Transverse momentum distributions inside the nucleon from Lattice QCD

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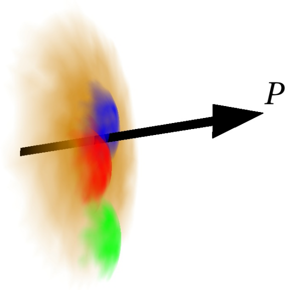
in collaboration with
Philipp Hägler (TU München), John Negele (MIT),
Andreas Schäfer (Univ. Regensburg),
and the LHP Collaboration

[Hägler et al. EPL88 61001 (2009)]
[Musch arXiv:0907.2381]

Emmy Noether-Programm
Deutsche Forschungsgemeinschaft
DFG

Jefferson Lab
Thomas Jefferson National Accelerator Facility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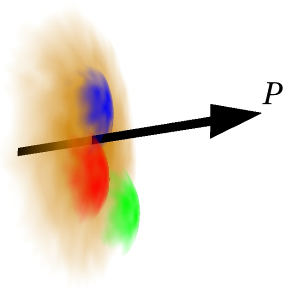
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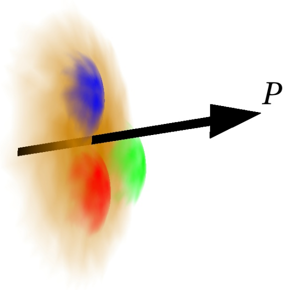
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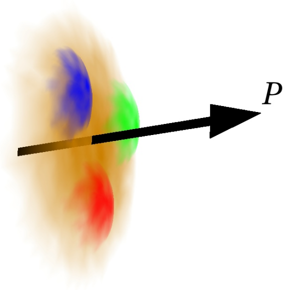
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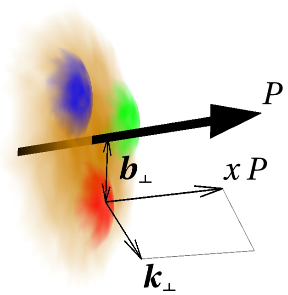
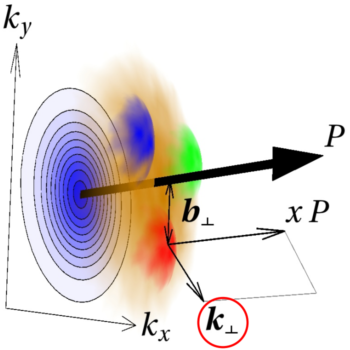

P
$\boldsymbol{b}_{\perp}$
$x\,P$
$\boldsymbol{k}_{\perp}$



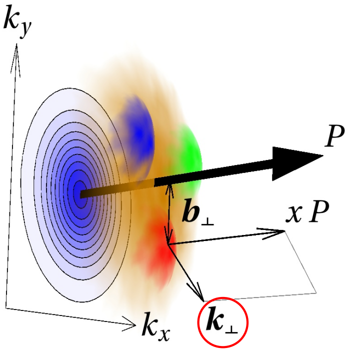
TMD PDFs

transverse **m**omentum **d**ependent **p**arton **d**istribution **f**unctions

e.g., $f_1(x, \boldsymbol{k}_\perp^2)$

$\Rightarrow$ quark density $\rho(\boldsymbol{k}_\perp)$.

- x (longitudinal momentum fraction) $\Rightarrow$ PDFs
- $x, \boldsymbol{b}_\perp$ (impact parameter) $\Rightarrow$ GPDs
- $x, \boldsymbol{k}_\perp$ (intrinsic transverse momentum) $\Rightarrow$ TMD PDFs



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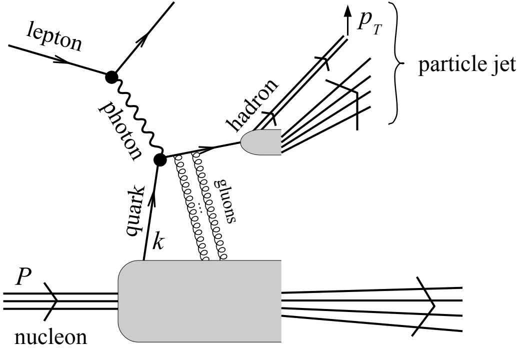
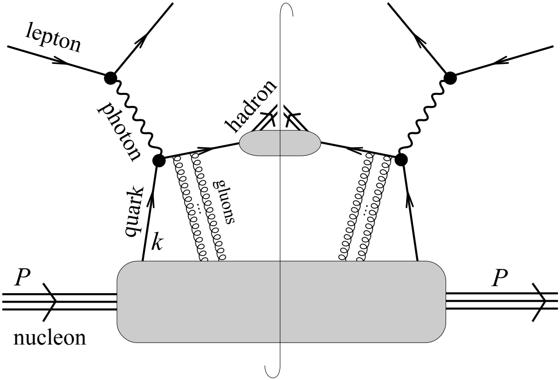
remember Constantia Alexandrou's talk this morning e.g., semi-inclusive DIS [Collins PLB 93], [Bacchetta et al. JHEP 07]



experiments sensitive to TMD PDFs

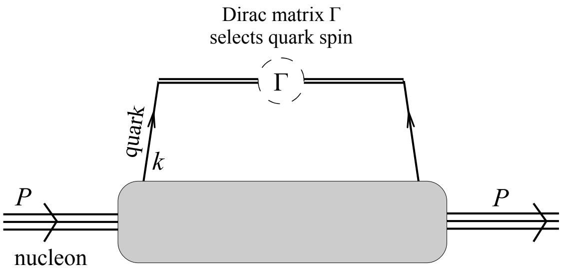
COMPASS (CERN), HERMES (DESY), JLab, RHIC (BNL), Fermilab, also planned at J-PARC, FAIR (GSI), NICA (JINR), ..., EIC (BNL/JLab?)

e.g., semi-inclusive DIS [Collins PLB 93], [Bacchetta et al. JHEP 07]



$$\frac{d\sigma}{d^3P_h\ d^3P_{l'}} \propto \underbrace{H(Q^2,\dots)}_{\text{hard part}} \int d^2\boldsymbol{k}_\perp\ \underbrace{f_1(x,\boldsymbol{k}_\perp,\dots)}_{\text{TMD PDF}}\ \underbrace{D_h(z,\boldsymbol{k}_\perp+\boldsymbol{q}_\perp,\dots)}_{\text{fragmentation f.}}$$

(no soft factor taken into account, see [Ji, Ma, Yuan PRD 71 (2005)])



$$\Phi^{[\Gamma]}(k, P, S) \;\equiv\; \text{“}\, \langle P, S|\, \bar{q}(k)\, \Gamma\, q(k)\, |P, S\rangle \,\text{”}$$

lightcone coor. $w^{\pm} = \frac{1}{\sqrt{2}}(w^0 \pm w^3)$, so $w = w^+ \hat{n}_+ + w^- \hat{n}_- + w_\perp$

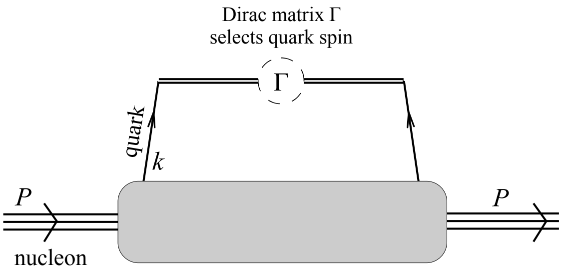
proton flies along z-axis: P^+ large, $P_\perp = 0$

parametrization in terms of TMD PDFs, example

$$\int dk^- \;\Phi^{[\gamma^+]}(k, P, S)\Big|_{k^+ = xP^+} = f_1(x, \boldsymbol{k}_\perp^2) - \frac{\boldsymbol{\epsilon}_{ij} \boldsymbol{k}_i \boldsymbol{S}_j}{m_N} f_{1T}^{\perp}(x, \boldsymbol{k}_\perp)$$

[Ralston, Soper NPB 1979], [Mulders, Tangerman NPB 1996], [Goeke, Metz, Schlegel PLB 2005]

definition of TMD PDFs ("basic" version)



$$\Phi^{[\Gamma]}(k,P,S) \equiv \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\, \langle P,S|\; \bar{q}(\ell)\,\Gamma\, \mathcal{U}\, q(0)\; |P,S\rangle$$

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[Ralston, Soper NPB 1979], [Mulders, Tangerman NPB 1996], [Goeke, Metz, Schlegel PLB 2005]

$\langle P| \;\; \overline{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0) \;\; |P\rangle$ is gauge invariant.

