Accurate computation of the demagnetization tensor

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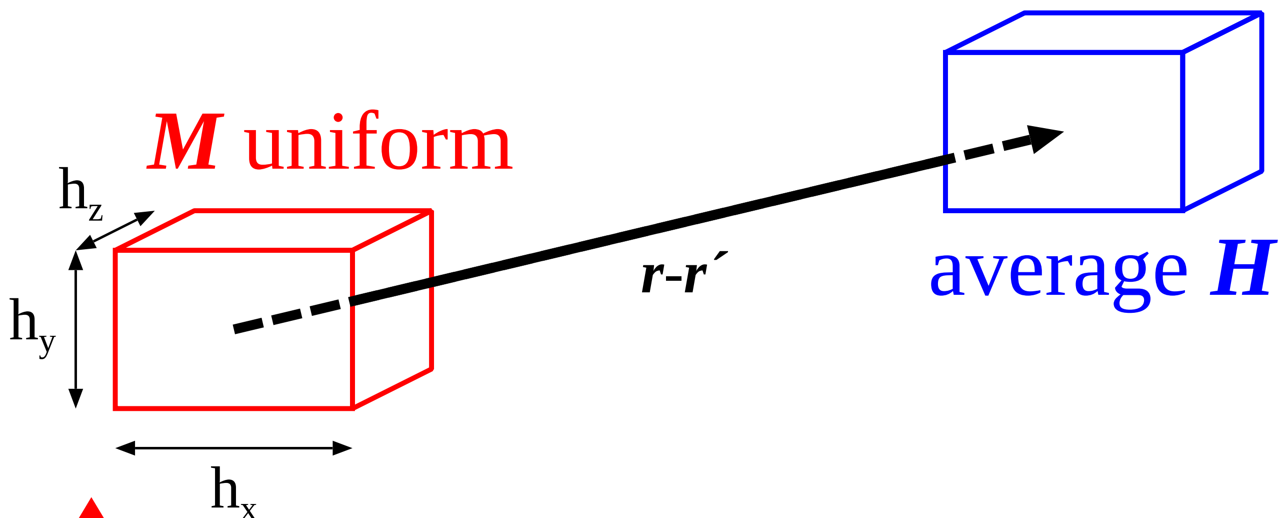
Demag tensor

$$\mathbf{H}(\mathbf{r}) = -N(\mathbf{r}-\mathbf{r}';\mathbf{h})\mathbf{M}(\mathbf{r}'),$$

$$N := \begin{pmatrix} N_{xx} & N_{xy} & N_{xz} \\ N_{xy} & N_{yy} & N_{yz} \\ N_{xz} & N_{yz} & N_{zz} \end{pmatrix}$$



Formulae

$$\begin{aligned} N_{xx}(\mathbf{r}) &= L[F;\mathbf{h}](\mathbf{r}) \\ N_{xy}(\mathbf{r}) &= L[G;\mathbf{h}](\mathbf{r}) \end{aligned}$$

where

$$L[\phi;\mathbf{h}](x,y,z) =$$
$$\sum_{\epsilon_1,\epsilon_2,\epsilon_3=-1}^{1} \frac{\gamma(\epsilon_1,\epsilon_2,\epsilon_3)}{4\pi h_x h_y h_z}\phi(x+\epsilon_1 h_x, y+\epsilon_2 h_y, z+\epsilon_3 h_z)$$

with

$$\gamma(\epsilon_1,\epsilon_2,\epsilon_3) = 8/(-2)^{(|\epsilon_1|+|\epsilon_2|+|\epsilon_3|)}$$

N_{xx} precursor F

$$
\begin{aligned}
F(x,y,z) &= (1/6)(2x^2 - y^2 - z^2)R \\
&+ (1/2)y(z^2 - x^2)\log(y+R) \\
&+ (1/2)z(y^2 - x^2)\log(z+R) \\
&- xyz\arctan(yz/xR)
\end{aligned}
$$

Here $R = \sqrt{x^2 + y^2 + z^2}$.

