Relativistic Description of Few-Nucleon Systems

LECTURES FOR THE 2007 PRAGUE SUMMER SCHOOL

Franz Gross JLab and W&M

Plan: Each lecture will be accompanied by questions and problems, some requiring original research

Jefferson Lab

Outline:

- **Lecture 1: Overview: discussion of relativistic methods**
  - *Two schools for the relativistic description of few nucleon systems will be described.*
  - *How do these approaches handle the problem of relativity and what are the advantages and disadvantages of each?*
- **Lecture 2: Theory: two and three nucleon systems**
  - *Introduction to the Covariant Spectator Theory.*
  - *How are the bound state and scattering equations obtained, what are the normalization conditions?*

Outline (continued)

- ★ **Lecture 3: Results: energies below the pion production threshold**
  - *New, high precision fits to np data below 350 MeV lab energy, and the relativistic properties of the deuteron and triton.*
  - *What do these new results tell us about the nature of nuclear forces?*
- ★ **Lecture 4: Electromagnetic interactions: gauge invariance and effective current operators**
  - *General method for doing gauge invariant calculations in systems composed of composite particles.*
  - *What can be learned from high energy electron scattering experiments?*

Lecture I:

Overview: Discussion of Relativistic Methods

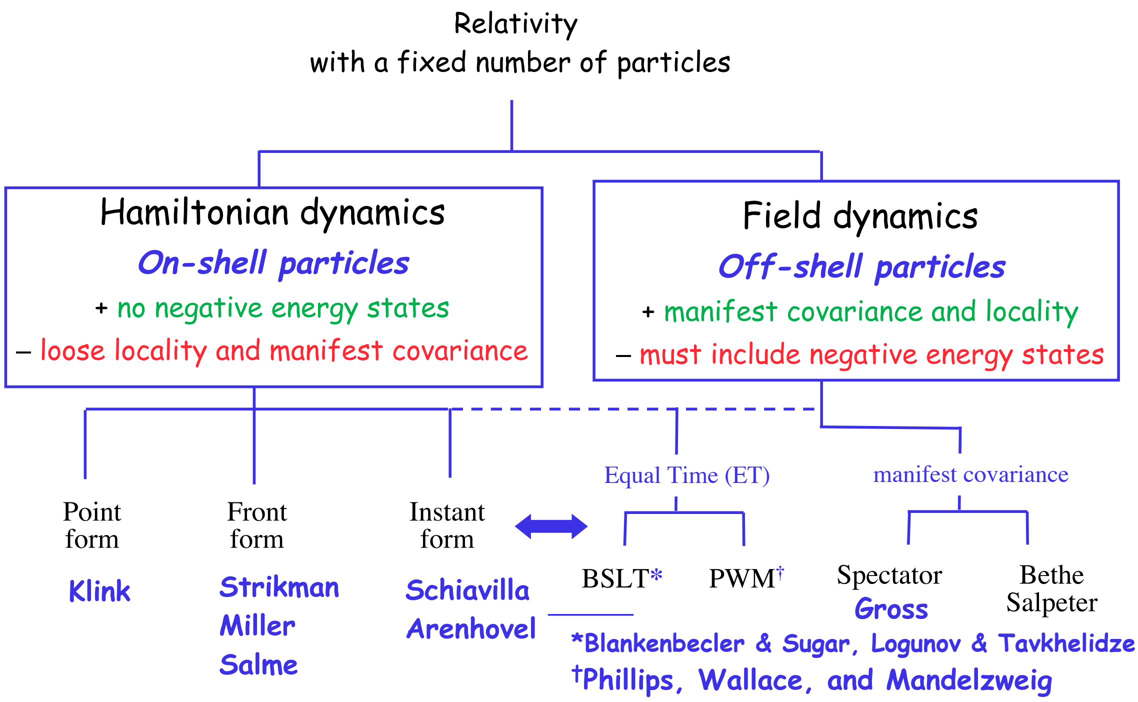
Outline

- ★ Overview of relativistic methods: "Two schools"
- ★ Field dynamics (also referred, in these lectures, to as "field form")
  - Relativistic interactions and equations in field theory
  - Introduction to Bethe-Salpeter (BS) and Covariant Spectator© (CS) equations
  - Description of bound states in field dynamics
- ★ Hamiltonian Dynamics
  - Basic theory in "instant form"
  - Comparison with field form
  - Poincaré transformations
  - Dirac's forms of dynamics
  - The Bakamjian-Thomas construction
  - The mass operator
- ★ Cluster separability
- ★ Conclusions

First -- why use a relativistic theory?

