Transverse Momentum Dependent Parton Distribution Functions with Domain Wall Valence Fermions

- class B proposal -

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[based on MILC lattices and LHPC propagators]
[method and first results arXiv:0907.2381, EPL88 61001 (2009)]

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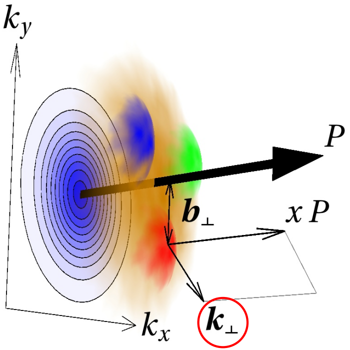
[e] Universität Regensburg, Germany

overview: parton densities



TMD PDFs

transverse **m**omentum **d**ependent **p**arton **d**istribution **f**unctions

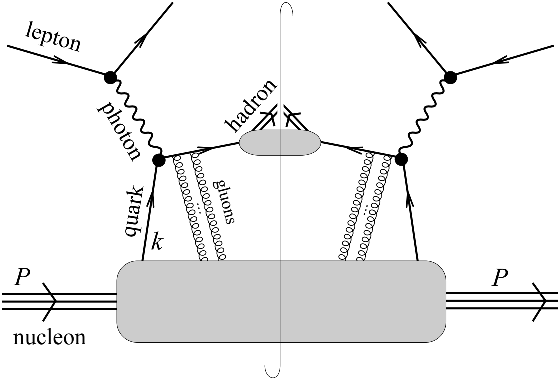
e.g., $f_1(x, \boldsymbol{k}_\perp^2)$

⇒ quark density $\rho(\boldsymbol{k}_\perp)$.

- x (longitudinal momentum fraction) ⇒ PDFs
- $x, \boldsymbol{b}_\perp$ (impact parameter) ⇒ GPDs
- $x, \boldsymbol{k}_\perp$ (intrinsic transverse momentum) ⇒ TMD PDFs

TMD PDFs from asymmetries in cross sections

e.g., semi-inclusive DIS [Collins PLB 93], [Bacchetta et al. JHEP 07]

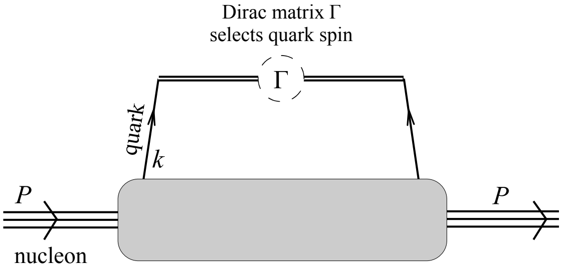


$$\frac{d\sigma}{d^3P_h\ d^3P_{l'}} \propto \underbrace{H(Q^2,\ldots)}_{\text{hard part}} \int d^2\boldsymbol{k}_\perp\ \underbrace{f_1(x,\boldsymbol{k}_\perp,\ldots)}_{\text{TMD PDF}}\ \underbrace{D_h(z,\boldsymbol{k}_\perp+\boldsymbol{q}_\perp,\ldots)}_{\text{fragmentation f.}}$$

experiments sensitive to TMD PDFs

COMPASS (CERN), HERMES (DESY), JLab, RHIC (BNL), Fermilab, also planned at J-PARC, FAIR (GSI), NICA (JINR), ..., EIC (BNL/JLab?)

definition of TMD PDFs (“basic” version)



$$\Phi^{[\Gamma]}(k,P,S) \;\equiv\; \text{“}\,\langle P,S|\,\bar{q}(k)\,\Gamma\,q(k)\,|P,S\rangle\,\text{”}$$

lightcone coor. $w^{\pm} = \frac{1}{\sqrt{2}}(w^0 \pm w^3)$, so $w = w^+\hat{n}_+ + w^-\hat{n}_- + w_\perp$

proton flies along z-axis: P^+ large, $P_\perp = 0$

parametrization in terms of TMD PDFs, example

$$\int dk^- \;\Phi^{[\gamma^+]}(k,P,S)\Big|_{k^+=xP^+} = f_1(x,\boldsymbol{k}_\perp^2) - \frac{\boldsymbol{\epsilon}_{ij}\boldsymbol{k}_i\boldsymbol{S}_j}{m_N} f_{1T}^{\perp}(x,\boldsymbol{k}_\perp)$$

[Ralston, Soper NPB 1979], [Mulders, Tangerman NPB 1996], [Goeke, Metz, Schlegel PLB 2005]

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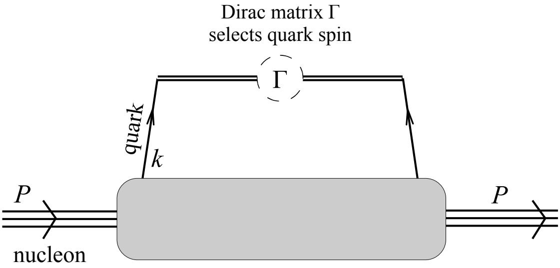

$$\Phi^{[\Gamma]}(k,P,S) \;\equiv\; \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\; \langle P,S|\;\bar{q}(\ell)\,\Gamma\,\enclose{circle}{\mathcal{U}}\,q(0)\;|P,S\rangle$$

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gauge link operator $\mathcal{U}$

$\langle P| \ \overline{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ \ |P\rangle$ is gauge invariant.

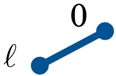
continuum

$$\mathcal{U} \ \equiv \ \mathcal{P}\exp\left(-ig\int_0^\ell d\xi^\mu A_\mu(\xi)\right)$$

along path from 0 to ℓ

- factorization in SIDIS :
  path runs to infinity and back

- ***simplification***:
  straight path (for first studies)



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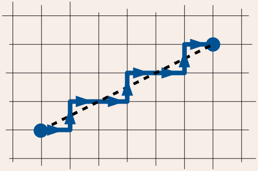
lattice



product of link variables

- factorization in SIDIS : path runs to infinity and back

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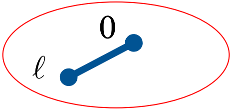
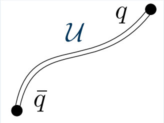


link renormalization

continuum renormalization of gauge links [Craigie, Dorn NPB185,204 (1981)]

smooth path



$$[\bar{q}\,\mathcal{U}\,q]_{\mathrm{ren}} = Z^{-1} \exp\left(-\delta\hat{m}\,\frac{l}{a}\right)\,[\bar{q}\,\mathcal{U}\,q]$$

l : the total length of the gauge link,

$\delta\hat{m}$: removes the power divergence $\sim 1/a$

static quark potential

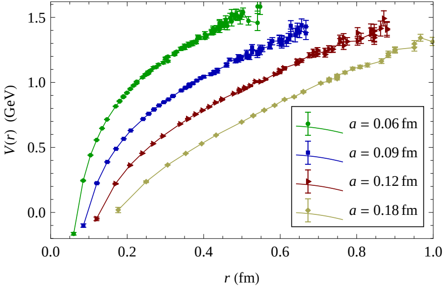
$$V_{\mathrm{ren}}(r) = V(r) + 2\,\delta\hat{m}/a$$

string [Lüscher,Symanzik,Weisz (1980)]

at large r: $V_{\mathrm{ren}}(r) \approx$ $V_{\mathrm{string}}(r) = \sigma r - \pi/12r + C$

method [Cheng PRD77,014511 (2008)]

determine $\delta\hat{m}$ from $V_{\mathrm{ren}}(0.7\text{ fm}) \overset{!}{=} V_{\mathrm{string}}(0.7\text{ fm})$



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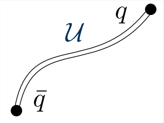

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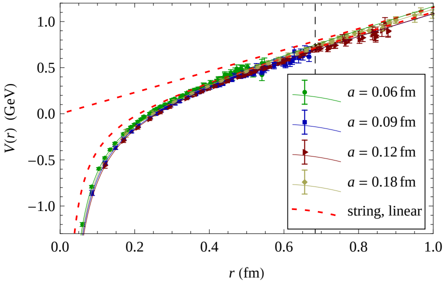

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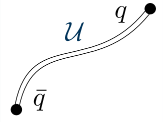

