V_{us} Calculation from Lattice QCD

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Outline

- Motivation/Background
- Review of Lattice Calculations
- New Dynamical Result from Hyperons
- Summary/Outlook

$|V_{us}|$ and the CKM Matrix

- Kobayashi and Maskawa (1973) propose extending Cabibbo's (1963) work with a mixing of three generations of quarks:

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

- Unitarity constraint: $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.$
- From the latest PDG (2006)
  - Well determined $|V_{ud}| = 0.97377(27)$
  - Very small $|V_{ub}| = 4.31(30) \times 10^{-3}$
  - Less known $|V_{us}| = 0.2257(21)(2006)$
    $|V_{us}| = 0.2196(23)(2003)$

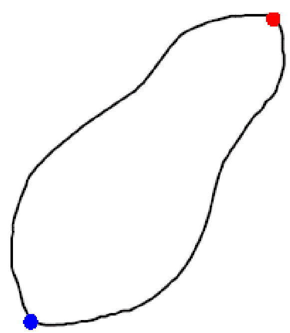
Background: Lattice 101

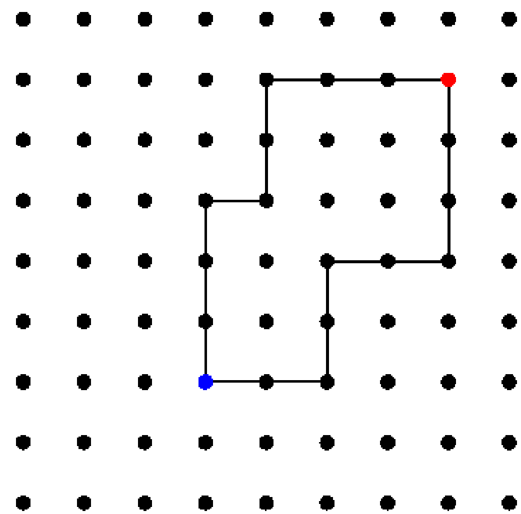
Lattice Gauge Theory

- Physical observables are calculated from the path integral

$$\langle \Omega | O | \Omega \rangle = \frac{1}{Z} \int [dU][d\overline{\psi}][d\psi] O(U, \overline{\psi}, \psi) e^{i \int d^4x [S_F(U, \overline{\psi}, \psi) + S_G(U)]}$$

- Lattice QCD is a discrete version of continuum QCD theory





- Use Monte Carlo integration combined with the "importance sampling" technique to calculate the path integral.

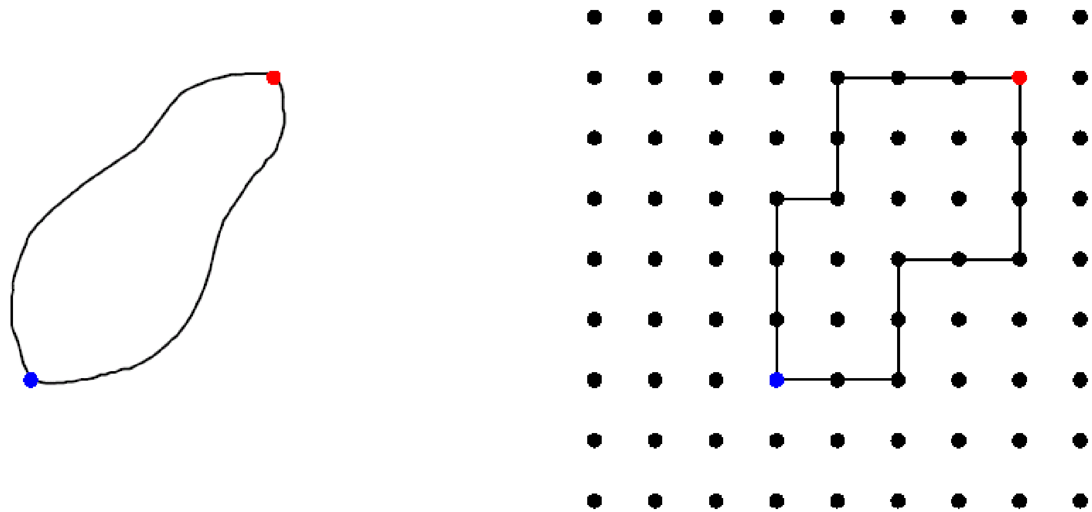
- Take $a \rightarrow 0$ and $V \rightarrow \infty$ for the "continuum limit"

Lattice Gauge Theory

- Physical observables are calculated from the path integral

$$\langle\Omega|O|\Omega\rangle = \frac{1}{Z}\int [dU][d\overline{\psi}][d\psi]O(U,\overline{\psi},\psi)e^{i\int d^4x[S_F(U,\overline{\psi},\psi)+S_G(U)]}$$

- Lattice QCD is a discrete version of continuum QCD theory



- Use Monte Carlo integration combined with the "importance sampling" technique to calculate the path integral.

