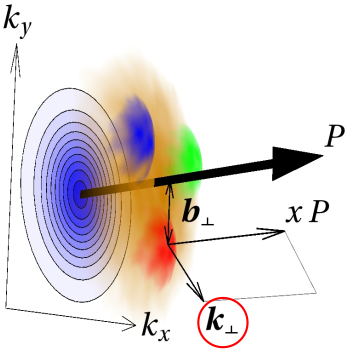
TMD PDFs on the Lattice

Bernhard Musch (Jefferson Lab)
in collaboration with
Philipp Hägler (TU München), John Negele (MIT),
Andreas Schäfer (Univ. Regensburg),
and the LHP Collaboration

[Hägler et al. EPL88 61001 (2009)]
[Musch arXiv:0907.2381]

Emmy Noether-Programm
Deutsche Forschungsgemeinschaft
DFG

Jefferson Lab
Thomas Jefferson National Accelerator Facility



TMD PDFs

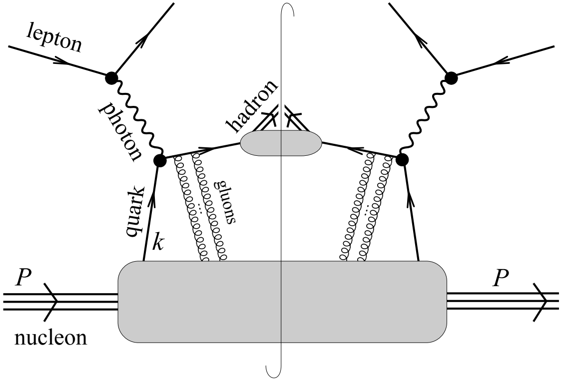
transverse **m**omentum **d**ependent **p**arton **d**istribution **f**unctions

e.g., $f_1(x, \boldsymbol{k}_\perp^2)$

$\Rightarrow$ quark density $\rho(\boldsymbol{k}_\perp)$.

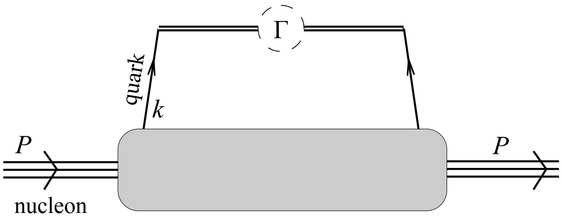
- x (longitudinal momentum fraction) $\Rightarrow$ PDFs
- $x, \boldsymbol{b}_\perp$ (impact parameter) $\Rightarrow$ GPDs
- $x, \boldsymbol{k}_\perp$ (intrinsic transverse momentum) $\Rightarrow$ TMD PDFs

e.g., [Collins PLB 93], [Bacchetta et al. JHEP 07]



$$\frac{d\sigma}{d^3P_h\ d^3P_{l'}} \propto \underbrace{L_{\mu\nu}}_{\substack{\text{lepton}\\\text{tensor}}} \quad \underbrace{W^{\mu\nu}}_{\substack{\text{hadron}\\\text{tensor}}}$$

$$W^{\mu\nu}_{\text{unpol.,LO}} \propto \underbrace{H(Q^2,\ldots)}_{\text{hard part}} \int d^2\boldsymbol{k}_\perp \ \underbrace{f_1(x,\boldsymbol{k}_\perp,\ldots)}_{\text{TMD PDF}} \ \underbrace{D_h(z,\boldsymbol{k}_\perp+\boldsymbol{q}_\perp,\ldots)}_{\text{fragmentation f.}}$$



$$\Phi^{[\Gamma]}(k, P, S) \equiv \text{“} \langle P, S|\, \bar{q}(k)\, \Gamma\, q(k)\, |P, S\rangle \text{”}$$

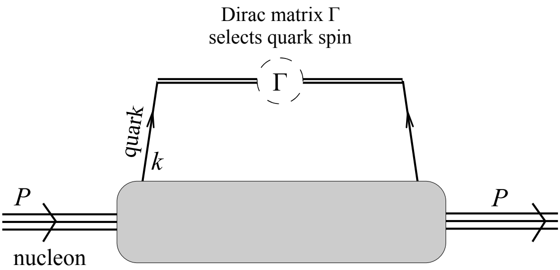
lightcone coor. $w^{\pm} = \frac{1}{\sqrt{2}}(w^0 \pm w^3)$, so $w = w^+ \hat{n}_+ + w^- \hat{n}_- + w_\perp$

proton flies along z-axis: P^+ large, $P_\perp = 0$

parametrization in terms of TMD PDFs, example

$$\int dk^- \; \Phi^{[\gamma^+]}(k, P, S)\Big|_{k^+ = xP^+} = f_1(x, \boldsymbol{k}_\perp^2) + \langle \text{spin dep. terms} \rangle$$

[Ralston, Soper NPB 1979], [Mulders, Tangerman NPB 1996], [Goeke, Metz, Schlegel PLB 2005]



$$\Phi^{[\Gamma]}(k,P,S) \equiv \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\, \langle P,S|\ \bar{q}(\ell)\,\Gamma\ \mathcal{U}\ q(0)\ |P,S\rangle$$

lightcone coor. $w^{\pm} = \frac{1}{\sqrt{2}}(w^0 \pm w^3)$, so $w = w^+\hat{n}_+ + w^-\hat{n}_- + w_\perp$

proton flies along z-axis: P^+ large, $P_\perp = 0$

parametrization in terms of TMD PDFs, example

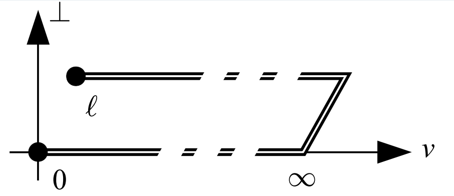
$$\int dk^-\ \Phi^{[\gamma^+]}(k,P,S)\Big|_{k^+=xP^+} = f_1(x,\boldsymbol{k}_\perp^2) + \langle\text{spin dep. terms}\rangle$$

[Ralston, Soper NPB 1979], [Mulders, Tangerman NPB 1996], [Goeke, Metz, Schlegel PLB 2005]

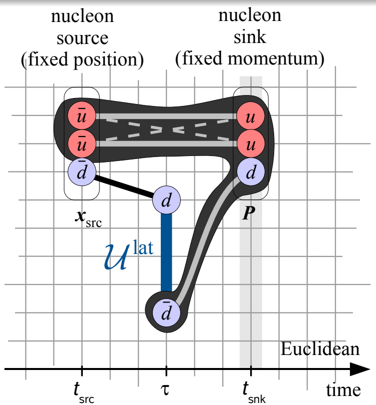
$$\mathcal{U} \equiv \mathcal{P} \exp\left(-ig\int_0^\ell d\xi^\mu A_\mu(\xi)\right) \qquad \text{along path from 0 to } \ell$$

$\Longrightarrow \quad \langle P|\ \overline{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P\rangle$ is gauge invariant.

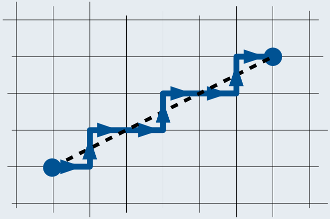
SIDIS / Drell Yan



$v = \hat{n}_-$ (lightlike), or slightly off, $v^- \gg v^+$



gauge link on lattice



For now, approximate **direct** gauge link, no soft factor.
$\Longrightarrow$ no T-odd structuers (Sivers, Boer-Mulders fcn.)

extract Lorentz-invariant amplitudes $\tilde{A}_i(\ell^2, \ell\cdot P)$

$$\langle P, S|\ \bar{q}(\ell)\,\gamma_\mu\,\mathcal{U}\,q(0)\ |P, S\rangle \quad = \quad 4\ \tilde{A}_2\ P_\mu\ +\ 4i\,m_N{}^2\ \tilde{A}_3\ \ell_\mu$$

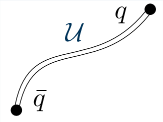
$\Rightarrow\ f_1(x, \boldsymbol{k}_\perp^2)$

Amplitudes are complex and fulfill $[\tilde{A}_i(\ell^2, \ell\cdot P)]^* = \tilde{A}_i(\ell^2, -\ell\cdot P)$.
Operator must not have temporal extent: $\ell^0 = \ell_4 = 0$.

continuum renormalization of gauge links

[Craigie, Dorn NPB185,204 (1981)]

smooth path



$$[\bar{q}\,\mathcal{U}\,q]_{\mathrm{ren}} = Z^{-1} \exp\left(-\delta\hat{m}\,\frac{l}{a} \right) [\bar{q}\,\mathcal{U}\,q]$$

l : the total length of the gauge link,

$\delta\hat{m}$: removes the power divergence $\sim 1/a$

static quark potential

$$V_{\mathrm{ren}}(r) = V(r) + 2\,\delta\hat{m}/a$$

string [Lüscher,Symanzik,Weisz (1980)]

