Hadronic Form Factors in Ads/QCD

A.V. Radyushkin

Old Dominion University and Jefferson Lab

in collaboration with H.R. Grigoryan (JLab & LSU)

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Quark counting rules

- Hadronic form factors: $(1/Q^2)^{n_q-1}$ counting rules
- Expectation: some fundamental/easily visible reason
- Most natural suspect: scale invariance
- Implementation: hard exchange in a theory with dimensionless coupling constant
- QCD: $(\alpha_s/Q^2)^{n_q-1}$
- Suppression: $F_\pi^{\text{as}}(Q^2) = \frac{2\alpha_s}{\pi} \cdot s_0/Q^2$

  $$\left[s_0 = 4\pi^2 f_\pi^2 \approx 0.67 \,\text{GeV}^2 \right]$$

- Looks like $\mathcal{O}(\alpha_s)$ correction to VMD's

$$F_\pi^{\text{VMD}}(Q^2) \sim 1/(1 + Q^2/m_\rho^2)$$

Quark counting rules

- Known: $\alpha_s/\pi \sim 0.1$ is penalty for an extra loop
- Growing consensus: pQCD gives small correction
- dominant contribution comes from soft terms modeled by GPDs $\mathcal{F}(x, Q^2)$ with exponential fall-off $e^{-Q^2 g(x)}$ for fixed x:

$$F_n(Q^2) \sim \int_0^1 e^{-(1-x)^2 Q^2/\Lambda^2} \underbrace{(1-x)^{2n-1}}_{\sim f_n(x)} dx \sim \left(\frac{\Lambda^2}{Q^2}\right)^{n-1}$$

AdS/QCD & Form Factors

- Form factor in AdS/QCD (from Brodsky & de Teramond)

$$F(Q^2) = \int_0^{1/\Lambda} \frac{d\zeta}{\zeta^3}\, \Phi_{P'}(\zeta) J(Q,\zeta) \Phi_P(\zeta)$$

- Nonnormalizable mode (for large Q)

$$J(Q,\zeta) = \zeta Q K_1(\zeta Q) \equiv \mathcal{K}(\zeta Q)$$

- Normalizable modes: $\Phi(\zeta) = C\zeta^2 J_L(\beta_{L,k}\zeta\Lambda) \equiv \zeta^2\phi(\zeta)$

$$\Rightarrow F(Q^2) = \int_0^{1/\Lambda} d\zeta\, \zeta\, \phi_{P'}(\zeta) J(Q,\zeta) \phi_P(\zeta)$$

- $\phi(\zeta)$ satisfies Dirichlet b.c. for $z = 1/\Lambda$
  $\phi(\zeta)$ is nonzero for $z = 0$ if $L = 0$ [$\phi(\zeta) \sim J_0(\beta_{0,k})$]

Hard-wall model for vector mesons (Erlich et al., Da Rold & Pomarol)

- AdS_5 metric with hard-wall

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2\right), \qquad 0 \le z \le z_0 = 1/\Lambda ,$$

- 5D gauge action for vector field

$$S_{\mathrm{AdS}} = -\frac{1}{4g_5^2}\int d^4x \, dz \, \sqrt{g}\, \mathrm{Tr}\left(F_{MN}F^{MN}\right)$$

- Field-strength tensor $F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]$
  $A_M = t^a A^a_M$, $t^a \in SU(2), a = 1,2,3; M, N = 0,1,2,3,z$
- AdS/QCD correspondence with 4D field $\tilde{A}_\mu(p)$

$$\tilde{A}_\mu(p,z) = \tilde{A}_\mu(p)\,\frac{V(p,z)}{V(p,\epsilon)}$$

Two-point Function

- Bulk-to-boundary propagator $V(p,z)$ satisfies

$$z\partial_z\left(\frac{1}{z}\,\partial_z V(p,z)\right)+p^2V(p,z)=0$$

with Neumann b.c. $\partial_z V(p,z_0)=0$
$\Rightarrow$ gauge invariant condition $F_{\mu z}(x,z_0)=0$

- Bilinear term of the action (after integration by parts)

$$S^{(2)}_{\rm AdS}=-\frac{1}{2g_5^2}\int\frac{d^4p}{(2\pi)^4}\tilde{A}^\mu(p)\tilde{A}_\mu(p)\left[\frac{1}{z}\,\frac{\partial_z V(p,z)}{V(p,\epsilon)}\right]_{z=\epsilon}$$

- 2-point function $\langle J_\mu J_\nu\rangle\sim\delta^2 S^{(2)}_{\rm AdS}/\delta A^\mu\delta A_\nu$
- Tensor structure

$$\int d^4x\; e^{ip\cdot x}\langle J^a_\mu(x)J^b_\nu(0)\rangle=\delta^{ab}\;\left(\eta_{\mu\nu}-p_\mu p_\nu/p^2\right)\;\Sigma(p^2)$$

Bound state expansion

- Solution for $V(p,z)$ with Neumann b.c.

$$V(p,z) = Pz\left[Y_0(Pz_0)J_1(Pz) - J_0(Pz_0)Y_1(Pz)\right]$$

- Two-point function

$$\Sigma(p^2) = \frac{\pi p^2}{2g_5^2}\left[Y_0(Pz) - J_0(Pz)\frac{Y_0(Pz_0)}{J_0(Pz_0)}\right]_{z=\epsilon\to 0}$$

- Kneser-Sommerfeld expansion

$$\frac{Y_0(Pz_0)J_0(Pz) - J_0(Pz_0)Y_0(Pz)}{J_0(Pz_0)}$$

$$= -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{J_0(\gamma_{0,n}z/z_0)}{[J_1(\gamma_{0,n})]^2(P^2z_0^2 - \gamma_{0,n}^2)}$$

Bound states

- Two-point function is given by sum of poles

$$\Sigma(p^2) = \frac{2p^2}{g_5^2 z_0^2} \sum_{n=1}^{\infty} \frac{[J_1(\gamma_{0,n})]^{-2}}{p^2 - M_n^2} \Rightarrow -\frac{p^2}{2g_5^2} \ln p^2$$

