AdS/QCD model as a tool for solving non-perturbative QCD problems

Hovhannes R. Grigoryan

Thomas Jefferson National Accelerator Facility
& Louisiana State University

Los Alamos National Laboratory

Los Alamos, NM

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Content

- Introduction into the AdS/QCD model.
- Vector meson form factors and associated observables.
- Pion form factor in the chiral limit.
- Anomalous pion decay and related form factor.
- *Baryon as skyrmion in the holographic model.*

- QFT is solved when all of its n-point Green's functions are known see, e.g., Streater & Wightman, "PCT, spin and statistics, and all that," 1989.
- Dyson-Schwinger equations allow, in principle, to find all the n-point Green's functions of the QFT.
- Method based on the DSEs may serve as a strong tool to study the nonperturbative QCD effects (see, e.g., Roberts & Williams, Prog. Part. Nucl. Phys.**33**, 477).
- The infinite tower of DSEs is hard to study for $n > 3$. However, the symmetries and the renormalizability of the theory can serve as a strong constraints on the ansatz for the omitted function(s).
- It is important to have an alternative nonperturbative approach to provide more insight into the form of these higher n-point functions. Lattice gauge theory is one of the examples.
- We will show that AdS/QCD model can serve as another continuous nonperturbative approach to QCD. We also believe that it can be used to complement the DSE approach for large n.

Formulation of Conjecture

The AdS/CFT correspondence conjectures the equivalence of gravity theory (Type IIB string theory) on $AdS_5 \times S_5$, and strongly coupled ($N = 4$ SYM) CFT_4. *(Maldacena, 1997)*

AdS/CFT says that for $\forall \mathcal{O}(x) \in \{CFT\ operator\}$,
$\exists! \phi(x,z) \in \{5D\ bulk\ field\}$ s.t. $\phi(x,0) = \phi_0(x)$, $x \in \partial AdS_5$.

Let $S_5[\phi_0(x)]$ is the gravity or string action of $\phi(x,z)$ with $\phi(x,0) = \phi_0(x)$, then the correspondence takes the form

$$\langle \exp(i \int d^4x \phi_0(x) \mathcal{O}(x)) \rangle_{CFT} = \exp(iS_5[\phi_0(x)]) \ ,$$

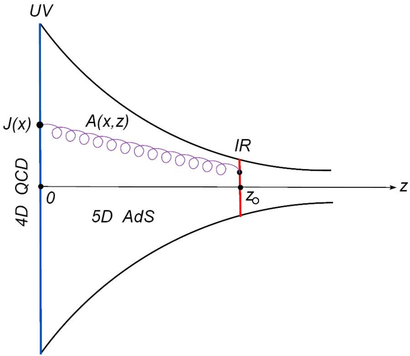
(Witten, 1998)

Addition of IR Brane

Addition of the IR brane, corresponds to deformation of the CFT leading to a breakdown of conformal invariance in the IR.

Now, we have both particles and S-matrix elements.

In particular, the KK-like gravitons in the gravity side can be interpreted in the 4D theory as resonances.



Initial Setup

We will use the (hard-wall) model proposed by Erlich, Katz, Son and Stephanov (EKSS) PRL95, 2005.

The slice of AdS_5 is defined according to:

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2\right), \qquad 0 < z \le z_0 \,,$$

$\eta_{\mu\nu} = \mathrm{Diag}\,(1,-1,-1,-1)$ and $z_0 \sim 1/\Lambda_{QCD}$ is the IR scale.

The holographic dictionary for vector sector is:

$$J^a_\mu(x) = \bar{q}\gamma_\mu t^a q(x) \leftrightarrow A^a_M(x,z)\,,$$

so that $A^a_M(x,0)$ is the source for $J^a_\mu(x)$.

The 5D Gauge Action

The 5D gauge action in AdS_5 space for the vector field is:

$$S_{AdS} = -\frac{1}{4g_5^2} \int d^4x\, dz\, \sqrt{g}\, Tr\left(F_{MN}F^{MN}\right) \;,$$

where $F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]$, $A = A^a t^a$, $(t^a \in SU(2),\ a = 1, 2, 3)$ and $M, N = 0, 1, 2, 3, z$.