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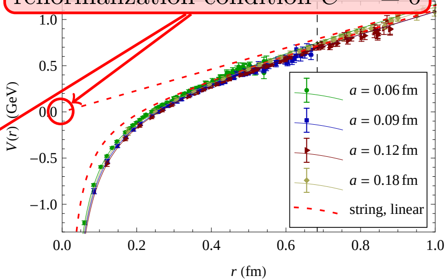
renormalization condition $C^{\text{ren}} = 0$



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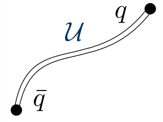

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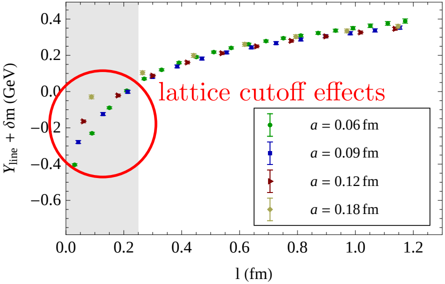
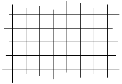


$$Y_{\mathrm{line}}(l) \equiv \frac{d}{dl}\ln\langle \mathrm{tr}\,\mathcal{U}\rangle_{(\text{Landau gauge})}$$

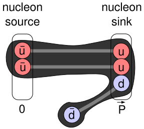
extracting nucleon structure from the lattice

Ingredients



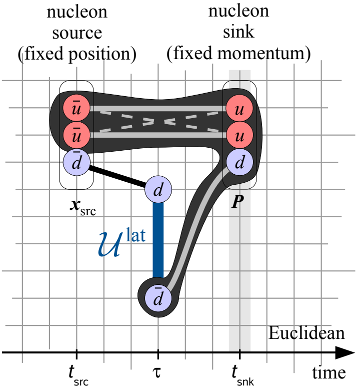
gauge configs. (MILC, staggered)

quark propagators (LHPC, DWF)



nucleon sequential propagators (LHPC, DWF)

Output : 3-point correlator $C_{3\text{pt}}$



form ratio $C_{3\text{pt}}/C_{2\text{pt}}$, take plateau

$$\Rightarrow \quad \langle P,S|\ \bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P,S\rangle$$

[We neglect "disconnected contributions" (absent for up minus down).]

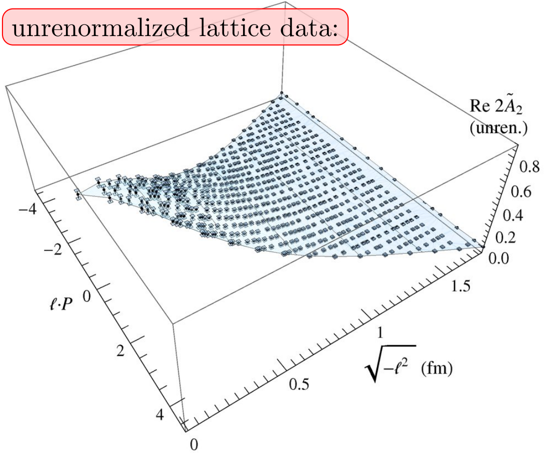
From 3-point functions to TMD PDFs

extract Lorentz-invariant amplitudes $\tilde{A}_i(\ell^2, \ell\cdot P)$, example :

$$\langle P,S|\ \overline{q}(\ell)\,\gamma_\mu\,\mathcal{U}\,q(0)\ |P,S\rangle\ =\ 4\tilde{A}_2 P_\mu\ +\ 4i\,m_N{}^2\ \tilde{A}_3\ \ell_\mu\ ,$$

$$f_1(x,\boldsymbol{k}_\perp^2) = \int \frac{d(\ell\cdot P)}{2\pi}\ e^{ix(\ell\cdot P)} \int \frac{d^2\boldsymbol{\ell}_\perp}{(2\pi)^2}\ e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{\ell}_\perp}\ 2\tilde{A}_2(\ell^2,\ell\cdot P)\ \Big|_{\ell^+=0}$$

unrenormalized lattice data:



$$\ell^2 \xleftrightarrow{\text{FT}} \boldsymbol{k}_\perp^2$$

$$\ell\cdot P \xleftrightarrow{\text{FT}} x$$

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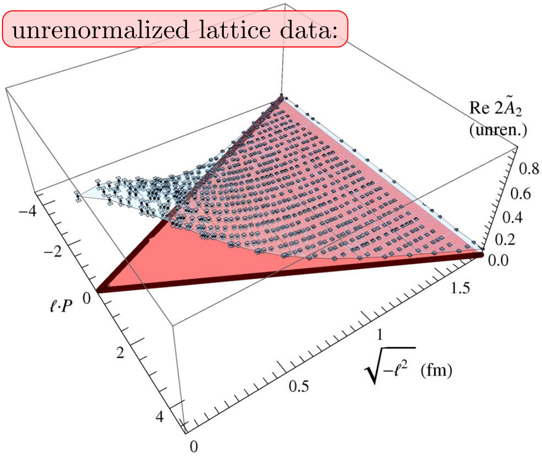

unrenormalized lattice data:

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$$\ell\cdot P \xleftrightarrow{\text{FT}} x$$

Euclidean lattice

$$\ell^0 = \ell_4 = 0$$

$$\Downarrow$$

$$\ell^2 \le 0,$$

$$|\ell\cdot P| \le |\boldsymbol{P}|\sqrt{-\ell^2}$$

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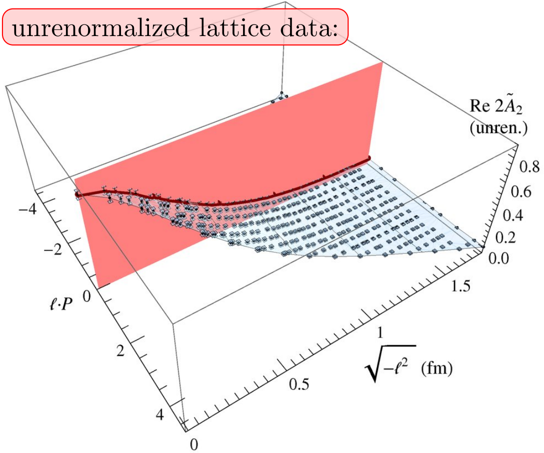

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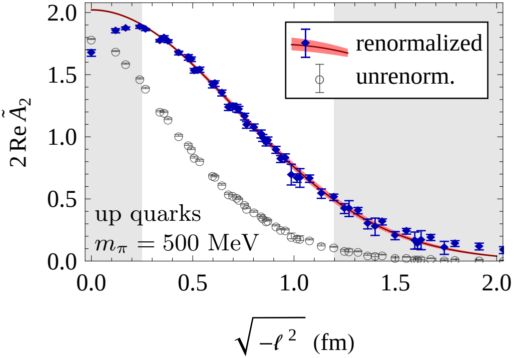
lowest x-moment of $f_1(x, \boldsymbol{k}_\perp^2)$

$$f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \equiv \int_{-1}^{1} dx\ f_1(x, \boldsymbol{k}_\perp^2) = \int \frac{d^2\ell_\perp}{(2\pi)^2}\ e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{\ell}_\perp}\ 2\,\tilde{A}_2(-\ell_\perp^{\,2}, 0)$$



fit function

$C_1\ \exp(-|\ell|^2/\sigma_1^2)$

Z-factor

$Z^{-1}\ C_1^{\text{up-down}} \overset{!}{=} 1$

multiplicative renormalization based on quark counting

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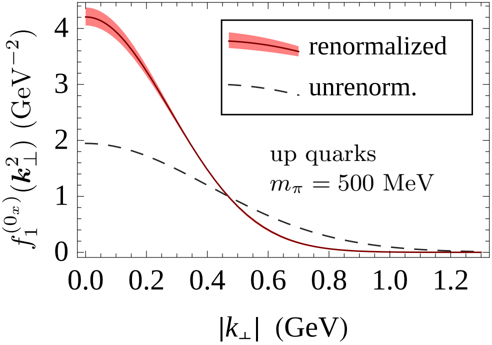

width of the distribution (RMS momentum):

$\langle \boldsymbol{k}_\perp^2 \rangle^{1/2} =$
$(391 \pm 8_{\text{stat}} \pm 27_{\text{sys}})$ MeV

compare phenomenology [Anselmino et al., PRD71, 074006 (2005)]:

$\langle \boldsymbol{k}_\perp^2 \rangle^{1/2} \approx 500$ MeV

(estimate, Gaussian Ansatz)