- ★ NOT because
  - of size of $(v/c)^2$ corrections (although they may be large in some applications)
  - it is more accurate (it may not be)
  - it is "better" than EFT (it complements EFT)
- ★ Use a covariant theory for the following reasons
  - Intellectual: to preserve an exact symmetry (Poncare' invariance)
  - Practical: to calculate boosts and Lorentz kinematics consistently to all orders (essential when energies are of the order of 1 GeV)
  - Consistent: to use field theory for guidance in the construction of
    - forces (2⇔3 body consistency)
    - currents consistent with forces
  - Conceptual: for "phenomenological economy", and to understand the non relativistic limit:
    - spin 1/2 particles (Dirac equation)
    - interpretation of L•S forces (covariant scalar-vector theory of N matter)
    - efficient one boson exchange models of NN forces (?)

Overview of relativistic methods: Two "schools"

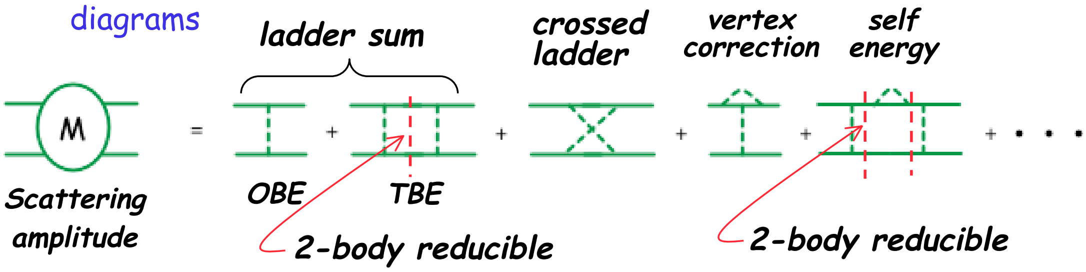


FIELD DYNAMICS

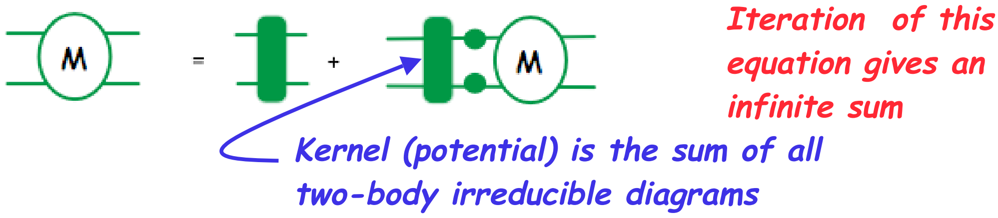
Relativistic interactions in field theory

★ Diagrammatic derivation for 2 body scattering:

- The exact scattering amplitude is the sum of all Feynman diagrams



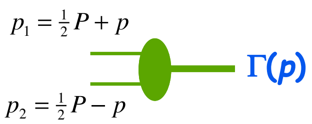
- Divide the sum into irreducible and 2-body reducible terms, and collect the irreducible terms into a kernel, which is iterated



Field Theory: How are bound states described?

★ **What is a bound state in field theory?**

★ **A bound state is a new particle (not in the Lagrangian) that arises because of the interactions. The vertex function Γ describes how it couples to the elementary particles in the Lagrangian:**



Notation:
P=total momentum (always conserved)
p relative momentum

★ **The bound state produces a *pole* in the scattering amplitude which does not correspond to one of the elementary particles in the theory:**

$$M = \Gamma(p')\frac{1}{\left(M_B{}^2 - P^2\right)}\overline{\Gamma}(p)$$

★ **If the bound state is not elementary, no single Feynman diagram will have the bound state pole; it must be generated from an infinite sum of Feynman diagrams, much as the geometric series generates a pole at z=1:**