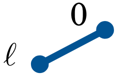
continuum

$$\mathcal{U} \;\equiv\; \mathcal{P}\exp\left(-ig\int_0^\ell d\xi^\mu A_\mu(\xi)\right)$$

along path from 0 to ℓ

- factorization in SIDIS :
  path runs to infinity and back

- ***simplification***:
  straight path (for first studies)



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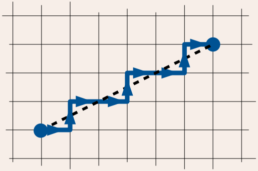
along path from 0 to ℓ

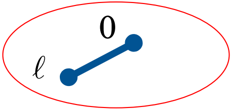
lattice



product of link variables

- factorization in SIDIS : path runs to infinity and back

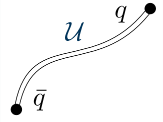
- ***simplification***: straight path (for first studies)



continuum

[Craigie, Dorn NPB185,204 (1981)]

smooth path



$$[\bar{q}\,\mathcal{U}\,q]_{\mathrm{ren}} = Z^{-1} \exp\left(-\delta\hat{m}\,\frac{l}{a}\right)\,[\bar{q}\,\mathcal{U}\,q]$$

$\delta\hat{m}$: removes the length dependent renorm. factor

static quark potential

$$V_{\mathrm{ren}}(r) = V(r) + 2\,\delta\hat{m}/a$$

string [Lüscher,Symanzik,Weisz (1980)]

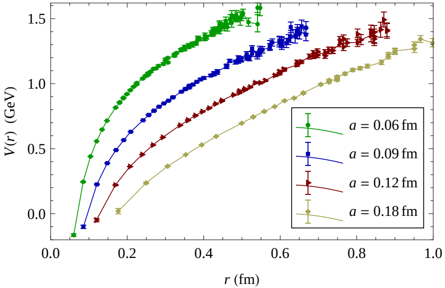
at large r: $V_{\mathrm{ren}}(r) \approx$

$$V_{\mathrm{string}}(r) = \sigma r - \pi/12r + 0$$

method [Cheng PRD77,014511 (2008)]

determine $\delta\hat{m}$ from

$$V_{\mathrm{ren}}(0.7\text{ fm}) \stackrel{!}{=} V_{\mathrm{string}}(0.7\text{ fm})$$



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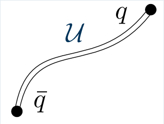

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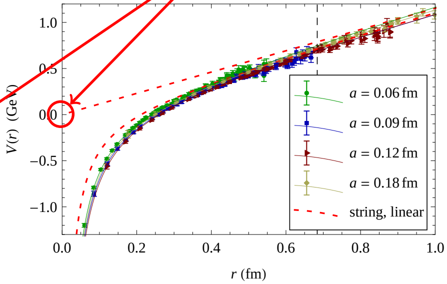
renormalization condition $C^{\text{ren}} = 0$



continuum

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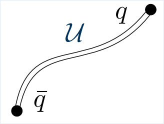

$$[\bar{q}\;\mathcal{U}\;q]_{\rm ren} = Z^{-1} \exp\left(-\delta\hat{m}\,\frac{l}{a}\right)\;[\bar{q}\;\mathcal{U}\;q]$$

$\delta\hat{m}$: removes the length dependent renorm. factor

static quark potential

$$V_{\rm ren}(r) = V(r) + 2\,\delta\hat{m}/a$$

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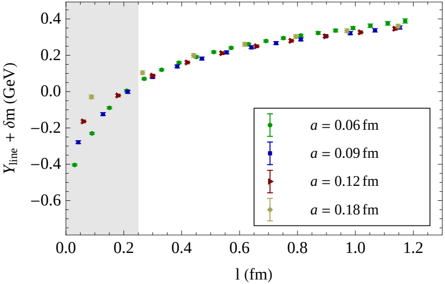
determine $\delta\hat{m}$ from

$V_{\rm ren}(0.7\text{ fm}) \stackrel{!}{=} V_{\rm string}(0.7\text{ fm})$

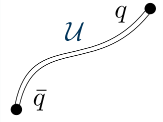


$$Y_{\rm line}(l) \equiv \frac{d}{dl}\ln\langle \mathrm{tr}\;\mathcal{U}\rangle_{(\text{Landau gauge})}$$

continuum

[Craigie, Dorn NPB185,204 (1981)]

smooth path



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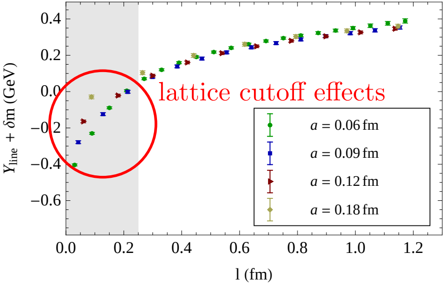

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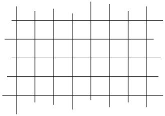
We employ the Chroma library [Edwards, Joo (2005)] to process



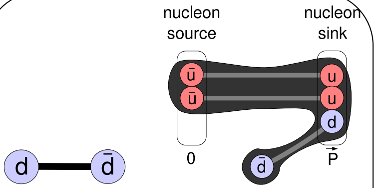
MILC gauge configurations

staggered Asqtad action,
2+1 flavors, $a \approx 0.124$ fm,
$m_\pi \approx 500,\ 610,$ and 760 MeV

[Orginos, Toussaint PRD (1999)]

+ finer MILC lattices
to test renormalization

[Aubin et al. PRD (2004)]
[Bazavov et al. 0903.3598]



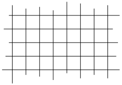
LHPC propagators

domain wall valence fermions,
m_π adjusted to staggered sea,

nucleon momenta:
$\boldsymbol{P} = 0$ and $|\boldsymbol{P}| = 500$ MeV

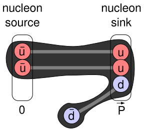
e.g., [Hägler et al. PRD (2008)]

Ingredients		Output : 3-point correlator $C_{3\text{pt}}$

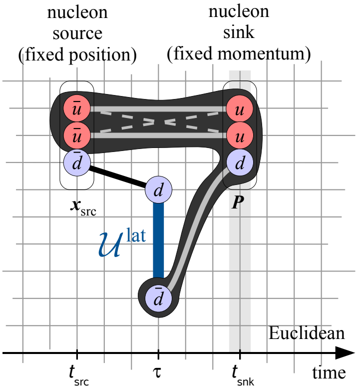


gauge configs.

quark propagators



nucleon sequential propagators



form ratio $C_{3\text{pt}}/C_{2\text{pt}}$, take plateau

$$\Rightarrow \quad \langle P, S|\ \overline{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P, S\rangle$$

[We neglect "disconnected contributions" (absent for up minus down).]

$$\Phi^{[\Gamma]}(k,P,S) \equiv \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\, \langle P,S|\ \bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P,S\rangle$$

isolation of Lorentz-invariant amplitudes

compare [Mulders, Tangerman NPB (1996)]

$$\langle P,S|\ \bar{q}(\ell)\,\gamma_\mu\,\mathcal{U}\,q(0)\ |P,S\rangle = 4\,\tilde{A}_2\,P_\mu + 4i\,m_N{}^2\,\tilde{A}_3\,\ell_\mu$$

$$\begin{aligned}\langle P,S|\ \bar{q}(\ell)\,\gamma_\mu\gamma^5\,\mathcal{U}\,q(0)\ |P,S\rangle = & -4\,m_N\,\tilde{A}_6\,S_\mu \\ & -4i\,m_N\,\tilde{A}_7\,P_\mu(\ell\cdot S) \\ & +4\,m_N{}^3\,\tilde{A}_8\,\ell_\mu(\ell\cdot S)\end{aligned}$$

$\langle P,S|\ \bar{q}(\ell)\ \ldots\ \mathcal{U}\,q(0)\ |P,S\rangle =$ further structures (9 amplitudes in total)

Transformation properties of the matrix element ($\dagger$, $\mathcal{P}$, $\mathcal{T}$) limit number of allowed structures. No $\mathcal{T}$-odd structures (Sivers function, ...) with straight gauge link.

The amplitudes fulfill $\tilde{A}_i(\ell^2,\ell\cdot P) = \left[\tilde{A}_i(\ell^2,-\ell\cdot P)\right]^*$.

