Relativistic Description of Few-Nucleon Systems

LECTURES: 2007 PRAGUE SUMMER SCHOOL

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Lecture II

Theory: Two and Three Nucleon Systems

- ★ **Introduction to the Covariant Spectator Theory.**
- ★ **How are the bound state and scattering equations obtained? What are the normalization conditions?**

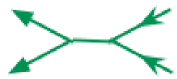
Jefferson Lab

Outline

- ★ Theoretical assumptions
- ★ CS equations for two-body systems
- ★ Equivalence of two-body BS and CS equations
- ★ Effective theory; estimate of the bound state mass
- ★ The one-body limit and cancellations
- ★ Freeze-out of nucleon resonances in the CS theory
- ★ Normalization condition for two-body relativistic bound states
- ★ Three-body CS equation
- ★ Conclusions

Theoretical Assumptions

- ★ Elementary particles (those in the Lagrangian) produce poles in the scattering amplitude



$$M \sim \frac{g^2}{m^2 - s}$$

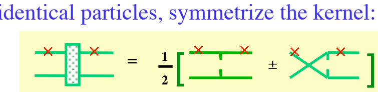
- ★ Nuclei are not elementary (comment: in some, very low energy EFT calculations, they may be treated as effective particles). No single Feynman diagram will have the bound state pole; it must be generated from an infinite sum of Feynman diagrams, much as the geometric series generates a pole:

$$z + z^2 + \cdots = \frac{z}{1-z}$$

  Therefore we must sum up infinite series of diagrams in order to treat nuclear bound states.

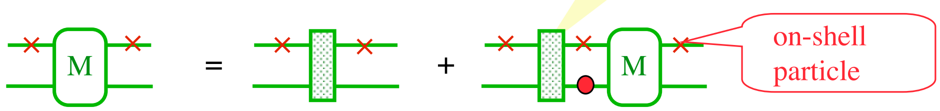
- ★ Nuclei arise from the *NN* (and *NNN*) interactions.
- ★ Nucleon resonances are frozen out (i.e. they do not needed to be treated dynamically, but can be put into the interaction).

for identical particles, symmetrize the kernel:

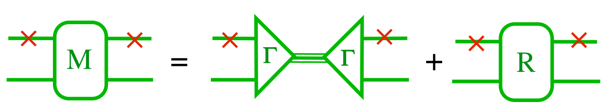


CS equations for 2-body systems

- ★ scattering amplitudes: an infinite sum of interactions *V*

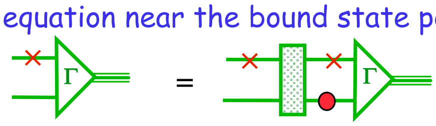


- ★ if a bound state exists, there is a pole in the scattering amplitude



residue: finite at the pole

- ★ equation for the bound state vertex functions: obtained from the scattering equation near the bound state pole



- ★ the bound state normalization condition follows from examination of the residue of the bound state pole

Equivalence of the two-body BS and CS equations

- In both cases, the two-body equation has the same form

$$M_{BS}(p',p;P)=V_{BS}(p',p;P)+\int V_{BS}(p',k;P)G_{BS}(k;P)M_{BS}(k,p;P)$$

$$M_{BS}(p',p;P)=V_{BS}(p',p;P)+\int M_{BS}(p',k;P)G_{BS}(k;P)V_{BS}(k,p;P)$$

$$M_{CS}(p',p;P)=V_{CS}(p',p;P)+\int V_{CS}(p',k;P)G_{CS}(k;P)M_{CS}(k,p;P)$$

Equate the amplitudes, and determine the relation between the kernels

$$M_{BS}=V_{BS}\left[1-G_{BS}V_{BS}\right]^{-1}=\left[1-V_{CS}G_{CS}\right]^{-1}V_{CS}=M_{CS}\Rightarrow$$

