Solving Large Graph Problems Using Tree Decompositions: A Computational Study

Presented by

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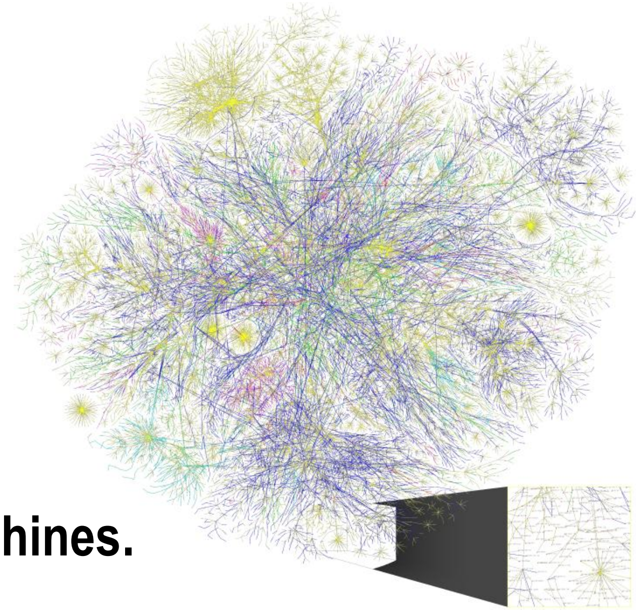
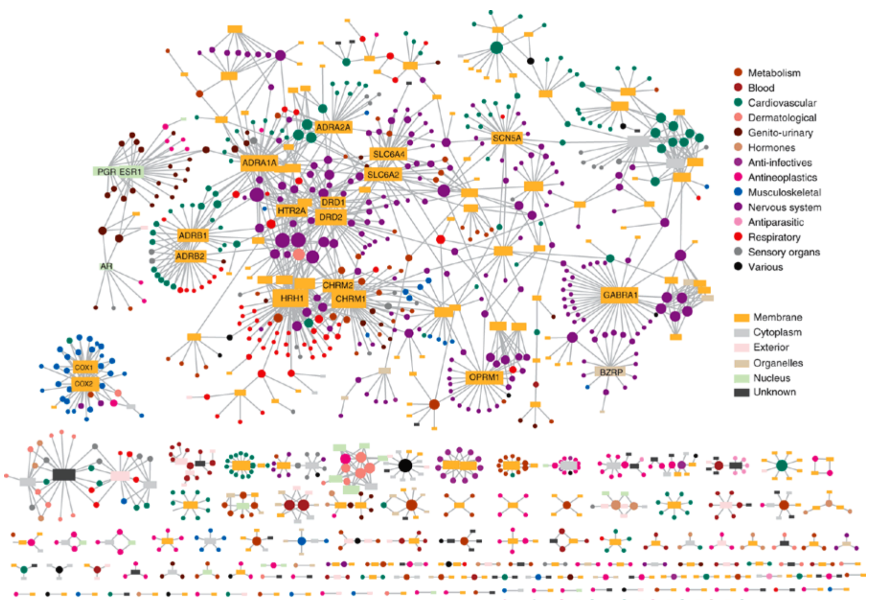
Background

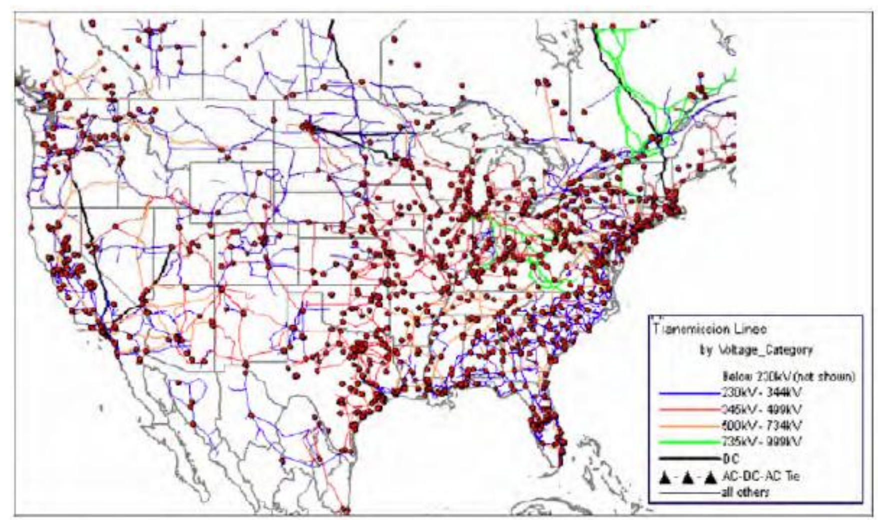
A partial map of the Internet, January 15 2005