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$\Rightarrow f_1(x,\boldsymbol{k}_\perp^2)$

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$\Rightarrow g_{1T}(x,\boldsymbol{k}_\perp^2)$

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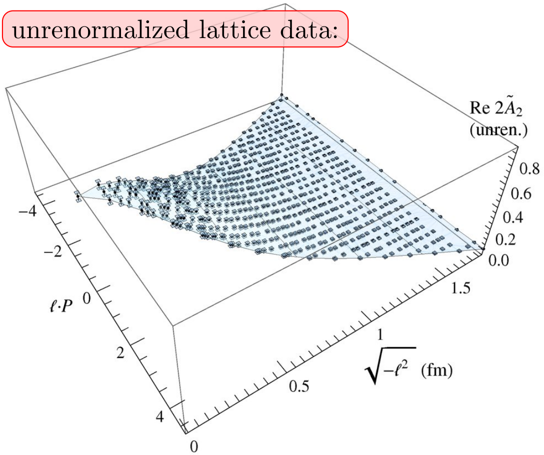
extract Lorentz-invariant amplitudes $\tilde{A}_i(\ell^2, \ell\cdot P)$, example :

$$\langle P,S|\ \overline{q}(\ell)\,\gamma_\mu\,\mathcal{U}\,q(0)\ |P,S\rangle \ =\ 4\tilde{A}_2 P_\mu\ +\ 4i\,m_N{}^2\ \tilde{A}_3\ \ell_\mu\ ,$$

$$f_1(x,\boldsymbol{k}_\perp^2) = \int \frac{d(\ell\cdot P)}{2\pi}\ e^{ix(\ell\cdot P)} \int \frac{d^2\boldsymbol{\ell}_\perp}{(2\pi)^2}\ e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{\ell}_\perp}\ 2\tilde{A}_2(\ell^2,\ell\cdot P)\ \Big|_{\ell^+=0}$$

unrenormalized lattice data:



$$\ell^2 \ \xleftrightarrow{\text{FT}}\ \boldsymbol{k}_\perp^2$$

$$\ell\cdot P \ \xleftrightarrow{\text{FT}}\ x$$

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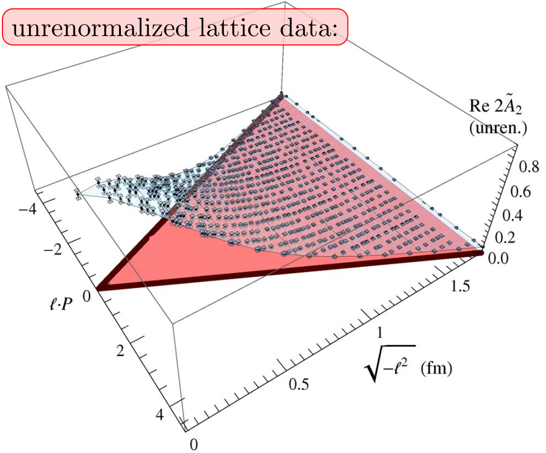

$$\ell^2 \xleftrightarrow{\text{FT}} \boldsymbol{k}_\perp^2$$

$$\ell\cdot P \xleftrightarrow{\text{FT}} x$$

Euclidean lattice

$$\ell^0 = \ell_4 = 0$$

$$\Downarrow$$

$$\ell^2 \leq 0,$$
$$|\ell\cdot P| \leq |\boldsymbol{P}|\sqrt{-\ell^2}$$

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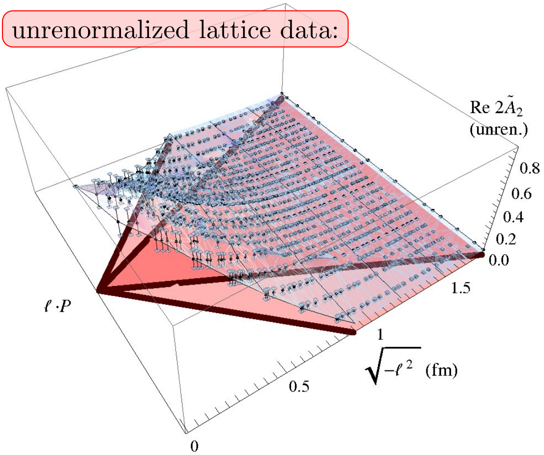

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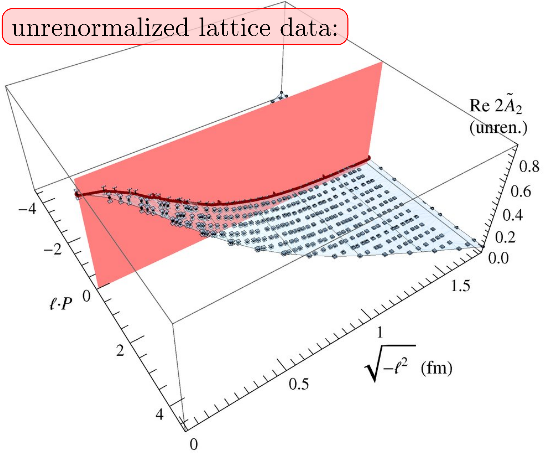
unrenormalized lattice data:



$$\ell^2 \;\overset{\text{FT}}{\longleftrightarrow}\; \boldsymbol{k}_\perp^2$$

$$\ell\cdot P \;\overset{\text{FT}}{\longleftrightarrow}\; x$$

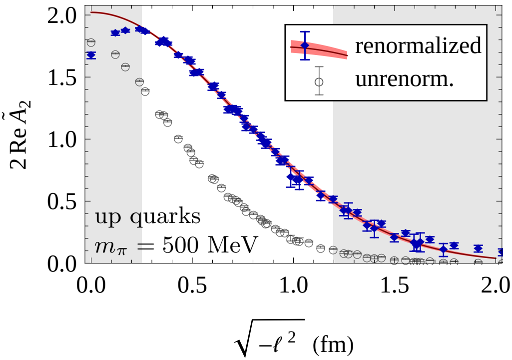
Euclidean lattice

$$\ell^0 = \ell_4 = 0$$

$$\Downarrow$$

$$\ell^2 \le 0,$$
$$|\ell\cdot P| \le |\boldsymbol{P}|\sqrt{-\ell^2}$$

$$f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \equiv \int_{-1}^{1} dx\ f_1(x, \boldsymbol{k}_\perp^2) = \int \frac{d^2\ell_\perp}{(2\pi)^2}\ e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{\ell}_\perp}\ 2\,\tilde{A}_2(-\ell_\perp^2, 0)$$



fit function

$C_1\ \exp(-|\ell|^2/\sigma_1^2)$

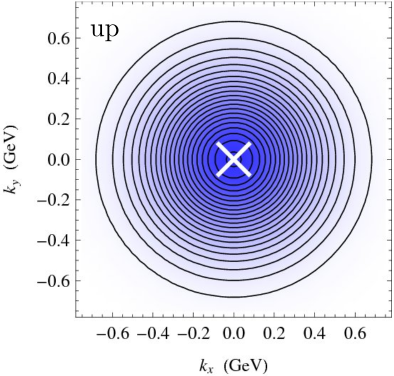
Z-factor

$Z^{-1}\ C_1^{\text{up-down}} \stackrel{!}{=} 1$

multiplicative renormalization based on quark counting

Density of unpolarized quarks (minus antiquarks)
in an unpolarized nucleon as a function of transverse momentum $\boldsymbol{k}_\perp$:

$$\rho_{UU}(\boldsymbol{k}_\perp) = \int_{-1}^{1} dx\ f_1(x, \boldsymbol{k}_\perp^2)$$



axially symmetric

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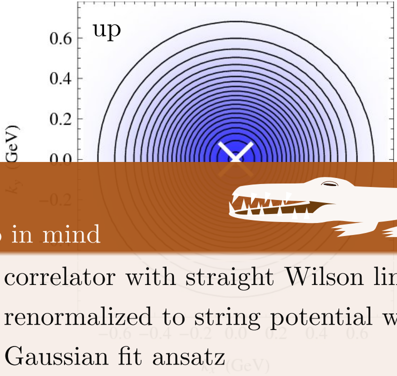
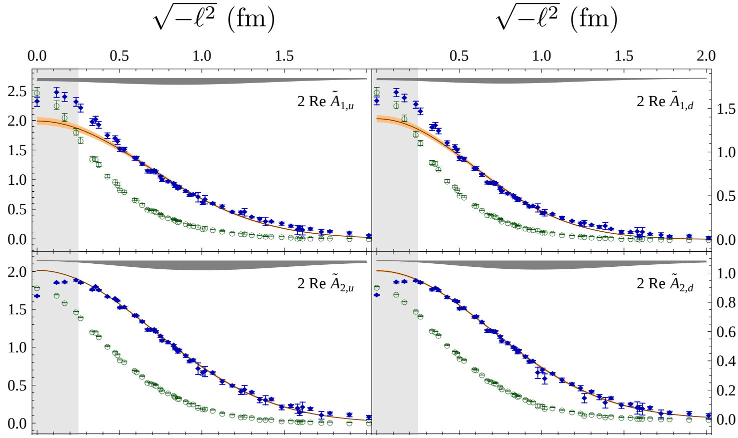