$$V_{BS}=V_{CS}+V_{CS}\left[G_{CS}-G_{BS}\right]V_{BS}$$

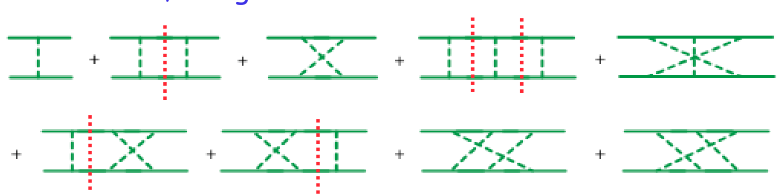
or

$$V_{CS}=V_{BS}+V_{BS}\left[G_{BS}-G_{CS}\right]V_{CS}$$

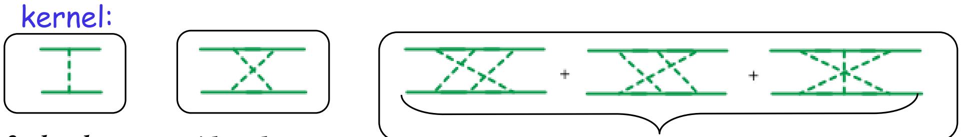
- The solutions of one equation are identical to the solutions of the other, provided the kernels are properly related

Equivalent summations of the *generalized ladder sum*

- To 6th order, the generalized ladder sum is

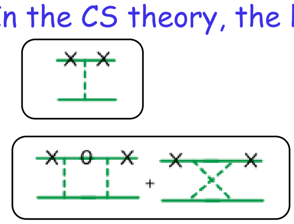


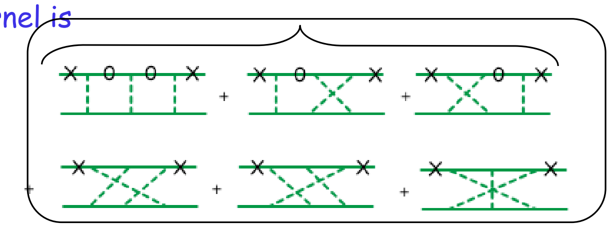
- In the BS theory, these terms require the following *irreducible* kernel:



2nd order ***4th order*** ***6th order***

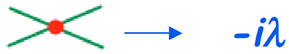
- In the CS theory, the kernel is

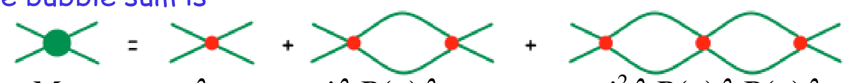




Effective theory; estimate of the bound state mass

- ★ Take an effective short range interaction (treated as a contact term)
- ★ The bubble sum is





$$M = \lambda + i\lambda B(s)\lambda + i^2\lambda B(s)\lambda B(s)\lambda + \cdots$$

$$= \lambda + i\lambda B(s) M$$

$$+\cdots$$

$$= \frac{\lambda}{1 - i\lambda B(s)}$$

a bound state of mass M_B exists if

$$\boxed{i\lambda B\left(M_B^2\right) = 1}$$

- ★ This means that all the Feynman diagrams in the series are the same size - the physics is non-perturbative.
- ★ Bound states arise in field theory from the infinite sum of Feynman diagrams.

Estimate: bound state mass in 1+1 dimensions (1)

- ★ Work in 1 time and 1 space dimensions ($p_0; p_z$) to remove divergences; most results carry over to 1+3 dimensions
- ★ The bubble in 1+1 dimensions is easy to calculate

$$iB(P^2) = i(-i)^4 \int \frac{d^2k}{(2\pi)^2} \left(\frac{1}{m_1^2 - \left(\tfrac{1}{2}P + k\right)^2 - i\varepsilon} \right) \left(\frac{1}{m_2^2 - \left(\tfrac{1}{2}P - k\right)^2 - i\varepsilon} \right) = i \int \frac{d^2k}{(2\pi)^2} \left(\frac{1}{A_1} \right) \left(\frac{1}{A_2} \right)$$

$$= i \int \frac{d^2k}{(2\pi)^2} \int_0^1 dx \frac{1}{\left(A_1 x + A_2(1-x)\right)^2} = i \int \frac{d^2k'}{(2\pi)^2} \int_0^1 dx \frac{1}{\left(m_1^2 x + m_2^2(1-x) - P^2 x(1-x) - k'^2 - i\varepsilon\right)^2}$$

$$= -\frac{1}{4\pi} \int_0^1 dx \frac{1}{\left(m_1^2 x + m_2^2(1-x) - P^2 x(1-x)\right)}$$

$$= -\frac{1}{2\pi\Delta} \left\{ \tan^{-1}\left(\frac{m_1^2 - m_2^2 + P^2}{\Delta} \right) - \tan^{-1}\left(\frac{m_1^2 - m_2^2 - P^2}{\Delta} \right) \right\}$$

where $\Delta^2 = \left(P^2 - (m_1 - m_2)^2\right)\left((m_1 + m_2)^2 - P^2\right)$