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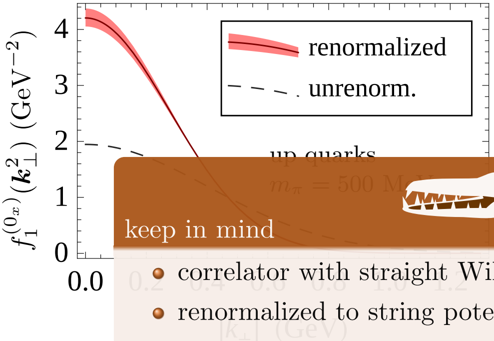

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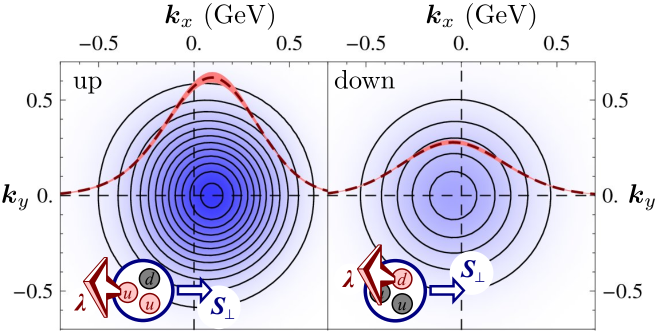
keep in mind

- correlator with straight Wilson line ("sW")
- renormalized to string potential with $C = 0$
- Gaussian fit ansatz
  ("wrong" at large-$\boldsymbol{k}_\perp$ [DIEHL, arXiv:0811.0774])
- $m_\pi \approx 500$ MeV

a polarized $\boldsymbol{k}_\perp$-dependent quark density

Density of quarks with positive helicity, $\lambda = 1$,
in a transversely polarized nucleon, $\boldsymbol{S}_\perp = (1, 0)$:

$$\rho_{TL}(\boldsymbol{k}_\perp; \boldsymbol{S}_\perp, \lambda) \equiv \frac{1}{2} \int dx \int dk^- \ \Phi^{[\gamma^+ \frac{1}{2}(\mathbb{1}+\gamma^5)]}(k, P, S_\perp)$$
$$= \frac{1}{2} f_1^{(0_x)}(\boldsymbol{k}_\perp^2) + \frac{\lambda}{2} \frac{\boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp}{m_N} g_{1T}^{(0_x)}(\boldsymbol{k}_\perp^2)$$



$\left(\begin{array}{c} m_\pi \approx 500 \text{ MeV, straight gauge link operator,} \\ \text{renormalization condition } C^{\text{ren}} = 0, \text{ Gaussian fit} \end{array} \right)$

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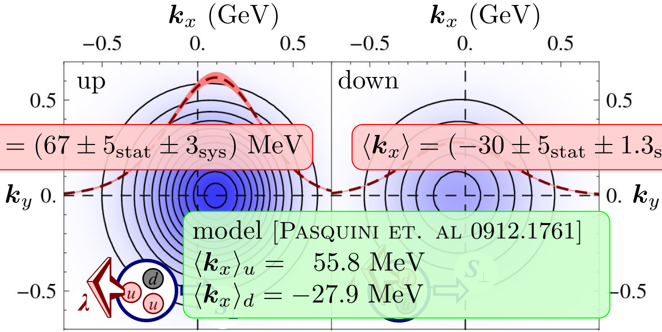


$\langle \boldsymbol{k}_x \rangle = (67 \pm 5_{\text{stat}} \pm 3_{\text{sys}})$ MeV

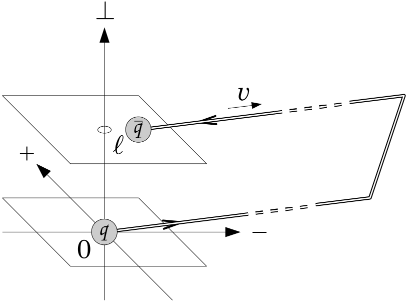
$\langle \boldsymbol{k}_x \rangle = (-30 \pm 5_{\text{stat}} \pm 1.3_{\text{sys}})$ MeV

model [Pasquini et. al 0912.1761]
$\langle \boldsymbol{k}_x \rangle_u = \quad 55.8$ MeV
$\langle \boldsymbol{k}_x \rangle_d = -27.9$ MeV

$\left(\begin{array}{c} m_\pi \approx 500 \text{ MeV, straight gauge link operator,} \\ \text{renormalization condition } C^{\text{ren}} = 0 \text{, Gaussian fit} \end{array} \right)$

staple-shaped gauge links ...

- ... appear in factorized SIDIS / Drell-Yan process
- are responsible for “time-reversal-odd” TMD PDFs, such as $f_{1T}^{\perp}$ (Sivers-function)

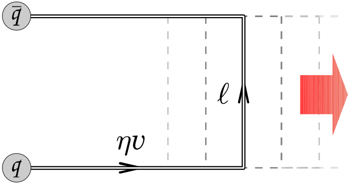


- gauge link = effective representation of struck quark (“final state interaction”)
- ⇒ (almost lightlike)

$$\zeta \equiv \frac{(v\cdot P)^2}{v^2} \to \pm\infty$$

- keep ζ finite to avoid “rapidity divergences”
- evolution equation in ζ [Collins, Soper NPB (1981)]

staple shaped links on the lattice



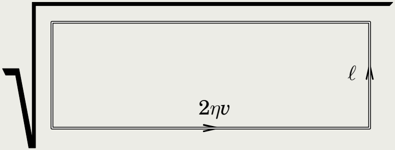
- v spatial $\Rightarrow$ $|\zeta| = \dfrac{(v\cdot P)^2}{|v|^2} \leq |\boldsymbol{P}_{\text{lat.}}|^2$
- look for plateaus at large $|\eta|$
- now 32 amplitudes $\tilde a_i(\ell^2, \ell\cdot P, v\cdot P; \eta, \zeta)$

Problem: need to subtract gauge link self-energy ($\to$ η-independence)

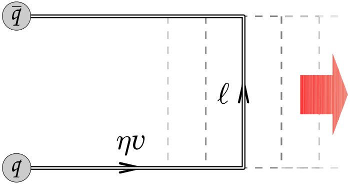
idea #1: modify definition of TMD PDFs [Collins PoS LC (2008)]

$$\Phi^{[\Gamma]}(k,P,S) \;\equiv\; \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\, \frac{\langle P,S|\;\bar q(\ell)\,\Gamma\,\mathcal{U}\,q(0)\;|P,S\rangle}{\widetilde{S}(\boldsymbol{\ell}_\perp,\ldots)}$$

with $\widetilde{S}$ obtained from a vacuum expectation value of gauge links, e.g.,



staple shaped links on the lattice



- v spatial $\Rightarrow$ $|\zeta| = \dfrac{(v\cdot P)^2}{|v|^2} \leq |\boldsymbol{P}_{\text{lat.}}|^2$
- look for plateaus at large $|\eta|$
- now 32 amplitudes $\tilde{a}_i(\ell^2, \ell\cdot P, v\cdot P; \eta, \zeta)$

Problem: need to subtract gauge link self-energy (→ η-independence)

idea #2: ratios of amplitudes → certain $\boldsymbol{k}_\perp$-moments

e.g., formally,

$$\langle \boldsymbol{k}_y \rangle_{TU} = -2m_N \boldsymbol{S}_x \lim_{\eta\to\infty} \frac{\tilde{a}_{12}(0,0,0;\eta,\zeta) + \ldots}{\tilde{a}_2(0,0,0;\eta,\zeta)} \propto \frac{\int dx \int d^2\boldsymbol{k}_\perp\, \boldsymbol{k}_\perp^2\, f_{1T}^\perp}{\int dx \int d^2\boldsymbol{k}_\perp\, f_1}$$

Sivers function causes average transverse quark momentum in y-direction in a transversely polarized nucleon (spin in x-direction).

