbb{N}$ and for all $z \in J$, $|(g^n)'(z)| \geq c\gamma^n$. For $t \in \mathbb{R}$ we consider the topological pressure defined by

$$P_z(t) = \lim_{n \to \infty} \frac{1}{n} \log P_z(n, t),$$

where

$$P_z(n,t) = \sum_{y \in g^{-n}(z)} |(g^n)'(y)|^{-t}.$$

The function $P(t) = P_z(t)$ as a function of t is independent of z, continuous, strictly decreasing, $\lim_{t\to-\infty} P(t) = +\infty$ and the following remarkable theorem holds.

Theorem 2.4 (Bowen's Formula). *Hausdorff dimension of J is the unique zero of $P(t)$.*

For more details and definitions concerning the thermodynamic formalism of conformal expanding repellers (initiated by Bowen and Ruelle) we refer the reader to [7] or [6].

In order to prove Proposition 2.5, i.e. to show that $\mathrm{HD}(J) > 1$, we use Bowen's formula and we observe that, from the definition of $P_z(n,t)$, it is enough to find a constant $C > 1$ such that, for all $z \in J$,

$$P_z(1,1) \geq C. \tag{5}$$

Proposition 2.5. *Let $f \in \mathcal{H}$. Then the Hausdorff dimension of the set of points in Julia set of f having bounded orbit is strictly greater than* 1.

Proof. Let N be a large number, $H = \{z \in \mathbb{C} : \mathrm{Re}\ z > N\}$. Observe that there exists U such that $\overline{U} \subset \{z : s - \pi < \mathrm{Im}\ z < s + \pi\}$ for some $s \in (-\pi, \pi]$, $\mathrm{Re}\ U > 0$, $f|_U$ is univalent and $f(U) = H$. Note that, since N is large, by Lemma 2.1 there exists $\gamma_N > 1$ such that, if $\mathrm{Re}\ z \geq N$, then

$$|F'(z)| = |f'(z)| > \gamma_N. \tag{6}$$

For every $M > N$ define

$$P(M) = \{z \in \overline{U} : N \leq \mathrm{Re}\ z \leq M\}.$$

Then, for $j \in \mathbb{Z}$, let $L_j : H \to U$ be defined by the formula

$$L_j(z) = (f|_U)^{-1}(z + 2\pi i j),$$

and let

$$Q_j(M) = L_j(P(M)). \tag{7}$$

The set P(M) and the family of functions

$$\{L_j\}_{j\in\mathcal{K}_M}$$

with

$$\mathcal{K}_M = \{j \in \mathbb{Z} : Q_j(M) \subset \mathrm{Int}P(M)\},$$

define a finite conformal iterated function system . By $J_1(M)$ we denote its limit set. The set $J_1(M)$ is forward $F-$invariant. From (6) and from the fact that the Julia set is the closure of the set of repelling periodic points it follows that

$$J_1(M) \subset J_F. \tag{8}$$

Next we need a condition for j which guarantees that $Q_j(M) \subset \mathrm{Int}P(M)$ (equivalently $j \in \mathcal{K}_M$) for all M large enough. Observe that

$$\mathcal{K}_M \subset \mathcal{K}_{M+1} \tag{9}$$

for all M large enough. To prove (9), let $j \in \mathcal{K}_M$ and let $z \in Q_j(M+1)\setminus Q_j(M)$. Note that, if we assume that $M > M_2 e^{(n-k)(N+1)}$, then we can be sure that Re $z > N+1$ (n and k are defined in section 2.1.1). Therefore, to get (9), it is enough to prove that Re $z < M+1$. Since

$$F(Q_j(M+1) \setminus Q_j(M)) = P(M+1) \setminus P(M),$$

it follows from Lemma 2.1 that $|F'(z)| \geq \frac{M_1}{M_2}|f(z)| \geq M$ and, then,

$$Q_j(M+1) \setminus Q_j(M) \subset B\Big(z, \frac{M_2 2\pi}{M_1 M}\Big) \subset B(z, 1).$$

But we know, that, for $y \in Q_j(M)$, Re $y \leq M$. This proves (9).

The next step is to prove that there exists $j_0 \in \mathbb{N}$ such that, for all $M \in \mathbb{N}$ large enough,

$$j_0, j_0+1, \ldots, e^{\lfloor M/2 \rfloor} \in \mathcal{K}_M. \tag{10}$$

Note that we can find j_0 such that, for every $j \geq j_0$, Re $Q_j(M) > N$. By Lemma 2.1 it is enough to take

$$j_0 = \left\lceil \frac{M_2 e^{(n-k)N} + 2\pi}{\pi} \right\rceil .$$

So, to prove (10) it remains to show that $j < e^{\lfloor M/2 \rfloor}$ implies

$$\text{Re } Q_j(M) \leq M.$$

Striving for a contradiction, suppose that $j < e^{\lfloor M/2 \rfloor}$ and there exists $z \in Q_j(M)$ such that Re $z > M$. Then by Lemma 2.1 we have

$$|f(z)| > M_1 e^{(n-k)M}. \tag{11}$$

Since $z \in Q_j(M)$, $f(z) \in P(M) + 2\pi i j$. Then the square of the distance from zero to the upper-right corner of $P(M) + 2\pi i j$ is greater than $|f(z)|^2$, i.e.

$$M^2 + (s + \pi + 2\pi j)^2 > |f(z)|^2.$$

By (11) and the assumption $j < e^{\lfloor M/2 \rfloor}$, it follows that

$$(M_1 e^{(n-k)M})^2 < M^2 + (s + \pi + 2\pi)^2 e^M.$$

Hence we have the required contradiction since for large M the inequality is false.

Finally observe that by Lemma 2.1, for $j \in \mathcal{K}_M$ and $z \in Q_j(M)$, the following is true

$$|F'(L_j(z + 2j\pi i))| \leq \frac{M_2}{M_1} |f(L_j(z + 2\pi i j))| \leq \frac{M_2}{M_1}(2j\pi + 2\pi + M).$$

Then

$$P_z(1,1) = \sum_{y \in F^{-1}(z) \cap J_1(M)} \frac{1}{|F'(y)|} = \sum_{j \in \mathcal{K}_M} |L_j'(z + 2j\pi i)| \geq \sum_{j=j_0}^{e^{\lfloor M/2 \rfloor}} \frac{1}{\frac{M_2}{M_1}(2j\pi + 2\pi + M)}.$$

Since, if M is large enough, the right side of this inequality can be as large as we want, and the proposition are proved. $\square$

APPENDIX A

TESTING 1 2 3

A.1. Section

Here is a sample section in an appendix ...

A.1.1. Proof of Lemma 2

Here is a sample subsection in an appendix ...

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