- **Large networks are becoming ubiquitous in many domains – biology, physics, chemistry, infrastructure, communications, and sociology**
- **Graph problems have high computational complexity and require excessive computation for large networks**
- **Hard to solve efficiently on distributed memory machines.**



Drug-Target Network.
Nature Biotechnology 25(10), October 2007



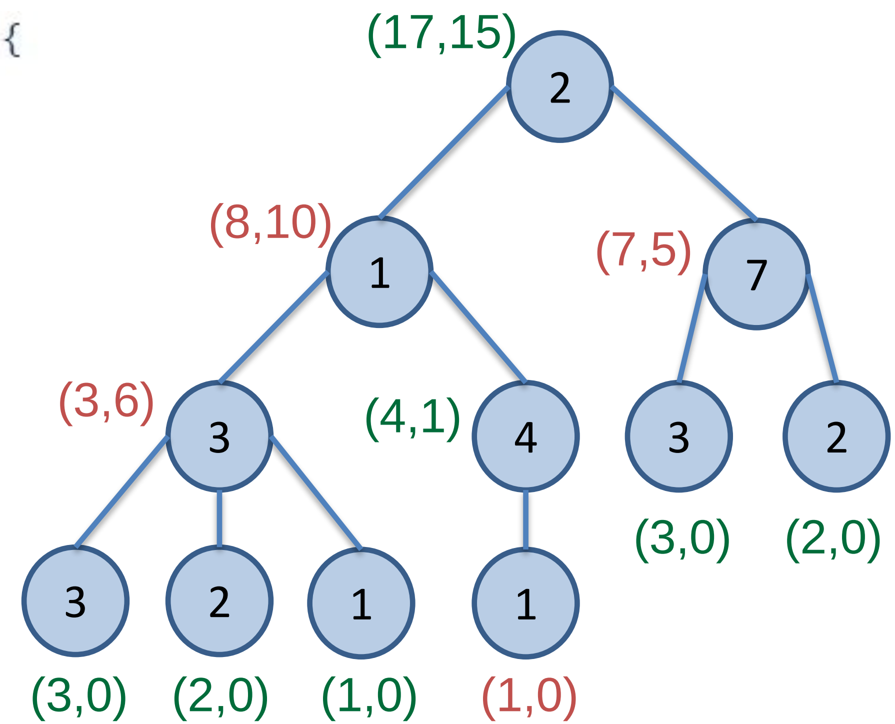
The US electric transmission system.
Courtesy North American Reliability Corporation.

Motivation: graph problems are easier on trees

- **Many NP-hard problems can be solved in polynomial time on trees (graphs with no cycles)**

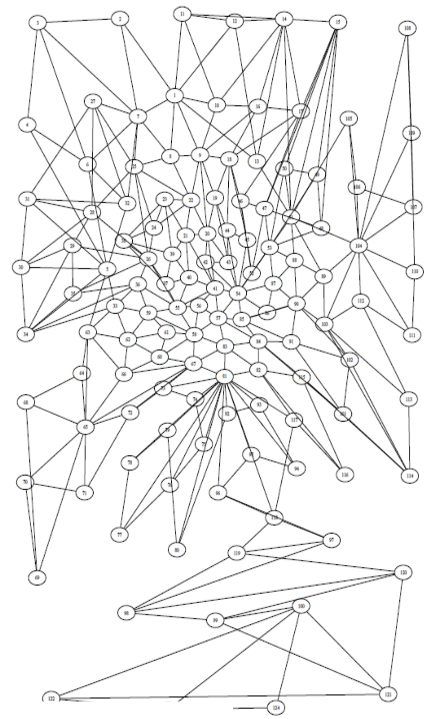
Example: Maximum Weighted Independent Set: Complexity O(|V|)