4D Global Chiral $SU(2) \leftrightarrow$ 5D Local Gauge $SU(2)$

We take our field to be non-Abelian, since later we are interested in calculating the 3-point function.

Equation of Motion

We work in $A_z = 0$ gauge with Fourier-transformed gauge field

$$A_\mu(q,z) = \tilde{A}_\mu(q)\frac{V(q,z)}{V(q,\epsilon)}|_{\epsilon\to 0}\ .$$

The boundary condition at $z = z_0$:

$\partial_z V(q,z_0) = 0 \Rightarrow F_{\mu z}(x,z_0) = 0$ gauge invariant condition.

The e.o.m. for the bulk-to-boundary propagator is

$$z\partial_z\left(\frac{1}{z}\partial_z V(q,z)\right) + q^2 V(q,z) = 0 \quad \Rightarrow$$

$$V(q,z) \propto qz(Y_0(qz_0)J_1(qz) - J_0(qz_0)Y_1(qz))\ ,$$

Two-point Function

The 2-point function defined from the relation:

$$\int d^4x\, e^{iq\cdot x}\langle J_\mu(x)J_\nu(0)\rangle = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Sigma(q^2)$$

AdS/QCD predicts for the scalar part of the 2-point function:

$$\Sigma(q^2) = -\frac{1}{g_5^2}\left(\frac{1}{z}\frac{\partial_z V(q,z)}{V(q,\epsilon)}\right)\Bigg|_{z=\epsilon\to 0} \Rightarrow \sum_{n=1}^{\infty}\frac{f_n^2}{q^2 - M_n^2}\ ,$$

where $M_n = \gamma_{0n}/z_0$ and

$$f_n^2 = \frac{2M_n^2}{g_5^2 z_0^2 J_1^2(\gamma_{0,n})}\ ,$$

since: $\langle 0|J_\mu^a|\rho_n^b\rangle = \delta^{ab} f_n \epsilon_\mu$.

Two-point Function

In the limit $qz_0 \gg 1$

$$\Sigma(q^2) = \frac{1}{2g_5^2} q^2 \ln(q^2\epsilon^2) \ ,$$

by matching with QCD, one finds (EKSS): $g_5^2 = 12\pi^2/N_c$.

For $N_c = 3$ to get $M_1 \equiv M_\rho^{\text{exp}} = 775.8$ MeV, we take $\frac{1}{z_0} = 323$ MeV .

As a result: $f_1 \equiv f_\rho = (392 \text{ MeV})^2$

N.B. $f_\rho{}^{\text{exp}} = (401 \pm 4 \text{ MeV})^2$ (PDG, 2007)

Three-point Function of Vector Currents

For the scalar part of $\langle J_a^\alpha(p_1) J_b^\beta(-p_2) J_c^\mu(q)\rangle$ the AdS/QCD predicts

$$T(p_1^2, p_2^2, Q^2) = \sum_{n,k=1}^{\infty} \frac{f_n f_k F_{nk}(Q^2)}{\left(p_1^2 - M_n^2\right)\left(p_2^2 - M_k^2\right)} ,$$

where

$$F_{nk}(Q^2) = \int_0^{z_0} \frac{dz}{z} \, \mathcal{J}(Q,z) \, \psi_n(z) \, \psi_k(z)$$

correspond to form factors for $n \to k$ transitions, where

$$\mathcal{J}(Q,z) \equiv \frac{V(iQ,z)}{V(iQ,\epsilon)}\Big|_{\epsilon \to 0} = Qz \left[K_1(Qz) + I_1(Qz) \frac{K_0(Qz_0)}{I_0(Qz_0)} \right] .$$

(HRG, Radyushkin, PLB650, 2007)