$$M = z + z^2 + z^3 + \bullet\bullet\bullet \quad = \frac{z}{1-z}$$

Relativistic scattering equations in field theory

- The scattering equation is

$$M(p',p;P)=V(p',p;P)+\int V(p',k;P)G(k;P)M(k,p;P)$$

where V is the kernel (i.e. potential) and G is the propagator

- If the kernel is *phenomenological*, this is field *dynamics* instead of field *theory*.
- The bound state equation follows by assuming the M matrix has a pole, and substituting

$$M(p',p;P)=\frac{\Gamma(p',M_B)\overline{\Gamma}(p,M_B)}{M_B^2-P^2}=V(p',p;P)$$
$$+\int V(p',k;P)G(k;P)\frac{\Gamma(k,M_B)\overline{\Gamma}(p,M_B)}{M_B^2-P^2}$$

extracting the pole part gives the bound state equation uniquely

$$\Gamma(p',M_B)=\int V(p',k;M_B)G(k;M_B)\Gamma(k,M_B)$$

- This equation also insures that the non-pole parts of the scattering amplitude do not contribute near the pole (next lecture)

Two field dynamical equations

- The Bethe-Salpeter (BS) propagator depends on all four components of the relative momentum, $\{k_0,\mathbf{k}\}$. For two scalar particles it is

$$G_{BS}(k;P)=\frac{1}{\left(m_1^2-p_1^2-\Sigma\left(p_1^2\right)-i\varepsilon\right)\left(m_2^2-p_2^2-\Sigma\left(p_2^2\right)-i\varepsilon\right)}\quad \textbf{with}\quad \begin{cases}p_1=\frac{1}{2}P+k\\ p_2=\frac{1}{2}P-k\end{cases}$$

- The Covariant Spectator© propagator depends on only three components of the relative momentum, $\mathbf{k}$. One particle is on-shell

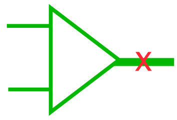
$$G_{CS}(k;P)=\frac{2\pi i\,\delta_+\left(m_1^2-\left(\frac{1}{2}P+k\right)^2\right)}{\left(m_2^2-p_2^2-\Sigma\left(p_2^2\right)-i\varepsilon\right)}=\frac{2\pi i\,\delta\left(k_0-E_1+\frac{1}{2}P_0\right)}{2E_1\left(E_2^2-\left(P_0-E_1\right)^2-\Sigma\left(p_2^2\right)-i\varepsilon\right)}$$

exercise: write the explicit form of these equations in a ϕ^4 theory

These equations both have a connection to field theory

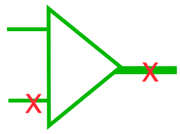
- ★ The Bethe-Salpeter amplitude is a well defined field theoretic matrix element:

$$\Psi(x_1,x_2)=\left\langle 0 \mid T\left(\psi(x_2)\psi(x_1)\right)\mid d\right\rangle$$



- ★ The Covariant Spectator© amplitude is *also* a well defined field theoretic amplitude:

$$\Psi(x_1)=\left\langle N \mid \psi(x_1)\mid d\right\rangle$$



- ★ Equations for the Bethe-Salpeter and the Spectator* amplitudes can be derived from field theory
  - Both are manifestly covariant under *all* Poincaré transformations (advantage)
  - Both incorporate negative energy (antiparticle) states (disadvantage)

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*O. W. Greenberg's "n-quantum approximation"

HAMILTONIAN DYNAMICS*

**B. D. Keister and W. N. Polyzou, Ad. in Nucl. Phys.* 20, 225 (1991)

Hamiltonian dynamics: basic theory (in “instant” form)

- Start with a Hilbert space of free particle states $\left(H_0 - E_i\right)\phi_i = 0$
- Interactions described by the interaction Hamiltonian, H_I

$$\boxed{\left(H_0 - E_i\right)\Psi_i = H_I \Psi_i} \quad \Rightarrow \quad \Psi_i = \phi_i + \frac{1}{H_0 - E_i} H_I \Psi_i$$

- Solve by iteration (perturbation theory) $H_{jk} = \left\langle \phi_j H_I \phi_k \right\rangle$

$$\Psi_i = \phi_i + \sum_{j \neq i} \phi_j \frac{1}{E_j - E_i} H_{ji} + \sum_{\substack{j' \neq i \\ j \neq i}} \phi_{j'} \left(\frac{1}{E_{j'} - E_i}\right) H_{j'j} \left(\frac{1}{E_j - E_i}\right) H_{ji} + \cdots$$