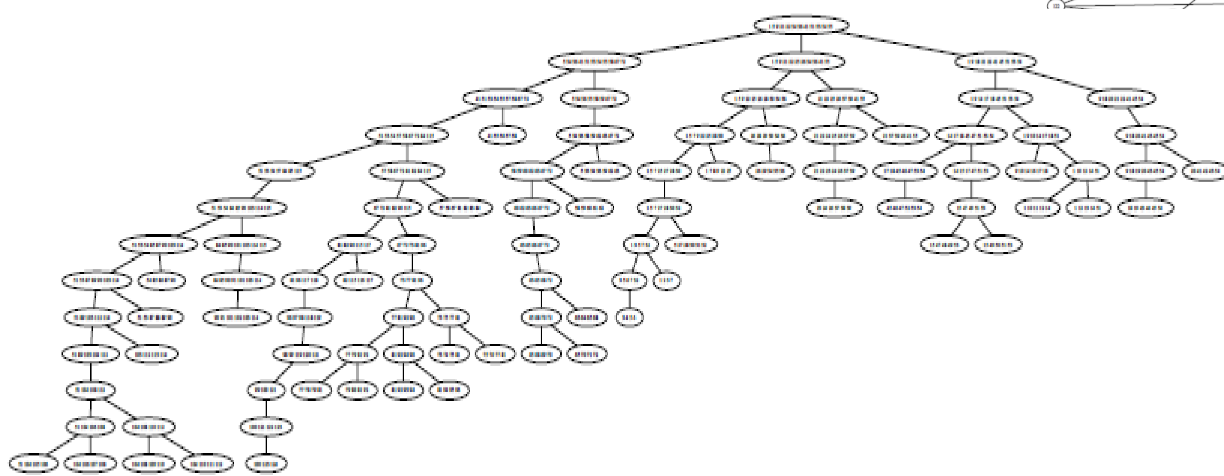
Maximum-weight independent set (Tree t, Node root) {
    For all nodes u of T in post-order {
        if u is a leaf {
            $M_{out}[u] = 0$
            $M_{in}[u] = w_u$
        }
        else {
            $M_{out}[u] = \sum_{v \in children(u)} \max(M_{out}[v], M_{in}[v])$
            $M_{in}[u] = w_u + \sum_{v \in children(u)} M_{out}[v]$
        }
    }
    return max(M_{out}[root], M_{in}[root])
}



Idea: generalize to “tree-like” graphs

- Tree decompositions are specialized maps of graphs onto trees, with subsets of V assigned to nodes of T.
- A graph is more “tree-like” if the subsets are all small.
- Many NP-hard decision/optimization problems are fixed-parameter tractable w.r.t. max subset size in map.
- This includes all problems expressible in second order monadic logic, including coloring, partial constraint satisfaction, maximum clique/independent set.