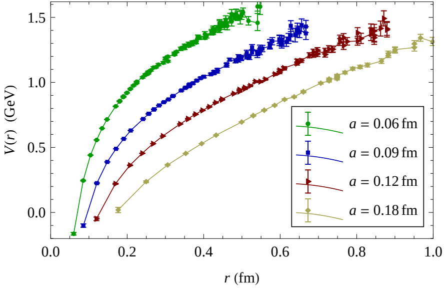
at large r: $V_{\mathrm{ren}}(r) \approx$

$V_{\mathrm{string}}(r) = \sigma r - \pi/12r + C$

method [Cheng PRD77,014511 (2008)]

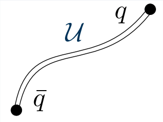
determine $\delta\hat{m}$ from

$V_{\mathrm{ren}}(0.7\ \mathrm{fm}) \stackrel{!}{=} V_{\mathrm{string}}(0.7\ \mathrm{fm})$



continuum renormalization of gauge links [Craigie, Dorn NPB185,204 (1981)]

smooth path



$$[\bar{q}\,\mathcal{U}\,q]_{\text{ren}} = Z^{-1} \exp\left(-\delta\hat{m}\,\frac{l}{a} \right) [\bar{q}\,\mathcal{U}\,q]$$

l : the total length of the gauge link,

$\delta\hat{m}$: removes the power divergence $\sim 1/a$

static quark potential

$$V_{\text{ren}}(r) = V(r) + 2\,\delta\hat{m}/a$$

string [Lüscher,Symanzik,Weisz (1980)]

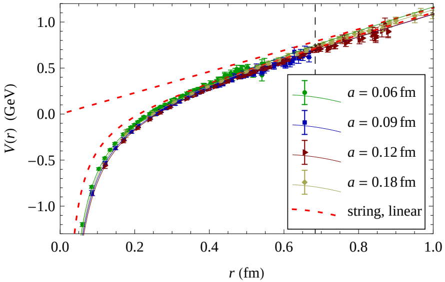
at large r: $V_{\text{ren}}(r) \approx$

$$V_{\text{string}}(r) = \sigma r - \pi/12r + C$$

method [Cheng PRD77,014511 (2008)]

determine $\delta\hat{m}$ from

$$V_{\text{ren}}(0.7\text{ fm}) \stackrel{!}{=} V_{\text{string}}(0.7\text{ fm})$$

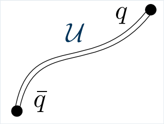


link renormalization

continuum renormalization of gauge links

[Craigie, Dorn NPB185,204 (1981)]

smooth path



$$[\bar{q}\,\mathcal{U}\,q]_{\text{ren}} = Z^{-1}\exp\left(-\delta\hat{m}\,\frac{l}{a}\right)\,[\bar{q}\,\mathcal{U}\,q]$$

l : the total length of the gauge link,

$\delta\hat{m}$: removes the power divergence $\sim 1/a$

static quark potential

$$V_{\text{ren}}(r) = V(r) + 2\,\delta\hat{m}/a$$

string [Lüscher,Symanzik,Weisz (1980)]

at large r: $V_{\text{ren}}(r) \approx$

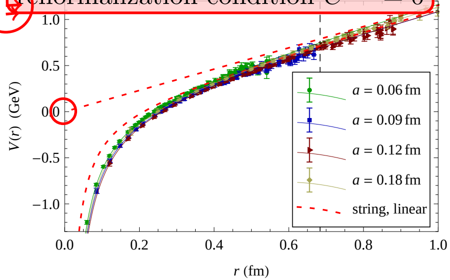
$V_{\text{string}}(r) = \sigma r - \pi/12r + 0$

method [Cheng PRD77,014511 (2008)]

determine $\delta\hat{m}$ from

$V_{\text{ren}}(0.7\text{ fm}) \overset{!}{=} V_{\text{string}}(0.7\text{ fm})$

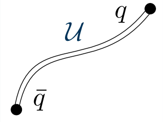
renormalization condition $C^{\text{ren}} = 0$



continuum renormalization of gauge links

[Craigie, Dorn NPB185,204 (1981)]

smooth path



$$[\bar{q}\,\mathcal{U}\,q]_{\text{ren}} = Z^{-1} \exp\left(-\delta\hat{m}\,\frac{l}{a} \right) [\bar{q}\,\mathcal{U}\,q]$$

l : the total length of the gauge link,

$\delta\hat{m}$: removes the power divergence $\sim 1/a$

static quark potential

$$V_{\text{ren}}(r) = V(r) + 2\,\delta\hat{m}/a$$

string [Lüscher, Symanzik, Weisz (1980)]

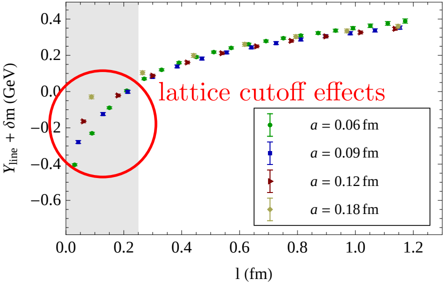
at large r: $V_{\text{ren}}(r) \approx$

$$V_{\text{string}}(r) = \sigma r - \pi/12r + 0$$

method [Cheng PRD77,014511 (2008)]

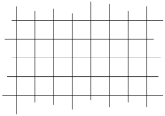
determine $\delta\hat{m}$ from

$$V_{\text{ren}}(0.7\text{ fm}) \overset{!}{=} V_{\text{string}}(0.7\text{ fm})$$



$$Y_{\text{line}}(l) \equiv \frac{d}{dl} \ln\langle \text{tr}\,\mathcal{U}\rangle_{(\text{Landau gauge})}$$

We employ the Chroma library [Edwards, Joo (2005)] to process



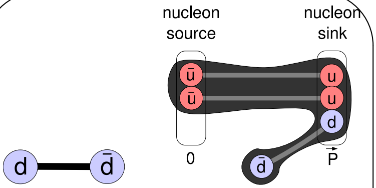
MILC gauge configurations

staggered Asqtad action,
2+1 flavors, $a \approx 0.124$ fm,
$m_\pi \approx 500,\ 610,$ and 760 MeV

[Orginos, Toussaint PRD (1999)]

+ finer MILC lattices
to test renormalization

[Aubin et al. PRD (2004)]
[Bazavov et al. 0903.3598]



LHPC propagators

domain wall valence fermions,
m_π adjusted to staggered sea,

nucleon momenta:
$\boldsymbol{P} = 0$ and $|\boldsymbol{P}| = 500$ MeV

e.g., [Hägler et al. PRD (2008)]

$$\Phi^{[\Gamma]}(k,P,S) \equiv \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\,\langle P,S|\;\bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\;|P,S\rangle$$

$$\Phi^{[\Gamma]}(x,\boldsymbol{k}_\perp;P,S) \equiv \int_{-\infty}^{\infty} dk^-\;\Phi^{[\Gamma]}(k;P,S)\,\Big|_{k^+=xP^+}$$

$$= \frac{1}{2(2\pi)^3}\int d\ell^-\;d^2\ell_\perp\;e^{ik\cdot\ell}\;\langle P,S|\;\bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\;|P,S\rangle\,\Big|_{\ell^+=0}$$

$$= \int \frac{d(\ell\cdot P)}{4\pi P^+}\,e^{ix(\ell\cdot P)}\int \frac{d^2\boldsymbol{\ell}_\perp}{(2\pi)^2}\,e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{\ell}_\perp}\;\langle P,S|\;\bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\;|P,S\rangle\,\Big|_{\ell^+=0}$$

Note: $\ell^2\Big|_{\ell^+=0} = -\boldsymbol{\ell}_\perp^2.$ $\quad x \longleftrightarrow \ell\cdot P \quad\quad \boldsymbol{k}_\perp^2 \longleftrightarrow \ell^2$