Estimate: bound state mass in 1+1 dimensions (2)

- Assume equal masses and weak binding: $m_1=m_2=m$; $P^2=4m^2-\delta^2$; $m \gg \delta$; $\Delta \cong 2m\delta$

$$iB\left(4m^2-\delta^2\right)=-\frac{1}{2\pi\Delta}\left\{\tan^{-1}\left(\frac{m_1^2-m_2^2+P^2}{\Delta}\right)-\tan^{-1}\left(\frac{m_1^2-m_2^2-P^2}{\Delta}\right)\right\}\simeq-\frac{1}{4m\delta}$$

- The binding energy is approximately

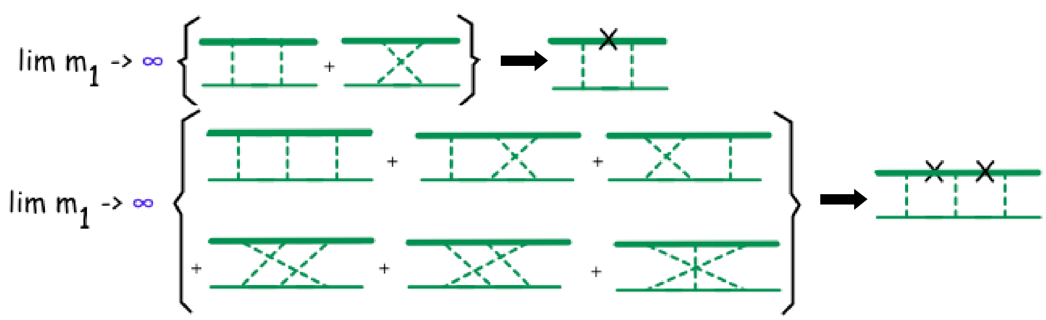
$$-\frac{\lambda}{4m\delta}\simeq 1 \quad\Rightarrow\quad \delta\simeq-\frac{\lambda}{4m}$$

- The contact term must be negative (attractive) for a bound state to exist.

exercise: work this out for 1+2 dimensions

The one-body limit

- If $m_1 \Rightarrow \infty$, the equation should reduce to a one-body equation for m_2 with a potential independent of the coordinates of m_1. This is the *one-body limit*.
- In scalar ϕ^3 theory, the generalized ladder sum has this property to *each* order. The proof is in my textbook "*Relativistic Quantum Mechanics and Field Theory*". Diagrammatically, for the 2nd and 4th orders



- For scalar theories in the $m_1 \Rightarrow \infty$ limit, the OBE approximation in CS theory gives the *exact* result for the generalized ladder sum.

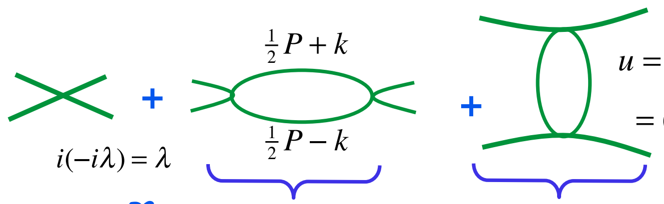
exercise: prove this

Cancellations: ϕ^4 theory in 1+1 dimensions

- ★ Study a simple example: ϕ^4 theory with one interaction $-i\lambda$

- ★ On shell scattering to 2nd order:

$\mathcal{M}$ =



$$u=\left(\tfrac{1}{2}P+p'-\left(\tfrac{1}{2}P-p\right)\right)^2 = (p'+p)^2$$

$s=P^2$ bubble $B(s)$ crossed bubble $B(u)$

- ★ $B(s)$ already evaluated previously:

$$B(s)=-\frac{\lambda^2}{2\pi\Delta}\left\{\tan^{-1}\left(\frac{m_1^2-m_2^2+P^2}{\Delta}\right)-\tan^{-1}\left(\frac{m_1^2-m_2^2-P^2}{\Delta}\right)\right\}$$