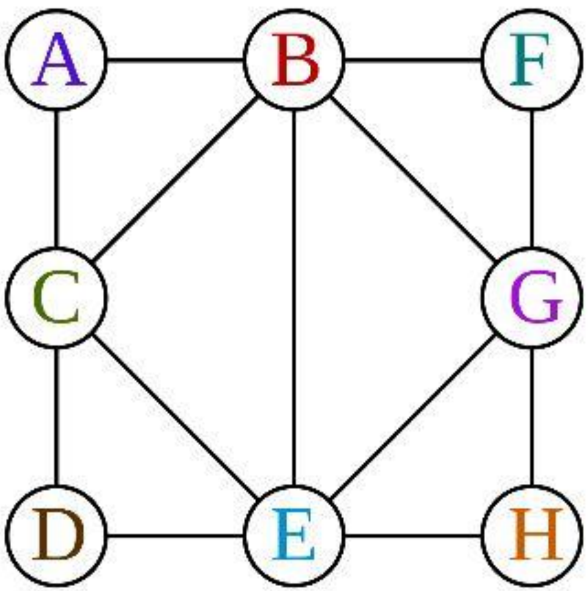


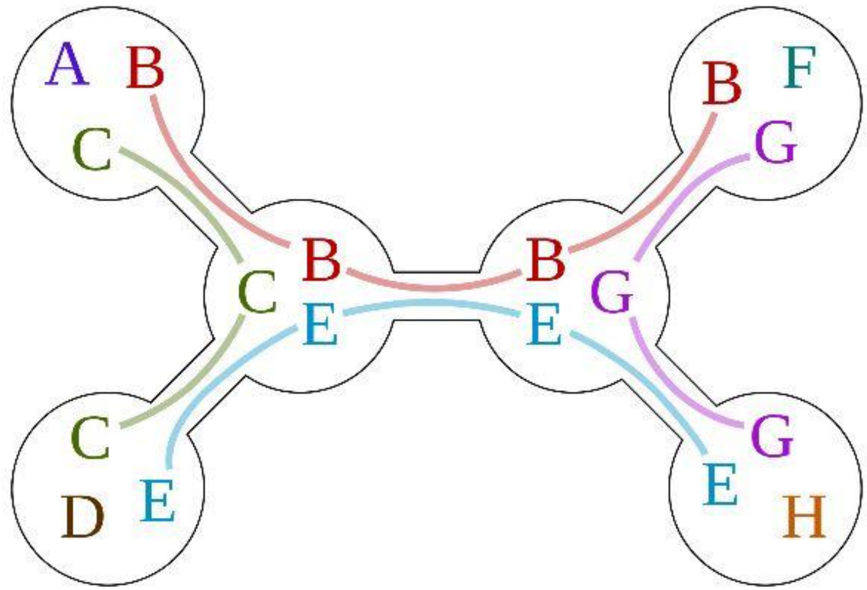


Tree Decompositions

Formally, a *tree decomposition* of a graph $G = (V, E)$ is a pair (X, T), where $X = \{X_1, \ldots, X_n\}$ is a collection of subsets of V and $T = (U, F)$ is a tree with $U = \{1, \ldots, n\}$, satisfying three conditions:

1. the union of the subsets X_i is equal to the vertex set V $(1 \le i \le n)$,
2. for every edge uv in G, $\{u, v\} \subseteq X_i$ for some $i \in [1, n]$, and
3. for every $v \in V$, if X_i and X_j contain v for some $i, j \in [1, n]$, then X_k also contains v for all k on the (unique) path in T connecting i and j. In other words, the set of nodes whose subsets contain v form a connected sub-tree of T.