- Masses: $M_n = \gamma_{0,n}/z_0$
- Positive residues $f_n^2 = \lim_{p^2 \to M_n^2} \left\{ (p^2 - M_n^2)\, \Sigma(p^2) \right\}$

$$f_n^2 = \frac{2M_n^2}{g_5^2 z_0^2 J_1^2(\gamma_{0,n})}$$

- Agrees with the usual definition $\langle 0 | J_\mu^a | \rho_n^b \rangle = \delta^{ab} f_n \epsilon_\mu$
- Matching with QCD result $\Sigma_{\rm QCD}(p^2) \Rightarrow -(N_c/24\pi^2) \ln p^2$ fixes g_5

$$g_5^2 = 12\pi^2/N_c$$

Three-Point Function

- Trilinear term of action calculated on $V(q,z)$ solution:

$$S_{\mathrm{AdS}}^{(3)} = -\frac{\epsilon_{abc}}{2g_5^2}\int d^4x \int_\epsilon^{z_0} \frac{dz}{z}\,\left(\partial_\mu A_\nu^a\right) A^{\mu,b} A^{\nu,c}$$

- 3-point correlator $\langle J_a^\alpha(p_1) J_b^\beta(-p_2) J_c^\mu(q)\rangle$
- Dynamical part has Mercedes-Benz form

$$W(p_1,p_2,q) \equiv \int_\epsilon^{z_0} \frac{dz}{z}\,\frac{V(p_1,z)}{V(p_1,\epsilon)}\,\frac{V(p_2,z)}{V(p_2,\epsilon)}\,\frac{V(q,z)}{V(q,\epsilon)}$$

- Bound state expansion

$$\frac{V(p,z)}{V(p,0)} \equiv \mathcal{V}(p,z) = -\sum_{n=1}^{\infty} \frac{g_5 f_n}{p^2 - M_n^2}\,\psi_n(z)$$

Wave functions of ψ type

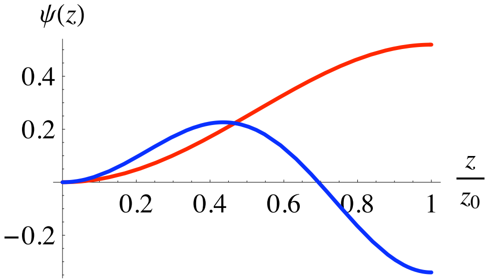
- Expansion over functions (Hong, Yoon and Strassler)

$$\psi_n(z) = \frac{\sqrt{2}}{z_0 J_1(\gamma_{0,n})} \, z J_1(M_n z)$$

- Obeying equation of motion with $p^2 = M_n^2$
- Satisfying $\psi_n(0) = 0$ and $\partial_z \psi_n(z_0) = 0$ at IR boundary
- Normalized according to

$$\int_0^{z_0} \frac{dz}{z} \, |\psi_n(z)|^2 = 1$$

Shape of ψ-type wave functions



- Do not look like bound state w.f. in quantum mechanics

EM current channel

- For spacelike q (with $q^2 = -Q^2$)

$$\mathcal{J}(Q,z) = Qz \left[K_1(Qz) + I_1(Qz) \frac{K_0(Qz_0)}{I_0(Qz_0)} \right]$$

- Bound-state expansion

$$\mathcal{J}(Q,z) = \sum_{m=1}^{\infty} \frac{g_5 f_m}{Q^2 + M_m^2} \, \psi_m(z)$$

- Infinite tower of vector mesons (Son, HJS)
- Transition form factors

$$F_{nk}(Q^2) = \int_0^{z_0} \frac{dz}{z} \, \mathcal{J}(Q,z) \, \psi_n(z) \, \psi_k(z)$$

Green's function formalism

- Green's function for equation of motion

$$G(p;z,z') = \sum_{n=1}^{\infty} \frac{\psi_n(z)\psi_n(z')}{p^2 - M_n^2}$$

- Two-point function

$$\Sigma(P^2) = \frac{1}{g_5^2}\left[\frac{1}{z'}\,\partial_{z'}\left[\frac{1}{z}\,\partial_z G(p;z,z')\right]\right]_{z,z'=\epsilon\to 0}$$

- Coupling constants

$$g_5 f_n = \left[\frac{1}{z}\,\partial_z \psi_n(z)\right]_{z=0}$$

Wave functions of ϕ type

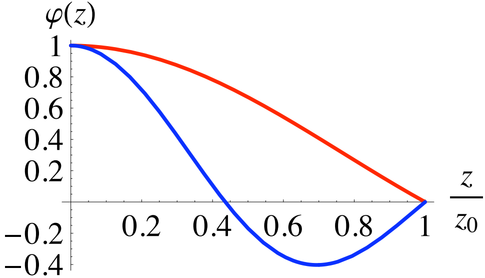
- Introducing ϕ wave functions

$$\phi_n(z) \equiv \frac{1}{M_n z}\,\partial_z \psi_n(z) = \frac{\sqrt{2}}{z_0 J_1(\gamma_{0,n})}\, J_0(M_n z)$$

- Give couplings $g_5 f_n / M_n$ as their values at the origin
- Satisfy Dirichlet b. c. $\phi_n(z_0) = 0$ at confinement radius
- Are normalized by

$$\int_0^{z_0} dz\, z\, |\phi_n(z)|^2 = 1$$

Shape of ϕ-type wave functions



- Are analogous to bound state wave functions in quantum mechanics
- Is it possible to write form factors in terms of ϕ functions!?