- ★ where $\Delta^2=\left(P^2-(m_1-m_2)^2\right)\left((m_1+m_2)^2-P^2\right)$

Interesting limits

- ★ $m_1=m_2=m$; $P^2=4m^2-\delta^2$; $u=\delta^2$, and $m \gg \delta$

$$B(s)\cong-\frac{\lambda^2}{2\pi m\delta}\tan^{-1}\left(\frac{2m}{\delta}\right)\simeq-\frac{\lambda^2}{2\pi m\delta}\left\{\frac{\pi}{2}-\frac{\delta}{2m}\right\};\quad B(u)\cong-\frac{\lambda^2}{2\pi m\delta}\left\{\frac{\delta}{2m}\right\}$$

$$B(s)+B(u)\cong-\frac{\lambda^2}{2\pi m\delta}\left\{\frac{\pi}{2}\right\}$$

Note the cancellation

- ★ $m_1 \gg m_2 \gg \delta$; $P^2=(m_1+m_2)^2-\delta^2$; $u=(m_1-m_2)^2+\delta^2$

$$B(s)\cong-\frac{\lambda^2}{4\pi\sqrt{m_1m_2}\,\delta}\left\{\tan^{-1}\left(\frac{m_1(m_1+m_2)}{\sqrt{m_1m_2}\,\delta}\right)+\tan^{-1}\left(\frac{m_2(m_1+m_2)}{\sqrt{m_1m_2}\,\delta}\right)\right\}\simeq-\frac{\lambda^2}{4\pi\sqrt{m_1m_2}\,\delta}\left\{\pi-\frac{\delta}{\sqrt{m_1m_2}}\right\}$$

$$B(u)\cong-\frac{\lambda^2}{4\pi\sqrt{m_1m_2}\,\delta}\left\{\tan^{-1}\left(\frac{m_1(m_1-m_2)}{\sqrt{m_1m_2}\,\delta}\right)-\tan^{-1}\left(\frac{m_2(m_1-m_2)}{\sqrt{m_1m_2}\,\delta}\right)\right\}\simeq-\frac{\lambda^2}{4\pi\sqrt{m_1m_2}\,\delta}\left\{\frac{\delta}{\sqrt{m_1m_2}}\right\}$$

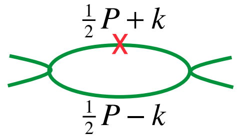
$$B(s)+B(u)\cong-\frac{\lambda^2}{4\pi\sqrt{m_1m_2}\,\delta}\{\pi\}$$

Note the cancellation

exercise: evaluate these bubbles in 1+2 dimensions

Evaluation of the CS bubble in 1+1 dimension (1)

- The CS bubble has particle #1 on-shell; there is no crossed bubble



- This can be written in the convenient form

$$
\begin{aligned}
C(s) &= i\lambda^2 \int \frac{d^2k}{(2\pi)^2}\left(\frac{1}{A_1 - i\varepsilon}\right)\left(\frac{1}{A_2 - A_1 - i\varepsilon}\right) \\
&= i\lambda^2 \int \frac{d^2k}{(2\pi)^2}\left(\frac{1}{E_1^2 - \left(\frac{1}{2}P + k_0\right)^2 - i\varepsilon}\right)\left(\frac{1}{m_2^2 - m_1^2 + 2Pk_0 - i\varepsilon}\right)
\end{aligned}
$$

$$
\underbrace{\left(E_1 - \tfrac{1}{2}P - k_0 - i\varepsilon\right)}\left(E_1 + \tfrac{1}{2}P + k_0 - i\varepsilon\right)
$$

only pole in the lower half-plane and hence this integral gives the exact CS result

Evaluation of the CS bubble in 1+1 dimension (2)

- This can be also be written

$$
C(s) = i\lambda^2 \int \frac{d^2k}{(2\pi)^2}\left(\frac{1}{A_1 - i\varepsilon}\right)\left(\frac{1}{A_2 - A_1 - i\varepsilon}\right) = i\lambda^2 \int \frac{d^2k}{(2\pi)^2}\int_{-\infty}^{1} dx \frac{1}{\left[\left(A_1 - i\varepsilon\right)x + \left(A_2 - 2i\varepsilon\right)(1-x)\right]^2}
$$

$$
= -\frac{\lambda^2}{4\pi}\int_{-\infty}^{1} dx \frac{1}{\left(m_1^2 x + m_2^2(1-x) - P^2 x(1-x)\right)} = -\frac{\lambda^2}{2\pi\Delta}\left\{\tan^{-1}\left(\frac{m_1^2 - m_2^2 + P^2}{\Delta}\right) + \frac{\pi}{2}\right\}
$$