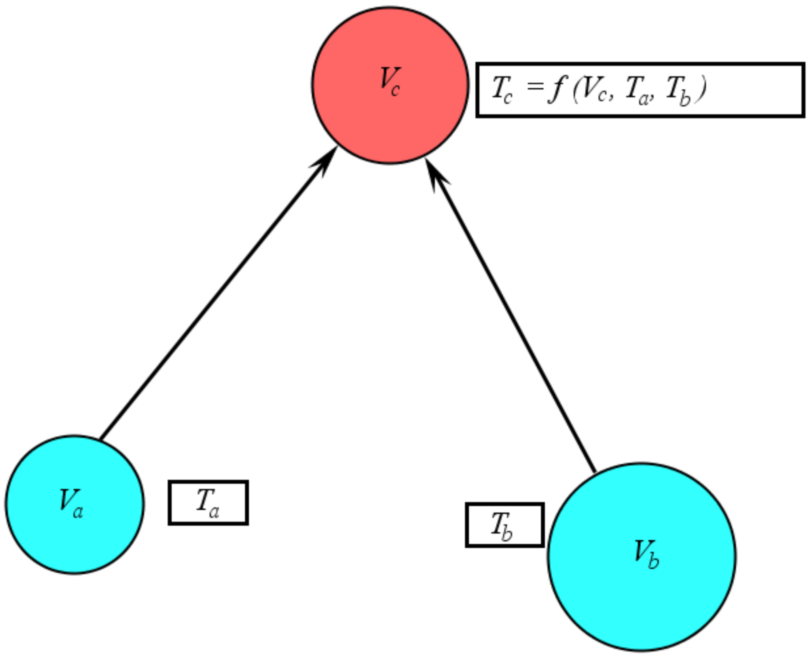




The *width* of a tree decomposition is max($|X_i|$-1), and the *treewidth* of a graph is the minimum width over all valid tree decompositions.

Dynamic Programming

- Solving decision/optimization problems uses DP on the tree decomposition.
- The general strategy is to root the tree and then work “up” from the leaves, solving sub-problems & storing partial solutions (tables) along the way, as in MWIS on a tree.



In a tree decomposition, computing the dynamic programming table at node *c* requires information about the vertices in the bag V_c and the children’s tables, T_a and T_b. The complexity of this computation can be exponential in $|V_c|$.

- Solving the sub-problems requires information about only a small part of the original graph, represented by the child nodes lower in the tree.
- The complexity of processing a specific node can be exponential in its bag size

Progress

"Tree decomposition based algorithms are a valuable alternative whenever the underlying graphs have small treewidth. As a rule of thumb, the typical border of practical feasibility lies somewhere below a treewidth of 20 for the underlying graph"
-Huffner, Niedermeier, Wernicke (2007)

- **Sequential code improvements include:**
  - **bitwise representation of vertex subsets that enable fast unions, intersections, quick elimination of families of non-independent subsets**
  - **storage of reduced dynamic programming tables using parent-child intersection properties (5-10% memory reduction on average test graph, more for sparser examples)**
  - **two-stage refinement technique for the decomposition to solve optimization problem with solution reconstruction makes two dynamic programming sweeps (demonstrated storage savings of up to an additional 80-85%)**
- **These improvements enabled the computation of MWIS on a 2 million node graph with a decomposition of over 1.8 million tree nodes, and on graphs with widths over 400.**

Some Implementation Details

- We represent a subset *S* in X_t as a single 128-bit word (type uint128_t) where a 1-bit in position *i* indicates that the *i*-th entry in X_t is in *S*
- The required union/intersection operations can be done via AND's and OR's
- The binary representation is convenient to rule out many of the 2^k possibilities in a single stroke (where $|X_t| = k$):
  - Suppose the edge *(7,11)* exists in *G*, so that any set *S* containing both *7* and *11* is not independent.
  - Let *{13,11,7,5,3,2,1}* be a bag, and consider what happens as we process subsets. We will encounter the set *{7,11}* as:

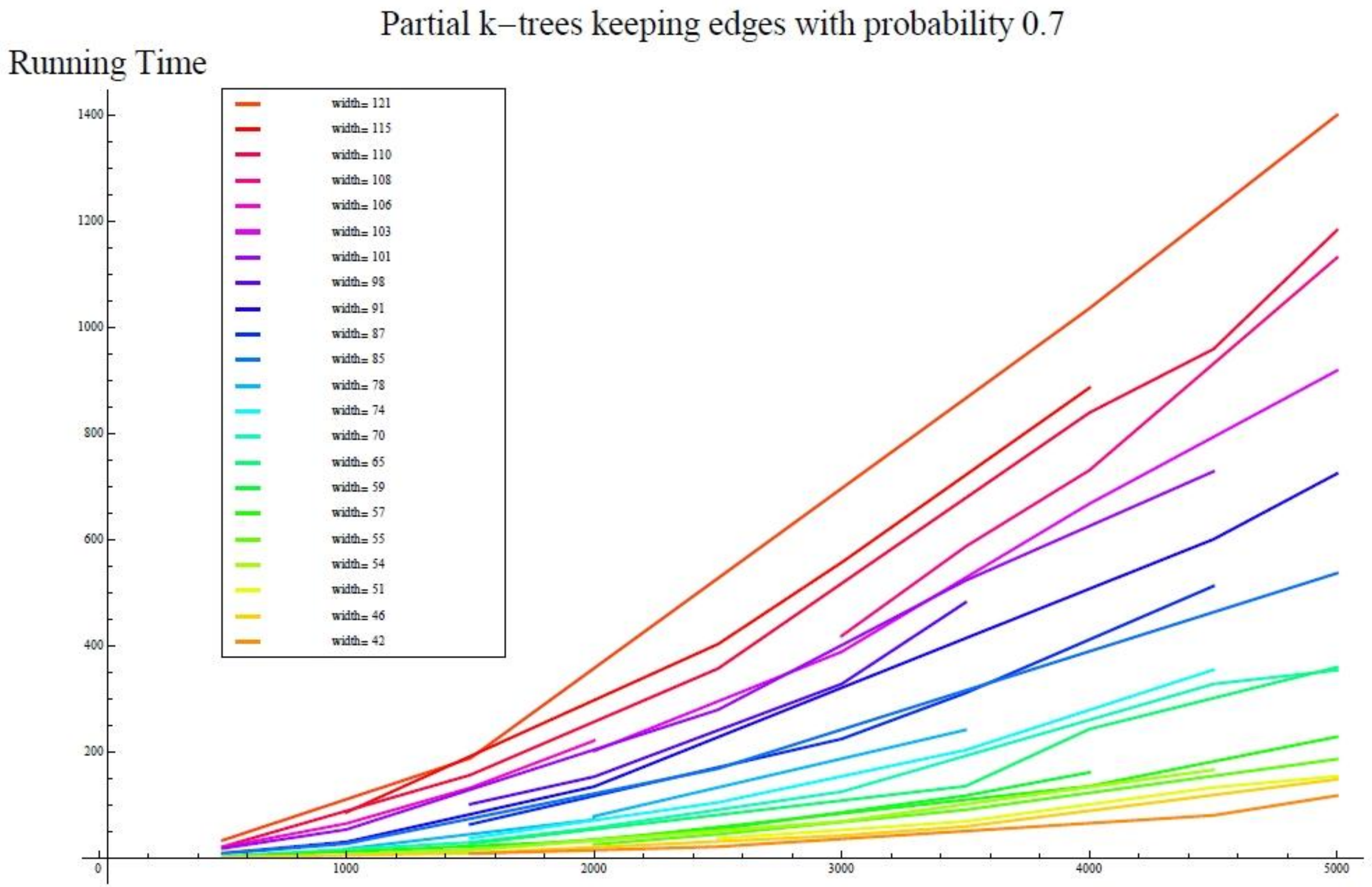
Bag	13	11	7	5	3	2	1
Mask	0	1	1	0	0	0	0

  - Now we know any mask of the form *011***** cannot represent an independent set and we eliminate $2^4 - 1$ additional possible subsets at once