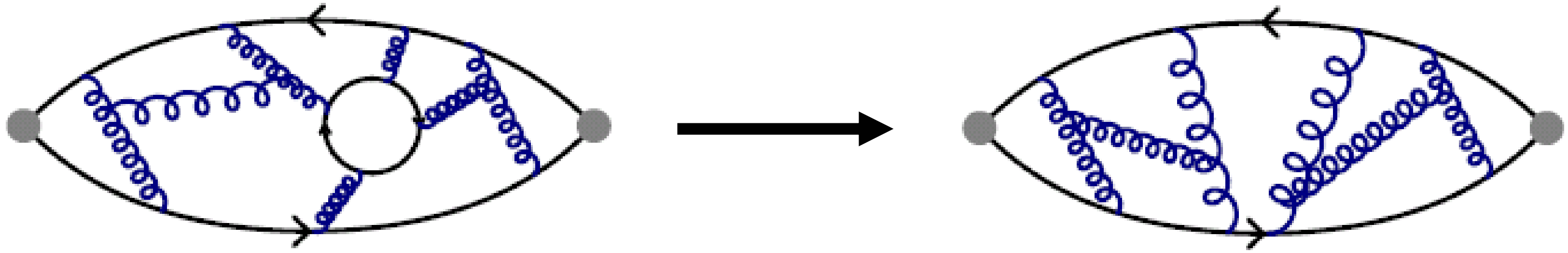
- Chiral extrapolation $m_\pi \to (m_\pi)_{phys}$

Lattice Actions

- (Improved) Staggered fermions (asqtad):
  - Relatively cheap for dynamical fermions (good)
  - Mixing among parities and flavours or "tastes"
  - Baryonic operators a nightmare — not suitable
- $\mathcal{O}(a)$-improved Wilson (Clover) fermions:
  - Moderate in cost
  - Chiral symmetry badly broken at non-zero lattice spacing
  - Operator mixing issues
- Chiral fermions (e.g., Domain-Wall/Overlap):
  - Automatically $O(a)$ improved, suitable for spin physics and weak matrix elements
  - Expensive
- Mixed actions:
  - Staggered sea (cheap) with Domain-Wall valence (chiral)
  - Match the sea Goldstone pion mass to the DWF pion

Quenched Approximation

- Correct Math $$\langle O \rangle = \frac{1}{Z}\int [dU][d\psi][d\overline{\psi}]e^{-S_F(U,\psi,\overline{\psi})-S_G(U)}O(U,\psi,\overline{\psi})$$ $$= \frac{1}{Z}\int [dU]\det M e^{-S_G(U)}O(U)$$
- Instead, take det M = constant.



- Historically due to the lack of computation power and improved algorithms
- Bad: Uncontrollable systematic error
- Good? Cheap exploratory studies to develop new methods

Review of Lattice Calculations

What can Lattice QCD do for V_{us}?

- Leptonic decay ratios $0.2234(^{+12}_{-31})$

- K_{l3} decay $0.2257(21)(2006)$

$$\bar{K} \to \pi l \nu$$

- Hyperon decays $0.2250(27)$

$$\Lambda \to p e^- \overline{\nu},\ \Sigma^- \to n e^- \overline{\nu},$$
$$\Xi^- \to \Lambda e^- \overline{\nu},\ \Xi^0 \to \Lambda^+ e^- \overline{\nu}$$

Leptonic Decays

- $K_{\mu 2}$ and $\pi_{\mu 2}$ decays

$$\left(\frac{|V_{us}|}{|V_{ud}|}\right)^2 = \left[\left(\frac{f_K}{f_\pi}\right)^2 \frac{M_K\left(1-m_\mu^2/M_K^2\right)^2}{M_\pi\left(1-m_\mu^2/M_\pi^2\right)^2}\left(1+\frac{\alpha}{\pi}\left(C_K-C_\pi\right)\right)\right]^{-1} \frac{\Gamma(K\to\mu\bar{\nu}_\mu)}{\Gamma(\pi\to\mu\bar{\nu}_\mu)}$$

- MILC collaboration (staggered, *C. Aubin et al., 2004*)

  f_K/f_π=1.210(4)(13)

- *W. Marciano, 2004* $|V_{us}|$ = 0.2219(25)
- Other full-QCD f_K/f_π available since 2004

  RBC+UKQCD *DWF*: f_K/f_π = 1.24(2)

  MILC 2006: f_K/f_π=1.208(2)($^{+7}/_{-14}$)

K_{l3} Decay

- Calculate $K \rightarrow \pi$ matrix element
- Lorentz invariance

$$\langle \pi(p') | V_\mu | K(p) \rangle = (p_\mu + p'_\mu) f_+(q^2) + (p_\mu - p'_\mu) f_-(q^2)$$

- Double ratio $\dfrac{\langle \pi | \bar{s}\gamma_0 u | K \rangle \langle K | \bar{u}\gamma_0 s | \pi \rangle}{\langle K | \bar{s}\gamma_0 s | K \rangle \langle \pi | \bar{u}\gamma_0 u | \pi \rangle} = [f_0(q^2_{\max})]^2 \dfrac{(M_K + M_\pi)^2}{4 M_K M_\pi}$