N_{xy} precursor G

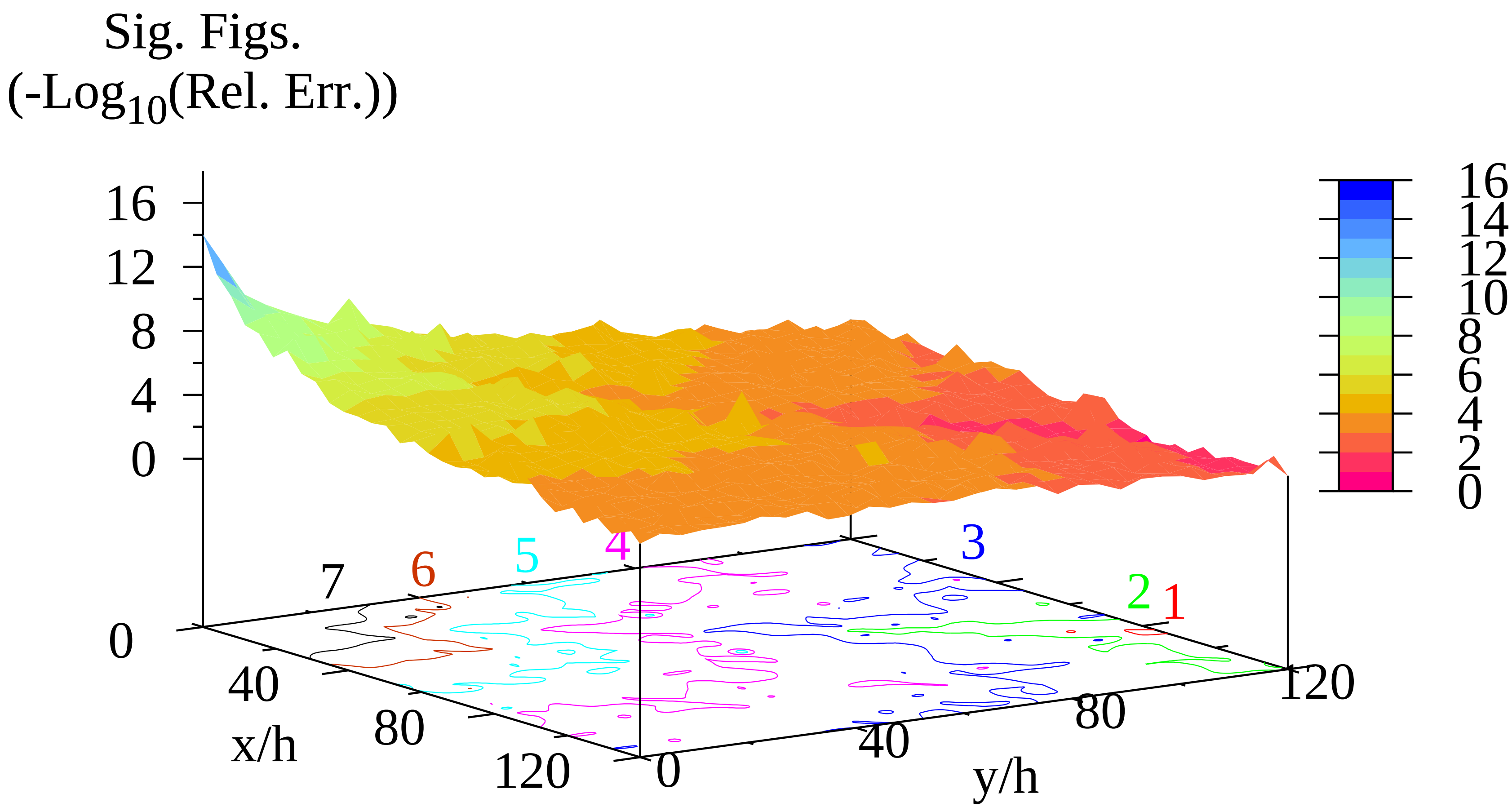
$$
\begin{aligned}
G(x,y,z) = {} & -(1/3)xyR + xyz\log(z+R) \\
& + (1/6)y(3z^2 - y^2)\log(x+R) \\
& + (1/6)x(3z^2 - x^2)\log(y+R) \\
& - (1/6)z^3\arctan(xy/zR) \\
& - (1/2)y^2 z\arctan(xz/yR) \\
& - (1/2)x^2 z\arctan(yz/xR)
\end{aligned}
$$

References

[1] M.E. Schabes and A. Aharoni, "Magnetostatic interaction fields for a 3-dimensional array of ferromagnetic cubes," *IEEE Trans. Magn.*, **23**, 3882–3888 (1987).

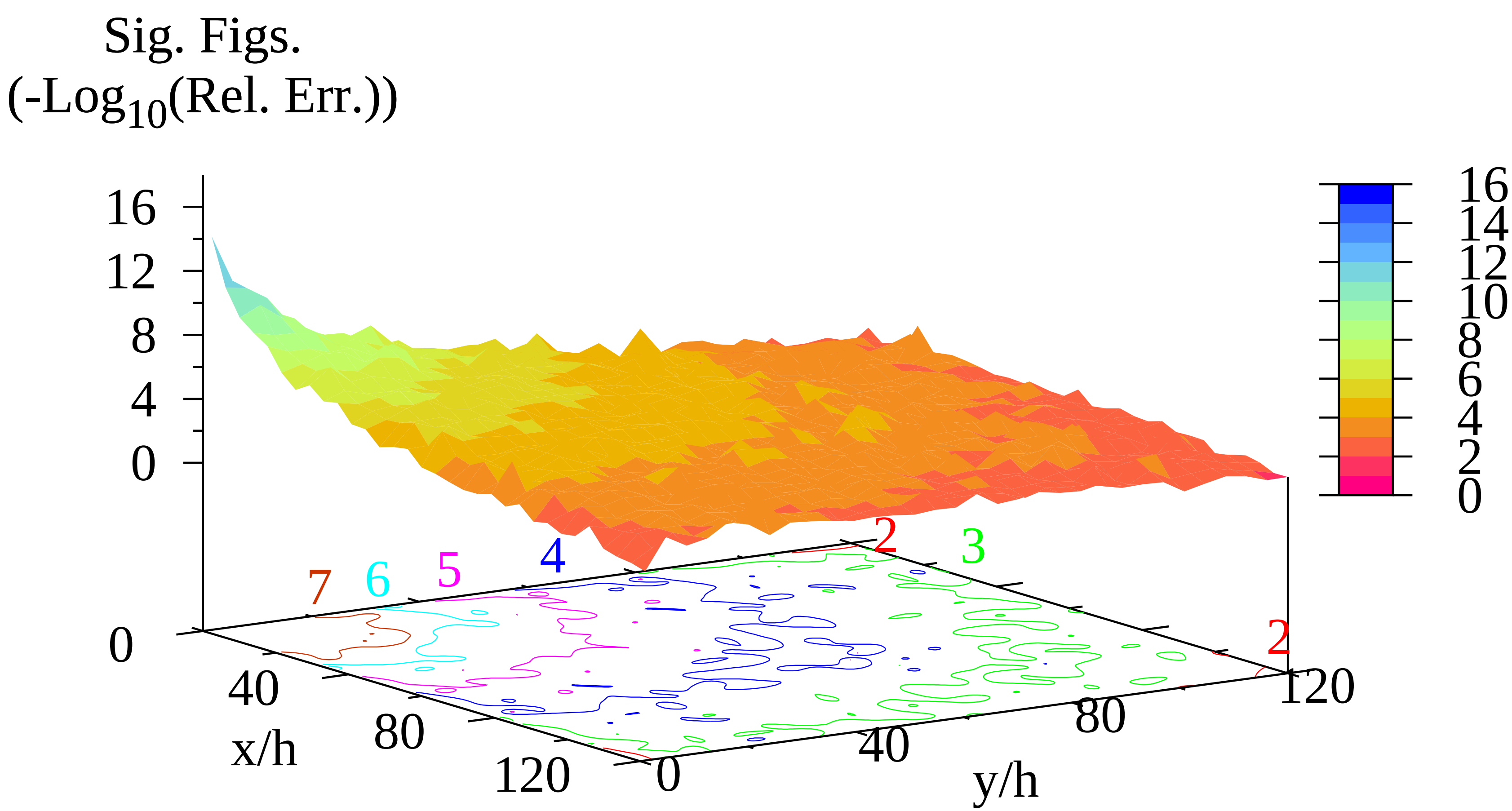
[2] A.J. Newell, W. Williams, and D.J. Dunlop, "A generalization of the demagnetizing tensor for nonuniform magnetization," *J. Geophysical Research-Solid Earth*, **98**, 9551–9555 (1993).