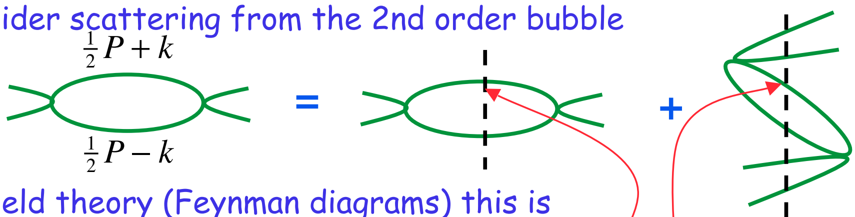
- The scattering amplitude is then

$$M_{ki} = \left\langle \phi_k H_I \Psi_i \right\rangle = H_{ki} + \sum_{j \neq i} H_{kj} \frac{1}{E_j - E_i} H_{ji} + \sum_{\substack{j' \neq i \\ j \neq i}} H_{kj'} \left(\frac{1}{E_{j'} - E_i}\right) H_{j'j} \left(\frac{1}{E_j - E_i}\right) H_{ji} + \cdots$$

exercise: check these relations

Comparison with Field form: ϕ^4 theory in 1+1 dimensions

- Consider scattering from the 2nd order bubble



- In field theory (Feynman diagrams) this is

$$B(s) = i \int \frac{d^2 k}{(2\pi)^2} \frac{\lambda^2}{\left(m_1^2 + \mathbf{k}_+^2 - \left(\frac{1}{2}P + k_0\right)^2 - i\varepsilon\right)\left(m_2^2 + \mathbf{k}_-^2 - \left(\frac{1}{2}P - k_0\right)^2 - i\varepsilon\right)}$$

$$= -\lambda^2 \int \frac{dk_z}{(2\pi)} \left(\frac{1}{4E_1 E_2}\right) \left(\frac{1}{E_1 + E_2 - P_0} + \frac{1}{E_1 + E_2 + 2P_0 - P_0}\right)$$

- Conclusion 1: Manifest covariance obtained when BOTH positive and negative energy contributions are included.
- Conclusion 2: ONE Feynman diagram is the sum of ALL possible time-ordered graphs.

Hamiltonian part

additional part for manifest covariance

The Poincaré group and Dirac forms of dynamics

★ The Poincaré group are unitary operators on the Hilbert space, with 10 generators: P^0, **P**, **J**, and **K**, satisfying the following 45 CR's:

$$[J^i, J^j] = i\varepsilon^{ijk} J^k, \quad [J^i, K^j] = i\varepsilon^{ijk} K^k, \quad [J^i, P^j] = i\varepsilon^{ijk} P^k, \quad [K^i, K^j] = -i\varepsilon^{ijk} J^k$$

$$[K^i, P^j] = -i\delta^{ij} P^0, \quad [K^i, P^0] = -iP^i, \quad [P^\mu, P^\nu] = 0, \quad [J^i, P^0] = 0$$

★ Forms of dynamics: The Poincaré group has three subgroups:

- The instant-form is based on the subgroup

$$[J^i, J^j] = i\varepsilon^{ijk} J^k, \quad [J^i, P^j] = i\varepsilon^{ijk} P^k, \quad [P^i, P^j] = 0$$

- The point-form is based on the Lorentz subgroup

$$[J^i, J^j] = i\varepsilon^{ijk} J^k, \quad [J^i, K^j] = i\varepsilon^{ijk} K^k, \quad [K^i, K^j] = -i\varepsilon^{ijk} J^k$$

- The front-form is based on the subgroup constructed from 7 generators

$$P^+ = P^0 + P^3, \quad \mathbf{P}_\perp = \{P^1, P^2\}, \quad J^3, \quad K^3, \quad \mathbf{E}_\perp = \mathbf{K}_\perp - \hat{\mathbf{z}} \times \mathbf{J}_\perp$$

exercise: prove that the commutation relations for these 7 generators close.