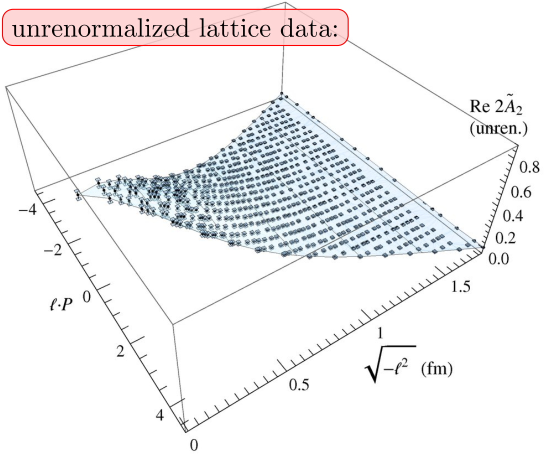
example: unpolarized case

$$f_1(x,\boldsymbol{k}_\perp^2) \equiv \Phi^{[\gamma^+]}(x,\boldsymbol{k}_\perp;P,S)$$

$$= \int \frac{d(\ell\cdot P)}{2\pi}\,e^{ix(\ell\cdot P)}\int \frac{d^2\boldsymbol{\ell}_\perp}{(2\pi)^2}\,e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{\ell}_\perp}\;2\tilde{A}_2(\ell^2,\ell\cdot P)\,\Big|_{\ell^+=0}$$

$$f_1(x, \boldsymbol{k}_\perp^2) \equiv \Phi^{[\gamma^+]}(x, \boldsymbol{k}_\perp; P, S)$$
$$= \int \frac{d(\ell \cdot P)}{2\pi}\, e^{ix(\ell\cdot P)} \int \frac{d^2\boldsymbol{\ell}_\perp}{(2\pi)^2}\, e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{\ell}_\perp}\, 2\tilde{A}_2(\ell^2, \ell\cdot P)\Big|_{\ell^+=0}$$

unrenormalized lattice data:

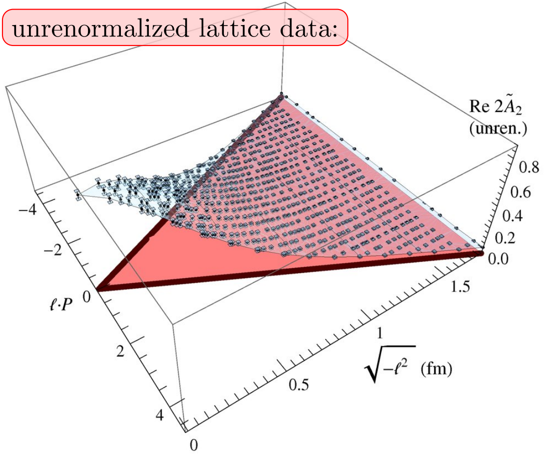


$\ell^2 \xleftrightarrow{\text{FT}} \boldsymbol{k}_\perp^2$

$\ell\cdot P \xleftrightarrow{\text{FT}} x$

$$f_1(x,\boldsymbol{k}_\perp^2) \equiv \Phi^{[\gamma^+]}(x,\boldsymbol{k}_\perp;P,S)$$
$$= \int \frac{d(\ell\cdot P)}{2\pi}\, e^{ix(\ell\cdot P)} \int \frac{d^2\boldsymbol{\ell}_\perp}{(2\pi)^2}\, e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{\ell}_\perp}\, 2\tilde{A}_2(\ell^2,\ell\cdot P)\Big|_{\ell^+=0}$$

unrenormalized lattice data:



$\ell^2 \xleftrightarrow{\text{FT}} \boldsymbol{k}_\perp^2$

$\ell\cdot P \xleftrightarrow{\text{FT}} x$

Euclidean lattice

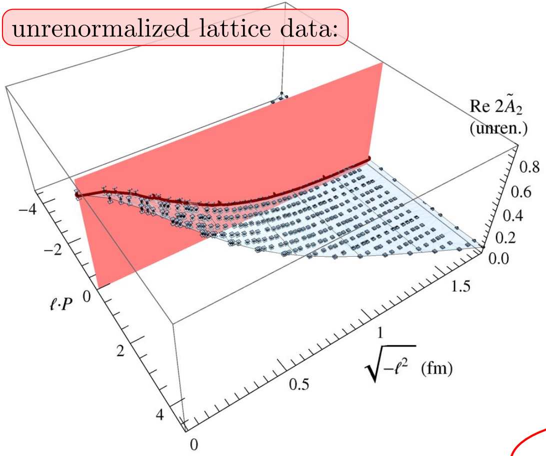
$\ell_4 = 0$

$\Downarrow$

$\ell^2 \le 0,$
$|\ell\cdot P| \le |\boldsymbol{P}|\sqrt{-\ell^2}$

$$f_1(x,\boldsymbol{k}_\perp^2) \equiv \Phi^{[\gamma^+]}(x,\boldsymbol{k}_\perp;P,S) = \int \frac{d(\ell\cdot P)}{2\pi}\, e^{ix(\ell\cdot P)} \int \frac{d^2\boldsymbol{\ell}_\perp}{(2\pi)^2}\, e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{\ell}_\perp}\, 2\tilde{A}_2(\ell^2,\ell\cdot P)\Big|_{\ell^+=0}$$

unrenormalized lattice data:



$$\ell^2 \overset{\text{FT}}{\longleftrightarrow} \boldsymbol{k}_\perp^2$$

$$\ell{\cdot}P \overset{\text{FT}}{\longleftrightarrow} x$$

Euclidean lattice

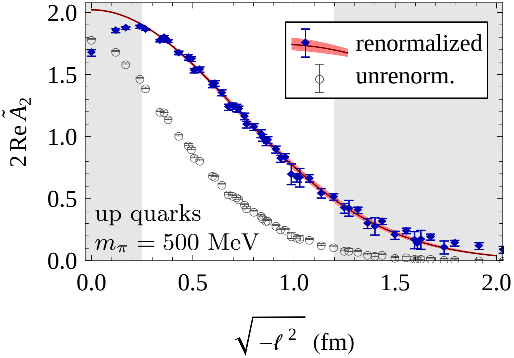
$$\ell_4 = 0$$

$$\Downarrow$$

$$\ell^2 \leq 0,$$
$$|\ell{\cdot}P| \leq |\boldsymbol{P}|\sqrt{-\ell^2}$$

Lowest x-moment of TMD PDFs

$$f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \equiv \int_{-1}^{1} dx\ f_1(x, \boldsymbol{k}_\perp^2) \equiv \int dx \int dk^-\ \Phi^{[\gamma^+]}(k, P, S)$$
$$= \int \frac{d^2\ell_\perp}{(2\pi)^2}\, e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{\ell}_\perp}\ 2\,\tilde{A}_2(-\ell_\perp^2, 0)$$



fit function

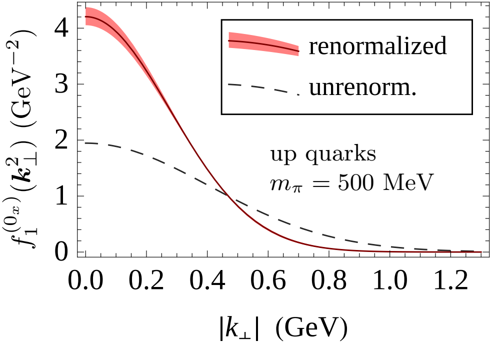
$C_1 \exp(-|\ell|^2/\sigma_1^2)$

Z-factor

$Z^{-1}\ C_1^{\text{up-down}} \overset{!}{=} 1$

multiplicative renormalization based on quark counting

$$f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \equiv \int_{-1}^{1} dx\ f_1(x, \boldsymbol{k}_\perp^2) \equiv \int dx \int dk^-\ \Phi^{[\gamma^+]}(k, P, S)$$
$$= \int \frac{d^2\ell_\perp}{(2\pi)^2}\ e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{\ell}_\perp}\ 2\,\tilde{A}_2(-\ell_\perp^2, 0)$$



width of the distribution (RMS momentum):

$\langle \boldsymbol{k}_\perp^2 \rangle^{1/2} =$ $(391 \pm 8_{\text{stat}} \pm 27_{\text{sys}})$ MeV

compare phenomenology [Anselmino et al., PRD71, 074006 (2005)]:

$\langle \boldsymbol{k}_\perp^2 \rangle^{1/2} \approx 500$ MeV

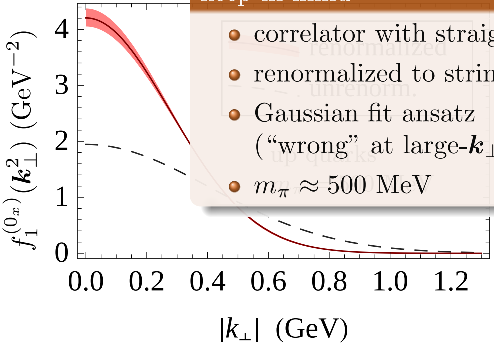
(estimate, Gaussian Ansatz)

$$f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \equiv \int_{-1}^{1} dx\ f_1(x, \boldsymbol{k}_\perp^2) \equiv \int dx \int dk^-\ \Phi^{[\gamma^+]}(k, P, S)$$

$$= \int \frac{d^2\ell_\perp}{(2\pi)^2}\ e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{\ell}_\perp}\ 2\,\tilde{A}_2(-\ell_\perp^2, 0)$$