Relative Error: N_{xx}



Direct numeric implementation,
cubic cells, z=0.

Relative Error: N_{xy}



Direct numeric implementation,
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Who cares?

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- Refining grid increases errors

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- Wrong physics (worse than cutoff)

So, why so bad?

$$L[\phi;\mathbf{h}](x,y,z) =$$

$$\sum_{\epsilon_1,\epsilon_2,\epsilon_3=-1}^{1} \frac{\gamma(\epsilon_1,\epsilon_2,\epsilon_3)}{4\pi h_x h_y h_z} \phi(x+\epsilon_1 h_x, y+\epsilon_2 h_y, z+\epsilon_3 h_z)$$

Then

$$L[\phi;\mathbf{h}](x,y,z) = -\delta_x^2 \circ \delta_y^2 \circ \delta_z^2 / 4\pi h_x h_y h_z$$

where

$$\delta_x[f](\mathbf{r}) = f(x+h/2,y,z) - f(x-h/2,y,z), \ \ldots$$

Catastrophic cancellation

$$f(x+h) = a_0 + a_1 h + a_2 h^2 + \ldots$$

For h=1/10:

a_0	6	5	4	3	2	1		
$+\ a_1 h$		6	5	4	3	2	1	
f(x+h)	7	1	9	7	5	3		
- f(x)	6	5	4	3	2	1		
		6	5	4	3	2		← 1 digit lost