Definition of generators

★ Finite transformations "generated" by the generators

$$T\psi(z) = \exp(-i P_\mu \cdot x^\mu)\psi(z) = \exp(i P_z a_z)\psi(z) = \exp(a_z \nabla_z)\psi(z)$$

$$= \psi(z) + a_z \frac{\partial}{\partial z}\psi(z) + \frac{1}{2} a_z^2 \frac{\partial^2}{\partial^2 z}\psi(z) + \cdots$$

$$= \psi(z + a_z)$$

Kinematic surfaces and generalized hamiltonia

- ★ Instant-form:

  States with definite momentum and spin (eigenstates of ***P*** and ***J***): defined on a surface connected by translations and rotations (the t=0 surface). P^0 and **K** are dynamical; evolving the states away from the t=0 surface

- ★ Point-form:

  States with definite four-velocity (eigenstates of ***J*** and ***K***): defined on a hyperboloid with $x_\mu x^\mu = 1$. The 4 components of P^μ are dynamical.

- ★ Front-form:

  States defined on a light-front, $x^- = t - z = 0$. The dynamical generators are $P^- = P^0 - P^3$, $F_\perp = K_\perp + z \times J_\perp$

Dirac Hamiltonian classifications

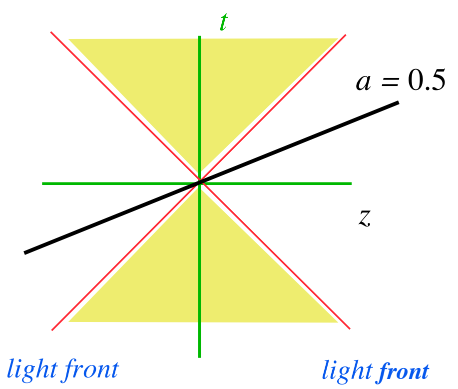
Some of the Poincaré transformations are kinematic; others involve the dynamics

★ Plane forms

$t - a\,z = 0$
$-1 \le a \le 1$

$a \le 1$: instant form — 6+4
$a = 1$: front form — 7+3

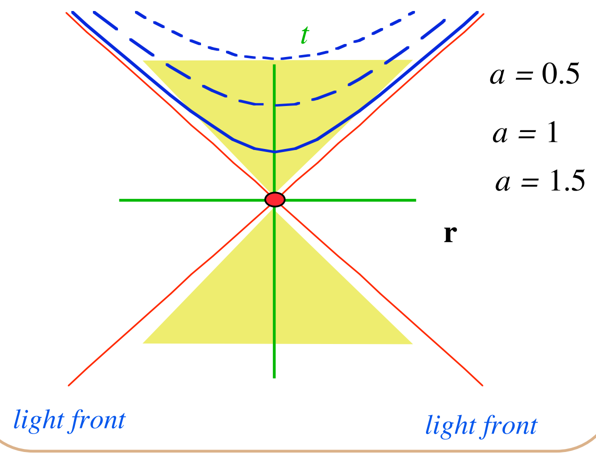
Limit not continuous



★ Hyperbolic forms

$t = \sqrt{(\mathbf{r}^2 + a^2)}$

$a = 0$: point form on the light cone
$a = \infty$: instant form



The Bakamjian-Thomas construction (in instant-form)*

★ The commutation relations can be automatically satisfied if the operators **P**, **J**, **K**, and $H = P^0$ are replaced by **P**, **r**, **s**, and M.

★ For a single particle, α, the generators are written in lower case:

$$\mathbf{p}_\alpha, \quad \mathbf{j}_\alpha = \mathbf{s}_\alpha + \mathbf{r}_\alpha \times \mathbf{p}_\alpha, \quad h_\alpha = \sqrt{m_\alpha^2 + \mathbf{p}_\alpha^2}, \quad \mathbf{k}_\alpha = -\tfrac{1}{2}\{h_\alpha, \mathbf{r}_\alpha\} - \frac{\mathbf{p}_\alpha \times \mathbf{s}_\alpha}{m_\alpha + h_\alpha}$$

with inverse relations

$$m_\alpha = \sqrt{h_\alpha^2 - \mathbf{p}_\alpha^2}, \quad \mathbf{r}_\alpha = -\tfrac{1}{2}\{h_\alpha^{-1}, \mathbf{k}_\alpha\} - \frac{\mathbf{p}_\alpha \times (h_\alpha \mathbf{j}_\alpha - \mathbf{p}_\alpha \times \mathbf{k}_\alpha)}{m_\alpha h_\alpha (m_\alpha + h_\alpha)},$$