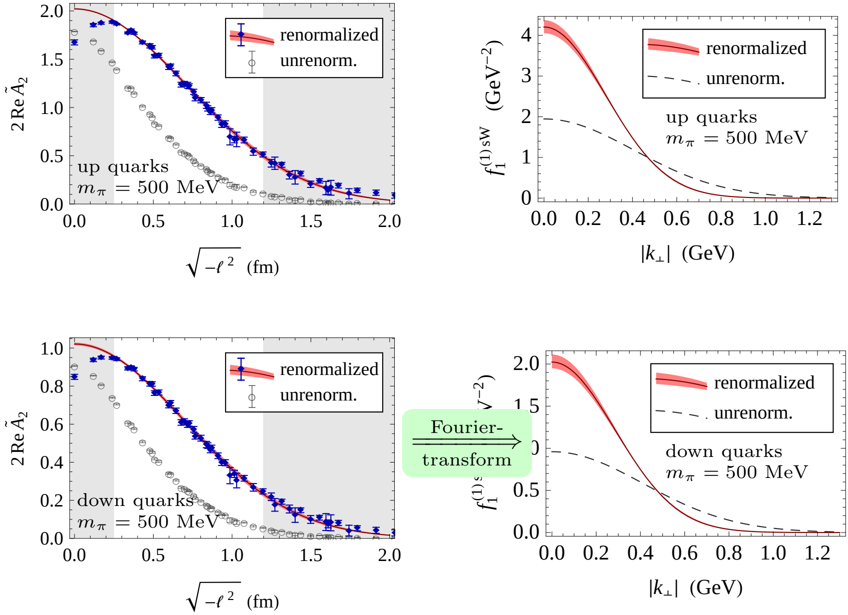
keep in mind

- correlator with straight Wilson line
- renormalized to string potential with $C = 0$
- Gaussian fit ansatz ("wrong" at large-$\boldsymbol{k}_\perp$ [Diehl, arXiv:0811.0774])
- $m_\pi \approx 500$ MeV

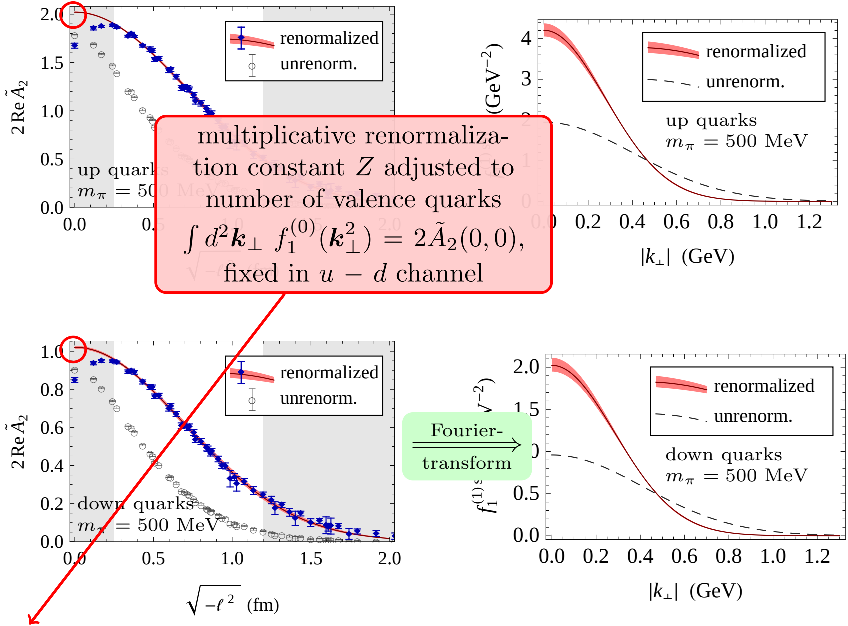


width of the distribution (RMS momentum): $\langle \boldsymbol{k}_\perp^2 \rangle^{1/2} = (391 \pm 8_{stat} \pm 27_{sys})$ MeV

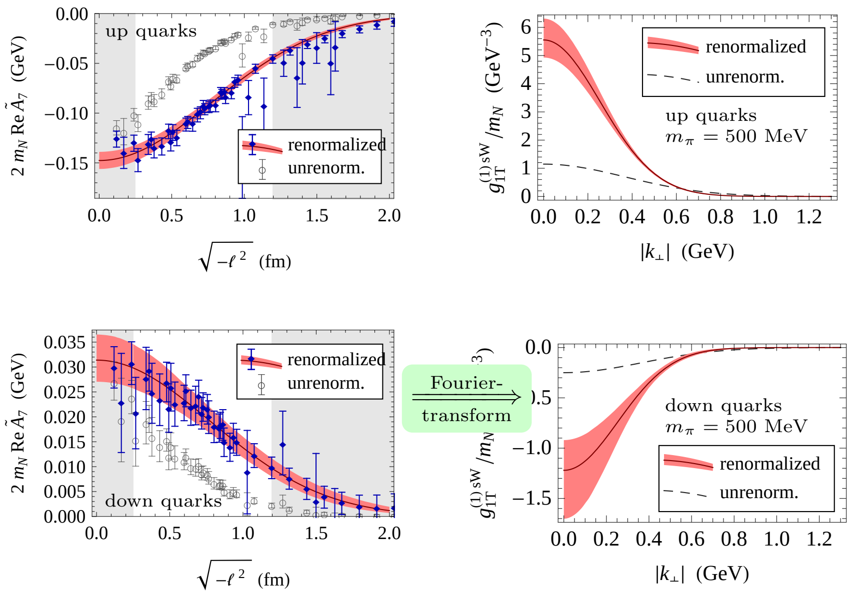
compare phenomenology [Anselmino et al., PRD71, 074006 (2005)]: $\langle \boldsymbol{k}_\perp^2 \rangle^{1/2} \approx 500$ MeV (estimate, Gaussian Ansatz)


renormalized
unrenorm.
up quarks
$m_\pi = 500$ MeV
$2\,\mathrm{Re}\,\tilde{A}_2$
$\sqrt{-\ell^2}$ (fm)
$f_1^{(1)\,\mathrm{sW}}$ (GeV^{-2})
$|k_\perp|$ (GeV)
down quarks
Fourier-
transform

$f_1^{(0_x)}$ from $\tilde{A}_2$

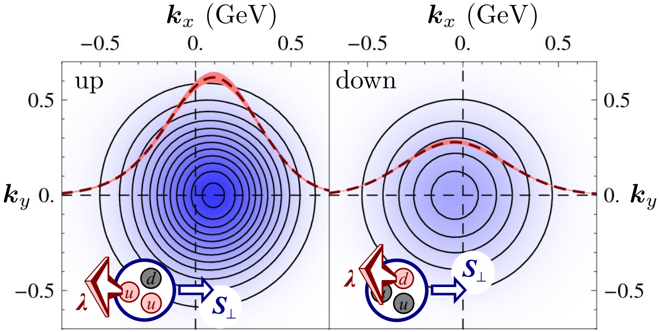


multiplicative renormalization constant Z adjusted to number of valence quarks
$\int d^2\boldsymbol{k}_\perp\ f_1^{(0)}(\boldsymbol{k}_\perp^2) = 2\tilde{A}_2(0,0)$,
fixed in $u - d$ channel


up quarks
renormalized
unrenorm.
$2\,m_N\,\mathrm{Re}\,\tilde{A}_7$ (GeV)
$\sqrt{-\ell^2}$ (fm)
renormalized
unrenorm.
up quarks
$m_\pi = 500$ MeV
$g_{1T}^{(1)\,\mathrm{sW}}/m_N$ (GeV^{-3})
$|k_\perp|$ (GeV)
renormalized
unrenorm.
down quarks
$2\,m_N\,\mathrm{Re}\,\tilde{A}_7$ (GeV)
$\sqrt{-\ell^2}$ (fm)
Fourier-
transform
down quarks
$m_\pi = 500$ MeV
renormalized
unrenorm.
$|k_\perp|$ (GeV)

Density of quarks with positive helicity, $\lambda = 1$,
in a transversely polarized nucleon, $\boldsymbol{S}_\perp = (1,0)$:

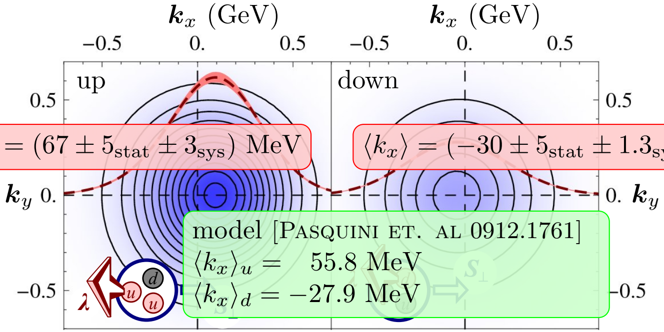
$$\rho_{TL}(\boldsymbol{k}_\perp; \boldsymbol{S}_\perp, \lambda) \equiv \frac{1}{2} \int dx \int dk^- \; \Phi^{[\gamma^+ \frac{1}{2}(\mathbb{1}+\gamma^5)]}(k, P, S_\perp)$$
$$= \frac{1}{2} f_1^{(0_x)}(\boldsymbol{k}_\perp^2) + \frac{\lambda}{2} \frac{\boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp}{m_N} g_{1T}^{(0_x)}(\boldsymbol{k}_\perp^2)$$



($m_\pi \approx 500$ MeV, straight gauge link operator,
renormalization condition $C^{\mathrm{ren}} = 0$, Gaussian fit)

Density of quarks with positive helicity, $\lambda = 1$,
in a transversely polarized nucleon, $\boldsymbol{S}_\perp = (1,0)$:

$$\rho_{TL}(\boldsymbol{k}_\perp; \boldsymbol{S}_\perp, \lambda) \equiv \frac{1}{2} \int dx \int dk^- \ \Phi^{[\gamma^+ \frac{1}{2}(\mathbb{1}+\gamma^5)]}(k, P, S_\perp)$$
$$= \frac{1}{2} f_1^{(0_x)}(\boldsymbol{k}_\perp^2) + \frac{\lambda}{2} \frac{\boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp}{m_N} g_{1T}^{(0_x)}(\boldsymbol{k}_\perp^2)$$



$\langle k_x \rangle = (67 \pm 5_{\text{stat}} \pm 3_{\text{sys}})$ MeV

$\langle k_x \rangle = (-30 \pm 5_{\text{stat}} \pm 1.3_{\text{sys}})$ MeV

model [Pasquini et. al 0912.1761]
$\langle k_x \rangle_u = \quad 55.8$ MeV
$\langle k_x \rangle_d = -27.9$ MeV

($m_\pi \approx 500$ MeV, straight gauge link operator,
renormalization condition $C^{\text{ren}} = 0$, Gaussian fit)

$$f^{(m_x, n_\perp)} \equiv \int_{-1}^{1} dx\, x^m \int d^2\boldsymbol{k}_\perp \left(\frac{\boldsymbol{k}_\perp^2}{2m_N^2} \right)^n f(x, \boldsymbol{k}_\perp^2)$$

Let us assume the amplitudes $\tilde{A}_i$ are sufficiently regular at $\ell^2 = 0$.

$$\langle \boldsymbol{k}_\perp \rangle_{\rho_{TL}} = \lambda \boldsymbol{S}_\perp m_N \frac{g_{1T}^{(0_x, 1_\perp)}}{f_1^{(0_x, 0_\perp)}} =$$

$$\lambda \boldsymbol{S}_\perp m_N \frac{\tilde{A}_7(0,0)}{\tilde{A}_2(0,0)} \stackrel{?}{=} \lim_{\ell^2 \to 0} \lambda \boldsymbol{S}_\perp m_N \frac{\tilde{A}_7(\ell^2, 0)}{\tilde{A}_2(\ell^2, 0)}$$

All self-energies from the gauge link cancel on the RHS
($\Rightarrow$ no dependence on the renormalization condition).
Similar to weighted asymmetries from experiment ($\rightarrow$ EIC):

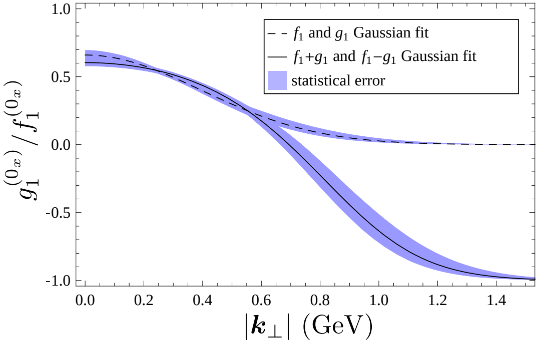
$$A_{LT}^{\frac{Q_T}{m_N} \cos(\phi_h - \phi_S)} = 2 \frac{\langle \frac{Q_T}{m_N} \cos(\phi_h - \phi_S) \rangle_{UT}}{\langle 1 \rangle_{UU}} \propto \frac{\sum_q e_q^2\, x\, g_{1T,q}^{(1_\perp)}(x)\, D_{1,q}(z)}{\sum_q e_q^2\, x\, f_{1,q}(x)\, D_{1,q}(z)}$$

[Boer, Mulders PRD 1998], [Bacchetta et al. arXiv:1003.1328]

$$f_1^{(0_x)}(\boldsymbol{k}_\perp^2) = C_0 \exp(-\boldsymbol{k}_\perp^2/\mu_0^2)$$
$$g_1^{(0_x)}(\boldsymbol{k}_\perp^2) = C_2 \exp(-\boldsymbol{k}_\perp^2/\mu_2^2)$$

vs.

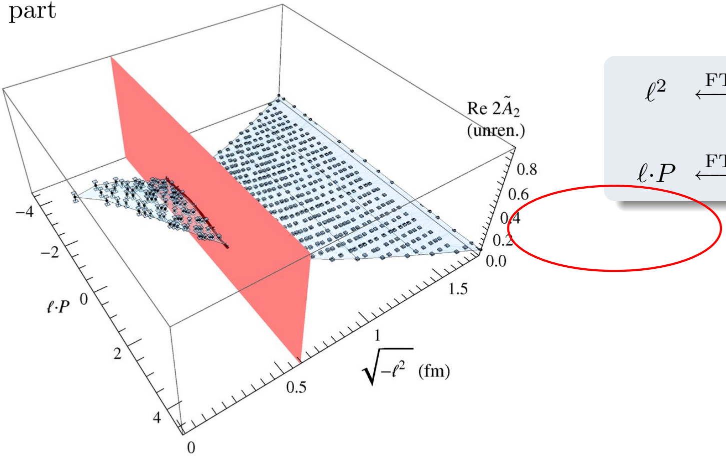
$$\rho_{LL}^{\pm}(\boldsymbol{k}_\perp) \equiv \tfrac{1}{2} f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \pm \tfrac{1}{2} g_1^{(0_x)}(\boldsymbol{k}_\perp^2)$$
$$\rho_{LL}^{+}(\boldsymbol{k}_\perp) = C_+ \exp(-\boldsymbol{k}_\perp^2/\mu_+^2)$$
$$\rho_{LL}^{-}(\boldsymbol{k}_\perp) = C_- \exp(-\boldsymbol{k}_\perp^2/\mu_-^2)$$



⇒ Asymptotic behavior at large $\boldsymbol{k}_\perp$ imposed by Gaussian ansatz; not a “lattice result”. Similar issues in analysis of experimental data.

x-dependence

real part



$$\ell^2 \overset{\text{FT}}{\longleftrightarrow} \boldsymbol{k}_\perp^2$$

$$\ell \cdot P \overset{\text{FT}}{\longleftrightarrow} x$$

factorization hypothesis

$$f_1(x, \boldsymbol{k}_\perp^2) \;\approx\; f_1(x)\;\; f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \;/\; \mathcal{N}$$

as in phenomenological applications,
e.g., Monte Carlo event generators

Then $\tilde{A}_2$ factorizes, too:

$$\tilde{A}_2(\ell^2, \ell\cdot P) = \tilde{A}_2^{\mathrm{norm}}(\ell\cdot P)\; \tilde{A}_2(\ell^2, 0).$$

To test this, we define

$$\boxed{\tilde{A}_2^{\mathrm{norm}}(\ell^2, \ell\cdot P) \equiv \frac{\tilde{A}_2(\ell^2, \ell\cdot P)}{\mathrm{Re}\ \tilde{A}_2(\ell^2, 0)}}$$

(needs no renormalization!)

If factorization holds, $\tilde{A}_2^{\mathrm{norm}}$ should be ℓ^2-independent.

factorization hypothesis

$$f_1(x, \boldsymbol{k}_\perp^2) \approx f_1(x) \; f_1^{(0_x)}(\boldsymbol{k}_\perp^2) \; / \; \mathcal{N}$$

as in phenomenological applications,
e.g., Monte Carlo event generators

Then $\tilde{A}_2$ factorizes, too:

$$\tilde{A}_2(\ell^2, \ell\cdot P) = \tilde{A}_2^{\text{norm}}(\ell\cdot P)\; \tilde{A}_2(\ell^2, 0).$$

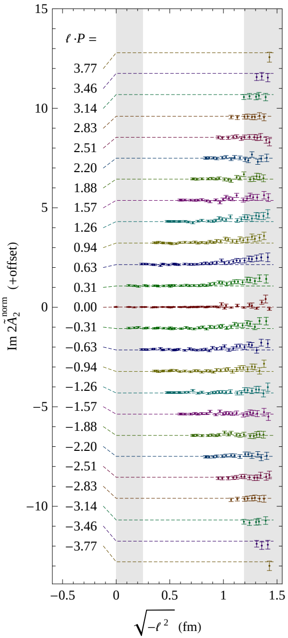
To test this, we define

$$\tilde{A}_2^{\text{norm}}(\ell^2, \ell\cdot P) \equiv \frac{\tilde{A}_2(\ell^2, \ell\cdot P)}{\text{Re}\; \tilde{A}_2(\ell^2, 0)}$$

(needs no renormalization!)