– Several other similar tricks are used to speed up the computations

Running Time Analysis

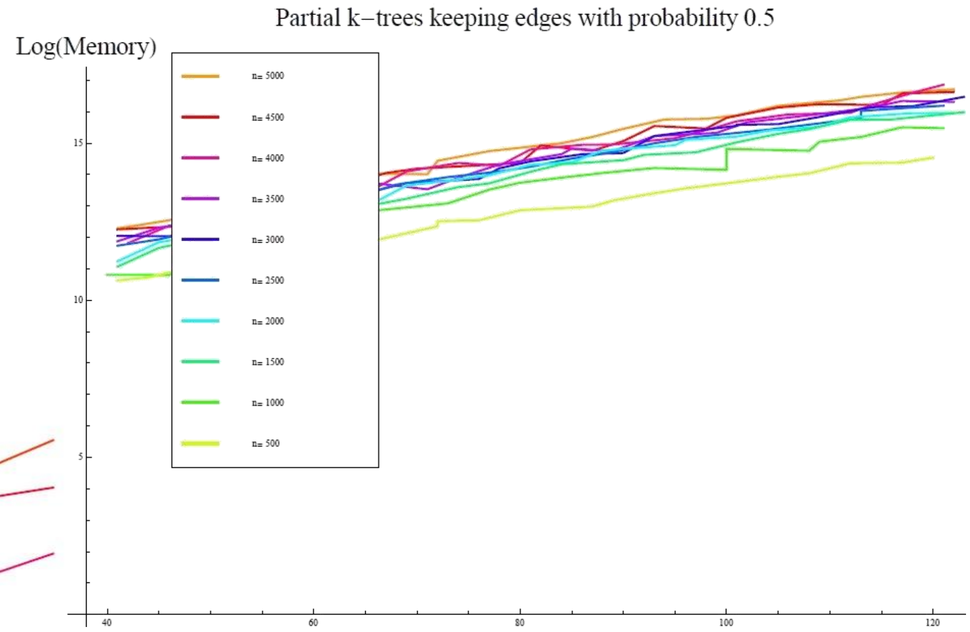
Demonstrated (approx) linear growth in running time with graph size for fixed width

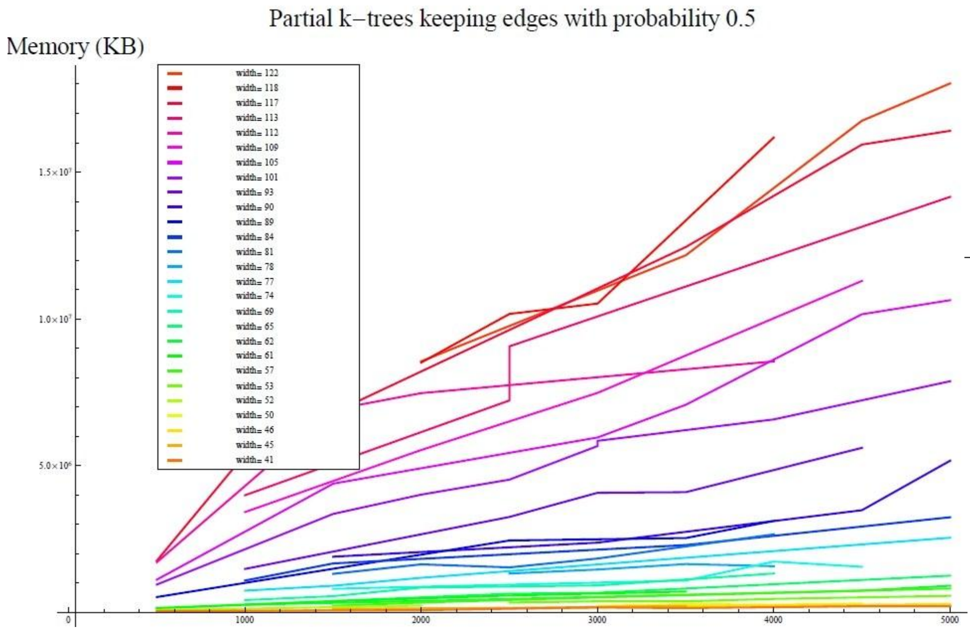


- Generated k-trees for k =50; 52; … ; 120 with n = 500; 1000; … 5000.
- Solved MWIS to optimality; recorded total time.
- Lines in the plot correspond to runs with a fixed decomposition width

Maximum Memory Usage

Memory usage also seems to grow linearly with graph size for fixed width, but exponentially with width for fixed graph size