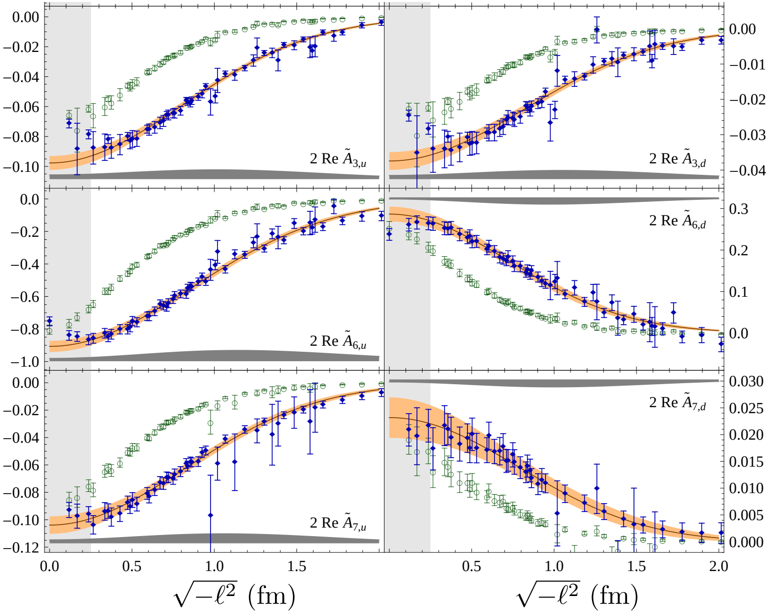
axially symmetric

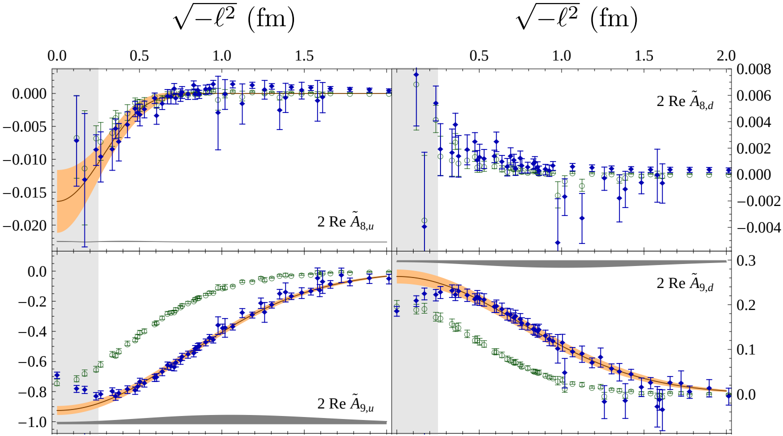
keep in mind

- correlator with straight Wilson line ("sW")
- renormalized to string potential with $C = 0$
- Gaussian fit ansatz
  ("wrong" at large-$\boldsymbol{k}_\perp$ [Diehl, arXiv:0811.0774])
- $m_\pi \approx 500$ MeV

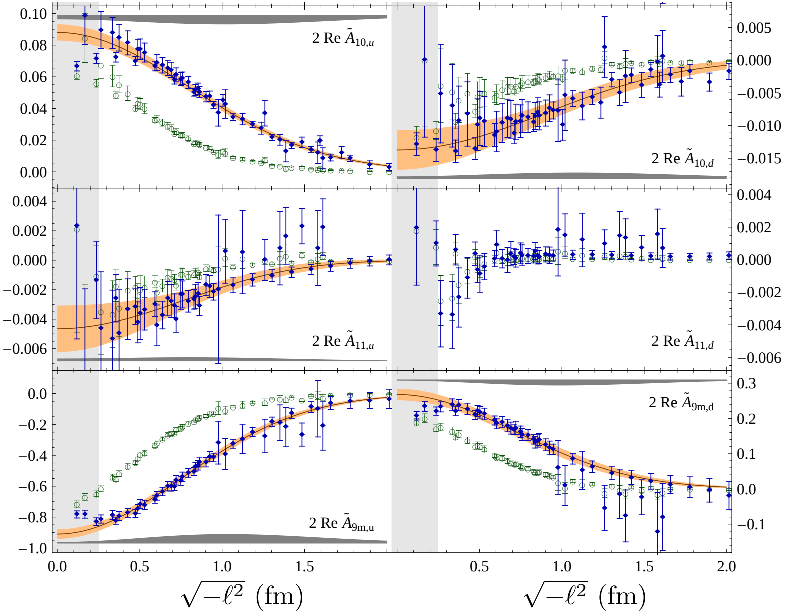


$\sqrt{-\ell^2}$ (fm)
$\sqrt{-\ell^2}$ (fm)
0.0
0.5
1.0
1.5
2.0
2.5
2 Re $\tilde{A}_{1,u}$
2 Re $\tilde{A}_{1,d}$
2 Re $\tilde{A}_{2,u}$
2 Re $\tilde{A}_{2,d}$
0.2
0.4
0.6
0.8


2 Re $\tilde{A}_{3,u}$
2 Re $\tilde{A}_{3,d}$
2 Re $\tilde{A}_{6,u}$
2 Re $\tilde{A}_{6,d}$
2 Re $\tilde{A}_{7,u}$
2 Re $\tilde{A}_{7,d}$
$\sqrt{-\ell^2}$ (fm)
$\sqrt{-\ell^2}$ (fm)



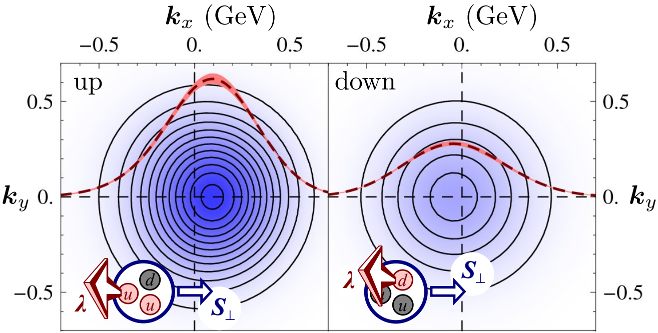
$\sqrt{-\ell^2}$ (fm)
$\sqrt{-\ell^2}$ (fm)
0.0
0.5
1.0
1.5
2.0
0.000
−0.005
−0.010
−0.015
−0.020
0.008
0.006
0.004
0.002
0.000
−0.002
−0.004
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0.3
0.2
0.1
0.0
2 Re $\tilde{A}_{8,u}$
2 Re $\tilde{A}_{8,d}$
2 Re $\tilde{A}_{9,u}$
2 Re $\tilde{A}_{9,d}$



2 Re $\tilde{A}_{10,u}$
2 Re $\tilde{A}_{10,d}$
2 Re $\tilde{A}_{11,u}$
2 Re $\tilde{A}_{11,d}$
2 Re $\tilde{A}_{9m,u}$
2 Re $\tilde{A}_{9m,d}$
$\sqrt{-\ell^2}$ (fm)
$\sqrt{-\ell^2}$ (fm)

Density of quarks with positive helicity, $\lambda = 1$,
in a transversely polarized nucleon, $\boldsymbol{S}_\perp = (1,0)$:

$$\rho_{TL}(\boldsymbol{k}_\perp; \boldsymbol{S}_\perp, \lambda) \equiv \frac{1}{2}\int dx \int dk^- \; \Phi^{[\gamma^+ \frac{1}{2}(\mathbb{1}+\gamma^5)]}(k, P, S_\perp)$$
$$= \frac{1}{2} f_1^{(0_x)}(\boldsymbol{k}_\perp^2) + \frac{\lambda}{2} \frac{\boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp}{m_N} \; g_{1T}^{(0_x)}(\boldsymbol{k}_\perp^2)$$



($m_\pi \approx 500$ MeV, straight gauge link operator,
renormalization condition $C^{\text{ren}} = 0$, Gaussian fit)

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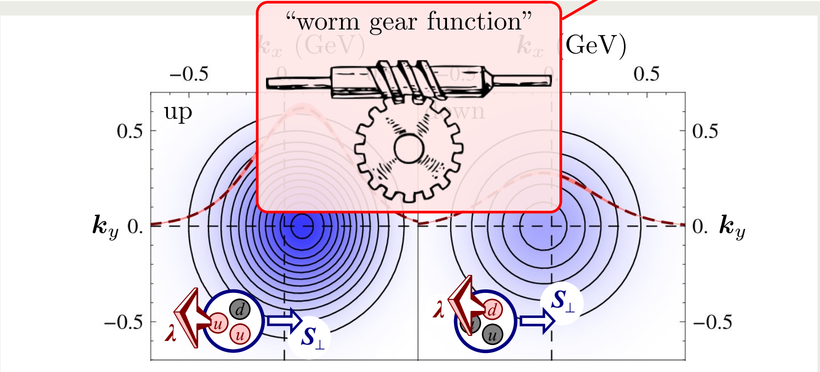

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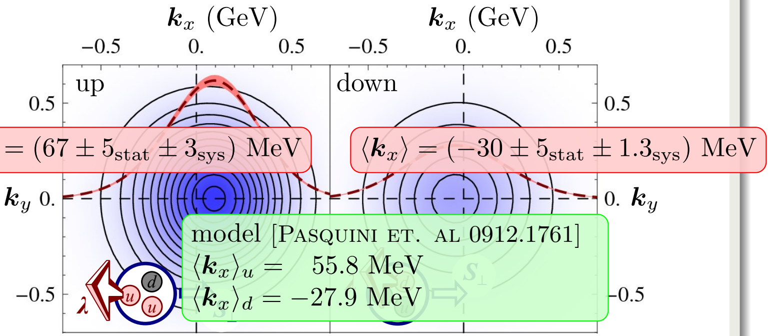
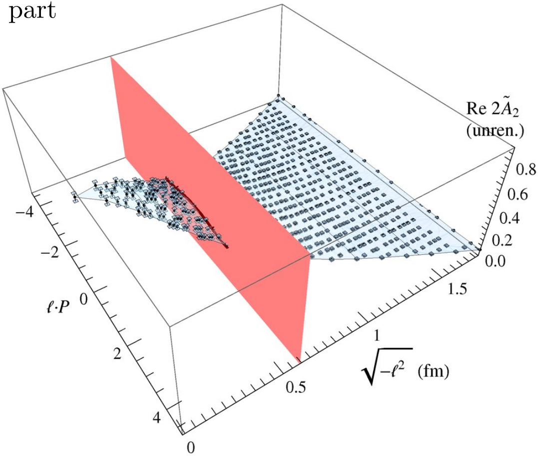