If factorization holds, $\tilde{A}_2^{\text{norm}}$ should be ℓ^2-independent.

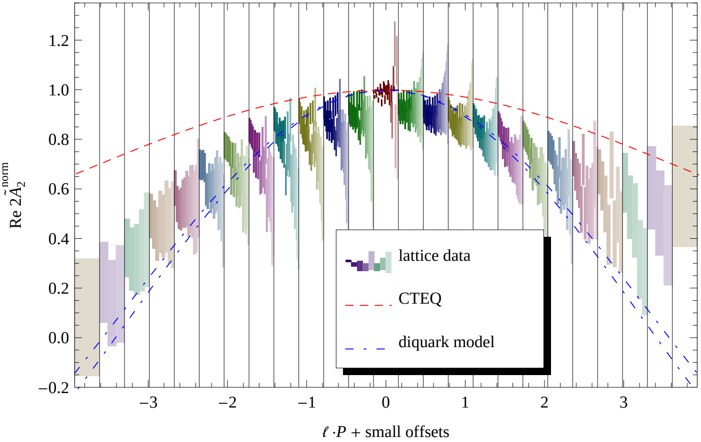
✓ within statistics



All our data for $\tilde{A}_2^{\text{norm}}(\ell^2, \ell\cdot P)$ at $m_\pi \approx 610$ MeV

qualitative comparison to

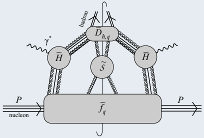
- a Fourier transform of $f_1(x)$ from CTEQ5 [Lai et al., EPJ C12, 375 (2000)]
- a scalar diquark model at $\sqrt{-\ell^2} = 0$ and 1 fm [JMR, NPA626, 937 (1997)]



Towards Extended Gauge Links

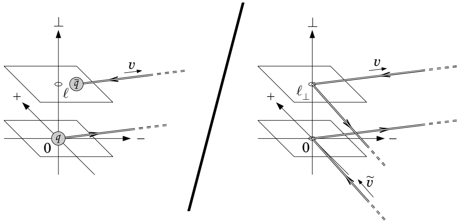
e.g., [Ji, Ma, Yuan PRD (2005)] :

$$W^{\mu\nu}_{\text{unpol.,LO}} \;\propto\; H \times f_1 \otimes D_h \otimes \underbrace{S}_{\text{soft factor}}$$



modified definition of TMD PDF correlator:

$$\Phi^{[\Gamma]}(k,P,S) \;\equiv\; \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\, \frac{\langle P,S|\;\bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\;|P,S\rangle}{\widetilde{S}(\boldsymbol{\ell}_\perp,\ldots)}$$



- gauge links slightly off lightcone: $v \neq \hat{n}_-$

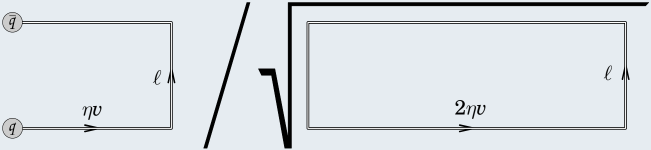
$\Rightarrow$ evolution eqn. in $\zeta \equiv (v\cdot P)^2/v^2$

- soft factor $\widetilde{S}$: vacuum expectation value of gauge link structure

subtraction factor

How to get rid of the gauge link self engergy $\exp(\delta m\, L)$?

Soft factor in TMD PDF correlator? Suggestion [Collins arXiv:0808.2665] :



Is this a meaningful definition of TMD PDFs?

prerequisite for quantitative lattice predictions

"To allow non-perturbative methods in QCD to be used to estimate parton densities, operator definitions of parton densities are needed that can be taken literally." [Collins arXiv:0808.2665 (2008)]

$\boldsymbol{k}_\perp$-moments from ratios of amplitudes ...

... may bridge the gap until we know more.

Example Sivers effect: $\langle \boldsymbol{k}_\perp \rangle_{\rho_{TU}}$ from $\tilde{A}_{12}/\tilde{A}_2$.
Self-energies cancel, no explicit subtraction factor needed.

Summary:

- We have explored ways to calculate intrinsic transverse momentum distributions in the nucleon with lattice QCD. We directly implement non-local operators on the lattice.
- First results are based on a simplified operator geometry (direct gauge link) and a Gaussian fit model, at $m_\pi \approx 500$ MeV:
    - We calculate the first Mellin moment of leading twist TMD PDFs $f_1^{(0)}(\boldsymbol{k}_\perp^2)$, $g_{1T}^{(0)}(\boldsymbol{k}_\perp^2)$, $h_{1L}^{\perp(1)}(\boldsymbol{k}_\perp^2)$ etc.
    - $\boldsymbol{k}_\perp$-densities of longitudinally polarized quarks in a transversely polarized proton are deformed, due to non-vanishing $g_{1T}^{(0)}$.
    - So far, no statistically significant incompatibility with factorization $f_1(x, \boldsymbol{k}_\perp^2) \approx f_1(x)\, f_1^{(0_x)}(\boldsymbol{k}_\perp^2)/\mathcal{N}$ detectable within the limited range of available lattice data.

Outlook:

- Beyond Gaussian fits:
  Matching to perturbative behavior at small ℓ, i.e., large $\boldsymbol{k}_\perp$.
- Study of non-straight gauge links similar as in SIDIS.
  Focus on selected $\boldsymbol{k}_\perp$-moments ($\leftrightarrow$ weighted asymmetries), until appropriate subtraction factors are better understood.

Backup Slides

$$\Phi^{[\Gamma]}(k,P,S) \equiv \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\, \langle P,S|\ \bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P,S\rangle$$

isolation of Lorentz-invariant amplitudes

compare [Mulders, Tangerman NPB (1996)]

$$\langle P,S|\ \bar{q}(\ell)\,\gamma_\mu\,\mathcal{U}\,q(0)\ |P,S\rangle = 4\,\tilde{A}_2\,P_\mu + 4i\,m_N{}^2\,\tilde{A}_3\,\ell_\mu$$

$$\begin{aligned}\langle P,S|\ \bar{q}(\ell)\,\gamma_\mu\gamma^5\,\mathcal{U}\,q(0)\ |P,S\rangle = & -4\,m_N\,\tilde{A}_6\,S_\mu \\ & -4i\,m_N\,\tilde{A}_7\,P_\mu(\ell\cdot S) \\ & +4\,m_N{}^3\,\tilde{A}_8\,\ell_\mu(\ell\cdot S)\end{aligned}$$

$$\langle P,S|\ \bar{q}(\ell)\,\ldots\,\mathcal{U}\,q(0)\ |P,S\rangle = \text{further structures (9 amplitudes in total)}$$

Transformation properties of the matrix element ($\dagger$, $\mathcal{P}$, $\mathcal{T}$) limit number of allowed structures. No $\mathcal{T}$-odd structures (Sivers function, ...) with straight gauge link.

The amplitudes fulfill $\tilde{A}_i(\ell^2,\ell\cdot P) = \left[\tilde{A}_i(\ell^2,-\ell\cdot P)\right]^*$.

$$\Phi^{[\Gamma]}(k,P,S) \equiv \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, e^{-ik\cdot\ell}\, \langle P,S|\ \bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P,S\rangle$$

isolation of Lorentz-invariant amplitudes

compare [Mulders, Tangerman NPB (1996)]

$$\langle P,S|\ \bar{q}(\ell)\,\gamma_\mu\,\mathcal{U}\,q(0)\ |P,S\rangle = 4\,\tilde{A}_2\,P_\mu + 4i\,{m_N}^2\,\tilde{A}_3\,\ell_\mu$$

$\Rightarrow f_1(x,\boldsymbol{k}_\perp^2)$

$$\langle P,S|\ \bar{q}(\ell)\,\gamma_\mu\gamma^5\,\mathcal{U}\,q(0)\ |P,S\rangle = -4\,m_N\,\tilde{A}_6\,S_\mu -4i\,m_N\,\tilde{A}_7\,P_\mu(\ell\cdot S) +4\,{m_N}^3\,\tilde{A}_8\,\ell_\mu(\ell\cdot S)$$

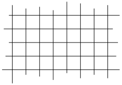
$\Rightarrow g_{1T}(x,\boldsymbol{k}_\perp^2)$

$\langle P,S|\ \bar{q}(\ell)\ \ldots\ \mathcal{U}\,q(0)\ |P,S\rangle =$ further structures (9 amplitudes in total)

Transformation properties of the matrix element ($\dagger$, $\mathcal{P}$, $\mathcal{T}$) limit number of allowed structures. No $\mathcal{T}$-odd structures (Sivers function, ...) with straight gauge link.