Eigenfunctions

The eigenfunctions $\psi_n(z)$ obey the equation

$$z\partial_z \left(\frac{1}{z} \partial_z \psi_n(z) \right) + M_n^2 \, \psi_n(z) = 0 \; ,$$

with the b.c.

$$\psi_n(0) = 0 \;\; , \;\; \partial_z \psi_n(z_0) = 0 \; ,$$

and normalized according to

$$\int_0^{z_0} \; \frac{dz}{z} \, |\psi_n(z)|^2 = 1 \; .$$

The tensor structure of the 3-pt function is

$$T^{\alpha\beta\mu} = \eta^{\alpha\mu}(q-p_1)^{\beta} - \eta^{\beta\mu}(p_2+q)^{\alpha} + \eta^{\alpha\beta}(p_1+p_2)^{\mu} \ .$$

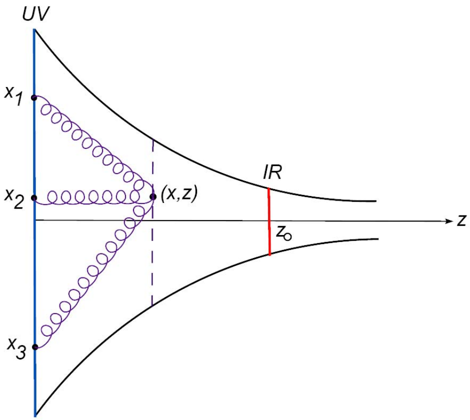


Figure: Schematic representation for the 3-point function

Wave functions

Define "ϕ wave functions" as

$$\phi_n(z) \equiv \frac{1}{M_n z}\, \partial_z \psi_n(z) \ ,$$

then:

$$\phi_n(0) = g_5 f_n / M_n \ , \quad \phi_n(z_0) = 0 \ ,$$

$$\int_0^{z_0} dz\, z\, |\phi_n(z)|^2 = 1 \ ,$$

$\Rightarrow$ ϕ w.f. are analogous of bound state w.f. in QM.

The form factor in terms of ϕ is

$$F_{nn}(Q^2) = \frac{1}{1 + Q^2/2M_n^2} \int_0^{z_0} dz\, z\, \mathcal{J}(Q, z)\, |\phi_n(z)|^2 \ .$$

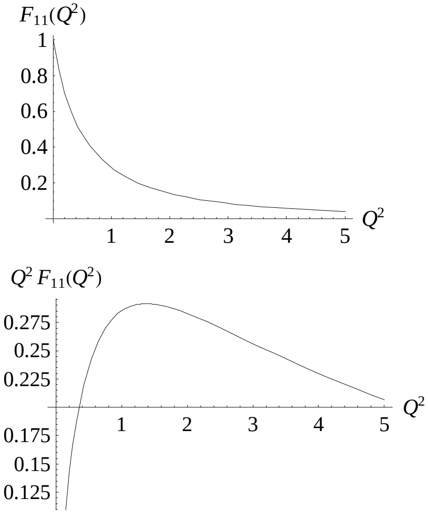


Figure: Plots of $F_{11}(Q^2)$ and $Q^2F_{11}(Q^2)$ as a function of Q^2 (GeV^2).

Form Factors

In general

$$\langle\rho^+(p_2,\epsilon')|J^\mu_{\mathrm{EM}}(0)|\rho^+(p_1,\epsilon)\rangle$$

$$= -\epsilon'_\beta\epsilon_\alpha\Big[\,\eta^{\alpha\beta}(p_1^\mu+p_2^\mu)\,G_1(Q^2)$$

$$+(\eta^{\mu\alpha}q^\beta-\eta^{\mu\beta}q^\alpha)(G_1(Q^2)+G_2(Q^2))$$

$$-\frac{1}{M^2}q^\alpha q^\beta(p_1^\mu+p_2^\mu)\,G_3(Q^2)\,\Big]\,.$$

AdS/QCD model predicts $G_1^{(n)}(Q^2)=G_2^{(n)}(Q^2)=F_{nn}(Q^2)$, and $G_3^{(n)}(Q^2)=0$ for form factors $G_i^{(n)}(Q^2)$ of n^{th} bound state.