($m_\pi \approx 500$ MeV, straight gauge link operator,
renormalization condition $C^{\rm ren} = 0$, Gaussian fit)

real part



$\ell^2 \xleftrightarrow{\text{FT}} \boldsymbol{k}_\perp^2$

$\ell{\cdot}P \xleftrightarrow{\text{FT}} x$

factorization hypothesis

$$f_1(x, \boldsymbol{k}_\perp^2) \;\approx\; f_1(x) \;\; f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \;/\; \mathcal{N}$$

as in phenomenological applications, e.g., Monte Carlo event generators

Then $\tilde{A}_2$ factorizes, too:

$$\tilde{A}_2(\ell^2, \ell\cdot P) = \tilde{A}_2^{\mathrm{norm}}(\ell\cdot P)\; \tilde{A}_2(\ell^2, 0).$$

To test this, we define

$$\boxed{\tilde{A}_2^{\mathrm{norm}}(\ell^2, \ell\cdot P) \equiv \frac{\tilde{A}_2(\ell^2, \ell\cdot P)}{\mathrm{Re}\;\tilde{A}_2(\ell^2, 0)}}$$

(needs no renormalization!)

If factorization holds, $\tilde{A}_2^{\mathrm{norm}}$ should be ℓ^2-independent.

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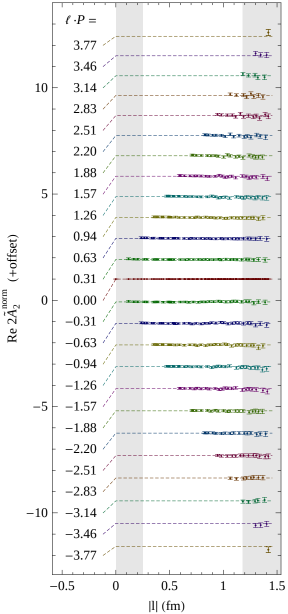
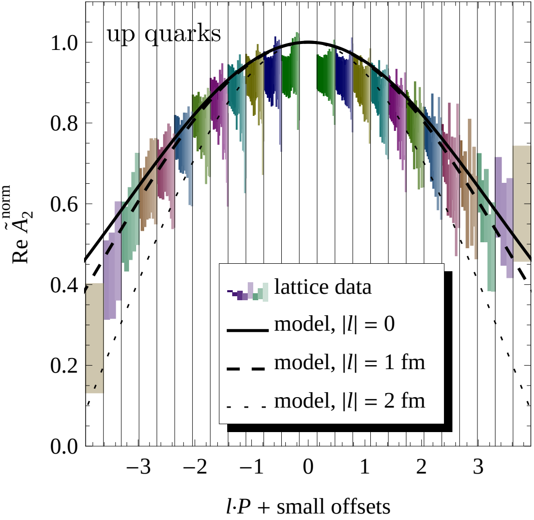
✓ within statistics



All our data for $\tilde{A}_2^{\mathrm{norm}}(\ell^2, \ell{\cdot}P)$ at $m_\pi \approx 600$ MeV

qualitative comparison to a scalar diquark model
[Bacchetta, Conti, Radici PRD (2008)] at $\sqrt{-\ell^2} = 0$, 1 and 2 fm



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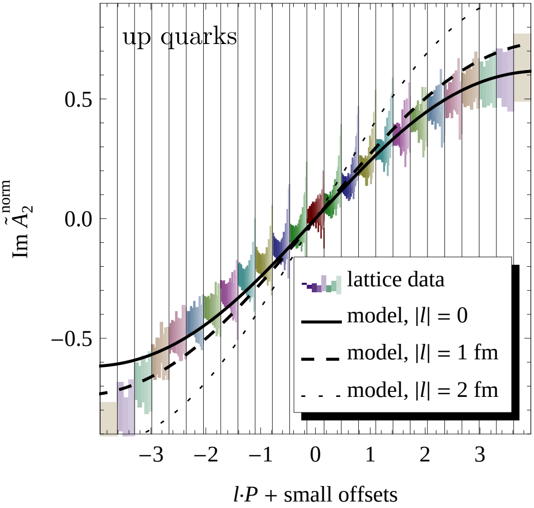
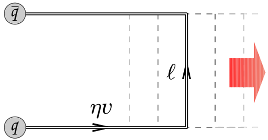


Summary:

- Lattice exploration of intrinsic quark momentum distributions in the nucleon.
- Manifestly non-local operators on the lattice.
- First results based on a a simplified operator geometry (direct gauge link) and a Gaussian fit model, at $m_\pi \approx 500$ MeV:
  - Obtaind x-integrated leading twist T-even TMD PDFs $f_1^{(0_x)}(\boldsymbol{k}_\perp^2)$, $g_{1T}^{(0_x)}(\boldsymbol{k}_\perp^2)$, $h_{1L}^{\perp(0_x)}(\boldsymbol{k}_\perp^2)$, ...
  - Observed deformed quark densities due to worm-gear functions.

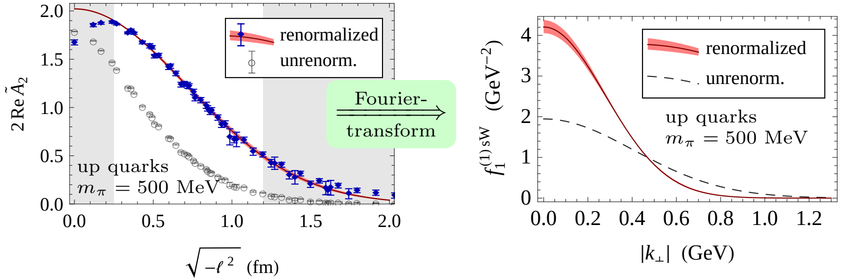
Outlook:

- Study of non-straight gauge links similar as in SIDIS.

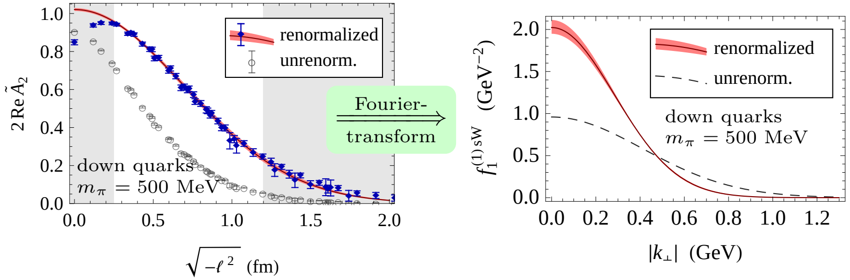


- Higher statistics needed to discuss factorization $f_1(x, \boldsymbol{k}_\perp^2) \approx f_1(x)\, f_1^{(0_x)}(\boldsymbol{k}_\perp^2)/\mathcal{N}$.
- Beyond Gaussian fits:
  Matching to perturbative behavior at small ℓ, i.e., large $\boldsymbol{k}_\perp$.

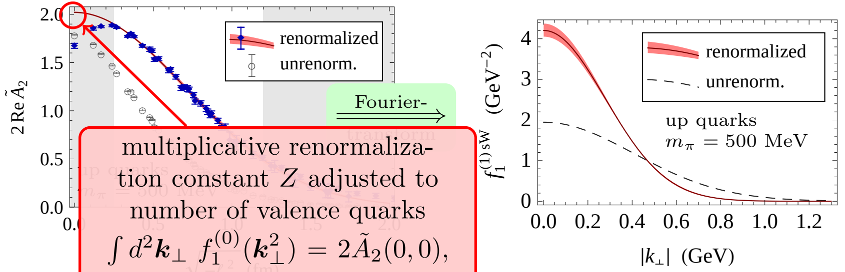
Backup Slides



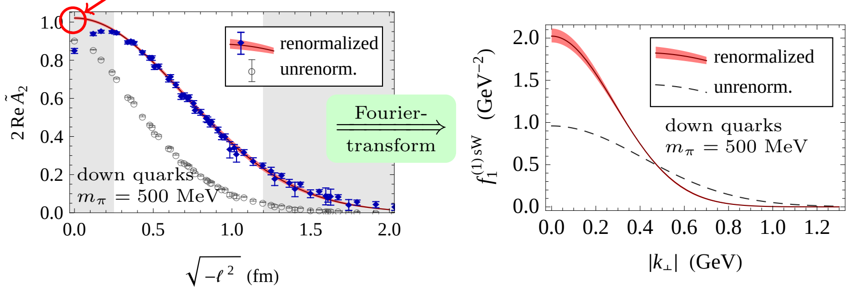
$2\,\mathrm{Re}\,\tilde{A}_2$
renormalized
unrenorm.
up quarks
$m_\pi = 500$ MeV
$\sqrt{-\ell^2}$ (fm)
Fourier-transform
$f_1^{(1)\,\mathrm{sW}}$ (GeV^{-2})
renormalized
unrenorm.
up quarks
$m_\pi = 500$ MeV
$|k_\perp|$ (GeV)