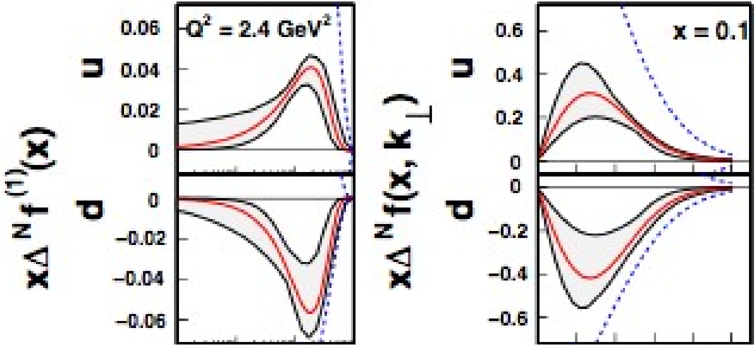
$$\langle \boldsymbol{k}_y \rangle_{TU} \underset{\eta\ \text{large}}{\approx} -2m_N \boldsymbol{S}_x \frac{\tilde{a}_{12}(\ell^2_{\text{min}},0,0;\eta,\zeta) + \ldots}{\tilde{a}_2(\ell^2_{\text{min}},0,0;\eta,\zeta)}$$

Self-energy cancels!

Sivers function from experiment

HERMES and COMPASS data

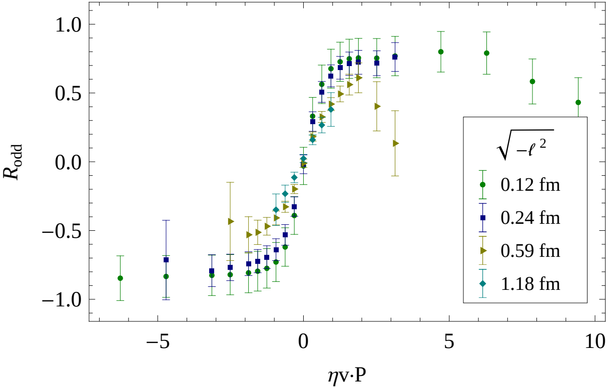
[Anselmino *et. al.* EPJ A (2009)]



$2\,\langle k_\perp\rangle_{TU} = 96^{+60}_{-28}$ MeV (up) $\quad \langle k_\perp\rangle_{TU} = -113^{+45}_{-51}$ MeV (down)

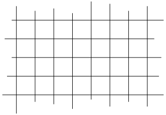
Test calculation: A $\mathcal{T}$-odd ratio from the lattice

$$R_{\text{odd}} = \frac{\tilde{a}_{12} + (\eta \frac{m_N^2 v_1}{P_1})\, \tilde{b}_8}{\tilde{a}_2}$$



Plateaus visible at large $|\eta|$. “Time-reversal odd” ↔ odd in $\eta v{\cdot}P$.
Part of the effect comes from the Sivers function $f_{1T}^{\perp}$!

input lattice data

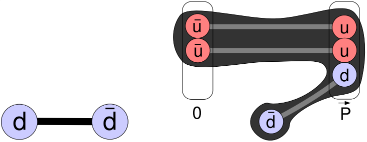


MILC gauge configurations

staggered Asqtad action,
2+1 flavors, $a \approx 0.12$ fm

[AUBIN *et. al.* PRD (2004)]

code built on top of CHROMA



LHPC propagators

domain wall valence fermions,
m_π adjusted to staggered sea,

nucleon at $|\boldsymbol{P}| = 0$ and 500 MeV,
source-sink separation $9a$,
8 measurements per config.

[BRATT *et. al.* arXiv:1001.3620]

$\hat{m}_{u,d}/\hat{m}_s$ (sea)	volume	m_π^{DWF} (MeV)	# conf.
0.007/0.050	$20^3 \times 64$	293	463
0.010/0.050	$28^3 \times 64$	356	274
0.010/0.050	$20^3 \times 64$	356	631
0.020/0.050	$20^3 \times 64$	495	486

request and goals

selection of link paths: in total ≈ 16000

- direct link analysis: ≈ 1000 choices of ℓ (as previously)
- staple link analysis: $\#\ell \approx 100$, $\#v = 4$, $\#\eta \approx 2 \times 18 \;\Rightarrow \approx 15000$

cost: 1.26 mill. J/Psi core hours, + 19 TB tape, + 4 TB work disk

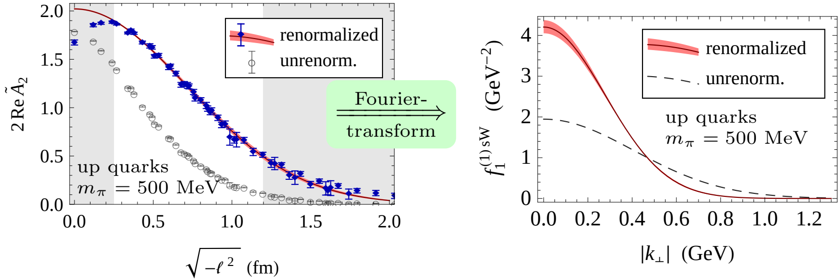
- ≈ 23 sec. per link path per config. on a JLab 6N core, ×1854 configs × 8 measurements × 16000 paths
- + smearing + chopping + analysis
- + (re)generation of some sequential propagators, including $|\boldsymbol{P}| = 1.0$ GeV at $m_\pi \approx 500$ MeV $\rightarrow |\zeta|_{\mathrm{max}} = (1\ \mathrm{GeV})^2$

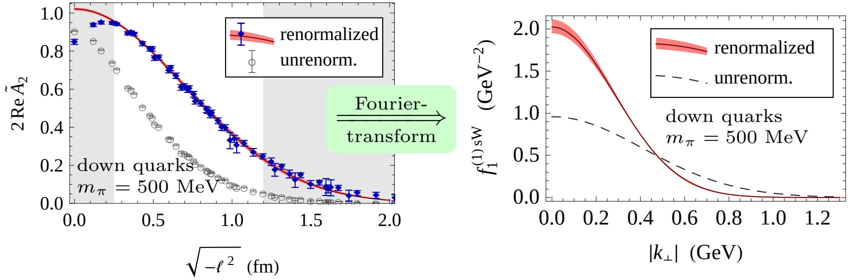
goals

- direct links: higher statistics, lower pion masses for
  - lowest x-moment of f_1, g_{1L}, g_{1T}, h_{1T}, $h_{1L}^\perp$, $h_{1T}^\perp \Rightarrow$ densities
  - study of correlations in x and $\boldsymbol{k}_\perp$
- staple shaped links:
  - x- and $\boldsymbol{k}_\perp$-moments; now including $\mathcal{T}$-odd functions such as $f_{1T}^\perp$
  - study of the effect of the link geometry