Form factors in terms of ϕ functions

- Elastic form factor

$$F_{nn}(Q^2) = \int_0^{z_0} \frac{dz}{z} \, \mathcal{J}(Q,z) \, |\psi_n(z)|^2$$

- Use e.o.m. for $J(Q,z)$ and ψ/ϕ connection

$$\phi_n(z) = \frac{1}{M_n z} \, \partial_z \psi_n(z) \quad , \quad \psi_n(z) = -\frac{z}{M_n} \, \partial_z \phi_n(z)$$

$$\Rightarrow F_{nn}(Q^2) = \int_0^{z_0} dz \, z \, \mathcal{J}(Q,z) \, |\phi_n(z)|^2$$

$$-\frac{Q^2}{2M_n^2} \int_0^{z_0} \frac{dz}{z} \, \mathcal{J}(Q,z) \, |\psi_n(z)|^2$$

$$= \frac{1}{1 + Q^2/2M_n^2} \int_0^{z_0} dz \, z \, \mathcal{J}(Q,z) \, |\phi_n(z)|^2$$

Form Factors

- Three form factors for vector mesons

$$\langle \rho^+(p_2,\epsilon')|J^\mu_{\rm EM}(0)|\rho^+(p_1,\epsilon)\rangle$$
$$= -\epsilon'_\beta \epsilon_\alpha \Big[\, \eta^{\alpha\beta}(p_1^\mu + p_2^\mu)\, G_1(Q^2)$$
$$+(\eta^{\mu\alpha} q^\beta - \eta^{\mu\beta} q^\alpha)(G_1(Q^2) + G_2(Q^2))$$
$$-\frac{1}{M^2} q^\alpha q^\beta (p_1^\mu + p_2^\mu)\, G_3(Q^2)\, \Big]$$

- Hard-wall model gives (also Son & Stephanov, HJS)

$$-\epsilon'_\beta \epsilon_\alpha \left[\eta_{\alpha\beta}(p_1+p_2)_\mu + 2(\eta_{\alpha\mu} q_\beta - \eta_{\beta\mu} q_\alpha)\right] F_{nn}(Q^2)$$

- Prediction: $G_1(Q^2) = G_2(Q^2) = F_{nn}(Q^2); G_3(Q^2) = 0$
- Moments: magnetic $\mu = 2$, quadrupole t $D = -1/M^2$, same result as for pointlike meson (Brodsky & Hiller)

+++ Form Factor

- +++ component of 3-point correlator gives combination

$$\mathcal{F}(Q^2) = G_1(Q^2) + \frac{Q^2}{2M^2} G_2(Q^2) - \left(\frac{Q^2}{2M^2}\right)^2 G_3(Q^2)$$

- Hard-wall model prediction

$$\mathcal{F}_{nn}(Q^2) = \int_0^{z_0} dz\, z\, \mathcal{J}(Q,z)\, |\phi_n(z)|^2$$

- Direct analogue of diagonal bound state form factors in quantum mechanics
- For ρ-meson, $\mathcal{F}(Q^2)$ coincides with (00) helicity transition that has leading $\sim 1/Q^2$ behavior in pQCD

Low-Q^2 behavior

- Free-field version $\mathcal{K}(Qz) \equiv zQK_1(Qz)$ vs. full propagator

$$\mathcal{J}(Q,z) = Qz\left[K_1(Qz) + I_1(Qz)\frac{K_0(Qz_0)}{I_0(Qz_0)}\right]$$

- $\mathcal{K}(Qz)$ has logarithmic branch singularity

$$\mathcal{K}(Qz) = 1 - \frac{z^2Q^2}{4}\left[1 - 2\gamma_E - \ln(Q^2z^2/4)\right] + \mathcal{O}(Q^4)$$

leading to incorrect infinite slope at $Q^2 = 0$

- $\mathcal{J}(Q,z)$ is analytic in Q^2

$$\mathcal{J}(Q,z)|_{Qz_0\ll 1} = 1 - \frac{z^2Q^2}{4}\left[1 - \ln\frac{z^2}{z_0^2}\right] + \mathcal{O}(Q^4)$$

- Low-Q^2 expansion for form factor

$$\mathcal{F}_{11}(Q^2) \approx 1 - 0.692\,\frac{Q^2}{M^2} + 0.633\,\frac{Q^4}{M^4} + \mathcal{O}(Q^6)$$

Comparison with constituent quark model

- CQM uses plane wave approximation $\Rightarrow$ similar to using $\mathcal{K}(Qz)$ in AdS/QCD
- Massless quarks $\Rightarrow$ infinite slope at $Q^2 = 0$
- Constituent masses $m_q \sim 300$ MeV: place lowest Q-channel singularity at $2m_q \sim m_\rho$
- AdS/QCD lesson: use massless quarks and current operator with Q-channel bound states

Electric Radius

- Electric G_C, magnetic G_M and quadrupole G_Q

$$\begin{aligned} G_C &= G_1 + Q^2/6M^2\, G_Q \;, \quad G_M = G_1 + G_2 \;, \\ G_Q &= \left(1 + Q^2/4M^2\right) G_3 - G_2 \;. \end{aligned}$$

- Hard-wall model prediction for electric form factor

$$G_C^{(n)}(Q^2) = \left(1 - \frac{Q^2}{6M^2}\right) F_{nn}(Q^2) = \frac{1 - Q^2/6M^2}{1 + Q^2/2M_n^2}\, \mathcal{F}_{nn}(Q^2)$$

$$\Rightarrow G_C^{(1)}(Q^2) \approx 1 - 1.359\, \frac{Q^2}{M^2} + 1.428\, \frac{Q^4}{M^4} + \mathcal{O}(Q^6)$$

- Electric radius of ρ-meson

$$\langle r_\rho^2 \rangle_C = 0.53\, \text{fm}^2$$

- Close to (0.54 fm^2) obtained within DSE (Bhagwat & Maris) and $m_\pi^2 \to 0$ limit of lattice calculations (Lasscock et al.)