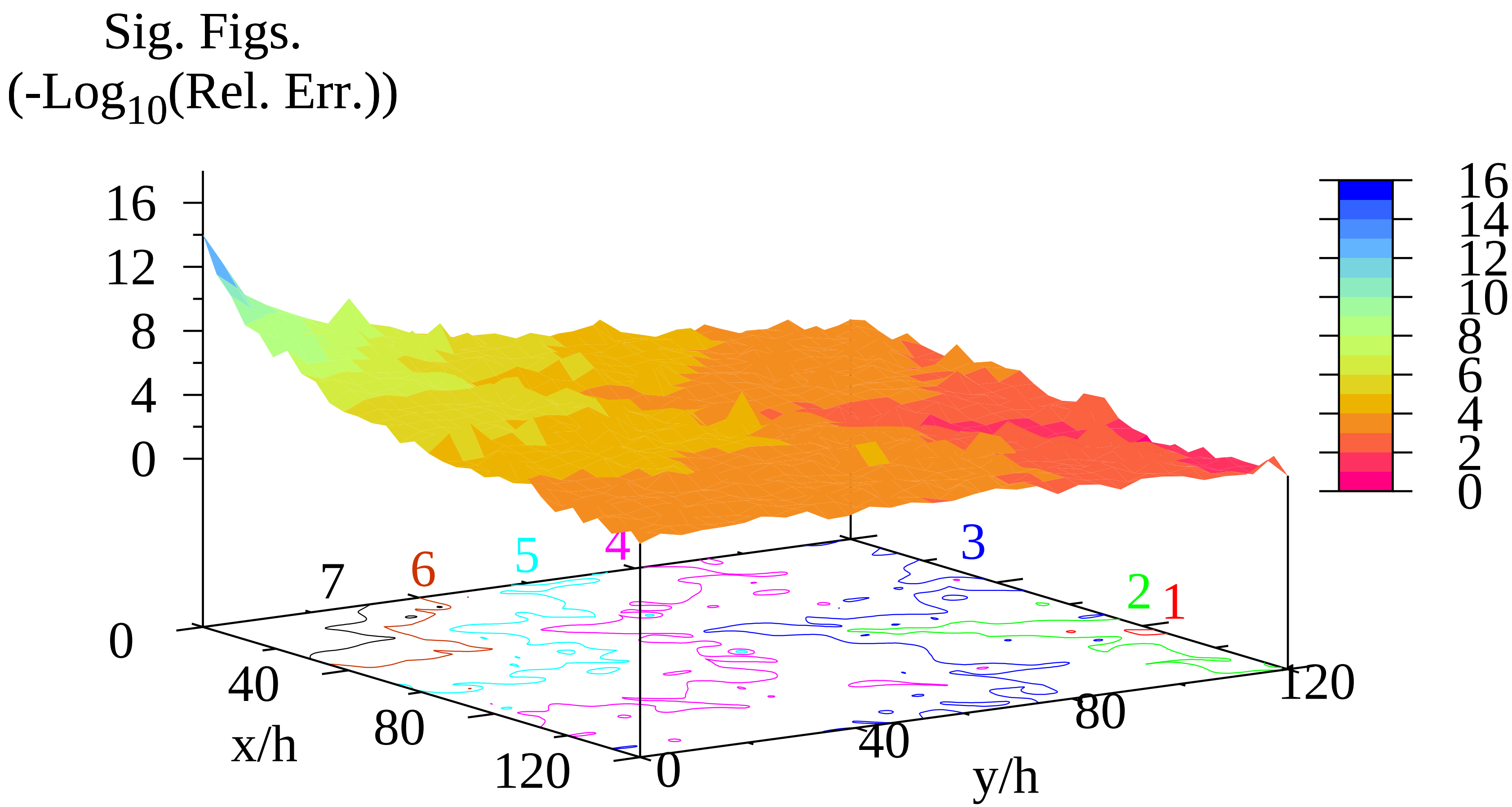
Catastrophic cancellation

$$f(x+h) = a_0 + a_1 h + a_2 h^2 + \ldots$$

For h=1/100:

a_0	6	5	4	3	2	1		
$+\ a_1 h$		6	5	4	3	2	1	
$f(x+h)$	6	6	0	8	6	4		
$-\ f(x)$	6	5	4	3	2	1		
			6	5	4	3		← 2 digits lost

Relative Error: N_{xx}



Direct numeric implementation, cubic cells, z=0.

What to do?

- Multiprecision library

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Algebraic manipulations

$$R(x+h) - R(x) = \sqrt{(x+h)^2 + a^2} - \sqrt{x^2 + a^2}$$

$$= \frac{\left((x+h)^2 + a^2\right) - \left(x^2 + a^2\right)}{\sqrt{(x+h)^2 + a^2} + \sqrt{x^2 + a^2}}$$

$$= \frac{2xh + h^2}{\sqrt{(x+h)^2 + a^2} + \sqrt{x^2 + a^2}}$$

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- Dipole approximation for far field
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- Solution: Some algebraic + higher order asymptotics

Tools and tricks

$$
\begin{aligned}
\log(A) - \log(B) &= \log(A/B) \\
&= \log\left(1 + (A - B)/B\right) \\
&= \log\left(1 + \epsilon\right)
\end{aligned}
$$

Tools and tricks

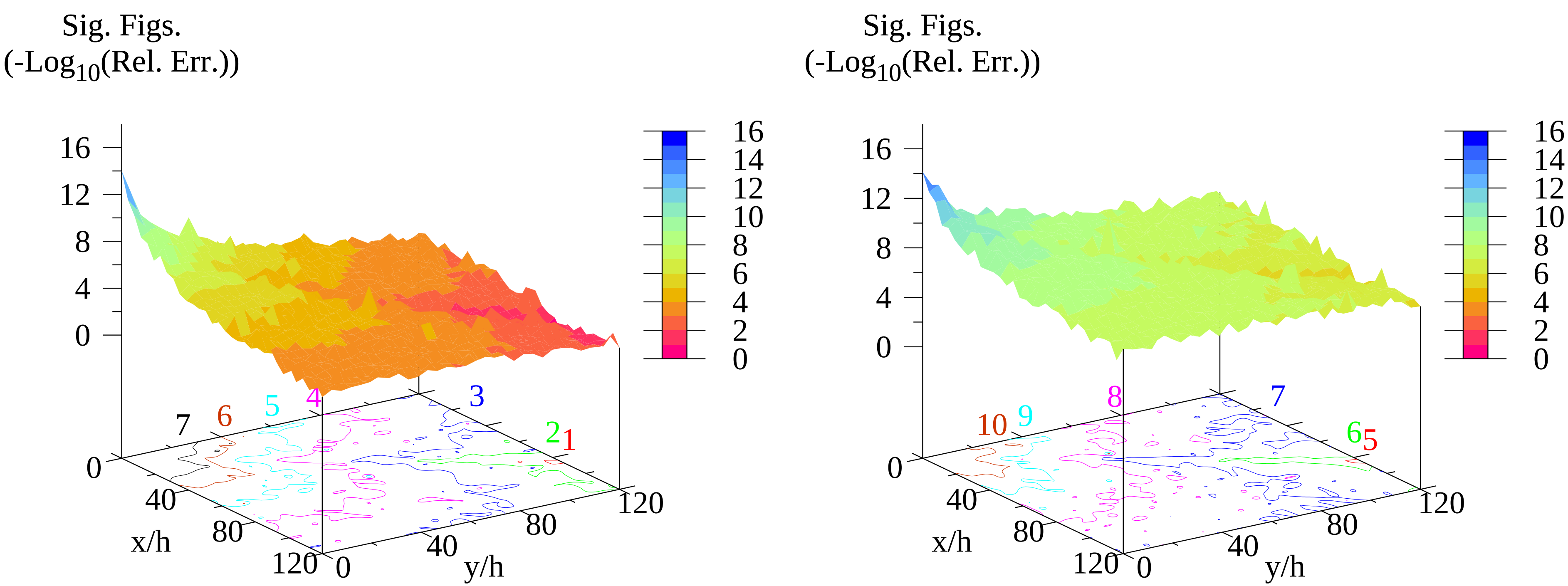
Arctan differences:

$$\arctan(A) - \arctan(B) = \arctan\left(\frac{A-B}{1+AB}\right)$$

Difference of products:

$$\delta^2(fg) = f\delta^2 g + g\delta^2 f + \Delta f \Delta g + \nabla f \nabla g$$