$$\mathbf{s}_\alpha = m_\alpha^{-1}(h_\alpha \mathbf{j}_\alpha - \mathbf{p}_\alpha \times \mathbf{k}_\alpha) - \frac{\mathbf{p}_\alpha(\mathbf{p}_\alpha \cdot \mathbf{j}_\alpha)}{m_\alpha(m_\alpha + h_\alpha)}$$

with non-zero commutators

$$\left[r_\alpha^i, p_\alpha^j\right] = i\delta^{ij}, \quad \left[s_\alpha^i, s_\alpha^j\right] = i\varepsilon^{ijk} s_\alpha^k$$

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*B. Bakamjian and H. L. Thomas, *Phys. Rev.* 92, 1300 (1953)

Bakamjian-Thomas for n>1

★ Proceed in 4 steps

1. Construct total P^μ, **J** and **K** by adding generators for each particle $P^\mu = \sum_\alpha p_\alpha^\mu, \quad \mathbf{J} = \sum_\alpha \mathbf{j}_\alpha, \quad \mathbf{K} = \sum_\alpha \mathbf{k}_\alpha$
2. Construct the operators $\mathbf{M}_0$, **R** and **S** (together with **P**, already constructed) using the inverse relations (previous slide)
3. Add the interactions to $\mathbf{M}_0$, $\mathbf{M} = \mathbf{M}_0 + \mathbf{V}$. Require that **V** commute with $\mathbf{M}_0$, **P**, **R** and **S**
4. Construct the new generators **H**, **J** and **K** as functions of **M**, **P**, **R** and **S**. This completes the construction. All interactions are in the mass operator.

Exercise: think about this and work through the relations

The mass operator

- To achieve manifest covariance *without* negative energy states, introduce the mass operator

$$M = M_0 + U = \left((H_0 + H_I)^2 - \mathbf{P}^2 \right)^{1/2} \quad \Rightarrow \quad M^2 = M_0^2 + V$$

where $V = H_0 H_I + H_I H_0 + H_I^2$

- Following the steps we used with the hamiltonian, we have

$$\left(M_0^2 - M_i^2\right)\Psi_i = V\,\Psi_i \quad \Rightarrow \quad \Psi_i = \phi_i + \frac{1}{M_0^2 - M_i^2} V\,\Psi_i$$

- Solving by iteration, the scattering amplitude becomes

$$M_{ki} = \langle \phi_k V \Psi_i \rangle = V_{ki} + \sum_{j\neq i} V_{kj} \frac{1}{M_j^2 - M_i^2} V_{ji} + \sum_{\substack{j'\neq i \\ j\neq i}} V_{kj'} \left(\frac{1}{M_{j'}^2 - M_i^2} \right) V_{j'j} \left(\frac{1}{M_j^2 - M_i^2} \right) V_{ji} + \cdots$$

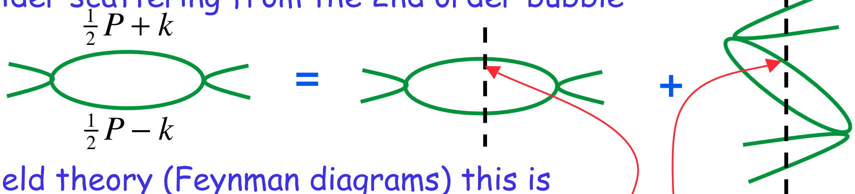
where, for the second order bubble

$$M_k^2 - M_i^2 = \left(E_+ + E_-\right)^2 - P_0^2$$

- This agrees with the covariant result.

Comparison with Field form: ϕ^4 theory in 1+1 dimensions

- Consider scattering from the 2nd order bubble



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$$= -\lambda^2 \int \frac{dk_z}{(2\pi)} \left(\frac{1}{4E_+E_-} \right) \left(\frac{1}{E_+ + E_- - P} + \frac{1}{E_+ + E_- + 2P - P} \right)$$