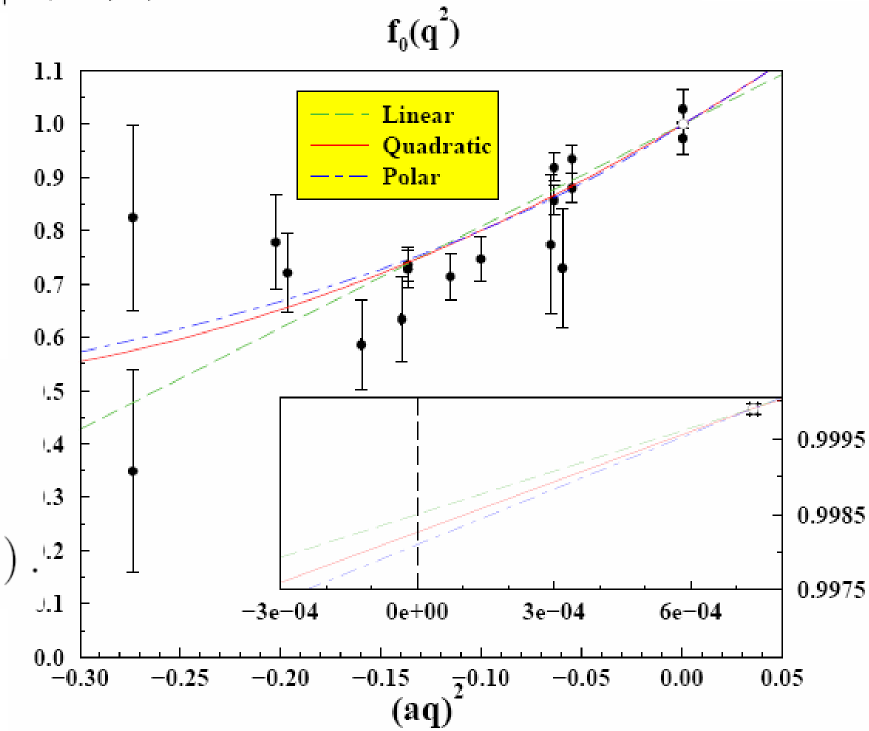
with

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2)$$

- Extrapolate to $q^2 = 0$

$$\begin{aligned} f_0(q^2) &= f_0^{(pol.)}(0)/(1 - \lambda_0^{(pol.)}\, q^2)\,, \\ f_0(q^2) &= f_0^{(lin.)}(0) \cdot (1 + \lambda_0^{(lin.)}\, q^2)\,, \\ f_0(q^2) &= f_0^{(quad.)}(0) \cdot (1 + \lambda_0^{(quad.)}\, q^2 + c_0\, q^4)\,. \end{aligned}$$

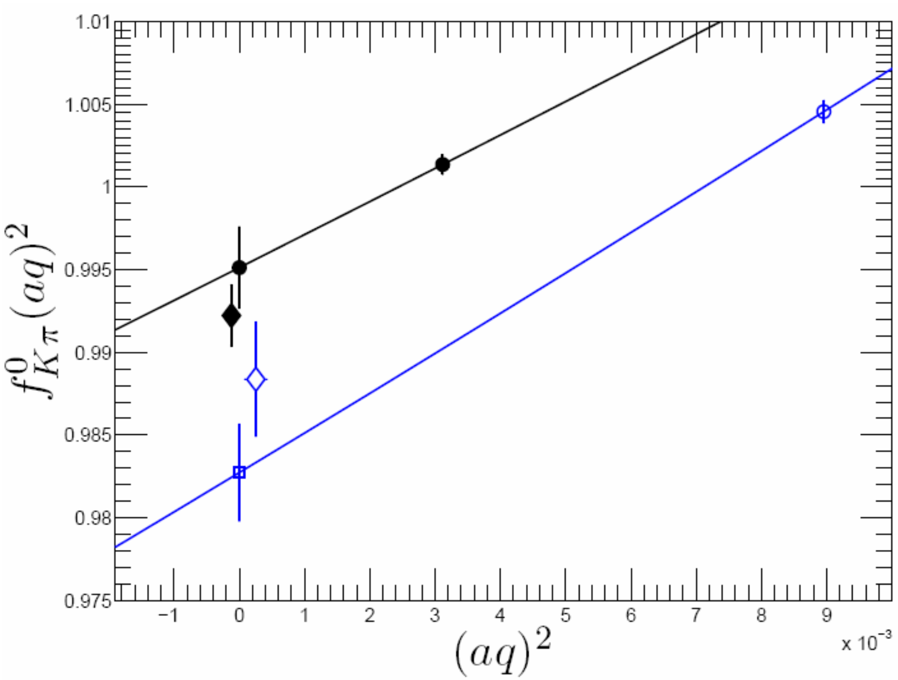
Guadagnoli et. al. (2004)



K_{l3} Decay

- Twisted-BC approach $\widetilde{\psi}(x+\hat{e}_j L) = e^{2\pi i\theta_j}\,\widetilde{\psi}(x)$

  momentum changes by $\widetilde{p}_j = \theta_j \frac{2\pi}{L} + n_j \frac{2\pi}{L}$
- Tune θ_j to cancel out the mass difference.
- No extrapolation in momentum is needed!