Algebraic recast (N_{xx})



Direct

Modified

Differences vs. Derivatives

$$
\begin{aligned}
\frac{\partial}{\partial x} &= (1/h_x)\delta_x + \ldots \\
\\
\delta_x^2 &= \frac{h_x^2}{2!}\frac{\partial^2}{\partial x^2} + \frac{h_x^4}{4!}\frac{\partial^4}{\partial x^4} + \frac{h_x^6}{6!}\frac{\partial^6}{\partial x^6} + \ldots \\
&= \cosh\left(h_x \frac{\partial}{\partial x}\right) - 1
\end{aligned}
$$

Asymptotic expansion

$$(-4\pi h_x h_y h_z)\, N_{xx}(\mathbf{r}) = (-4\pi h_x h_y h_z)\, L\left[F;\mathbf{h}\right](\mathbf{r})$$

$$= \quad \delta_x^2 \circ \delta_y^2 \circ \delta_z^2 \left[F\right](\mathbf{r})$$

$$= \quad \left(\cosh\left(h_x \partial/\partial x\right) - 1\right) \circ \left(\cosh\left(h_y \partial/\partial y\right) - 1\right) \circ \left(\cosh\left(h_z \partial/\partial z\right) - 1\right) F(\mathbf{r})$$

Asymptotic expansion

$$(-4\pi h_x h_y h_z)\, N_{xx}(\mathbf{r})$$

$$= \left(\frac{\cosh\left(h_x \partial/\partial x\right) - 1}{h_x^2\, \partial^2/\partial x^2}\right) \circ \left(\frac{\cosh\left(h_y \partial/\partial y\right) - 1}{h_y^2\, \partial^2/\partial y^2}\right)$$

$$\circ \left(\frac{\cosh\left(h_z \partial/\partial z\right) - 1}{h_z^2\, \partial^2/\partial z^2}\right)$$

$$\circ \left(h_x^2 h_y^2 h_z^2 \frac{\partial^6}{\partial x^2 \partial y^2 \partial z^2}\right) F(\mathbf{r})$$

Surprise!

$$\frac{\partial^6 F}{\partial x^2 \partial y^2 \partial z^2}(\mathbf{r}) = \frac{3x^2 - R^2}{R^5}$$

(dipole field)

$$\frac{\partial^6 G}{\partial x^2 \partial y^2 \partial z^2}(\mathbf{r}) = \frac{3xy}{R^5}$$

Asymptotics

$$N_{xx} = \frac{h_x h_y h_z}{-4\pi}\left[\frac{(3x^2/R^2)-1}{R^3} + \frac{\mathbf{h}_2^T A_5 \mathbf{r}_4}{R^5} + \frac{\mathbf{h}_4^T A_7 \mathbf{r}_6}{R^7}\right] + O(1/R^9)$$

where, for example,

$$\mathbf{h}_2^T A_5 \mathbf{r}_4 = \begin{pmatrix} h_x^2 \\ h_y^2 \\ h_z^2 \end{pmatrix}^T \begin{pmatrix} -8 & -3 & -3 & 24 & 24 & -6 \\ 4 & 4 & -1 & -27 & 3 & 3 \\ 4 & -1 & 4 & 3 & -27 & 3 \end{pmatrix} \begin{pmatrix} x^4 \\ y^4 \\ z^4 \\ x^2y^2 \\ x^2z^2 \\ y^2z^2 \end{pmatrix} \cdot \frac{1}{R^4}$$

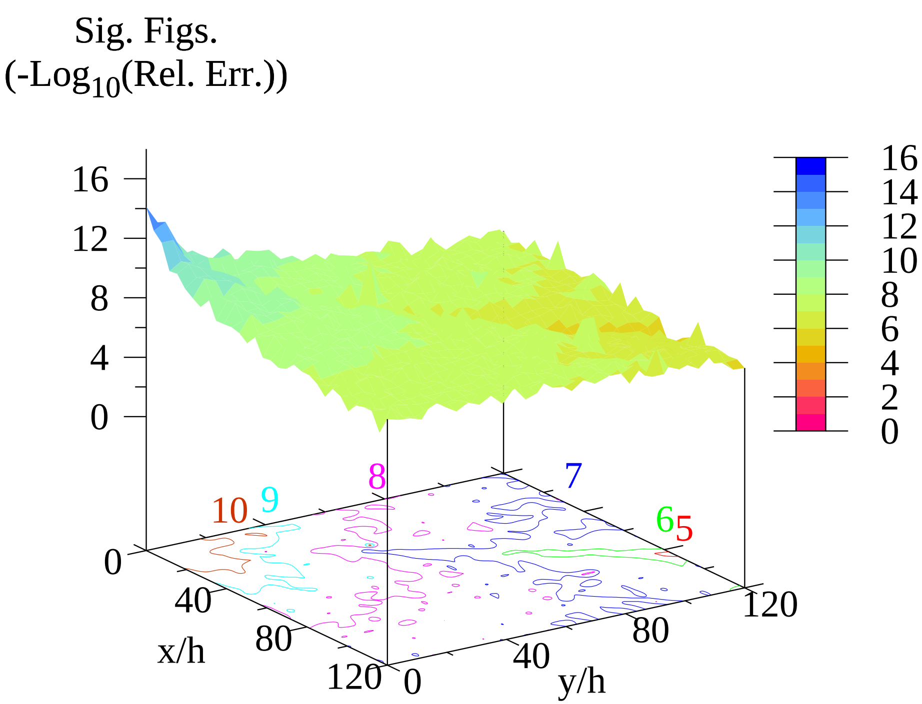
Alternative asymptotics

$$N_{xx}(\mathbf{r}) = (-1/4\pi h_x h_y h_z)\, \delta_x^2 \circ \delta_y^2 \circ \delta_z^2 \,[F]\,(\mathbf{r})$$