- Interesting limits (as before)
  - $m_1 \gg m_2 \gg \delta$; $P^2=(m_1+m_2)^2-\delta^2$ $\quad C(s) \Rightarrow -\frac{\lambda^2}{4\pi\sqrt{m_1 m_2 \delta}}\{\pi\} \cong B(s) + B(u)$

    The correction $\left(-\frac{\delta}{\sqrt{m_1 m_2}}\frac{m_2}{m_1}\right)$ is much smaller than the term cancelled by *B(u)*.
  - $m_1 = m_2 = m$; $P^2 = 4m^2 - \delta^2$ $\quad C(s) \Rightarrow -\frac{\lambda^2}{4\pi m\delta}\left\{\pi - \frac{\delta}{m}\right\} \simeq B(s)$

- Conclusion: the CS equation (in the *scalar case* when $m_1 \to \infty$) builds in the cancellations.

research exercise: this bubble diverges in 1+2 dimensions; how can it be regularized?

Freeze-out of nucleon resonances (in the CS theory)

- Nucleon resonances can be excited when the mass of the off-shell nucleon becomes bigger than $(m + m_\pi)^2$.
- However, in the CS theory, the mass of the off shell nucleon is bounded from above. For two nucleon scattering at lab energy of $W > 2m$ (with **k** the internal relative nucleon three-momentum),

$$\rho = (W-k)^2 - m^2 = W^2 - 2W\sqrt{m^2+\mathbf{k}^2} < W(W-2m)$$

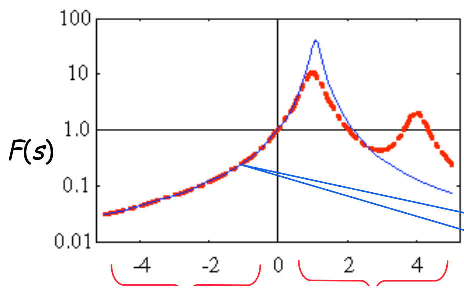
- Hence, nucleon resonances are not *explicitly* excited unless

$$W > 2m + m_\pi$$

- This is fundamentally different from Hamiltonian dynamics, where they are excited for *all W*. The internal momentum must only be larger than a minimum value

$$2E(k) > 2m + m_\pi \quad \Rightarrow \quad \mathbf{k}^2 > m m_\pi + \tfrac{1}{4} m_\pi^2$$

Resonances frozen out because "left hides right"



$$\rho = (W-k)^2 - m^2 \le W(W-2m)$$

left right

$$F_1(\rho) = \frac{1.033^2 + 0.03}{(1.033-\rho)^2 + 0.03}$$

$$F_2(\rho) = \frac{1.1\left(1 - \frac{0.2}{16.1}\right)}{(1-\rho)^2 + 0.1} + \frac{0.2}{(4-\rho)^2 + 0.1}$$

- Compare the "left-hand-side" of two resonance structures
- Under certain conditions they are indistinguishable
- in this case, the two functions agree on the left-hand side to 1%!
- LESSON:

THE RIGHT-HAND NUCLEON RESONANCE STRUCTURE CANNOT BE INFERRED UNIQUELY FROM THE LEFT-HAND STRUCTURE

- Low energy NN scattering does not "see" the resonances

Relativistic normalization condition (1)

- The normalization condition for the bound state vertex function also follows from the scattering equation. First find the nonlinear forms of the equation:

$$M = V + \int VGM = V + \int \overline{M}GM - \iint \overline{M}\,\overline{G}VGM$$

$$\overline{M} = V + \int \overline{M}\,\overline{G}\,V$$

- Then substitute the pole part of M and expand (away from the pole, $i\varepsilon$ -> 0 and $G = \overline{G}$:

$$\frac{\Gamma\overline{\Gamma}}{M_B^2 - P^2} = V + \int \frac{\Gamma\overline{\Gamma}}{M_B^2 - P^2}\left\{G + \frac{\partial G}{\partial M_B^2}\left(P^2 - M_B^2\right)\right\}\frac{\Gamma\overline{\Gamma}}{M_B^2 - P^2}$$

$$-\iint \frac{\Gamma\overline{\Gamma}}{M_B^2 - P^2}\left\{GVG + \left[\frac{\partial G}{\partial M_B^2}VG + G\frac{\partial V}{\partial M_B^2}G + GV\frac{\partial G}{\partial M_B^2}\right]\left(P^2 - M_B^2\right)\right\}\frac{\Gamma\overline{\Gamma}}{M_B^2 - P^2}$$