Vector meson dominance patterns

- Generalized VMD representation (Son, HYS)

$$F_{11}(Q^2) = \sum_{m=1}^{\infty} \frac{F_{m,11}}{1 + Q^2/M_m^2} \quad , \quad \mathcal{F}_{11}(Q^2) = \sum_{m=1}^{\infty} \frac{\mathcal{F}_{m,11}}{1 + Q^2/M_m^2}$$

- Overlap integrals

$$F_{m,11} = 4 \int_0^1 d\xi \, \xi^2 \, \frac{J_1(\gamma_{0,m}\xi)\, J_1^2(\gamma_{0,1}\xi)}{\gamma_{0,m} J_1^2(\gamma_{0,m}) J_1^2(\gamma_{0,1})}$$

$$\mathcal{F}_{m,11} = 4 \int_0^1 d\xi \, \xi^2 \, \frac{J_1(\gamma_{0,m}\xi)\, J_0^2(\gamma_{0,1}\xi)}{\gamma_{0,m} J_1^2(\gamma_{0,m}) J_1^2(\gamma_{0,1})}$$

- Relation between two VMD patterns

$$F_{m,11} = \frac{\mathcal{F}_{m,11}}{1 - M_m^2/2M_1^2}$$

Low-Q^2 Sum Rules

- Charge Sum Rules

$$\sum_{m=1}^{\infty} F_{m,11} = 1 \quad , \quad \sum_{m=1}^{\infty} \mathcal{F}_{m,11} = 1$$

- For F_{11}: $1 = 1.237 - 0.239 + 0.002 + \ldots$
- For $\mathcal{F}_{11}$: $1 = 0.619 + 0.391 - 0.012 + 0.002 + \ldots$
- Slopes are given by sums

$$\sum_{m=1}^{\infty} F_{m,11}/M_m^2 \quad \text{or} \quad \sum_{m=1}^{\infty} \mathcal{F}_{m,11}/M_m^2$$

- For F_{11}: $1.192 = 1.237 - 0.045 + \ldots$
- For $\mathcal{F}_{11}$: $0.692 = 0.619 + 0.074 - 0.001 + \ldots$
- Two-resonance dominance (also HYS)

Large-Q^2 Sum Rule

- Asymptotically $F_{11}(Q^2)$ is suppressed by $1/Q^2$ compared to $\mathcal{F}_{11}(Q^2)$ since

$$F_{11}(Q^2) = \frac{\mathcal{F}_{11}(Q^2)}{1 + Q^2/2M_1^2}$$

- Known: $\mathcal{F}_{11}(Q^2) \sim M^2/Q^2$ (SJB/GdT, A.R.)
- Superconvergence/conspiracy (HYS) relation

$$\sum_{m=1}^{\infty} \frac{M_m^2}{M_1^2} F_{m,11} = 0$$

- Numerically: $0 = 1.237 - 1.261 + 0.027 + \ldots$ (agrees with HYS)
- Again, sum rule is saturated by first two states

Large-Q^2 behavior of $\mathcal{F}_{11}(Q^2)$

- Asymptotic normalization of $\mathcal{F}_{11}(Q^2)$ given by

$$\frac{1}{Q^2}\sum_{m=1}^{\infty} M_m^2 F_{m,11} \equiv \mathcal{A}\,\frac{M_1^2}{Q^2}$$

  with $\mathcal{A} = 2.566$ (A.R.) strongly exceeds naïve VMD expectation $\mathcal{A} = 1$

- Numerically: $2.566 = 0.619 + 2.061 - 0.150 + 0.054 + \dots$, result is dominated by second bound state

- Sum rule

$$\sum_{m=1}^{\infty} M_m^2 \mathcal{F}_{m,11} = |\phi_1(0)|^2 \int_0^{\infty} d\chi\, \chi^2\, K_1(\chi) = 2\,|\phi_1(0)|^2$$

Soft-Wall model

- Take model with z^2 barrier (Karch et al.)
- Equation for bulk-to-boundary propagator $V(p,z)$

$$\partial_z \left[\frac{1}{z}\, e^{-\kappa^2 z^2} \partial_z V \right] + p^2 \frac{1}{z}\, e^{-\kappa^2 z^2} V = 0$$

- Solution normalized to 1 for $z = 0$:

$$\mathcal{V}(p,z) = a \int_0^1 dx\, x^{a-1} \, \exp\left[-\frac{x}{1-x}\, \kappa^2 z^2 \right] \; ,$$

where $a = -p^2/4\kappa^2$. Integrating by parts:

$$\mathcal{V}(p,z) = \kappa^2 z^2 \int_0^1 \frac{dx}{(1-x)^2}\, x^a \, \exp\left[-\frac{x}{1-x}\, \kappa^2 z^2 \right]$$

ψ Wave Functions

- Generating function for Laguerre polynomials $L_n^1(\kappa^2 z^2)$

$$\frac{1}{(1-x)^2}\,\exp\left[-\frac{x}{1-x}\,\kappa^2 z^2\right] = \sum_{n=0}^{\infty} L_n^1(\kappa^2 z^2)\, x^n$$

- Representation analytically continuable for $a < 0$

$$\mathcal{V}(p,z) = \kappa^2 z^2 \sum_{n=0}^{\infty} \frac{L_n^1(\kappa^2 z^2)}{a+n+1} = \sum_{n=0}^{\infty} \frac{g_5 f_n}{M_n^2 - p^2}\,\psi_n(z)$$

has poles at locations $p^2 = 4(n+1)\kappa^2 \equiv M_n^2$

- ψ wave functions

$$\psi_n(z) = z^2 \sqrt{\frac{2}{n+1}}\, L_n^1(\kappa^2 z^2)$$

ϕ Wave Functions

- Coupling constants

$$g_5 f_n = \frac{1}{z} e^{-\kappa^2 z^2} \partial_z \psi_n(z) \Big|_{z=\epsilon\to 0} = \sqrt{8(n+1)}\kappa^2$$

- ϕ wave functions

$$\phi_n(z) = \frac{1}{M_n z} e^{-\kappa^2 z^2} \partial_z \psi_n(z) = \frac{2}{M_n} e^{-\kappa^2 z^2} L_n^0(\kappa^2 z^2)$$

$$\phi_0(z) = \sqrt{2} e^{-\kappa^2 z^2} \quad , \quad \phi_1(z) = \sqrt{2} e^{-\kappa^2 z^2} (1 - \kappa^2 z^2)$$