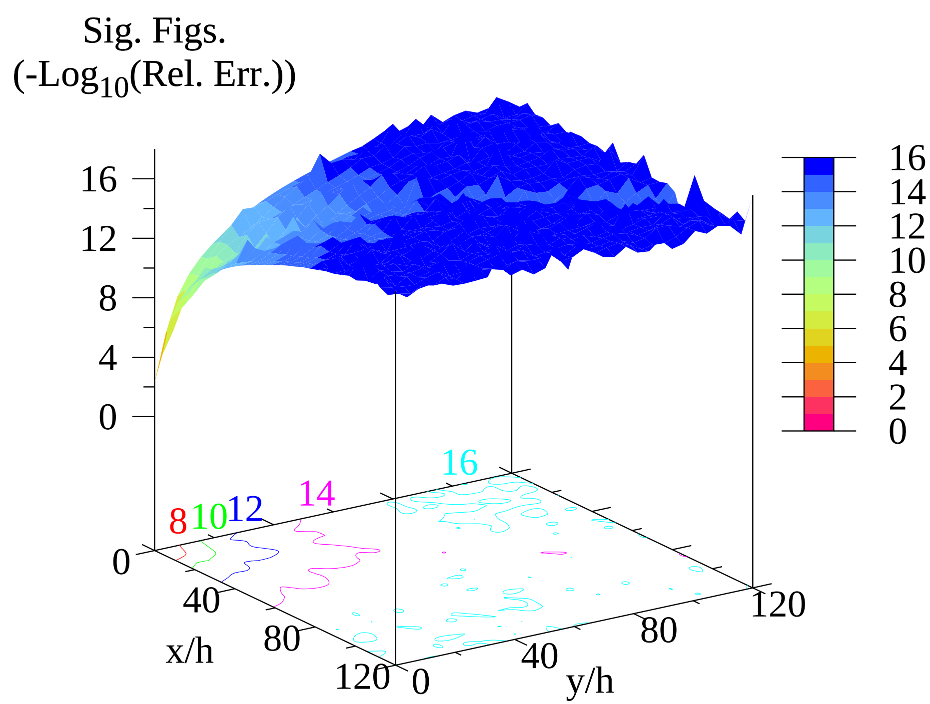
$$\begin{aligned} &= (-1/4\pi h_x h_y h_z) \cdot \delta_x^2 \\ &\quad \circ \left(\cosh\left(h_y \partial/\partial y\right) - 1\right) \circ \left(\cosh\left(h_z \partial/\partial z\right) - 1\right) F(\mathbf{r}) \end{aligned}$$

$$\begin{aligned} &= -\frac{h_y h_z}{4\pi h_x} \cdot \delta_x^2 \left[\frac{1}{R} + \frac{\mathbf{h}_{yz,2}^T B_3 \mathbf{r}_2}{R^3} + \frac{\mathbf{h}_{yz,4}^T B_5 \mathbf{r}_4}{R^5} + \frac{\mathbf{h}_{yz,6}^T B_7 \mathbf{r}_6}{R^7} \right] \\ &\quad + O(1/R^{11}) \end{aligned}$$