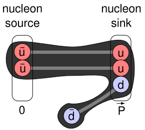
The amplitudes fulfill $\tilde{A}_i(\ell^2,\ell\cdot P) = \left[\tilde{A}_i(\ell^2,-\ell\cdot P)\right]^*$.

Ingredients	Output : 3-point correlator $C_{3\text{pt}}$

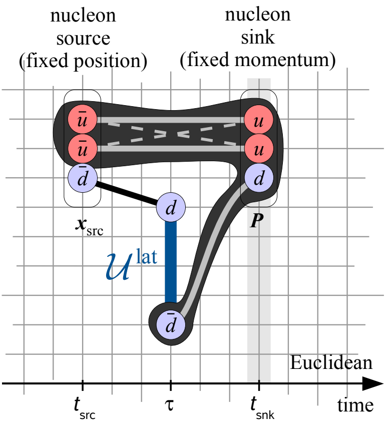


gauge configs.

quark propagators



nucleon sequential propagators



[We neglect "disconnected contributions" (absent for up minus down).]

ratio of correlators far away from nucleon source and sink

$$\frac{C_{3\mathrm{pt}}(\tau;\Gamma,\ell,P)}{C_{2\mathrm{pt}}(P)} \xrightarrow{t_{\mathrm{src}}\ll\tau\ll t_{\mathrm{sink}}} \text{const. (“plateau value”)},$$

$$\Downarrow$$

$$\text{access to } \langle P,S|\ \bar{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P,S\rangle$$

Γ	$\frac{1}{2}C^{\mathrm{ren}}_{3\mathrm{pt}}(\tau;\Gamma,\boldsymbol{\ell},\boldsymbol{P})/C_{2\mathrm{pt}}(\boldsymbol{P})$ (LHPC projectors)
$\mathbb{1}$	$\frac{m_N}{E(P)}\tilde{A}_1$
$-\gamma_4\gamma_5$	$i m_N \tilde{A}_7\,\ell_z$
γ_4	$\tilde{A}_2$
$\frac{1}{2}[\gamma_2,\gamma_4]$	$\frac{1}{E(P)}\tilde{A}_9\,P_x + \frac{i m_N^2}{E(P)}\tilde{A}_{10}\,\ell_x + \frac{m_N^2}{E(P)}\tilde{A}_{11}\,(\ell_z)^2 P_x$
...	...

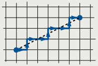
ratio of correlators far away from nucleon source and sink

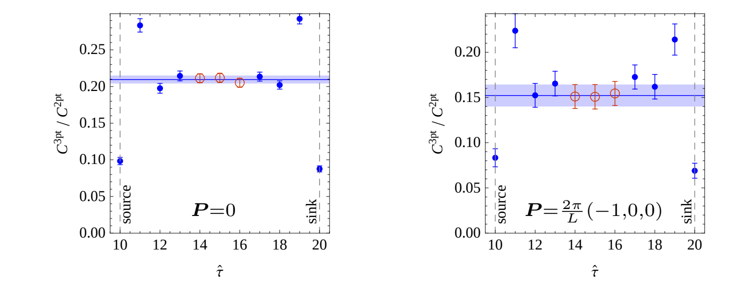
$$\frac{C_{3\mathrm{pt}}(\tau;\Gamma,\ell,P)}{C_{2\mathrm{pt}}(P)} \xrightarrow{t_{\mathrm{src}}\ll\tau\ll t_{\mathrm{sink}}} \text{const. (“plateau value”)},$$

$$\Downarrow$$

access to $\langle P,S|\ \overline{q}(\ell)\,\Gamma\,\mathcal{U}\,q(0)\ |P,S\rangle$

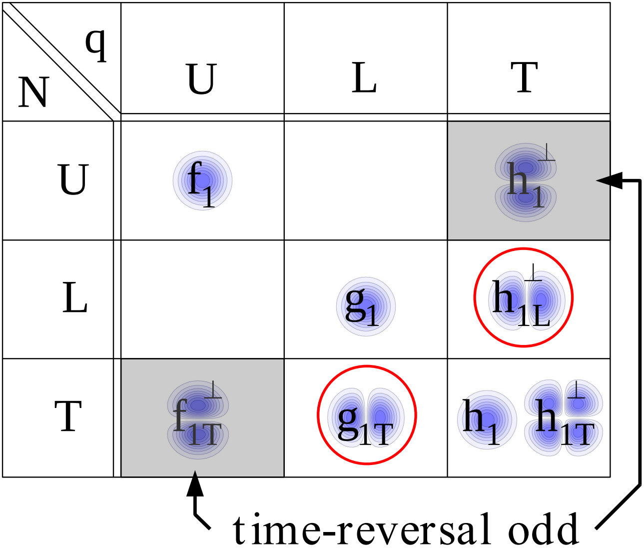
example plateau plots at $m_\pi \approx 600$ MeV

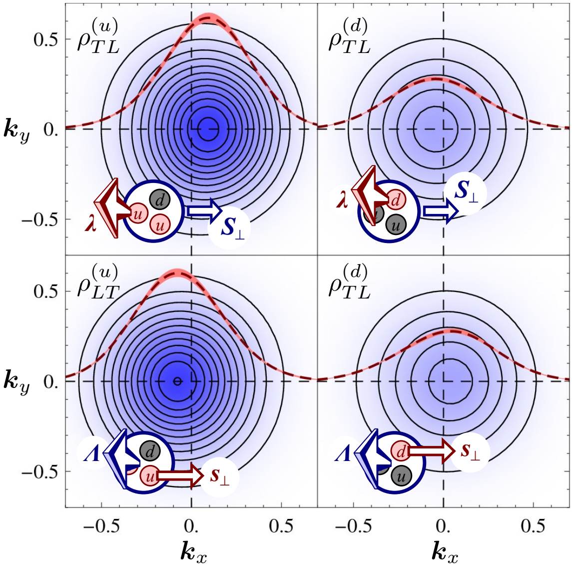
for $\Gamma = \gamma_4$ ($\Rightarrow \tilde{A}_2$), with HYP smeared gauge link $\mathcal{U} =$  :



N \ q	U	L	T
U	f_1		$h_1^{\perp}$
L		g_1	$h_{1L}^{\perp}$
T	$f_{1T}^{\perp}$	g_{1T}	h_1 $h_{1T}^{\perp}$

time-reversal odd


q
N
U
L
T
U
f_1
$h_1^\perp$
L
g_1
$h_{1L}^\perp$
T
$f_{1T}^\perp$
g_{1T}
h_1
$h_{1T}^\perp$
time-reversal odd



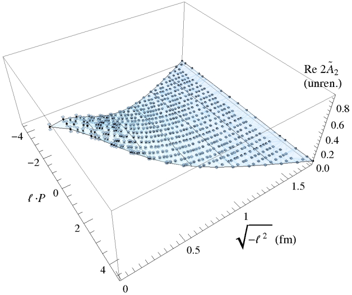
Diplole deformations

$\rho_{TL} : \sim \lambda\, \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp \;\; g_{1T}$
$\rho_{TL} : \sim \Lambda\, \boldsymbol{k}_\perp \cdot \boldsymbol{s}_\perp \;\; h_{1L}^{\perp}$

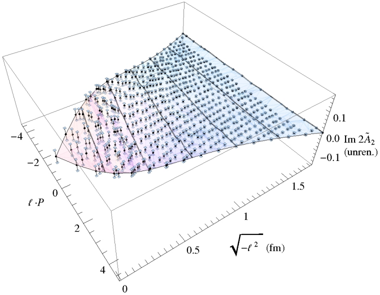
The corresponding dipole structures $\sim \lambda\, \boldsymbol{b}_\perp \cdot \boldsymbol{S}_\perp$, $\sim \Lambda\, \boldsymbol{b}_\perp \cdot \boldsymbol{s}_\perp$ for impact parameter densities (from GPDs) are ruled out by symmetries.