With 1.5 times the statistics, the results are compatible with the conventional ones

J.M. Flenn et al. (2007)

$K_{\ell 3}$ Decay

- Chiral extrapolation: Ademollo-Gatto Theorem
  - SU(3) symmetry-breaking Hamiltonian

$$H' = \frac{1}{\sqrt{3}}\left(m_s - \frac{m_d + m_u}{2}\right)\bar{q}\lambda^8 q$$

  - There is no first-order correction $O(H')$ to $f_1(0)$; thus

$$f_1(0) = f_1^{SU(3)}(0) + O(H'^2)$$

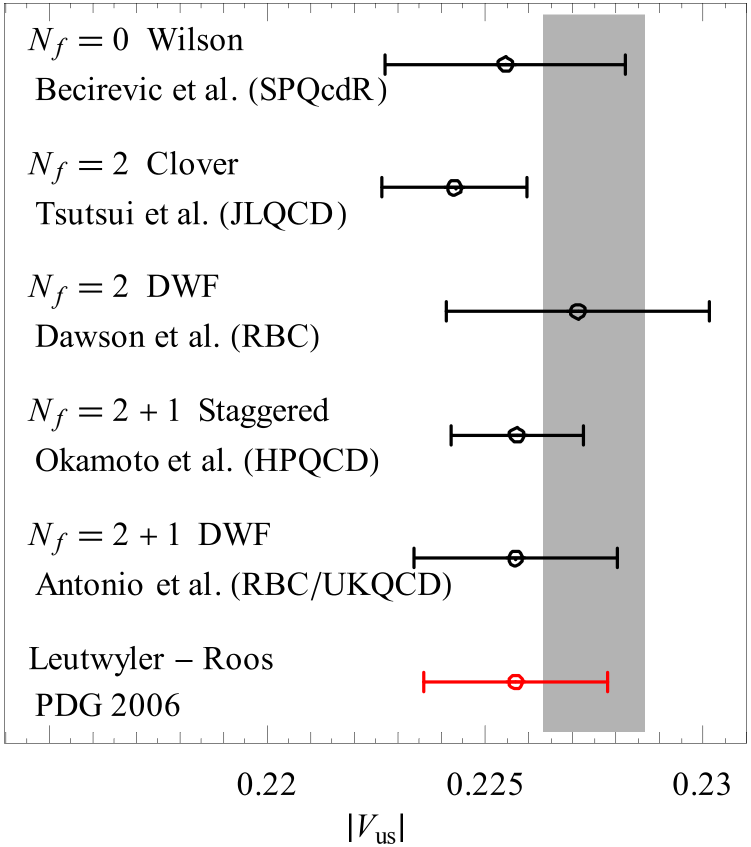
  - Common choice of observable for H' : $M_K^2 - M_\pi^2$
  - Step I: $R(M_K, M_\pi) = \dfrac{1 - |f'(0)|}{a^4(M_K^2 - M_\pi^2)^2}$
  - Step II: $R(M_K, M_\pi) = b_0 + b_1 a^2 (M_K^2 + M_\pi^2)$
- Obtain $|V_{us}|$ from

$$\Gamma(K_{\ell 3}) = \frac{G_F^2 M_K^5}{128\pi^3}|V_{us}|^2 S_{\rm ew}|f_+^{K^0\pi^-}(0)|^2 C_K^2 I_K^\ell(\lambda_i)\left[1 + \delta_{SU(2)}^K + \delta_{\rm em}^{K\ell}\right]^2$$

K_{l3} Decay

- Summary of K_{l3} calculations
  - Use common $|f_+ V_{us}| = 0.2169(9)$ from PDG 2006

Group	N_f	S_f	M_π (GeV)
SPQcdR	0	Wilson	0.500–1.000
JLQCD	2	Clover	0.440–0.960*
RBC	2	DWF	0.475–0.700
HPQCD	2+1	Staggered	0.500–0.700
RBC/UKQCD	2+1	DWF	0.390–0.700



Hyperon Decays

- Matrix element of the hyperon β-decay process $B_1 \to B_2 e^- \overline{\nu}$

$$\mathcal{M} = \frac{G_s}{\sqrt{2}} \overline{u}_{B_2} (O_\alpha^{\rm V} + O_\alpha^{\rm A}) u_{B_1} \overline{u}_e \gamma^\alpha (1+\gamma_5) v_\nu$$

with

$$O_\alpha^{\rm V} = f_1(q^2)\gamma^\alpha + \frac{f_2(q^2)}{M_{B_1}} \sigma_{\alpha\beta} q^\beta + \frac{f_3(q^2)}{M_{B_2}} q_\alpha$$

$$O_\alpha^{\rm A} = \left(g_1(q^2)\gamma^\alpha + \frac{g_2(q^2)}{M_{B_1}} \sigma_{\alpha\beta} q^\beta + \frac{g_3(q^2)}{M_{B_2}} q_\alpha \right) \gamma_5$$