N_{xx}: *Near and far*

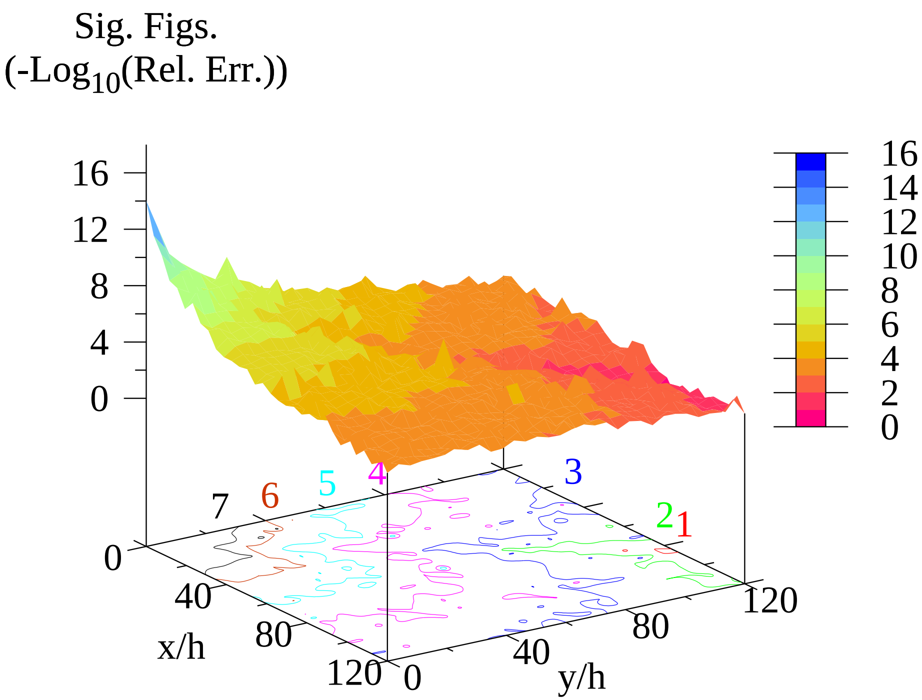


Algebraic

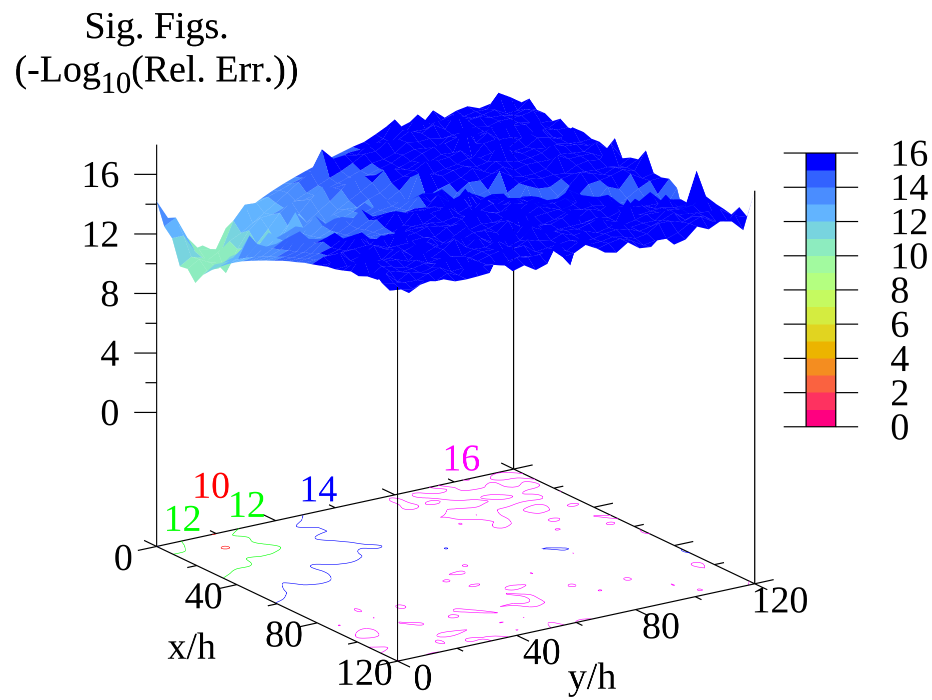


Asymptotic, $O(1/R^8)$

Results: N_{xx}

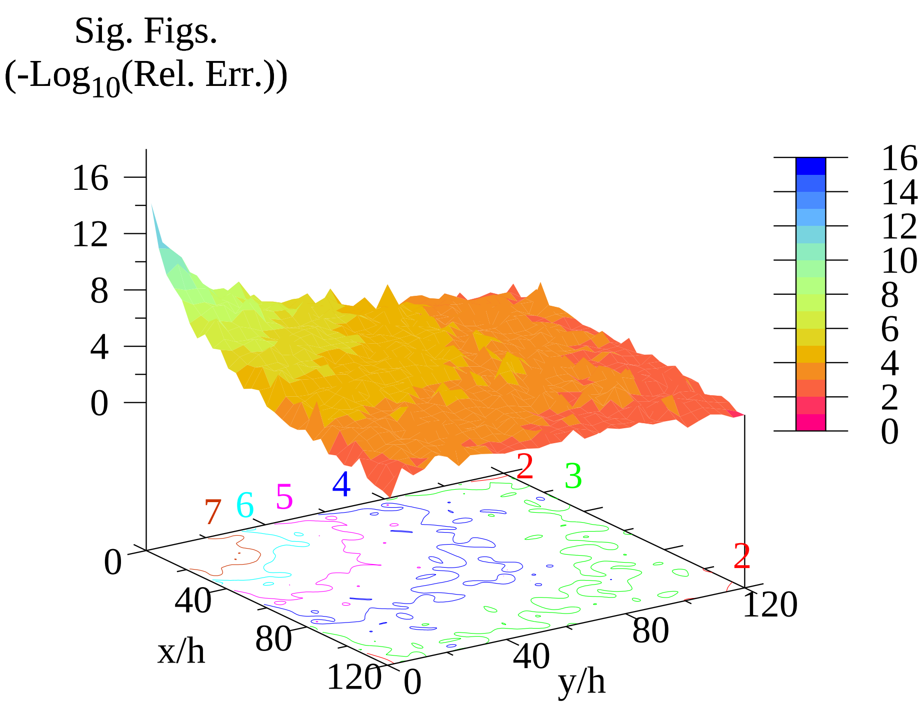


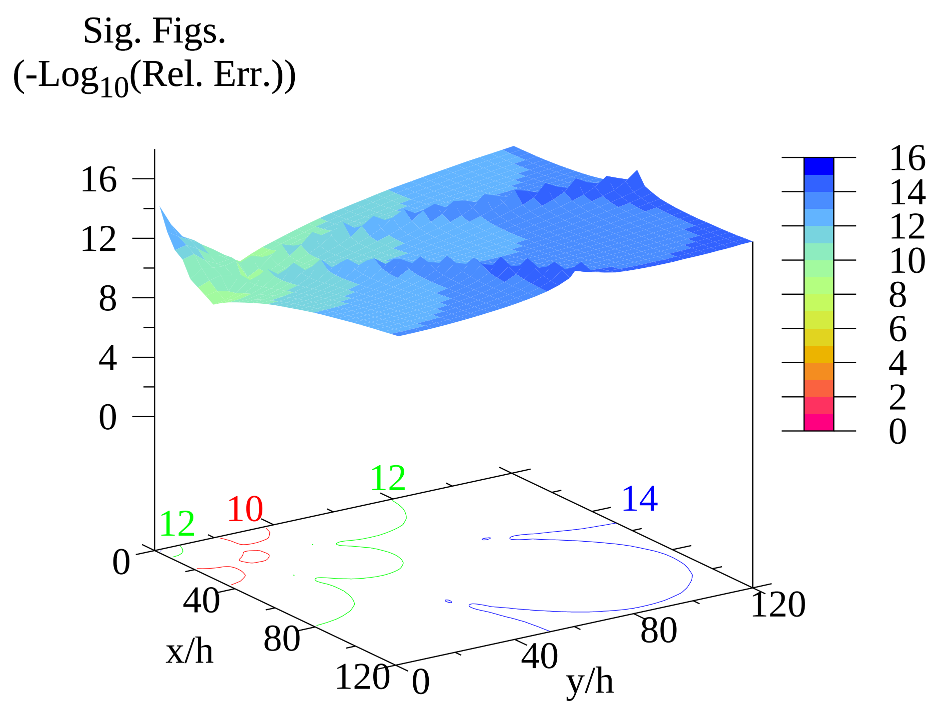
Original



Algebraic+Asymptotic, crossover at $R/h = 20$.