- Inverse relation between ψ and ϕ wave functions

$$\psi_n(z) = -\frac{1}{M_n} z e^{\kappa^2 z^2} \partial_z \phi_n(z)$$

Form Factors

- Normalization conditions (with $\kappa = 1$)

$$\int_0^\infty dz\, z\, e^{-z^2}\, \psi_m(z)\, \psi_n(z) = \delta_{mn}$$

$$\int_0^\infty dz\, z\, e^{z^2}\, \phi_m(z)\, \phi_n(z) = \delta_{mn}$$

- Elastic form factors

$$F_{nn}(Q^2) = \int_0^\infty \frac{dz}{z}\, e^{-z^2} \mathcal{J}(Q,z)\, |\psi_n(z)|^2$$

$$= \frac{1}{1+Q^2/2M_n^2} \int_0^\infty dz\, z\, e^{z^2}\, \mathcal{J}(Q,z)\, |\phi_n(z)|^2$$

$$\equiv \frac{1}{1+Q^2/2M_n^2}\, \mathcal{F}_{nn}(Q^2)$$

ρ-Meson Dominance

- Form factor of the lowest state

$$\mathcal{F}_{00}(Q^2) = 2\int_0^\infty dz\, z\, e^{-z^2}\, \mathcal{J}(Q,z)$$

- Using representation for $\mathcal{J}(Q,z)$ with $a = Q^2/4$ gives

$$\mathcal{F}_{00}(Q^2) = \frac{1}{1+a} = \frac{1}{1+Q^2/M_0^2}$$

- Exact vector dominance is due to overlap integral

$$\mathcal{F}_{m,00} \equiv 2\int_0^\infty dz\, z^3\, e^{-z^2}\, L_m^1(z^2) = \delta_{m0}$$

- For $F_{00}(Q^2)$ two states contribute

$$F_{00}(Q^2) = \frac{1}{(1+a)(1+a/2)} = \frac{2}{1+Q^2/M_0^2} - \frac{1}{1+Q^2/M_1^2}$$

Electric Radius in Soft-Wall Model

- Small-Q^2 expansion for electric form factor

$$G_C^0(Q^2) = \left[1 - \frac{5}{3}\frac{Q^2}{M_0^2} + 2\frac{Q^4}{M_0^4} + \mathcal{O}(Q^6)\right]$$

- Electric radius of ρ-meson

$$\langle r_\rho^2 \rangle_C = 0.655\,\mathrm{fm}^2$$

- Larger than in hard-wall model ($0.53\,\mathrm{fm}^2$)
- For higher excitations, the slope

$$\frac{d}{dQ^2} F_{nn}(Q^2)\bigg|_{Q^2=0} \equiv -\frac{S_n}{M_0^2}$$

increases logarithmically

$$S_n \approx \ln(n+1) + \frac{2}{3} + \frac{5}{4(n+1)}$$

- While size increases linearly $\langle r_n^2 \rangle \sim n/m_\rho^2$

Large-Q^2 behavior

- Large-Q^2 behavior of $\mathcal{F}$ form factor

$$\mathcal{F}_{nn}(Q^2) = \int_0^\infty dz\, z\, \mathcal{J}(Q,z)\, \Phi_n(z)\Phi_n(z)$$

- follows from $\mathcal{J}(Q,z) \to zQK_1(zQ) \equiv \mathcal{K}(zQ)$ as $Q \to \infty$ and $\mathcal{K}(zQ) \sim e^{-zQ}$, so that only $z \sim 1/Q$ work

$$\mathcal{F}_{nn}(Q^2) \to \frac{\Phi_n^2(0)}{Q^2} \int_0^\infty d\chi\, \chi^2\, K_1(\chi) = \frac{2\,\Phi_n^2(0)}{Q^2}$$

- In hard-wall model:

$$\Phi_0^{\rm H}(0) = \frac{\sqrt{2} M_\rho}{\gamma_{0,1} J_1(\gamma_{0,1})} \approx 1.133\, M_\rho \Rightarrow \mathcal{F}_\rho^{\rm H}(Q^2) \to \frac{2.56 m_\rho^2}{Q^2}$$

- In soft-wall model:

$$\Phi_0^{\rm S}(0) = \frac{M_\rho}{\sqrt{2}} \approx 0.707\, M_\rho \Rightarrow \mathcal{F}_\rho^{\rm S}(Q^2) \to \frac{m_\rho^2}{Q^2}$$

Action including χSB

- Full action of hard-wall model

$$S^B_{\mathrm{AdS}} = \mathrm{Tr} \int d^4x \int_0^{z_0} dz \left[\frac{1}{z^3} (D^M X)^\dagger (D_M X) + \frac{3}{z^5} X^\dagger X - \frac{1}{8 g_5^2 z} \left(B^{MN}_{(L)} B_{(L)MN} + B^{MN}_{(R)} B_{(R)MN} \right) \right]$$

- $DX = \partial X - i B_{(L)} X + i X B_{(R)}$, $B_{(L,R)} = V \pm A$,
  $X(x,z) = v(z) U(x,z)/2$,
  Chiral field: $U(x,z) = \exp\left[2 i t^a \pi^a(x,z)\right]$, $t^a = \sigma^a/2$
  Pion field: $\pi^a(x,z)$
  $v(z) = (m_q z + \sigma z^3)$ with $m_q \sim$ quark mass, $\sigma \sim$ condensate
- Longitudinal component of axial field

$$A^a_{\parallel\, M}(x,z) = \partial_M \psi^a(x,z)$$

gives another pion field $\psi^a(x,z)$

Pion wave function Ψ

- Model satisfies Gell-Mann–Oakes–Renner relation $m_\pi^2 \sim m_q$
- Chiral limit $m_q = 0$, then $\pi(z) = -1$
- Analytic result for $\Psi(z) \equiv \psi(z) - \pi(z)$ (Da Rold & Pomarol)