- The vector form factor $f_1(0)$ links to $|V_{us}|$

$$\Gamma = G_F^2 |V_{us}|^2 \frac{\Delta m^5}{60\pi^3} (1+\delta_{\rm rad}) \times \left[\left(1 - \frac{3}{2}\beta\right) \left(|f_1|^2 + |g_1|^2\right) + \frac{6}{7}\beta^2 \left(|f_1|^2 + 2|g_1|^2 + {\rm Re}(f_1 f_2^\star) + \frac{2}{3}|f_2^2| \right) + \delta_{q^2} \right]$$

with g_1/f_1 (exp) and f_2/f_1 (SU(3) value)

Hyperon Decay Experiments

- Experiments: CERN WA2, Fermilab E715, BNL AGS, Fermilab KTeV, CERN NA48
- Summary *N. Cabibbo et al. 2003*

  with f_2/f_1 and f_1 at the SU(3) limit

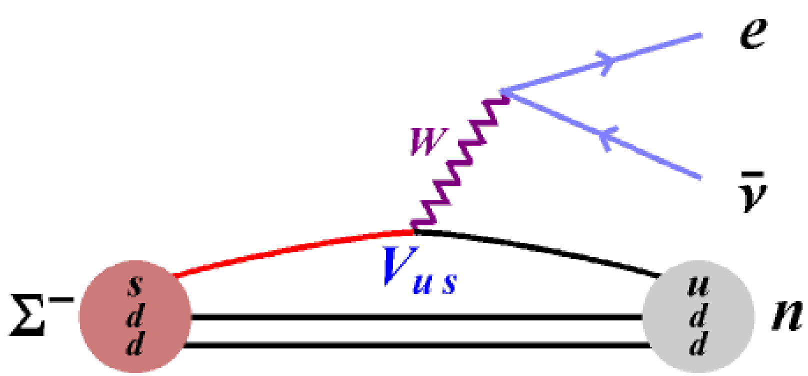
Decay	Rate (μs^{-1})	g_1/f_1	V_{us}
$\Lambda \to p e^- \bar{\nu}$	3.161(58)	0.718(15)	0.2224 ± 0.0034
$\Sigma^- \to n e^- \bar{\nu}$	6.88(24)	−0.340(17)	0.2282 ± 0.0049
$\Xi^- \to \Lambda e^- \bar{\nu}$	3.44(19)	0.25(5)	0.2367 ± 0.0099
$\Xi^0 \to \Sigma^+ e^- \bar{\nu}$	0.876(71)	1.32(+.22/−.18)	0.209 ± 0.027
Combined	—	—	0.2250 ± 0.0027

PDG 2006 number

- Better g_1/f_1 from lattice calculations?

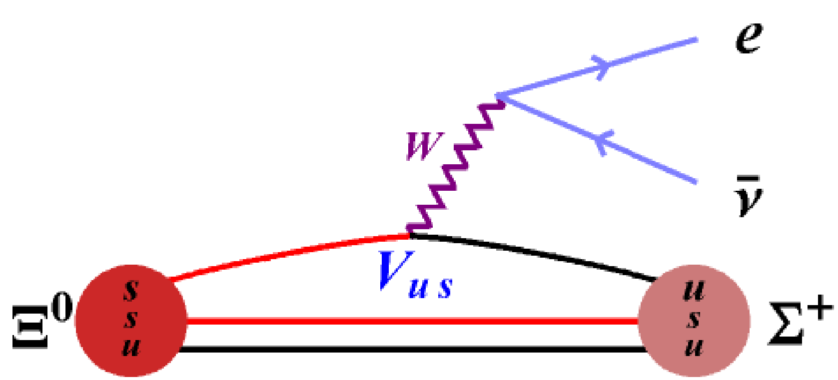
$|V_{us}|$ — Lattice Hyperons

- Two quenched calculations, different channels



- Pion mass > 700 MeV
- $f_1(0) = -0.988(29)_{\text{stat.}}$
- $|V_{us}| = 0.230(5)_{\text{exp}}(7)_{\text{lat}}$

Guadagnoli et al.



- Pion mass ≈ 530–650 MeV
- $f_1(0) = 0.953(24)_{\text{stat}}$
- $|V_{us}| = 0.219(27)_{\text{exp}}(5)_{\text{lat}}$

Sasaki et al.

No systematic error estimate from **quenching** effects!

Parameters

- This calculation:
  - Mixed action (staggered sea with DWF valence)
  - Pion mass range: 360–700 MeV
  - Strange-strange Goldstone fixed at 763(2) MeV
  - Volume fixed at 2.6 fm
  - $a \approx 0.125$ fm, $L_s = 16$, $M_5 = 1.7$
  - HYP-smeared gauge, box size of $20^3 \times 32$

Label	m_π (MeV)	m_K (MeV)	$\Sigma^- \to n$ conf.
m010	358(2)	605(2)	600
m020	503(2)	653(2)	420
m030	599(1)	688(2)	561
m040	689(2)	730(2)	306

Construction

- Two-point function

$$\Gamma^{NN}_{AB}(t_i,t_f,\vec{p}\,;T) \to a^6 Z^N_B(p) Z^N_A(p) \sum_s T_{\alpha\beta}\bar{u}_\alpha(\vec{p},s)u_\beta(\vec{p},s) \times \frac{m_N}{E_N(\vec{p})} e^{-(t_f-t_i)E_N(\vec{p})} = \left(\frac{E_N(\vec{p})+m_N}{2E_N(\vec{p})}\right) e^{-(t_f-t_i)E_N(\vec{p})}$$