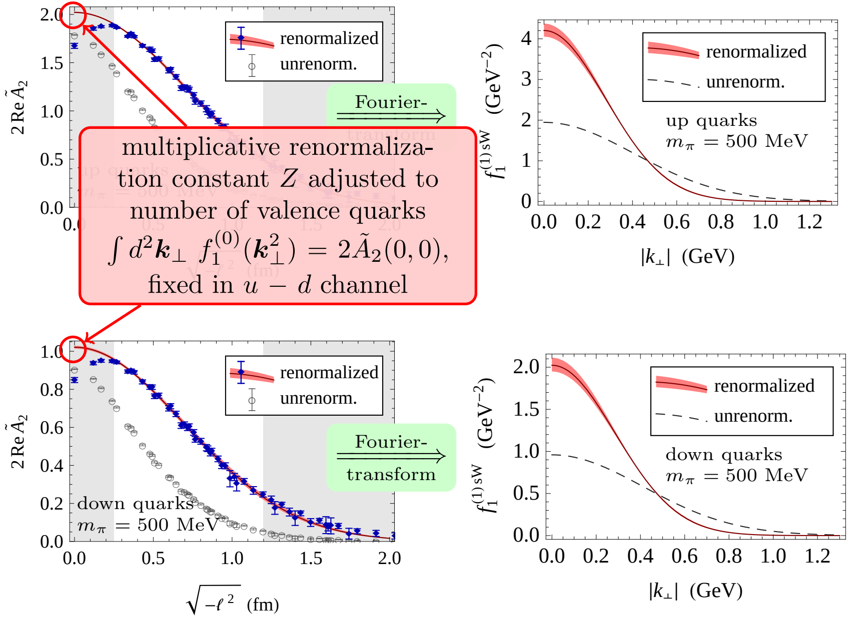
Backup Slides

$f_1^{(0_x)}$ from $\tilde{A}_2$





$f_1^{(0_x)}$ from $\tilde{A}_2$



multiplicative renormalization constant Z adjusted to number of valence quarks

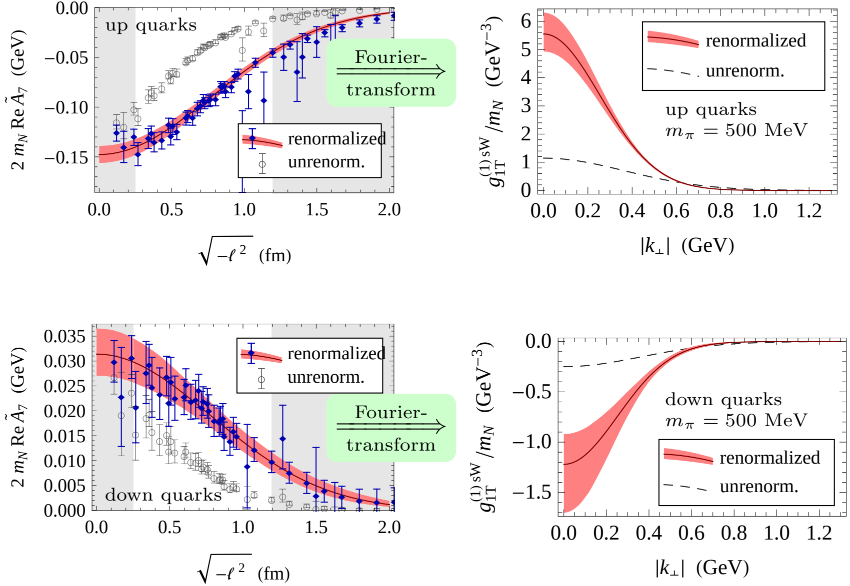
$$\int d^2\boldsymbol{k}_\perp\; f_1^{(0)}(\boldsymbol{k}_\perp^2) = 2\tilde{A}_2(0,0),$$

fixed in $u - d$ channel

$$\xRightarrow[\text{transform}]{\text{Fourier-}}$$

$$\xRightarrow[\text{transform}]{\text{Fourier-}}$$

$g_{1T}^{(0_x)}$ from $\tilde{A}_7$



$\boldsymbol{k}_\perp$-moments, weighted asymmetries

$$f^{(m_x,n_\perp)} \equiv \int_{-1}^{1} dx\, x^m \int d^2\boldsymbol{k}_\perp \left(\frac{\boldsymbol{k}_\perp^2}{2m_N^2}\right)^n f(x,\boldsymbol{k}_\perp^2)$$

Let us assume the amplitudes $\tilde{A}_i$ are sufficiently regular at $\ell^2 = 0$.

$$\langle \boldsymbol{k}_\perp \rangle_{\rho_{TL}} = \lambda \boldsymbol{S}_\perp m_N \frac{g_{1T}^{(0_x,1_\perp)}}{f_1^{(0_x,0_\perp)}} =$$

$$\lambda \boldsymbol{S}_\perp m_N \frac{\tilde{A}_7(0,0)}{\tilde{A}_2(0,0)} \overset{?}{=} \lim_{\ell^2\to 0} \lambda \boldsymbol{S}_\perp m_N \frac{\tilde{A}_7(\ell^2,0)}{\tilde{A}_2(\ell^2,0)}$$

All self-energies from the gauge link cancel on the RHS
(⇒ no dependence on the renormalization condition).
Similar to weighted asymmetries from experiment (→ EIC):

$$A_{LT}^{\frac{Q_T}{m_N}\cos(\phi_h-\phi_S)} = 2\frac{\langle \frac{Q_T}{m_N}\cos(\phi_h-\phi_S)\rangle_{UT}}{\langle 1\rangle_{UU}} \propto \frac{\sum_q e_q^2\, x\, g_{1T,q}^{(1_\perp)}(x)\, D_{1,q}(z)}{\sum_q e_q^2\, x\, f_{1,q}(x)\, D_{1,q}(z)}$$

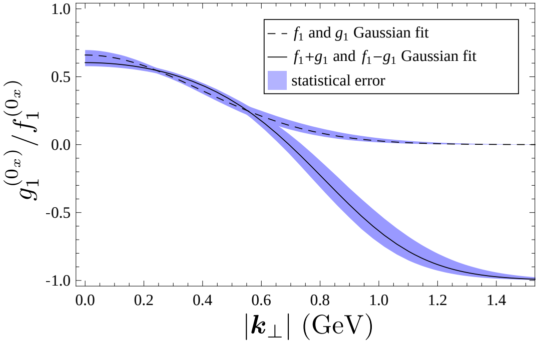
[Boer, Mulders PRD 1998], [Bacchetta et al. arXiv:1003.1328]

testing Gaussian parametrization

$$f_1^{(0_x)}(\boldsymbol{k}_\perp^2) = C_0 \exp(-\boldsymbol{k}_\perp^2/\mu_0^2)$$
$$g_1^{(0_x)}(\boldsymbol{k}_\perp^2) = C_2 \exp(-\boldsymbol{k}_\perp^2/\mu_2^2)$$

vs.

$$\rho_{LL}^{\pm}(\boldsymbol{k}_\perp) \equiv \tfrac{1}{2} f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \pm \tfrac{1}{2} g_1^{(0_x)}(\boldsymbol{k}_\perp^2)$$
$$\rho_{LL}^{+}(\boldsymbol{k}_\perp) = C_+ \exp(-\boldsymbol{k}_\perp^2/\mu_+^2)$$
$$\rho_{LL}^{-}(\boldsymbol{k}_\perp) = C_- \exp(-\boldsymbol{k}_\perp^2/\mu_-^2)$$

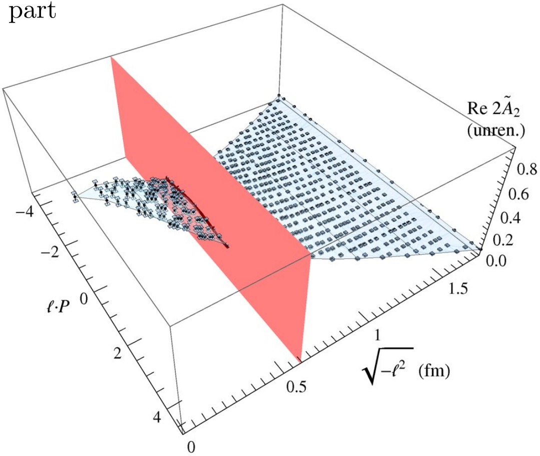


⇒ Asymptotic behavior at large $\boldsymbol{k}_\perp$ imposed by Gaussian ansatz; not a "lattice result". Similar issues in analysis of experimental data.

x-dependence

$\ell\cdot P$-dependence in $\tilde{A}_2$

real part



$$\ell^2 \xleftrightarrow{\text{FT}} \boldsymbol{k}_\perp^2$$

$$\ell\cdot P \xleftrightarrow{\text{FT}} x$$

$(x, k_\perp)$-factorization hypothesis

factorization hypothesis

$$f_1(x, \boldsymbol{k}_\perp^2) \;\approx\; f_1(x) \;\; f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \; / \; \mathcal{N}$$

as in phenomenological applications,
e.g., Monte Carlo event generators

Then $\tilde{A}_2$ factorizes, too:

$$\tilde{A}_2(\ell^2, \ell\cdot P) = \tilde{A}_2^{\text{norm}}(\ell\cdot P)\; \tilde{A}_2(\ell^2, 0).$$

To test this, we define

$$\boxed{\tilde{A}_2^{\text{norm}}(\ell^2, \ell\cdot P) \equiv \frac{\tilde{A}_2(\ell^2, \ell\cdot P)}{\text{Re}\;\tilde{A}_2(\ell^2, 0)}}$$

(needs no renormalization!)

If factorization holds, $\tilde{A}_2^{\text{norm}}$ should be ℓ^2-independent.

$(x, k_\perp)$-factorization hypothesis

factorization hypothesis

$$f_1(x, \boldsymbol{k}_\perp^2) \;\approx\; f_1(x) \;\; f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \;/\; \mathcal{N}$$

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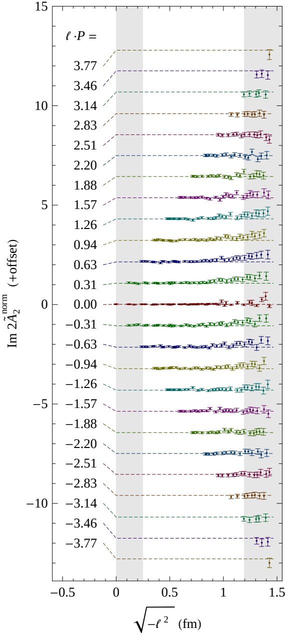
(needs no renormalization!)