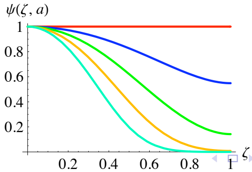
$$\Psi(z) = z\,\Gamma(2/3)\left(\frac{\alpha}{2}\right)^{1/3}\left[I_{-1/3}\left(\alpha z^3\right) - I_{1/3}\left(\alpha z^3\right)\frac{I_{2/3}\left(\alpha z_0^3\right)}{I_{-2/3}\left(\alpha z_0^3\right)}\right]$$

where $\alpha = g_5\sigma/3$

- $\Psi(z)$ satisfies $\Psi(0) = 1$, Neumann b.c. $\Psi'(z_0) = 0$ and

$$f_\pi^2 = -\frac{1}{g_5^2}\left(\frac{1}{z}\partial_z\Psi(z)\right)_{z=\epsilon\to 0}$$

$\Psi(z) \to \psi(\zeta, a)$
$\zeta \equiv z/z_0$
$a \equiv \alpha z_0^3$



$a = 0$
$a = 1$
$a = 2.26$
$a = 5$
$a = 10$

Pion wave function Φ

- Conjugate wave function

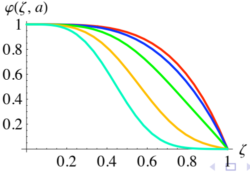
$$\Phi(z) = -\frac{1}{g_5^2 f_\pi^2}\left(\frac{1}{z}\,\partial_z \Psi(z)\right) = -\frac{2}{s_0}\,\left(\frac{1}{z}\,\partial_z \Psi(z)\right)$$

- As usual, $s_0 = 4\pi^2 f_\pi^2 \approx 0.67\,\text{GeV}^2$
- Analytic result for $\Phi(z) \to \phi(\zeta, a)$

$$\phi(\zeta,a) = \frac{3\,\zeta^2}{g_5^2 f_\pi^2}\,\Gamma\left[\frac{2}{3}\right]\frac{a^{4/3}}{2^{1/3}}\left[-I_{2/3}\left(a\zeta^3\right) + I_{-2/3}\left(a\zeta^3\right)\frac{I_{2/3}\left(a\right)}{I_{-2/3}\left(a\right)}\right]$$

- $\Phi(z)$ satisfies $\Phi(0) = 1$ and Dirichlet b.c. $\Phi'(z_0) = 0$

$\Phi(z) \to \phi(\zeta, a)$
$\zeta \equiv z/z_0$
$a \equiv \alpha z_0^3$



$a = 0$
$a = 1$
$a = 2.26$
$a = 5$
$a = 10$

Parameters of model

- z_0 is fixed through ρ-meson mass: $z_0 = z_0^\rho = (323 \text{ MeV})^{-1}$
- From $\Phi(0) = 1$, it follows that

$$g_5^2 f_\pi^2 = 3 \cdot 2^{1/3} \frac{\Gamma(2/3)}{\Gamma(1/3)} \frac{I_{2/3}\left(\alpha z_0^3\right)}{I_{-2/3}\left(\alpha z_0^3\right)} \alpha^{2/3}$$

- Experimental f_π is obtained for $\alpha = (424\,\text{MeV})^3$
- Then $a \equiv \alpha z_0^3$ equals $2.26 \equiv a_0$
- Note: $I_{2/3}(a)/I_{-2/3}(a) \approx 1$ for $a \gtrsim 1$
  $\Rightarrow$ value of f_π is basically determined by α alone

Pion Form Factor

- In terms of $\Psi(z)$:

$$F_\pi(Q^2) = \frac{1}{g_5^2 f_\pi^2} \int_0^{z_0} dz\, z\, \mathcal{J}(Q,z) \left[\left(\frac{\partial_z \Psi}{z} \right)^2 + \frac{g_5^2 v^2}{z^4} \Psi^2(z) \right]$$

- Normalization can be checked from

$$F_\pi(Q^2) = - \int_0^{z_0} dz\, \mathcal{J}(Q,z)\, \partial_z \Big(\Psi(z)\, \Phi(z) \Big)$$

that gives

$$F_\pi(0) = - \int_0^{z_0} dz\, \partial_z \Big(\Psi(z)\, \Phi(z) \Big) = \Psi(0)\, \Phi(0) = 1$$

Pion Charge Radius

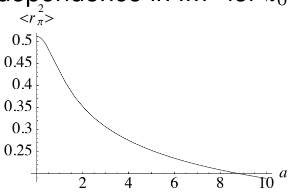
- Pion charge radius for small α is determined by z_0

$$\langle r_\pi^2 \rangle = \frac{4}{3}\, z_0^2 \left\{ 1 - \frac{a^2}{4} + \mathcal{O}(a^4) \right\}$$

- Pion charge radius for large $a \equiv \alpha z_0^3$

$$\langle r_\pi^2 \rangle \Big|_{a \gtrsim 2} \approx \frac{\Gamma(1/3)}{2^{4/3}\Gamma(2/3)} \left(\frac{1}{\alpha} \right)^{2/3} \left[1 + \frac{2}{3} \ln \left(\frac{a}{0.566} \right) \right]$$

- a-dependence in fm^2 for $z_0 = z_0^\rho$



Experimentally
$\langle r_\pi^2 \rangle \approx 0.45\,\mathrm{fm}^2$
Model value
$\langle r_\pi^2 \rangle|_{a=2.26} \approx 0.34\,\mathrm{fm}^2$
is too small

Pion Charge Radius

- In terms of f_π:

$$\langle r_\pi^2 \rangle \Big|_{a \gtrsim 2} = \frac{3}{2\pi^2 f_\pi^2} + \frac{1}{2\pi^2 f_\pi^2} \ln\left(\frac{\alpha z_0^3}{2.54}\right)$$