- Three-point function

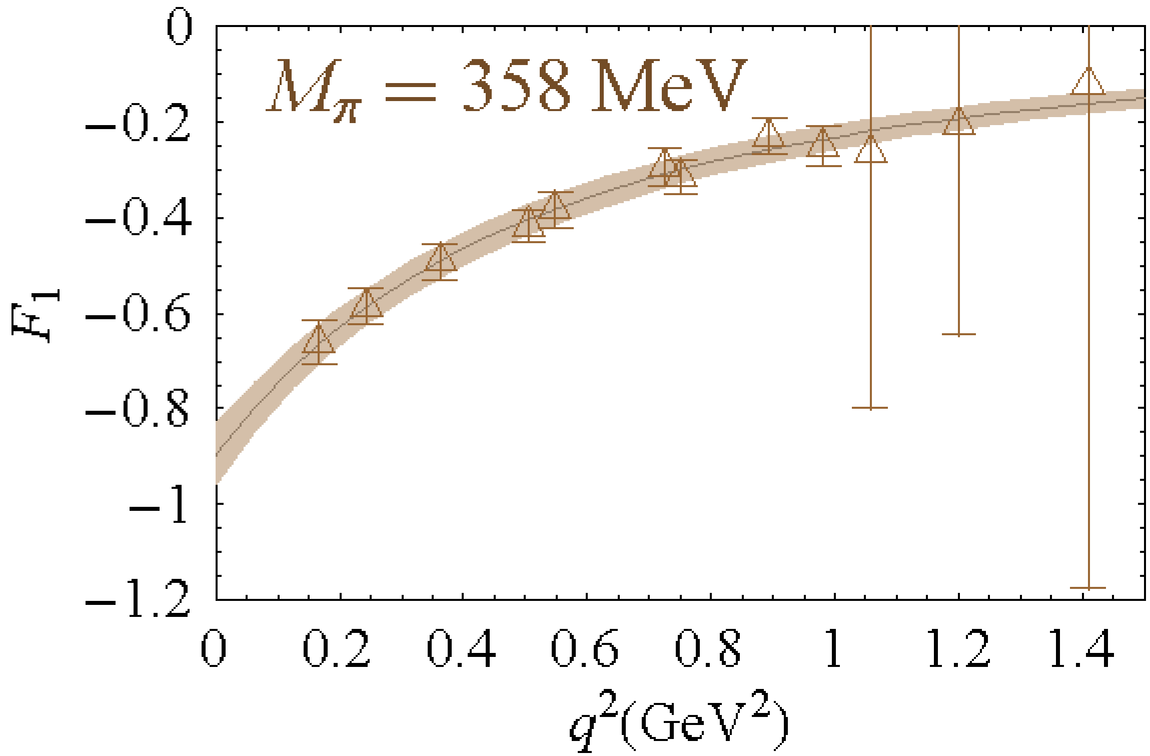
$$\Gamma^{NN}_{\mu,AB}(t_i,t,t_f,\vec{p}_i,\vec{p}_f;T) \to = \frac{m_N^2}{E_N(\vec{p}_f)E_N(\vec{p}_i)} e^{-(t_f-t)E_N(\vec{p}_f)} e^{-(t-t_i)E_N(\vec{p}_i)} \sum_{s,s'} T_{\alpha\beta} Z_B(p_f) u_\beta(\vec{p}_f,s') \langle N(\vec{p}_f,s')\,|j_\mu(0)|\,N(\vec{p}_i,s)\rangle \bar{u}_\alpha(\vec{p}_i,s) Z_A(p_i)$$

Ratio cancels out *t* and *Z* dependence

$$R_{j_\mu} = \frac{Z_V \Gamma^{\Sigma N}_{\mu,GG}(t_i,t,t_f,\vec{p}_i,\vec{p}_f;T)}{\Gamma^{NN}_{GG}(t_i,t_f,\vec{p}_f;T)} \sqrt{\frac{\Gamma^{\Sigma\Sigma}_{PG}(t,t_f,\vec{p}_i;T)}{\Gamma^{NN}_{PG}(t,t_f,\vec{p}_f;T)}} \times \sqrt{\frac{\Gamma^{NN}_{GG}(t_i,t,\vec{p}_f;T)}{\Gamma^{\Sigma\Sigma}_{GG}(t_i,t,\vec{p}_i;T)}} \sqrt{\frac{\Gamma^{NN}_{PG}(t_i,t_f,\vec{p}_f;T)}{\Gamma^{\Sigma\Sigma}_{PG}(t_i,t_f,\vec{p}_i;T)}},$$