If factorization holds, $\tilde{A}_2^{\text{norm}}$ should be ℓ^2-independent.

✓ within statistics

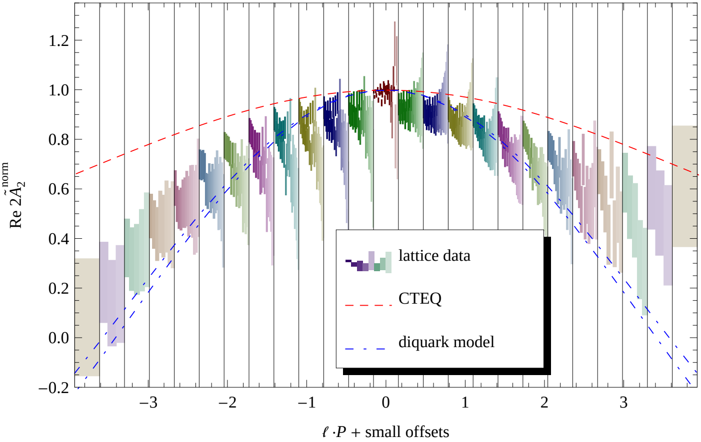


global $\ell\cdot P$-behavior

All our data for $\tilde{A}_2^{\mathrm{norm}}(\ell^2, \ell\cdot P)$ at $m_\pi \approx 610$ MeV

qualitative comparison to

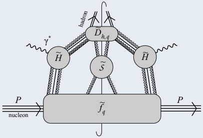
- a Fourier transform of $f_1(x)$ from CTEQ5 [Lai et al., EPJ C12, 375 (2000)]
- a scalar diquark model at $\sqrt{-\ell^2} = 0$ and 1 fm [JMR, NPA626, 937 (1997)]



SIDIS beyond the “basic” ansatz

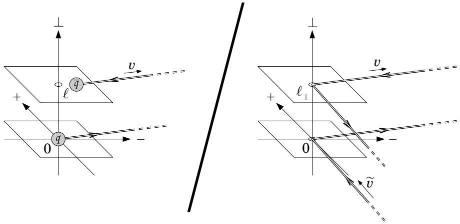
e.g., [Ji, Ma, Yuan PRD (2005)] :

$$W^{\mu\nu}_{\text{unpol.,LO}} \quad \propto \quad H \times f_1 \otimes D_h \otimes \underbrace{S}_{\text{soft factor}}$$



modified definition of TMD PDF correlator:

$$\Phi^{[\Gamma]}(k,P,S) \;\equiv\; \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\; e^{-ik\cdot\ell}\; \frac{\langle P,S|\;\bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\;|P,S\rangle}{\tilde{S}(\boldsymbol{\ell}_\perp,\ldots)}$$



- gauge links slightly off lightcone: $v \neq \hat{n}_-$

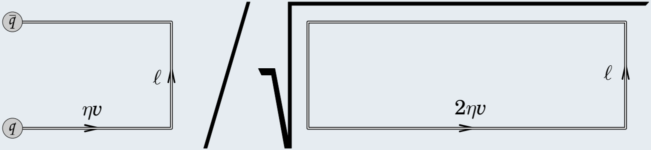
⇒ evolution eqn. in $\zeta \equiv (v\cdot P)^2/v^2$

- soft factor $\tilde{S}$: vacuum expectation value of gauge link structure

subtraction factor

How to get rid of the gauge link self engergy $\exp(\delta m\, L)$?

Soft factor in TMD PDF correlator? Suggestion [Collins arXiv:0808.2665] :



Is this a meaningful definition of TMD PDFs?

prerequisite for quantitative lattice predictions

"To allow non-perturbative methods in QCD to be used to estimate parton densities, operator definitions of parton densities are needed that can be taken literally." [Collins arXiv:0808.2665 (2008)]

$\boldsymbol{k}_\perp$-moments from ratios of amplitudes ...

... bridge the gap until we know more.

Example Sivers effect: $\langle \boldsymbol{k}_\perp \rangle_{\rho_{TU}}$ from $\tilde{A}_{12}/\tilde{A}_2$.
Self-energies cancel, no explicit subtraction factor needed.

parametrization of the matrix elements

$$\Phi^{[\Gamma]}(k,P,S) \equiv \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\ \langle P,S|\ \bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P,S\rangle$$

isolation of Lorentz-invariant amplitudes

compare [Mulders, Tangerman NPB (1996)]

$$\langle P,S|\ \bar{q}(\ell)\,\gamma_\mu\,\mathcal{U}\,q(0)\ |P,S\rangle = 4\,\tilde{A}_2\,P_\mu + 4i\,m_N{}^2\,\tilde{A}_3\,\ell_\mu$$

$$\begin{aligned}\langle P,S|\ \bar{q}(\ell)\,\gamma_\mu\gamma^5\,\mathcal{U}\,q(0)\ |P,S\rangle = &-4\,m_N\,\tilde{A}_6\,S_\mu\\ &-4i\,m_N\,\tilde{A}_7\,P_\mu(\ell\cdot S)\\ &+4\,m_N{}^3\,\tilde{A}_8\,\ell_\mu(\ell\cdot S)\end{aligned}$$

$\langle P,S|\ \bar{q}(\ell)\ \ldots\ \mathcal{U}\,q(0)\ |P,S\rangle =$ further structures (9 amplitudes in total)

Transformation properties of the matrix element ($\dagger$, $\mathcal{P}$, $\mathcal{T}$) limit number of allowed structures. No $\mathcal{T}$-odd structures (Sivers function, ...) with straight gauge link.

The amplitudes fulfill $\tilde{A}_i(\ell^2,\ell\cdot P) = \left[\tilde{A}_i(\ell^2,-\ell\cdot P)\right]^*$.

parametrization of the matrix elements

$$\Phi^{[\Gamma]}(k,P,S) \;\equiv\; \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\; e^{-ik\cdot\ell}\;\langle P,S|\;\bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\;|P,S\rangle$$

isolation of Lorentz-invariant amplitudes

compare [Mulders, Tangerman NPB (1996)]

$$\langle P,S|\;\bar{q}(\ell)\,\gamma_\mu\,\mathcal{U}\,q(0)\;|P,S\rangle \;=\; 4\,\tilde{A}_2\,P_\mu \;+\; 4i\,m_N{}^2\;\tilde{A}_3\;\ell_\mu$$

$\Rightarrow\; f_1(x,\boldsymbol{k}_\perp^2)$

$$\begin{aligned}\langle P,S|\;\bar{q}(\ell)\,\gamma_\mu\gamma^5\,\mathcal{U}\,q(0)\;|P,S\rangle \;=\; &-4\,m_N\;\tilde{A}_6\;S_\mu\\ &-4i\,m_N\,\tilde{A}_7\,P_\mu(\ell\cdot S)\\ &+4\,m_N{}^3\;\tilde{A}_8\;\ell_\mu(\ell\cdot S)\end{aligned}$$

$\Rightarrow\; g_{1T}(x,\boldsymbol{k}_\perp^2)$

$$\langle P,S|\;\bar{q}(\ell)\;\ldots\;\mathcal{U}\,q(0)\;|P,S\rangle \;=\; \text{further structures (9 amplitudes in total)}$$

Transformation properties of the matrix element ($\dagger$, $\mathcal{P}$, $\mathcal{T}$) limit number of allowed structures. No $\mathcal{T}$-odd structures (Sivers function, …) with straight gauge link.