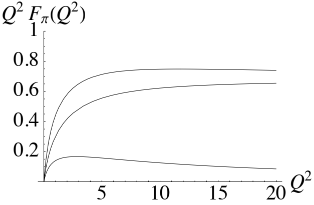
- Compare to Nambu-Jona-Lasinio model

$$\langle r_\pi^2 \rangle_{\rm NJL} = \underbrace{\frac{3}{2\pi^2 f_\pi^2}}_{0.34\rm fm^2} + \underbrace{\frac{1}{8\pi^2 f_\pi^2} \ln\left(\frac{m_\sigma^2}{m_\pi^2}\right)}_{0.11\rm fm^2}$$

Pion Form Factor at Large Q^2

- Form factor in terms of $\Psi(z)$ and $\Phi(z)$:

$$F_\pi(Q^2) = \int_0^{z_0} dz\, z\, \mathcal{J}(Q,z) \left[g_5^2 f_\pi^2 \Phi^2(z) + \frac{9\alpha^2}{g_5^2 f_\pi^2}\, z^2\, \Psi^2(z) \right]$$



- Total (in GeV^2)
- Φ^2 term
- Ψ^2 term

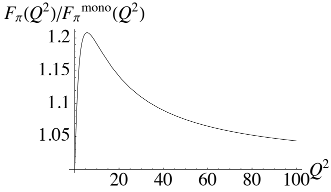
- For large Q, only $z \sim 1/Q$ work:

$$F_\pi(Q^2) \to \frac{2\, g_5^2 f_\pi^2 \Phi^2(0)}{Q^2} = \frac{4\pi^2 f_\pi^2}{Q^2} \equiv \frac{s_0}{Q^2}$$

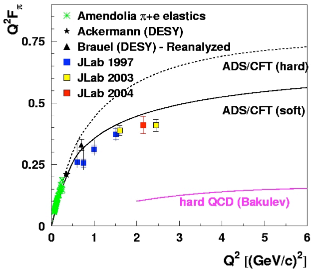
Pion Form Factor

- Monopole interpolation:

$$F_\pi^{\rm mono}(Q^2) = \frac{1}{1+Q^2/s_0}$$



- Comparison with experiment



- pQCD:

$$F_\pi^{\rm pQCD}(Q^2) \to \frac{2\alpha_s}{\pi} \cdot \frac{s_0}{Q^2}$$

$$\sim 0.2\, F_\pi^{\rm AdS/QCD}(Q^2)$$

Anomalous Amplitude

- $\pi^0\gamma^*\gamma^*$ form factor

$$\int \langle \pi, p | T\left\{ J^{\mu}_{\mathrm{EM}}(x)\, J^{\nu}_{\mathrm{EM}}(0)\right\} |0\rangle e^{-iq_1 x} d^4x$$
$$= \epsilon^{\mu\nu\alpha\beta} q_{1\,\alpha} q_{2\,\beta}\, F_{\gamma^*\gamma^*\pi^0}\left(Q_1^2, Q_2^2\right)$$

$p = q_1 + q_2$ and $q^2_{1,2} = -Q^2_{1,2}$

- For real photons in QCD is fixed by axial anomaly

$$F_{\gamma^*\gamma^*\pi^0}(0,0) = \frac{N_c}{12\pi^2 f_\pi}$$

Extending AdS/QCD Model

- Need to have isoscalar fields $\Rightarrow$ gauging $U(2)_L \otimes U(2)_R$

$$\mathcal{B}_\mu = t^a B^a_\mu + \mathbb{1}\,\frac{\hat{B}_\mu}{2}$$

- 4D currents correspond to following 5D gauge fields

$$J^{A,a}_\mu(x) \to A^a_\mu(x,z)$$
$$J^{\{I=0\}}_\mu(x) \to \hat{V}_\mu(x,z)$$
$$J^{\{I=1\},a}_\mu(x) \to V^a_\mu(x,z)$$

- Need Chern-Simons term

$$S^{(3)}_{\rm CS}[\mathcal{B}] = k\,\frac{N_c}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\,{\rm Tr}\int d^4x\,dz\,(\partial_z\mathcal{B}_\mu)\left[\mathcal{F}_{\nu\rho}\mathcal{B}_\sigma + \mathcal{B}_\nu\mathcal{F}_{\rho\sigma}\right]$$

(axial gauge $B_z = 0$)

Chern-Simons term

- Take $S_{\rm CS}^{\rm AdS}[\mathcal{B}_L, \mathcal{B}_R] = S_{\rm CS}^{(3)}[\mathcal{B}_L] - S_{\rm CS}^{(3)}[\mathcal{B}_R]$ with $\mathcal{B}_{L,R} = \mathcal{V} \pm \mathcal{A}$, and keep only $A^a_{\parallel\sigma} = \partial_\sigma \psi^a$

$$S_{\rm CS}^{\rm AdS} = k\,\frac{N_c}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4x \Big\{ -\Big[\psi^a (\partial_\rho V^a_\mu)\,(\partial_\sigma \hat{V}_\nu)\Big]\Big|_{z=z_0} + 3\int_0^{z_0} dz\,(\partial_z \psi^a)\,(\partial_\rho V^a_\mu)\,\Big(\partial_\sigma \hat{V}_\nu\Big)\Big\}$$

- After adding compensating surface term

$$S^{\rm anom} = k\,\frac{N_c}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4x \int_0^{z_0} dz\,(\partial_z \psi^a)\,(\partial_\rho V^a_\mu)\,\Big(\partial_\sigma \hat{V}_\nu\Big)$$

- Structure similar to $\pi\omega\rho$ interaction term

$$\mathcal{L}_{\pi\omega\rho} = \frac{N_c g^2}{8\pi^2 f_\pi}\epsilon^{\mu\nu\alpha\beta}\pi^a\,(\partial_\mu \rho^a_\nu)\,(\partial_\alpha \omega_\beta)\ , \qquad (1)$$

in hidden local symmetries approach (cf. Fujiwara et al., Meissner)

Conforming to QCD Anomaly

- “Bare” form factor

$$K_{\text{bare}}^{\text{AdS}}(Q_1^2, Q_2^2) = -\frac{k}{2}\int_0^{z_0} \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\partial_z\psi(z)\,dz$$

- QCD axial anomaly corresponds to $K^{\text{QCD}}(0,0) = 1$
- Calculation for “bare” form factor gives

$$K_{\text{bare}}(0,0) = -\frac{k}{2}\int_0^{z_0} \partial_z\psi(z)\,dz = -\frac{k}{2}\,\psi(z_0) = \frac{k}{2}\left[1 - \Psi(z_0)\right]$$

- On IR boundary $z = z_0$:

$$\Psi(z_0, a) = \frac{\sqrt{3}\,\Gamma\left(2/3\right)}{\pi I_{-2/3}(a)}\left(\frac{1}{2a^2}\right)^{1/3} \Rightarrow \Psi(z_0, a = 2.26) = 0.14$$

Artifact of hard-wall b.c.