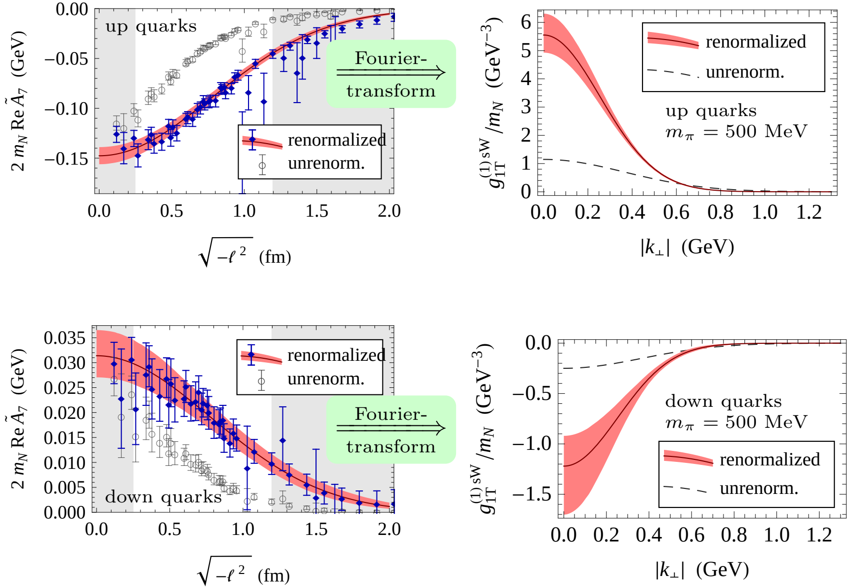
$2\,\mathrm{Re}\,\tilde{A}_2$
renormalized
unrenorm.
down quarks
$m_\pi = 500$ MeV
$\sqrt{-\ell^2}$ (fm)
Fourier-transform
$f_1^{(1)\,\mathrm{sW}}$ (GeV^{-2})
renormalized
unrenorm.
down quarks
$m_\pi = 500$ MeV
$|k_\perp|$ (GeV)



multiplicative renormalization constant Z adjusted to number of valence quarks
$$\int d^2\boldsymbol{k}_\perp\, f_1^{(0)}(\boldsymbol{k}_\perp^2) = 2\tilde{A}_2(0,0),$$
fixed in $u-d$ channel



$g_{1T}^{(0_x)}$ from $\tilde{A}_7$

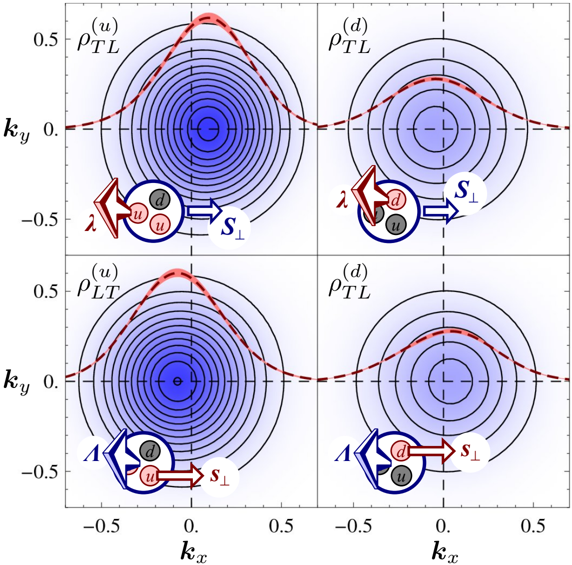


N \ q	U	L	T
U	f_1		$h_1^{\perp}$
L		g_1	$h_{1L}^{\perp}$
T	$f_{1T}^{\perp}$	g_{1T}	h_1 $h_{1T}^{\perp}$

time-reversal odd

N \ q	U	L	T
U	f_1		$h_1^\perp$
L		g_1	$h_{1L}^\perp$
T	$f_{1T}^\perp$	g_{1T}	h_1 $h_{1T}^\perp$

time-reversal odd



Diplole deformations

$\rho_{TL} : \sim \lambda\, \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp\ g_{1T}$
$\rho_{TL} : \sim \Lambda\, \boldsymbol{k}_\perp \cdot \boldsymbol{s}_\perp\ h^\perp_{1L}$

The corresponding dipole structures $\sim \lambda\, \boldsymbol{b}_\perp \cdot \boldsymbol{S}_\perp$, $\sim \Lambda\, \boldsymbol{b}_\perp \cdot \boldsymbol{s}_\perp$ for impact parameter densities (from GPDs) are ruled out by symmetries.

[Hägler, Musch, Negele, Schäfer EPL 88, 61001 (2009)]

$$f^{(m_x,n_\perp)} \equiv \int_{-1}^{1} dx\, x^m \int d^2\boldsymbol{k}_\perp \left(\frac{\boldsymbol{k}_\perp^2}{2m_N^2}\right)^n f(x,\boldsymbol{k}_\perp^2)$$

Let us assume the amplitudes $\tilde{A}_i$ are sufficiently regular at $\ell^2 = 0$.

$$\langle \boldsymbol{k}_\perp \rangle_{\rho_{TL}} = \lambda \boldsymbol{S}_\perp m_N \frac{g_{1T}^{(0_x,1_\perp)}}{f_1^{(0_x,0_\perp)}} =$$

$$\lambda \boldsymbol{S}_\perp m_N \frac{\tilde{A}_7(0,0)}{\tilde{A}_2(0,0)} \stackrel{?}{=} \lim_{\ell^2\to 0} \lambda \boldsymbol{S}_\perp m_N \frac{\tilde{A}_7(\ell^2,0)}{\tilde{A}_2(\ell^2,0)}$$

All self-energies from the gauge link cancel on the RHS
($\Rightarrow$ no dependence on the renormalization condition).
Similar to weighted asymmetries from experiment ($\to$ EIC):

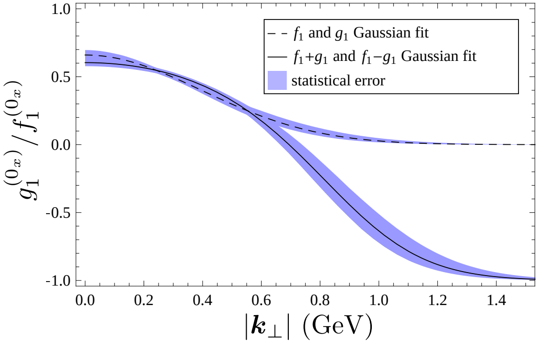
$$A_{LT}^{\frac{Q_T}{m_N}\cos(\phi_h-\phi_S)} = 2\,\frac{\langle \frac{Q_T}{m_N}\cos(\phi_h-\phi_S)\rangle_{UT}}{\langle 1\rangle_{UU}} \propto \frac{\sum_q e_q^2\, x\, g_{1T,q}^{(1_\perp)}(x)\, D_{1,q}(z)}{\sum_q e_q^2\, x\, f_{1,q}(x)\, D_{1,q}(z)}$$

[Boer, Mulders PRD 1998], [Bacchetta et al. arXiv:1003.1328]

$$f_1^{(0_x)}(\boldsymbol{k}_\perp^2) = C_0 \exp(-\boldsymbol{k}_\perp^2/\mu_0^2)$$
$$g_1^{(0_x)}(\boldsymbol{k}_\perp^2) = C_2 \exp(-\boldsymbol{k}_\perp^2/\mu_2^2)$$

vs.

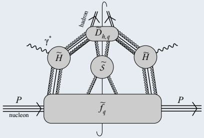
$$\rho_{LL}^{\pm}(\boldsymbol{k}_\perp) \equiv \tfrac{1}{2} f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \pm \tfrac{1}{2} g_1^{(0_x)}(\boldsymbol{k}_\perp^2)$$
$$\rho_{LL}^{+}(\boldsymbol{k}_\perp) = C_+ \exp(-\boldsymbol{k}_\perp^2/\mu_+^2)$$
$$\rho_{LL}^{-}(\boldsymbol{k}_\perp) = C_- \exp(-\boldsymbol{k}_\perp^2/\mu_-^2)$$



$\Rightarrow$ Asymptotic behavior at large $\boldsymbol{k}_\perp$ imposed by Gaussian ansatz; not a “lattice result”. Similar issues in analysis of experimental data.