Momentum Extrapolation

- Fit to the dipole form



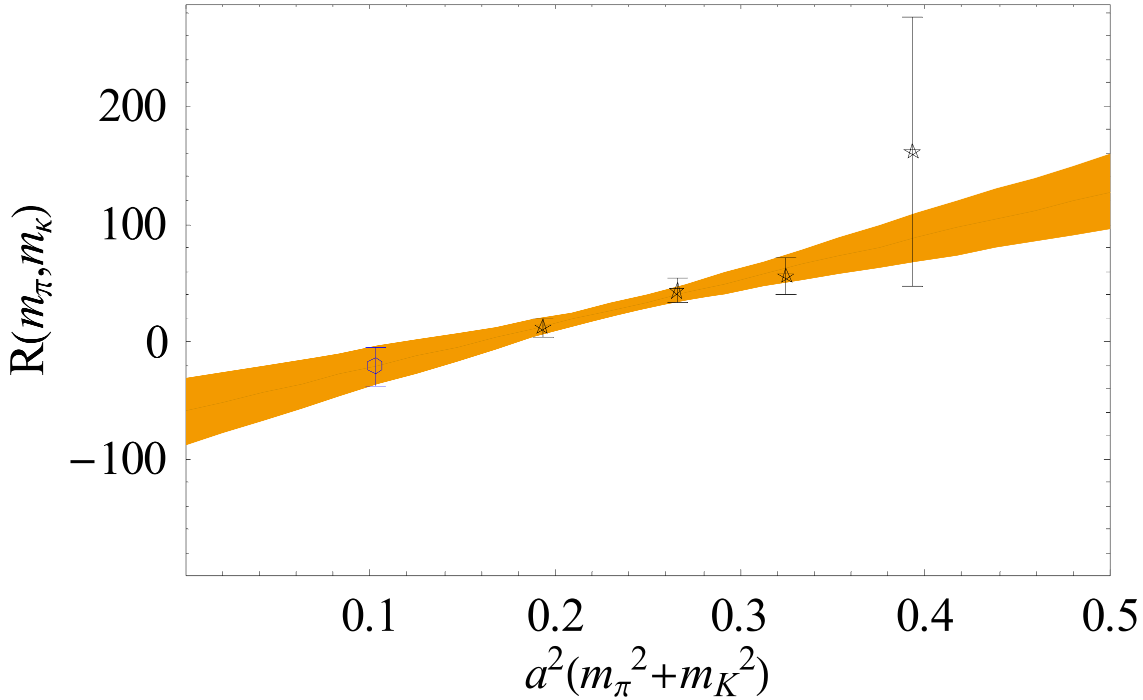
Mass Dependence — I

- Construct an Ademollo-Gatto ratio

$$R(M_K, M_\pi) = \frac{1 - f_1(0)}{a^4(M_K^2 - M_\pi^2)^2}$$

and extrapolate mass dependence as

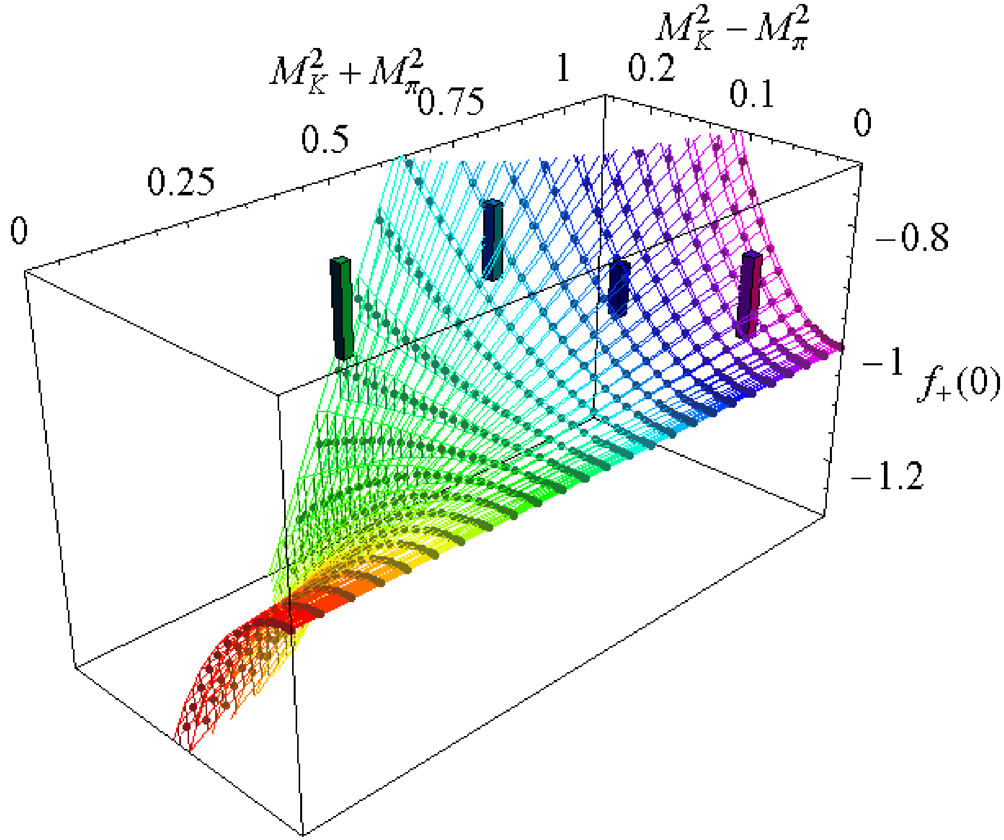
$$R(M_K, M_\pi) = b + ca^2(M_K^2 + M_\pi^2)$$