Model for Anomalous Form Factor

- Take $k = 2$ and add surface term

$$\begin{aligned} K(Q_1^2, Q_2^2) &= \Psi(z_0)\mathcal{J}(Q_1, z_0)\mathcal{J}(Q_2, z_0) \\ &- \int_0^{z_0} \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\, \partial_z \Psi(z)\, dz \end{aligned}$$

- Extra term rapidly decreases with Q_1 and/or Q_2

$$\mathcal{J}(Q, z_0) = \frac{1}{I_0(Qz_0)}$$

- Effects of fixing $\Psi(z_0) \neq 0$ artifact are wiped out for large $Q_{1,2}$

One real and one highly virtual photon

- For large Q_1 and/or Q_2

$$K(Q_1^2, Q_2^2) \simeq \frac{s_0}{2} \int_0^{z_0} \mathcal{J}(Q_1, z) \mathcal{J}(Q_2, z) \, \Phi(z) \, z \, dz$$

- One real photon:

$$K(0, Q^2) \to \frac{\Phi(0) s_0}{2Q^2} \int_0^{\infty} d\chi \, \chi^2 \, K_1(\chi) = \frac{s_0}{Q^2}$$

- In pQCD:

$$K^{\mathrm{pQCD}}(0, Q^2) = \frac{s_0}{3Q^2} \int_0^1 \frac{\varphi_\pi(x)}{x} \, dx \equiv \frac{s_0}{3Q^2} \, I^\varphi$$

- Coincides with AdS/QCD model if $I^\varphi = 3$,
  e.g., for $\varphi_\pi(x) = 6x(1-x)$ (asymptotic DA)

Equal large photon virtualities

- For large Q_1 and/or Q_2

$$K(Q_1^2, Q_2^2) \simeq \frac{s_0}{2} \int_0^{z_0} \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\, \Phi(z)\, z\, dz$$

- Equal photon virtualities:

$$K(Q^2, Q^2) \to \frac{\Phi(0) s_0}{Q^2} \int_0^{\infty} d\chi\, \chi^3\, [K_1(\chi)]^2 = \frac{s_0}{3Q^2}$$

- pQCD result does not depend on pion DA

$$K^{\mathrm{pQCD}}(Q^2, Q^2) = \frac{s_0}{3} \int_0^1 \frac{\varphi_\pi(x)\, dx}{xQ^2 + (1-x)Q^2} = \frac{s_0}{3Q^2}$$

- and coincides with AdS/QCD model!

Non-equal large photon virtualities

- Take $Q_1^2 = (1+\omega)Q^2$ and $Q_2^2 = (1-\omega)Q^2$
- Leading-order pQCD gives in this case

$$K^{\text{pQCD}}(Q_1^2, Q_2^2) = \frac{s_0}{3Q^2} \int_0^1 \frac{\varphi_\pi(x)\, dx}{1+\omega(2x-1)} \equiv \frac{s_0}{3Q^2} I^\varphi(\omega)$$

- AdS/QCD model gives

$$\frac{\Phi(0)s_0}{2Q^2} \sqrt{1-\omega^2} \int_0^\infty d\chi\, \chi^3\, K_1(\chi\sqrt{1+\omega}) K_1(\chi\sqrt{1-\omega})$$

$$= \left(\frac{s_0}{3Q^2}\right) \left\{ \frac{3}{4\omega^3} \left[2\omega - (1-\omega^2) \ln\left(\frac{1-\omega}{1+\omega}\right)\right]\right\}$$

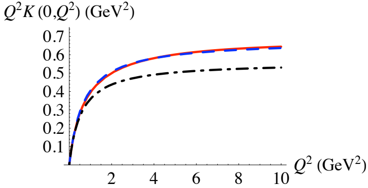
- $\{\ldots\}$ coincides with pQCD $I^\varphi(\omega)$ for $\varphi(x) = 6x(1-x)$

Comparison with data

- Brodsky-Lepage interpolation

$$K^{\mathrm{BL}}(0,Q^2) = \frac{1}{1+Q^2/s_0} \tag{2}$$

- Our model (red) is very close to BL interpolation (blue)



- CLEO data represented by black dash-dotted line
- NLO pQCD fits data. Fits give DA's with $I^{\varphi} \approx 3$

Summary

- Form factors of vector mesons:
  1. Charge radius agrees with DSE and lattice calculations
  2. Lessons for constituent quark models
  3. (00) light-front helicity form factor $\mathcal{F}(Q^2)$ indeed behaves like $1/Q^2$ for large Q^2
  4. Existence of GVMD representation has nothing to do with asymptotic large-Q^2 behavior
  5. Exact ρ-dominance for $\mathcal{F}(Q^2)$ in soft-wall model
- Pion form factor
  1. Charge radius too small compared to experimental
  2. Large-Q^2 asymptotics is s_0/Q^2 vs. pQCD $(2\alpha_s/\pi)s_0/Q^2$
  3. Overshoots data: AdS/QCD pion is too small again
- Anomalous amplitude
  1. Extension to $U(2)_L \otimes U(2)_R$ and Chern-Simons term
  2. Fixing normalization by conforming to QCD anomaly
  3. Large-Q^2 behavior coincides with pQCD calculations for asymptotic pion DA