[Hägler, Musch, Negele, Schäfer EPL 88, 61001 (2009)]

$2 \text{ Re } \tilde{A}_2(\ell^2, \ell \cdot P)$

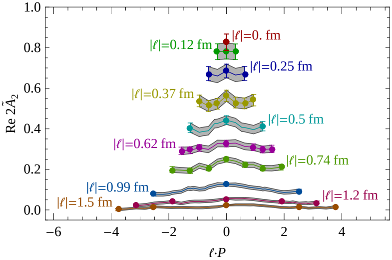


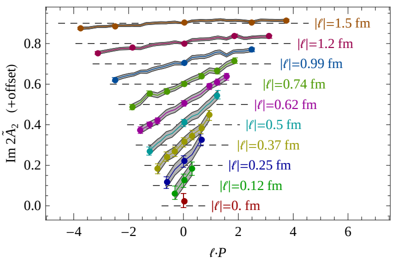
$2 \text{ Im } \tilde{A}_2(\ell^2, \ell \cdot P)$



2 Re $\tilde{A}_2(\ell^2, \ell\cdot P)$

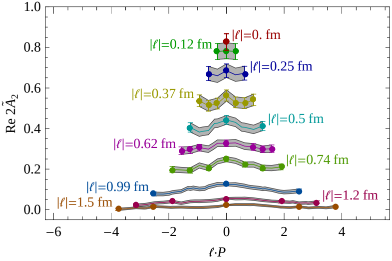
2 Im $\tilde{A}_2(\ell^2, \ell\cdot P)$

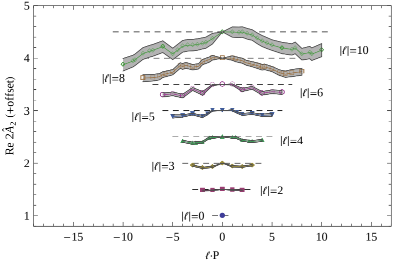




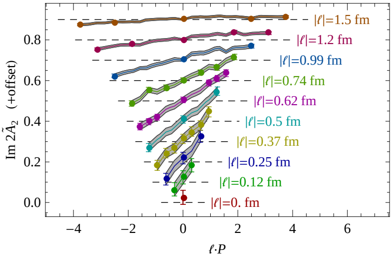
$$2\ \mathrm{Re}\ \tilde{A}_2(\ell^2, \ell \cdot P)$$

$$\mathrm{Re}\ \tilde{A}_2^{\mathrm{norm}} = \frac{\mathrm{Re}\ \tilde{A}_2(\ell^2, \ell \cdot P)}{\mathrm{Re}\ \tilde{A}_2(\ell^2, 0)}$$

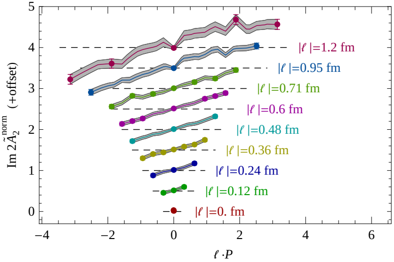


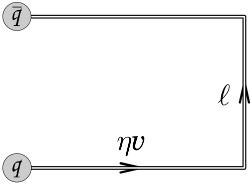


$$2\ \mathrm{Im}\ \tilde{A}_2(\ell^2, \ell\cdot P)$$



$$\mathrm{Im}\ \tilde{A}_2^{\mathrm{norm}} = \frac{\mathrm{Im}\ \tilde{A}_2(\ell^2, \ell\cdot P)}{\mathrm{Re}\ \tilde{A}_2(\ell^2, 0)}$$





32 Lorentz-invariant amplitudes [Goeke,Metz,Schlegel PLB618,90 (2005)]

$$A_i\left(k^2, k\cdot P, \frac{v\cdot k}{|v\cdot P|}, \frac{v^2}{|v\cdot P|^2}, \frac{v\cdot P}{|v\cdot P|}\right) = A_i\left(k^2, k\cdot P, \underbrace{\frac{v\cdot k}{|v\cdot P|}}_{\approx\, x}, \zeta^{-1}, \mathrm{sgn}(v\cdot P)\right)$$

Links approaching light cone: $v \to \hat{n}_- \Rightarrow \zeta \to \infty$. For large ζ, the evolution with ζ is known [Collins,Soper NPB194,445 (1981)].

$$\left.\begin{array}{r}(v^0, v^1, v^2, v^3)\\ \text{future pointing } v\\ \text{TMD PDFs for SIDIS}\end{array}\right\} \xrightarrow{\mathcal{T}} \left\{\begin{array}{l}(-v^0, v^1, v^2, v^3)\\ \text{past pointing } v\\ \text{TMD PDFs for Drell-Yan}\end{array}\right.$$

The transformation property of the matrix elements under time reversal provides relations:

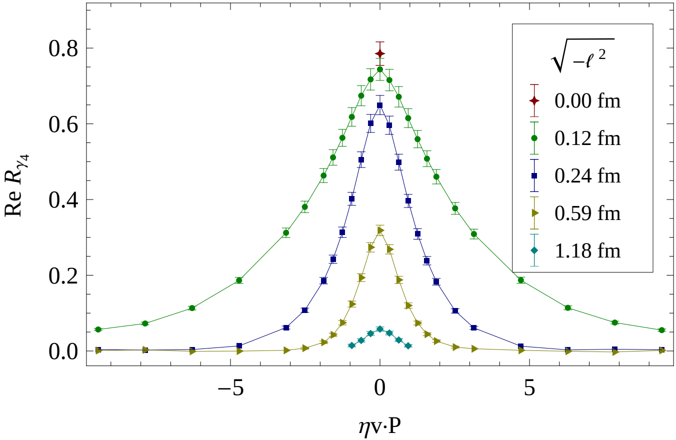
Example of a $\mathcal{T}$-even amplitude:

$$A_2\left(k^2, k\cdot P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, 1\right) = A_2\left(k^2, k\cdot P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, -1\right)$$
$$\Downarrow$$
$$f_1^{(\text{SIDIS})}(x, \boldsymbol{k}_\perp; \zeta, \ldots) = f_1^{(\text{Drell-Yan})}(x, \boldsymbol{k}_\perp; \zeta, \ldots)$$

Example of a $\mathcal{T}$-odd amplitude: ($\rightarrow$ Sivers function $f_{1T}^\perp$)

$$A_{12}\left(k^2, k\cdot P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, 1\right) = -A_{12}\left(k^2, k\cdot P, \tfrac{v\cdot k}{v\cdot P}, \zeta^{-1}, -1\right)$$
$$\Downarrow$$
$$f_{1T}^{\perp(\text{SIDIS})}(x, \boldsymbol{k}_\perp; \zeta, \ldots) = -f_{1T}^{\perp(\text{Drell-Yan})}(x, \boldsymbol{k}_\perp; \zeta, \ldots)$$

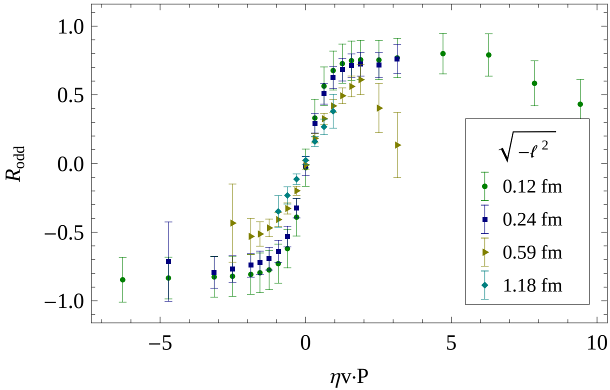
$$\tilde{A}_2\left(\ell^2, \ell\cdot P, \frac{v\cdot\ell}{|v\cdot P|}, \zeta^{-1}, \mathrm{sgn}(v\cdot P)\right) \equiv \lim_{\eta\to\infty} \tilde{a}_2(\ell^2, \ell\cdot P, \eta v\cdot\ell, -\eta^2, \eta v\cdot P)$$



But $\tilde{a}_2 = \mathrm{Re}\, R_{\gamma_4}$ always vanishes for large η!
Reason: power divergence suppresses $\tilde{a}_2 \sim \exp(-\delta m\, \eta)$.

$$R_{\text{odd}} = \frac{\tilde{a}_{12} + (\eta \frac{m_N^2 v_1}{P_1})\,\tilde{b}_8}{\tilde{a}_2}$$

$$\xrightarrow{\pm\eta v\cdot P \text{ large}} \frac{\tilde{A}_{12}(\ell^2,0,0,\zeta^{-1},\pm1) + \left(\frac{m_N}{P_1}\right)^2\ \tilde{B}_8(\ell^2,0,0,\zeta^{-1},\pm1)}{\tilde{A}_2(\ell^2,0,0,\zeta^{-1},\pm1)}$$



Part of the effect comes from the Sivers function $f_{1T}^{\perp}$ via $\tilde{A}_{12}$!