Mass Dependence — II

- Do a mass extrapolation as

$$f_1(0) = -1 + (b_0 + b_1 a^2 (M_K^2 + M_\pi^2)) \times a^4 (M_K^2 - M_\pi^2)^2$$



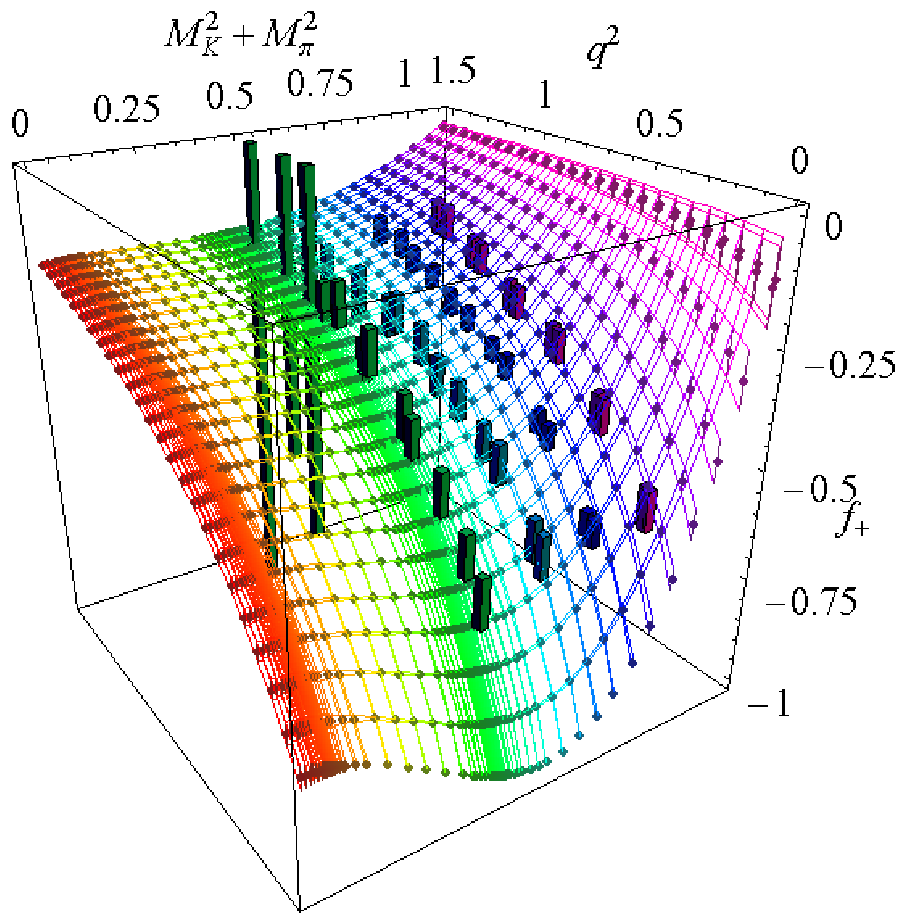
Simultaneous Fit

- Combine the momentum and mass extrapolation into one fitting form

$$f_+(q^2) = \frac{1 + \left(M_K^2 - M_\pi^2\right)^2 \left(A_1 + A_2\left(M_K^2 + M_\pi^2\right)\right)}{\left(1 - \frac{q^2}{M_0 + M_1\left(M_K^2 + M_\pi^2\right)}\right)^2}$$

Simultaneous Fit

- Combined momentum and mass dependence



$f_1(0) = -0.88(15)$ (Preliminary)

A Remark

- Current $|V_{us}|$ number from hyperons *N. Cabibbo et al. 2003* used f_2/f_1 at the SU(3) limit and g_2 taken to be zero
  - f_2/f_1

Guadagnoli et al.	−0.85(45),
This Work	–0.94(4), consistent w/ Exp.: –0.96(7)
SU(3) limit	−1.297

  - $g_2 = 0$? affects the actual g_1 values

    Experiment good at $|g_1/f_1 - 0.237 g_2/f_1|$

    Guadagnoli et al $g_2/f_1 = 0.37(14)$

  This could increase the $|f_1 V_{us}|$

- Will the revised calculation in $|V_{us}|$ be consistent with the result from K_{l3}?

Summary/Outlook

- Taking f_K/f_π (MILC 2006) from lattice,

$$|V_{us}| = 0.2226(^{+26}/_{-15})$$

- f_+ from best lattice calculation (RBC+UKQCD 2+1f) with proper systematic error estimation

$$|V_{us}| = 0.2257(23)$$

- More work needs to be done in hyperon channels
  - Preliminary result from Lin-Orginos is consistent with the previous calculation
  - We need much higher statistics for a lighter-pion mass calculation (compared with the quenched one)
  - Change the calculation to Clover action to nail down this problem
  - Higher precision g_1/f_1: Will make the $|V_{us}|$ equivalently or better than the one from K_{l3} channel