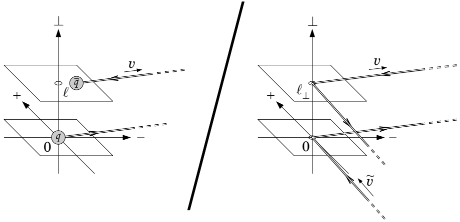
e.g., [Ji, Ma, Yuan PRD (2005)] :

$$W^{\mu\nu}_{\text{unpol.,LO}} \quad \propto \quad H \times f_1 \otimes D_h \otimes \underbrace{S}_{\text{soft factor}}$$



modified definition of TMD PDF correlator:

$$\Phi^{[\Gamma]}(k,P,S) \;\equiv\; \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\, \frac{\langle P,S|\;\bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\;|P,S\rangle}{\tilde{S}(\boldsymbol{\ell}_\perp,\ldots)}$$



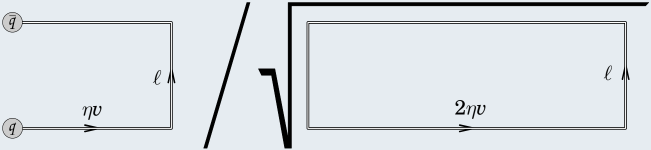
- gauge links slightly off lightcone: $v \neq \hat{n}_-$

$\Rightarrow$ evolution eqn. in $\zeta \equiv (v{\cdot}P)^2/v^2$

- soft factor $\tilde{S}$: vacuum expectation value of gauge link structure

How to get rid of the gauge link self engergy $\exp(\delta m\, L)$?

Soft factor in TMD PDF correlator? Suggestion [Collins arXiv:0808.2665] :



Is this a meaningful definition of TMD PDFs?

prerequisite for quantitative lattice predictions

"To allow non-perturbative methods in QCD to be used to estimate parton densities, operator definitions of parton densities are needed that can be taken literally." [Collins arXiv:0808.2665 (2008)]

$\boldsymbol{k}_\perp$-moments from ratios of amplitudes ...

... bridge the gap until we know more.

Example Sivers effect: $\langle \boldsymbol{k}_\perp \rangle_{\rho_{TU}}$ from $\tilde{A}_{12}/\tilde{A}_2$.
Self-energies cancel, no explicit subtraction factor needed.

ratio of correlators far away from nucleon source and sink

$$\frac{C_{3\text{pt}}(\tau;\Gamma,\ell,P)}{C_{2\text{pt}}(P)} \xrightarrow{t_{\text{src}} \ll \tau \ll t_{\text{sink}}} \text{const. (“plateau value”)},$$

$$\Downarrow$$

access to $\langle P,S|\ \bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P,S\rangle$

Γ	$\frac{1}{2}C^{\text{ren}}_{3\text{pt}}(\tau;\Gamma,\boldsymbol{\ell},\boldsymbol{P})/C_{2\text{pt}}(\boldsymbol{P})$ (LHPC projectors)
$\mathbb{1}$	$\frac{m_N}{E(P)}\,\tilde{A}_1$
$-\gamma_4\gamma_5$	$i m_N\,\tilde{A}_7\,\ell_z$
γ_4	$\tilde{A}_2$
$\frac{1}{2}[\gamma_2,\gamma_4]$	$\frac{1}{E(P)}\,\tilde{A}_9\,P_x + \frac{i m_N^2}{E(P)}\,\tilde{A}_{10}\,\ell_x + \frac{m_N^2}{E(P)}\,\tilde{A}_{11}\,(\ell_z)^2 P_x$
$\ldots$	$\ldots$

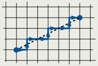
ratio of correlators far away from nucleon source and sink

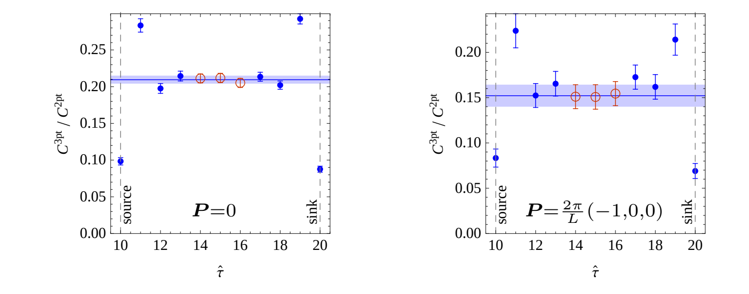
$$\frac{C_{3\mathrm{pt}}(\tau;\Gamma,\ell,P)}{C_{2\mathrm{pt}}(P)} \xrightarrow{t_{\mathrm{src}}\ll\tau\ll t_{\mathrm{sink}}} \text{const. (“plateau value”)},$$

$$\Downarrow$$

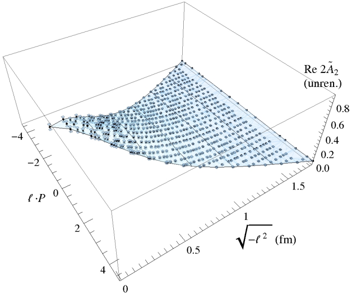
access to $\langle P,S|\ \overline{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P,S\rangle$

example plateau plots at $m_\pi \approx 600$ MeV

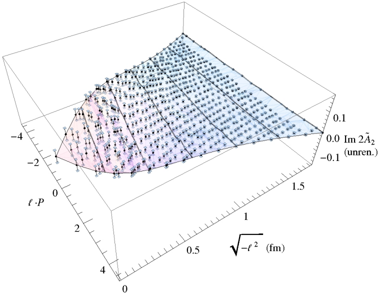
for $\Gamma = \gamma_4$ ($\Rightarrow \tilde{A}_2$), with HYP smeared gauge link $\mathcal{U}$ =  :



2 Re $\tilde{A}_2(\ell^2, \ell \cdot P)$

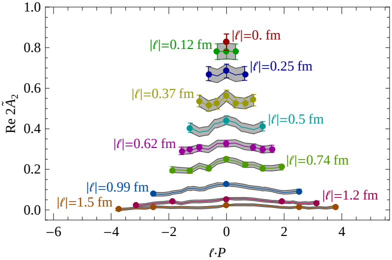


2 Im $\tilde{A}_2(\ell^2, \ell \cdot P)$

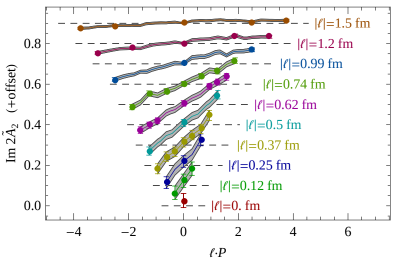


$\ell\cdot P$ - dependence of $\tilde{A}_2(\ell^2, \ell\cdot P)$

$2\ \mathrm{Re}\ \tilde{A}_2(\ell^2, \ell\cdot P)$

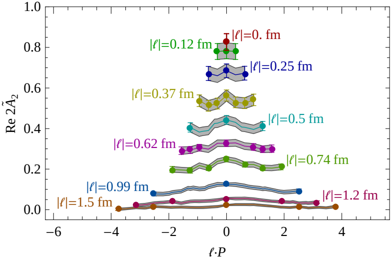


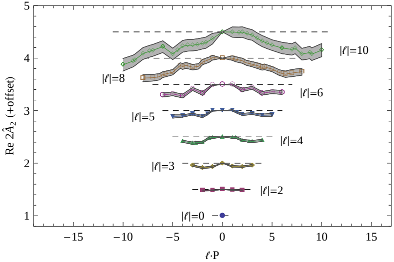
$2\ \mathrm{Im}\ \tilde{A}_2(\ell^2, \ell\cdot P)$



$$2 \text{ Re } \tilde{A}_2(\ell^2, \ell \cdot P)$$

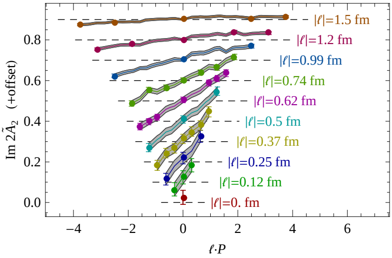
$$\text{Re } \tilde{A}_2^{\text{norm}} = \frac{\text{Re } \tilde{A}_2(\ell^2, \ell \cdot P)}{\text{Re } \tilde{A}_2(\ell^2, 0)}$$

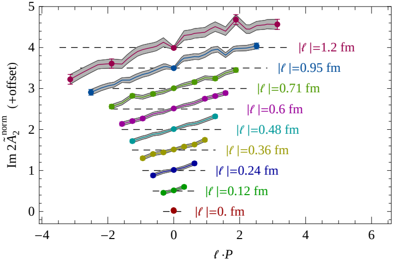


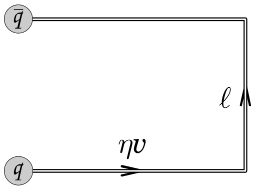


$$2 \text{ Im } \tilde{A}_2(\ell^2, \ell \cdot P)$$

$$\text{Im } \tilde{A}_2^{\text{norm}} = \frac{\text{Im } \tilde{A}_2(\ell^2, \ell \cdot P)}{\text{Re } \tilde{A}_2(\ell^2, 0)}$$







32 Lorentz-invariant amplitudes [Goeke,Metz,Schlegel PLB618,90 (2005)]

$$A_i\left(k^2, k\cdot P, \frac{v\cdot k}{|v\cdot P|}, \frac{v^2}{|v\cdot P|^2}, \frac{v\cdot P}{|v\cdot P|}\right) = A_i\left(k^2, k\cdot P, \underbrace{\frac{v\cdot k}{|v\cdot P|}}_{\approx\, x}, \zeta^{-1}, \mathrm{sgn}(v\cdot P)\right)$$