The amplitudes fulfill $\tilde{A}_i(\ell^2,\ell\cdot P) = \left[\tilde{A}_i(\ell^2,-\ell\cdot P)\right]^*$.

transfer matrix formalism

ratio of correlators far away from nucleon source and sink

$$\frac{C_{3\mathrm{pt}}(\tau;\Gamma,\ell,P)}{C_{2\mathrm{pt}}(P)} \xrightarrow{t_{\mathrm{src}}\ll\tau\ll t_{\mathrm{sink}}} \text{const. ("plateau value")},$$

$$\Downarrow$$

access to $\langle P,S|\ \overline{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P,S\rangle$

Γ	$\frac{1}{2}C^{\mathrm{ren}}_{3\mathrm{pt}}(\tau;\Gamma,\boldsymbol{\ell},\boldsymbol{P})/C_{2\mathrm{pt}}(\boldsymbol{P})$ (LHPC projectors)
$\mathbb{1}$	$\frac{m_N}{E(P)}\,\tilde{A}_1$
$-\gamma_4\gamma_5$	$im_N\,\tilde{A}_7\,\ell_z$
γ_4	$\tilde{A}_2$
$\frac{1}{2}[\gamma_2,\gamma_4]$	$\frac{1}{E(P)}\,\tilde{A}_9\,P_x + \frac{im_N^2}{E(P)}\,\tilde{A}_{10}\,\ell_x + \frac{m_N^2}{E(P)}\,\tilde{A}_{11}\,(\ell_z)^2 P_x$
$\ldots$	$\ldots$

transfer matrix formalism

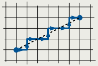
ratio of correlators far away from nucleon source and sink

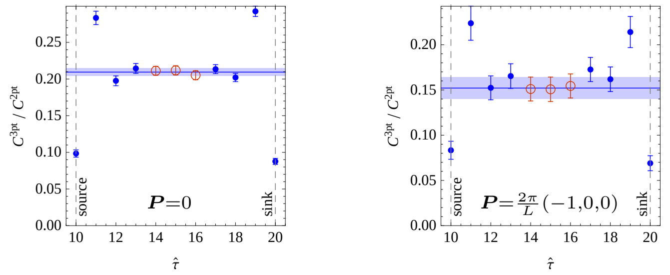
$$\frac{C_{3\mathrm{pt}}(\tau;\Gamma,\ell,P)}{C_{2\mathrm{pt}}(P)} \xrightarrow{t_{\mathrm{src}}\ll\tau\ll t_{\mathrm{sink}}} \text{const. (“plateau value”)},$$

$$\Downarrow$$

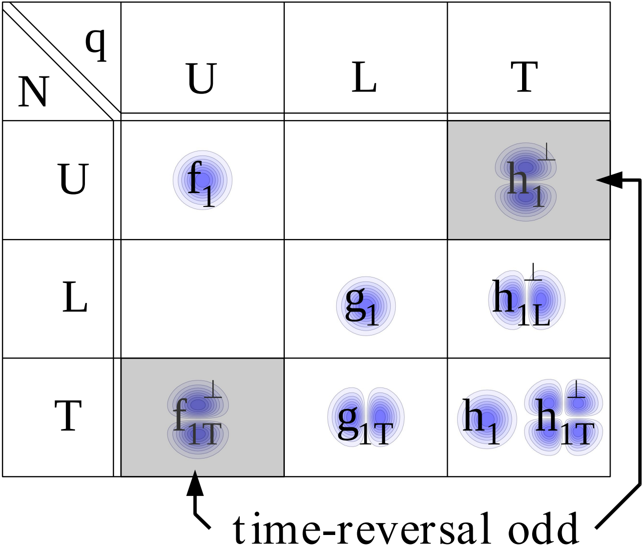
$$\text{access to } \langle P,S|\ \bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P,S\rangle$$

example plateau plots at $m_\pi \approx 600$ MeV

for $\Gamma = \gamma_4$ ($\Rightarrow \tilde{A}_2$), with HYP smeared gauge link $\mathcal{U} =$  :



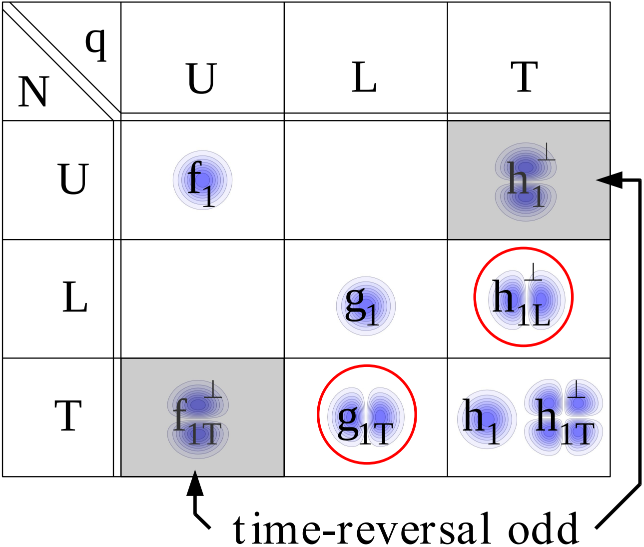
leading-twist TMD PDFs



N \ q	U	L	T
U	f_1		$h_1^\perp$
L		g_1	$h_{1L}^\perp$
T	$f_{1T}^\perp$	g_{1T}	h_1 $h_{1T}^\perp$

time-reversal odd

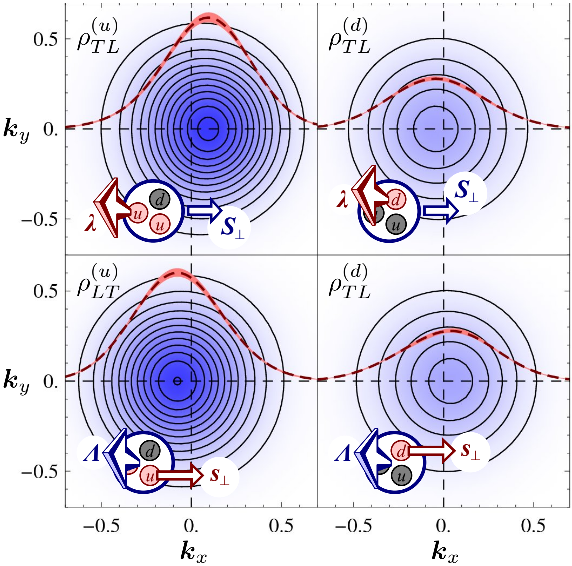
leading-twist TMD PDFs



N \ q	U	L	T
U	f_1		$h_1^\perp$
L		g_1	$h_{1L}^\perp$
T	$f_{1T}^\perp$	g_{1T}	h_1 $h_{1T}^\perp$

time-reversal odd

“genuine” signs of intrinsic quark momentum



Diplole deformations

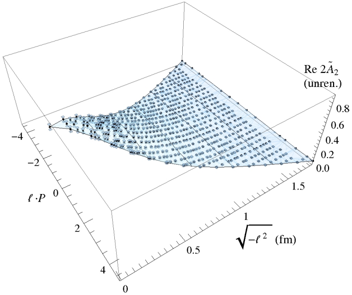
$\rho_{TL} : \sim \lambda\, \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp \;\; g_{1T}$
$\rho_{TL} : \sim \Lambda\, \boldsymbol{k}_\perp \cdot \boldsymbol{s}_\perp \;\; h^\perp_{1L}$

The corresponding dipole structures $\sim \lambda\, \boldsymbol{b}_\perp \cdot \boldsymbol{S}_\perp$, $\sim \Lambda\, \boldsymbol{b}_\perp \cdot \boldsymbol{s}_\perp$ for impact parameter densities (from GPDs) are ruled out by symmetries.

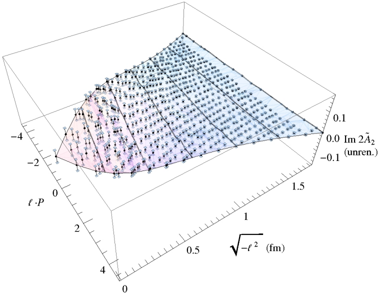
[Hägler, Musch, Negele, Schäfer EPL 88, 61001 (2009)]