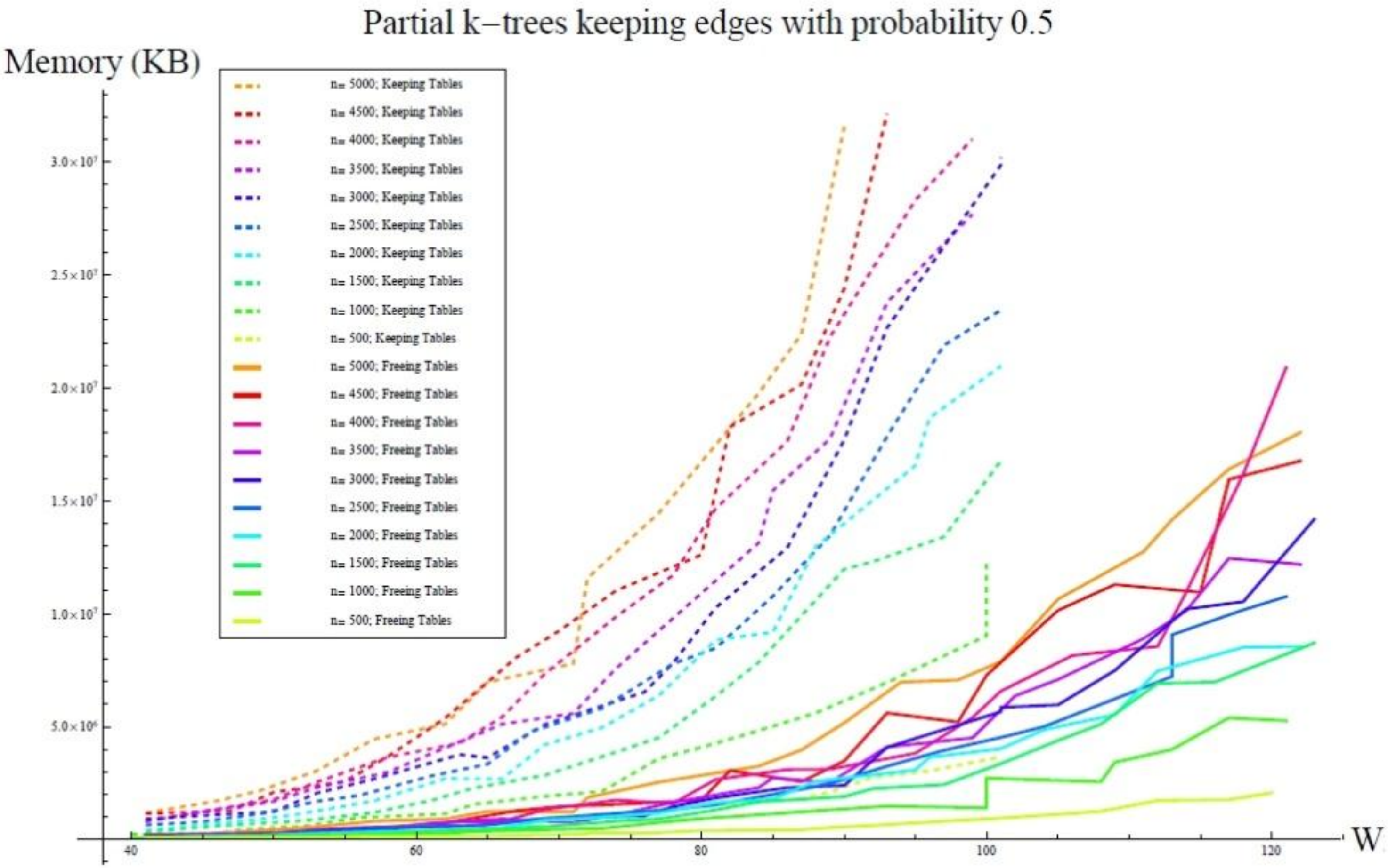




- Generated k-trees for k =50; 52; … ; 120 with n = 500; 1000; … 5000.
- Solved MWIS to optimality; recorded total time.
- Lines in the left plot correspond to fixed width, right plot correspond to fixed size

Memory Savings from Reconstruction Pruning

Second DP sweep on reduced decomposition reduces consumption drastically



- Generated k-trees for k =50; 52; … ; 120 with n = 500; 1000; … 5000.
- Solved MWIS to optimality; recorded memory high water mark.
- Negligible additional computation time required for second DP pass.

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