Results: N_{xy}

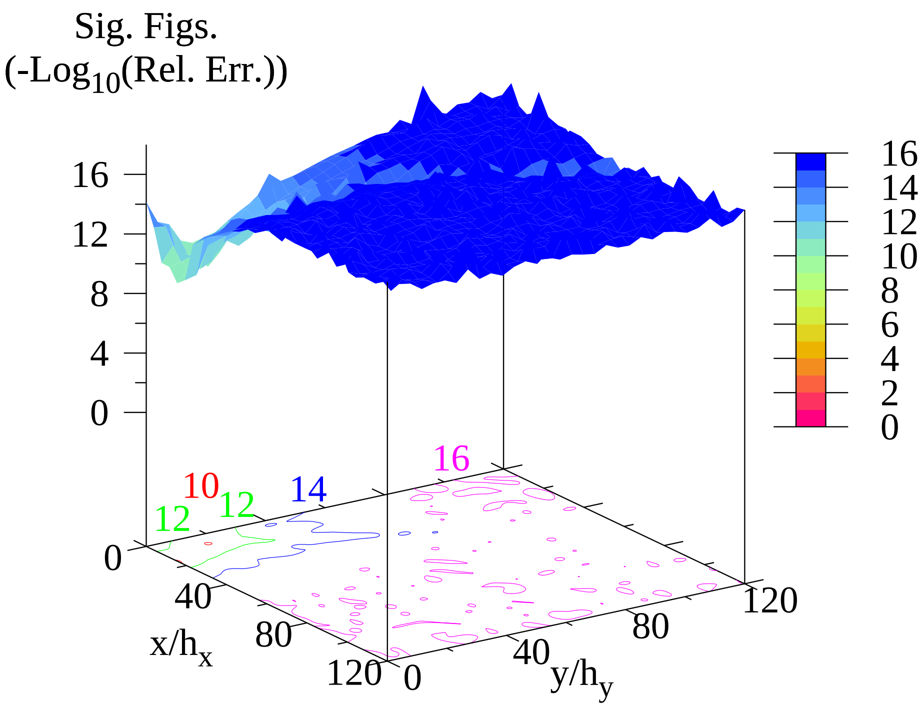




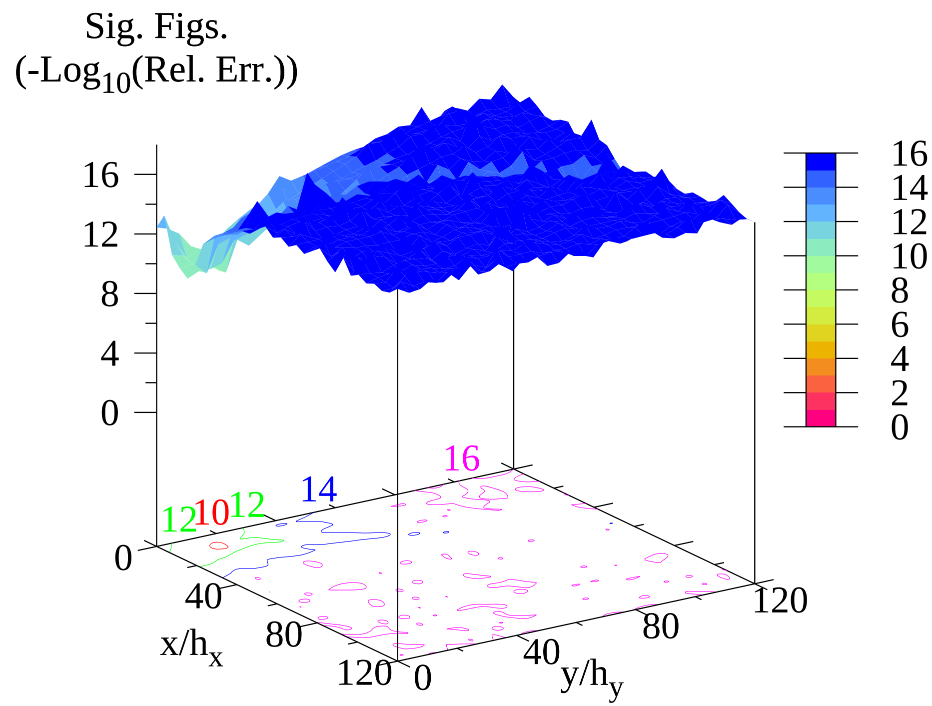
Original

Algebraic+Asymptotic, crossover at $R/h = 20$.

N_{xx}, 3:2:1 cell geometry



Offset $z/h_z = 0$



Offset $z/h_z = 10$

Conclusions

- Algebraic + asymptotics $\Rightarrow$ 10 digits
- Worse accuracy at mid-field transition (20h)
- + “long double” >12 digits?
- High order asymptotics allow MP library use restricted to near field
- Valid for arbitrary rectangular prisms