$\ell \cdot P$ - dependence of $\tilde{A}_2(\ell^2, \ell \cdot P)$

$2 \text{ Re } \tilde{A}_2(\ell^2, \ell \cdot P)$

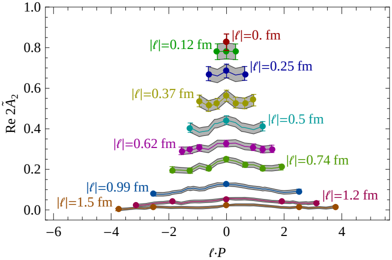


$2 \text{ Im } \tilde{A}_2(\ell^2, \ell \cdot P)$

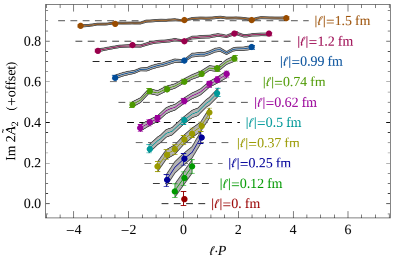


$\ell\cdot P$ - dependence of $\tilde{A}_2(\ell^2, \ell\cdot P)$

$2\ \mathrm{Re}\ \tilde{A}_2(\ell^2, \ell\cdot P)$

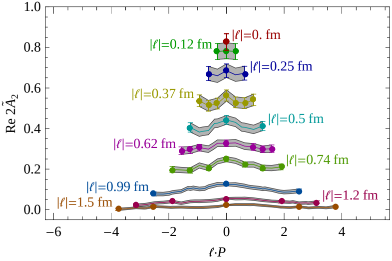


$2\ \mathrm{Im}\ \tilde{A}_2(\ell^2, \ell\cdot P)$

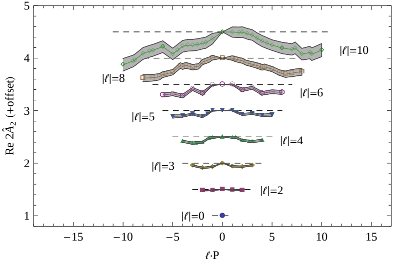


effect of normalization with amplitude at $\ell\cdot P = 0$

$$2\ \mathrm{Re}\ \tilde{A}_2(\ell^2, \ell\cdot P)$$



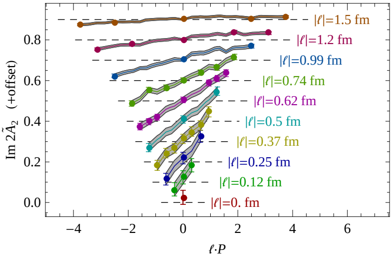
$$\mathrm{Re}\ \tilde{A}_2^{\mathrm{norm}} = \frac{\mathrm{Re}\ \tilde{A}_2(\ell^2, \ell\cdot P)}{\mathrm{Re}\ \tilde{A}_2(\ell^2, 0)}$$

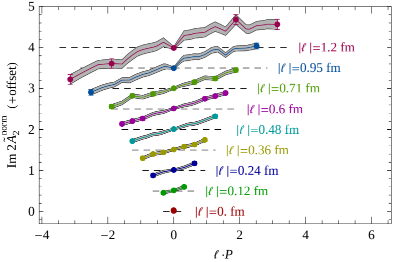


effect of normalization with amplitude at $\ell\cdot P = 0$

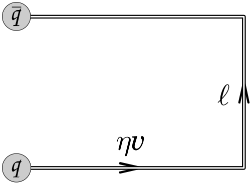
$$2\ \mathrm{Im}\ \tilde{A}_2(\ell^2, \ell\cdot P)$$

$$\mathrm{Im}\ \tilde{A}_2^{\mathrm{norm}} = \frac{\mathrm{Im}\ \tilde{A}_2(\ell^2, \ell\cdot P)}{\mathrm{Re}\ \tilde{A}_2(\ell^2, 0)}$$





staple-shaped gauge links



32 Lorentz-invariant amplitudes [Goeke,Metz,Schlegel PLB618,90 (2005)]

$$A_i\left(k^2, k\cdot P, \frac{v\cdot k}{|v\cdot P|}, \frac{v^2}{|v\cdot P|^2}, \frac{v\cdot P}{|v\cdot P|}\right) = A_i\left(k^2, k\cdot P, \underbrace{\frac{v\cdot k}{|v\cdot P|}}_{\approx\, x}, \zeta^{-1}, \mathrm{sgn}(v\cdot P)\right)$$

Links approaching light cone: $v \to \hat{n}_- \Rightarrow \zeta \to \infty$. For large ζ, the evolution with ζ is known [Collins,Soper NPB194,445 (1981)].

time reversal $\mathcal{T}$

$$\left.\begin{array}{r}(v^0,v^1,v^2,v^3)\\ \text{future pointing } v\\ \text{TMD PDFs for SIDIS}\end{array}\right\}\xrightarrow{\mathcal{T}}\left\{\begin{array}{l}(-v^0,v^1,v^2,v^3)\\ \text{past pointing } v\\ \text{TMD PDFs for Drell-Yan}\end{array}\right.$$

The transformation property of the matrix elements under time reversal provides relations:

Example of a $\mathcal{T}$-even amplitude:

$$A_2\Big(k^2, k\cdot P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, 1\Big) = A_2\Big(k^2, k\cdot P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, -1\Big)$$

$$\Downarrow$$

$$f_1^{(\text{SIDIS})}(x, \boldsymbol{k}_\perp; \zeta, \ldots) = f_1^{(\text{Drell-Yan})}(x, \boldsymbol{k}_\perp; \zeta, \ldots)$$

Example of a $\mathcal{T}$-odd amplitude: ($\rightarrow$ Sivers function $f_{1T}^\perp$)

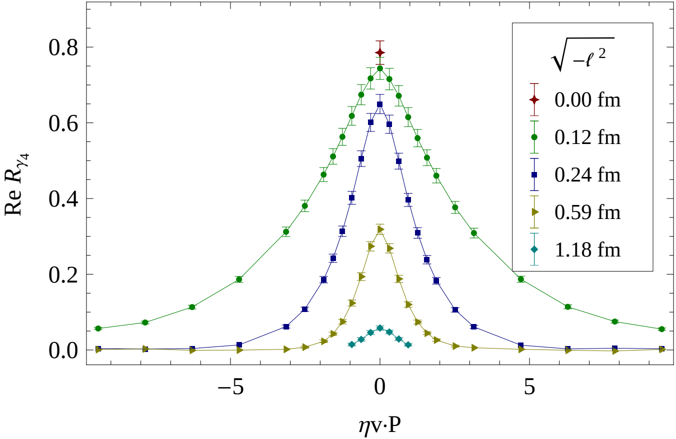
$$A_{12}\Big(k^2, k\cdot P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, 1\Big) = -A_{12}\Big(k^2, k\cdot P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, -1\Big)$$

$$\Downarrow$$

$$f_{1T}^{\perp(\text{SIDIS})}(x, \boldsymbol{k}_\perp; \zeta, \ldots) = -f_{1T}^{\perp(\text{Drell-Yan})}(x, \boldsymbol{k}_\perp; \zeta, \ldots)$$

$\tilde{A}_2$ from the lattice for extended gauge links

$$\tilde{A}_2\left(\ell^2, \ell\cdot P, \frac{v\cdot\ell}{|v\cdot P|}, \zeta^{-1}, \mathrm{sgn}(v\cdot P)\right) \equiv \lim_{\eta\to\infty} \tilde{a}_2(\ell^2, \ell\cdot P, \eta v\cdot\ell, -\eta^2, \eta v\cdot P)$$

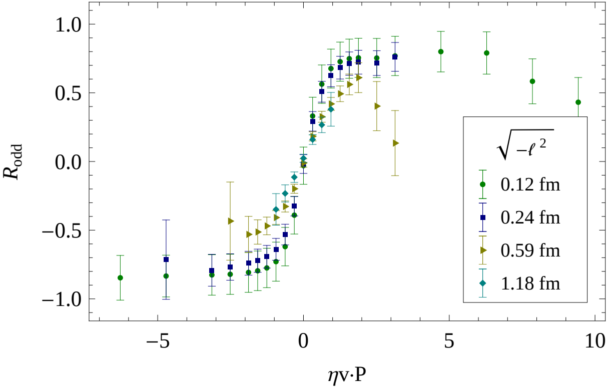


But $\tilde{a}_2 = \mathrm{Re}\, R_{\gamma_4}$ always vanishes for large η!
Reason: power divergence suppresses $\tilde{a}_2 \sim \exp(-\delta m\, \eta)$.

A $\mathcal{T}$-odd ratio from the lattice

$$R_{\text{odd}} = \frac{\tilde{a}_{12} + \left(\eta \frac{m_N^2 v_1}{P_1}\right)\tilde{b}_8}{\tilde{a}_2}$$

$$\xrightarrow{\pm\eta v\cdot P \text{ large}} \frac{\tilde{A}_{12}(\ell^2,0,0,\zeta^{-1},\pm1) + \left(\frac{m_N}{P_1}\right)^2 \tilde{B}_8(\ell^2,0,0,\zeta^{-1},\pm1)}{\tilde{A}_2(\ell^2,0,0,\zeta^{-1},\pm1)}$$



Part of the effect comes from the Sivers function $f_{1T}^{\perp}$ via $\tilde{A}_{12}$!