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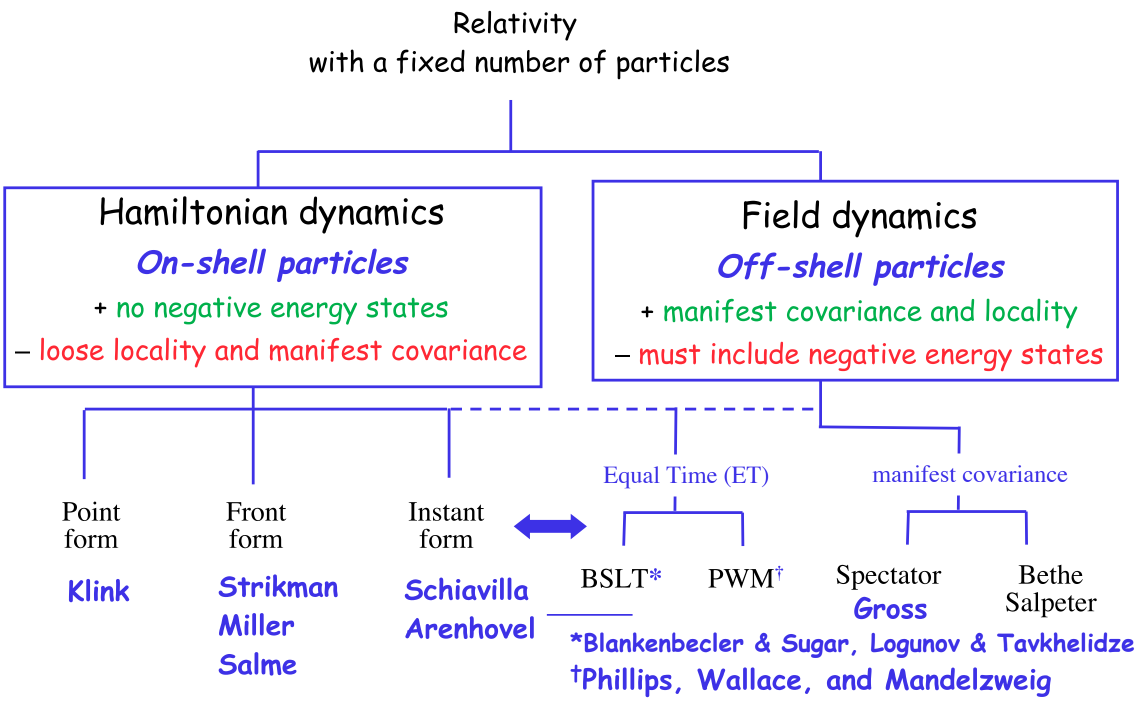
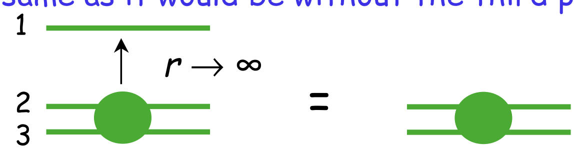
Overview of relativistic methods: Two "schools"



Cluster separability -- 3-body example

- ★ Definition: when one particle is far away, the interaction between the other two is the same as it would be without the third particle



- ★ If $P = p_1 + p_2 + p_3 = 0$, and $p_1 \neq 0$, then the 23 amplitude is in a moving frame. The boost depends on the mass of the 2-body system.
- ★ Hamiltonian dynamics is off-energy shell, $E_2 + E_3 \neq \sqrt{M_{23}^2 + \mathbf{p}_1^2}$. The energies of particles and subsystems do not match the free particle energies, and under boosts the cluster property is not easy to implement.
- ★ Field dynamics is off-mass shell, $p_0 \neq \sqrt{m^2 + \mathbf{p}^2}$. Energy is conserved so boosts and cluster properties are easily satisfied, but off-mass shell $\Rightarrow$ negative energy states.

Research study: How is separability handled by the two schools; Can you support my claim that here off-mass shell techniques work better?

Conclusions and comparison

- Hamiltonian dynamics
  - Advantages:
    - Real quantum mechanics
    - No negative energy states
  - Disadvantages
    - More ambiguities; no direct connection to field theory
    - Difficulties with cluster separability (?)
- Field Dynamics
  - Advantages
    - Manifest covariance and cluster separability easily implemented
    - Close connection to field theory guides the construction of interactions and currents
  - Disadvantages
    - Not conventional quantum mechanics; a new approach (if you think that 1951 is new?) still requiring conceptual development
    - Singularities, and need to treat negative energy states is more work

Exercise: What's your opinion?