Links approaching light cone: $v \to \hat{n}_- \Rightarrow \zeta \to \infty$. For large ζ, the evolution with ζ is known [Collins,Soper NPB194,445 (1981)].

$$\left.\begin{array}{r}(v^0, v^1, v^2, v^3)\\ \text{future pointing } v\\ \text{TMD PDFs for SIDIS}\end{array}\right\} \xrightarrow{\mathcal{T}} \left\{\begin{array}{l}(-v^0, v^1, v^2, v^3)\\ \text{past pointing } v\\ \text{TMD PDFs for Drell-Yan}\end{array}\right.$$

The transformation property of the matrix elements under time reversal provides relations:

Example of a $\mathcal{T}$-even amplitude:

$$A_2\left(k^2, k{\cdot}P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, 1\right) = A_2\left(k^2, k{\cdot}P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, -1\right)$$

$$\Downarrow$$

$$f_1^{(\text{SIDIS})}(x, \boldsymbol{k}_\perp; \zeta, \ldots) = f_1^{(\text{Drell-Yan})}(x, \boldsymbol{k}_\perp; \zeta, \ldots)$$

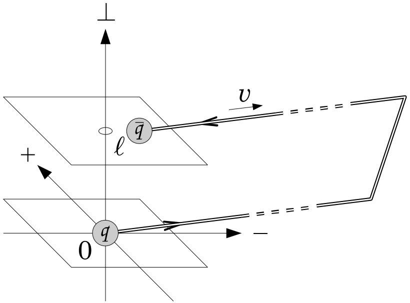
Example of a $\mathcal{T}$-odd amplitude: ($\to$ Sivers function $f_{1T}^\perp$)

$$A_{12}\left(k^2, k{\cdot}P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, 1\right) = -A_{12}\left(k^2, k{\cdot}P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, -1\right)$$

$$\Downarrow$$

$$f_{1T}^{\perp(\text{SIDIS})}(x, \boldsymbol{k}_\perp; \zeta, \ldots) = -f_{1T}^{\perp(\text{Drell-Yan})}(x, \boldsymbol{k}_\perp; \zeta, \ldots)$$

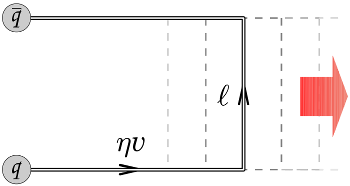
- ... appear in factorized SIDIS / Drell-Yan process
- are responsible for "time-reversal-odd" TMD PDFs, such as $f_{1T}^{\perp}$ (Sivers-function)



- gauge link = effective representation of struck quark ("final state interaction")
- $\Rightarrow$ (almost lightlike)

$$\zeta \equiv \frac{(v\cdot P)^2}{v^2} \to \pm\infty$$

- keep ζ finite to avoid "rapidity divergences"
- evolution equation in ζ [Collins, Soper NPB (1981)]

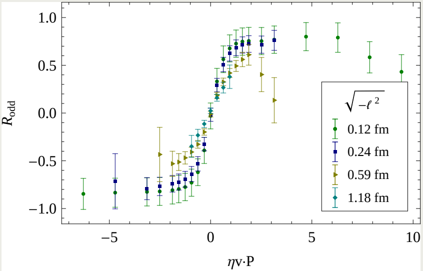


- v spatial $\Rightarrow$ $|\zeta| = \dfrac{(v\cdot P)^2}{|v|^2} \leq |\boldsymbol{P}_{\text{lat.}}|^2$
- look for plateaus at large $|\eta|$
- now 32 amplitudes

[Goeke, Metz, Schlegel PLB (2005)]

$\tilde{a}_i(\ell^2, \ell\cdot P, v\cdot P; \eta, \zeta)$, $\tilde{b}_i(\ldots)$

Test calculation: a time reversal odd ratio of ampitudes



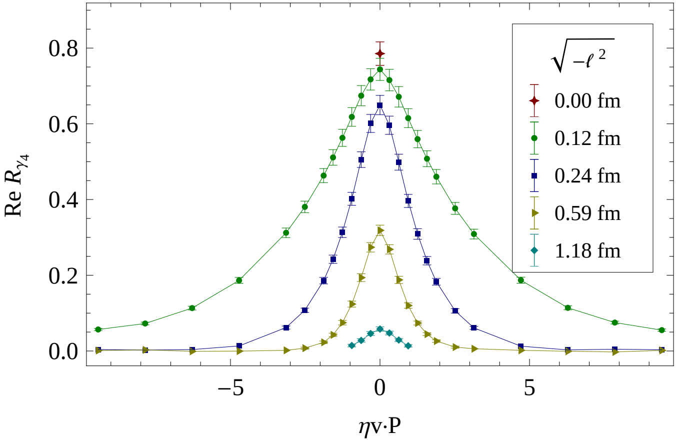
$$R_{\text{odd}} = -\frac{\tilde{a}_{12} - (\eta \frac{m_N^2 v_1}{P_1})\, \tilde{b}_8}{\tilde{a}_2}$$

Plateaus visible at large $|\eta|$.

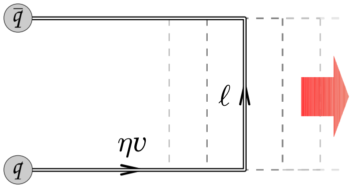
"Time-reversal odd" $\leftrightarrow$ odd in $\eta v\cdot P$.

Part of the effect comes from the Sivers function $f_{1T}^{\perp}$!

$$\tilde{A}_2\left(\ell^2, \ell\cdot P, \frac{v\cdot\ell}{|v\cdot P|}, \zeta^{-1}, \mathrm{sgn}(v\cdot P)\right) \equiv \lim_{\eta\to\infty} \tilde{a}_2(\ell^2, \ell\cdot P, \eta v\cdot\ell, -\eta^2, \eta v\cdot P)$$



But $\tilde{a}_2 = \mathrm{Re}\, R_{\gamma_4}$ always vanishes for large η!
Reason: power divergence suppresses $\tilde{a}_2 \sim \exp(-\delta m\, \eta)$.



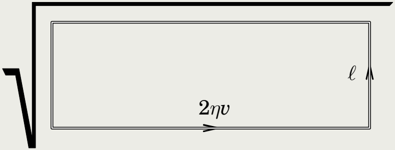
- v spatial $\Rightarrow$ $|\zeta| = \dfrac{(v\cdot P)^2}{|v|^2} \leq |\boldsymbol{P}_{\text{lat.}}|^2$
- look for plateaus at large $|\eta|$
- now 32 amplitudes $\tilde{a}_i(\ell^2, \ell\cdot P, v\cdot P; \eta, \zeta)$

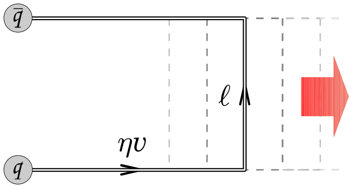
Problem: need to subtract gauge link self-energy ($\rightarrow$ η-independence)

idea #1: modify definition of TMD PDFs [Collins PoS LC (2008)]

$$\Phi^{[\Gamma]}(k,P,S) \;\equiv\; \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\, \frac{\langle P,S|\;\bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\;|P,S\rangle}{\widetilde{S}(\boldsymbol{\ell}_\perp,\ldots)}$$

with $\widetilde{S}$ obtained from a vacuum expectation value of gauge links, e.g.,





- v spatial $\Rightarrow$ $|\zeta| = \dfrac{(v\cdot P)^2}{|v|^2} \leq |\boldsymbol{P}_{\text{lat.}}|^2$
- look for plateaus at large $|\eta|$
- now 32 amplitudes $\tilde{a}_i(\ell^2, \ell\cdot P, v\cdot P; \eta, \zeta)$

Problem: need to subtract gauge link self-energy ($\rightarrow$ η-independence)

idea #2: ratios of amplitudes $\rightarrow$ certain $\boldsymbol{k}_\perp$-moments

e.g., formally,

$$\langle \boldsymbol{k}_y \rangle_{TU} = -2m_N \boldsymbol{S}_x \lim_{\eta\to\infty} \frac{\tilde{a}_{12}(0,0,0;\eta,\zeta) + \ldots}{\tilde{a}_2(0,0,0;\eta,\zeta)} \propto \frac{\int dx \int d^2\boldsymbol{k}_\perp\, \boldsymbol{k}_\perp^2\, f_{1T}^\perp}{\int dx \int d^2\boldsymbol{k}_\perp\, f_1}$$

Sivers function causes average transverse quark momentum in y-direction in a transversely polarized nucleon (spin in x-direction).

$$\langle \boldsymbol{k}_y \rangle_{TU} \underset{\eta\ \text{large}}{\approx} -2m_N \boldsymbol{S}_x \frac{\tilde{a}_{12}(\ell^2_{\text{min}},0,0;\eta,\zeta) + \ldots}{\tilde{a}_2(\ell^2_{\text{min}},0,0;\eta,\zeta)}$$

Self-energy cancels!