Relativistic normalization condition (2)

- The double poles give the bound state equation (again)
- The single poles give the normalization condition:

$$\frac{\Gamma\overline{\Gamma}}{M_B^2 - P^2} = \int \frac{\Gamma\overline{\Gamma}}{M_B^2 - P^2}\left\{\frac{\partial G}{\partial M_B^2}\left(P^2 - M_B^2\right)\right\}\frac{\Gamma\overline{\Gamma}}{M_B^2 - P^2}$$

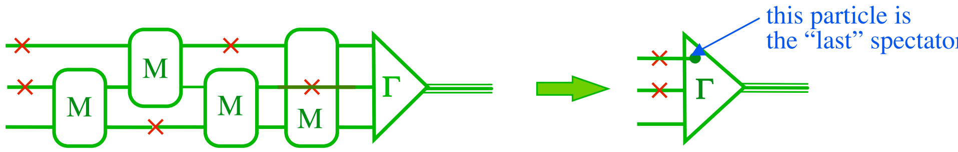
$$-\iint \frac{\Gamma\overline{\Gamma}}{M_B^2 - P^2}\left\{\left[\frac{\partial G}{\partial M_B^2}VG + G\frac{\partial V}{\partial M_B^2}G + GV\frac{\partial G}{\partial M_B^2}\right]\left(P^2 - M_B^2\right)\right\}\frac{\Gamma\overline{\Gamma}}{M_B^2 - P^2}$$

$$1 = \int \overline{\Gamma}\left\{\frac{\partial G}{\partial M_B^2}\right\}\Gamma + \iint \overline{\Gamma}\left\{\frac{\partial V}{\partial M_B^2}\right\}\Gamma$$

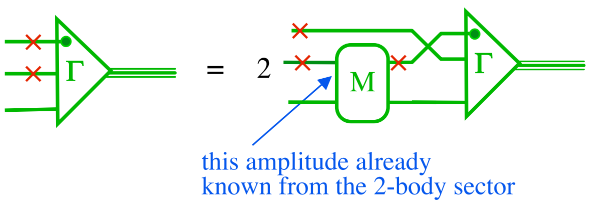
exercise: work through these details

CS equations for three-body systems*

★ Define three-body vertex functions for each possibility



★ Then three body Faddeev-like equations emerge automatically. For identical particles they are:



*Alfred Stadler, FG, and Michael Frank, Phys. Rev. C **56**, 2396 (1997)

Applications of the CS theory

★ Gauge invariance can be treated exactly (lecture 4)

★ Excellent fits to the NN data below 350 MeV (with $\chi^2 \approx 1.06$ - lecture 3)

★ Excellent description of the 3N binding energy with no explicit three body force (lecture 3)

★ Excellent fit to all deuteron form factors to $Q^2 \sim 6$ GeV^2 with one free parameter in the current (lecture 4)

★ Satisfactory description of πN scattering and various quark model calculations (not discussed)

★ Exploratory study of d(e,e'p)n in Born approximation*

★ To do (work in progress)

- photodisintegration and electrodisintegration of 2 and 3 body nuclei

***J. Adam Jr., FG, S.Jeschonnek, P.Ulmer, and J.W.Van Orden, PRC 66**: 044003 (2002).

Conclusions

- ★ Few body nuclei are composite systems. They must be described non-perturbatively ⇒ integral equations for amplitudes in *p* space.
- ★ The features of a relativistic description depend on the formalism. In Field form -- *all* generators are kinematic at the cost of negative energy states (twice as many degrees of freedom).
- ★ Physics depends on whether or not nucleon resonances are explicitly excited (recall: "left hides right").
- ★ A theoretically sound description of few-body reactions requires FSI and MEC and *NNN* forces consistent with the two-body dynamics assumed. We will return to this in the subsequent lectures.
- ★ The CS theory can serve as a framework for the use of any method. Take nonrelativistic limit to interpret correspondence with relativistic theory.

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