Form Factors

Electric G_C, magnetic G_M and quadrupole G_Q form factors are

$$G_C^{(n)}(Q^2) \equiv G_1^{(n)} + \frac{Q^2}{6M_n^2} G_Q^{(n)} = \left(1 - \frac{Q^2}{6M_n^2}\right) F_{nn}(Q^2) \ ,$$
$$G_M^{(n)}(Q^2) \equiv G_1^{(n)} + G_2^{(n)} = 2F_{nn}(Q^2) \ ,$$
$$G_Q^{(n)}(Q^2) \equiv \left(1 + \frac{Q^2}{4M_n^2}\right) G_3^{(n)} - G_2^{(n)} = -F_{nn}(Q^2) \ .$$

For ρ meson ($n = 1$)

$$e \equiv G_C^{(1)}(0) = 1 \ ,$$
$$\mu \equiv G_M^{(1)}(0) = 2 \ ,$$
$$DM_\rho^2 \equiv G_Q^{(1)}(0) = -1 \ .$$

these are canonical values for a vector particle *(Brodsky, PRD46, 1992)*.

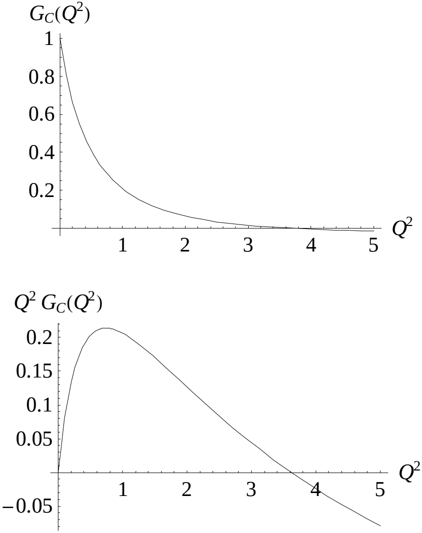


Figure: Plots of $G_C(Q^2)$ and $Q^2 G_C(Q^2)$ as a function of Q^2 (GeV^2).

Light-Cone Formalism vs. Holography

The light-cone form factor is:

$$\mathcal{F}(Q^2) \equiv G_1(Q^2) + \frac{Q^2}{2M^2}\, G_2(Q^2) - \left(\frac{Q^2}{2M^2}\right)^2 G_3(Q^2)$$

$$= \int_0^{z_0} dz\, z\, \mathcal{J}(Q,z)\, |\phi_n(z)|^2 \, .$$

This is a "$+\,+\,+$" component of the 3-point correlator obtained by convoluting it with $n_\alpha n_\beta n_\mu$, where $n^2 = 0$, $(np_1) = 1$, $(nq) = 0$, *(Radyushkin, PLB642, 2006)*.

Low-Q^2 behavior.

The electric form factor in the $Qz_0 \ll 1$ limit is

$$G_C^{(1)}(Q^2) \approx 1 - 1.359\,\frac{Q^2}{M^2} + 1.428\,\frac{Q^4}{M^4} + \mathcal{O}(Q^6)\ .$$

For the electric radius of the ρ-meson this gives

$$\langle r_\rho^2 \rangle_C = 0.53\,\mathrm{fm}^2\ .$$

This value is very close to the results from DSE approach *(Bhagwat, 2006)* and lattice calculations *(Lasscock, 2006)*.

VMD pattern

The $F_{11}(Q^2)$ form factor can be written in the generalized VMD representation

$$F_{11}(Q^2) = \sum_{m=1}^{\infty} \frac{F_{m,11}}{1 + Q^2/M_m^2} \ ,$$

with the coefficients $F_{m,11} = \{1.237, \ -0.239, \ 0.002, \ ...\}$.

At $Q^2 = 0$ the normalization of the ρ meson form factor is almost completely saturated by the first two bound states:

$$F_{11}(0) = F_{1,11} + F_{2,11} + \ ... = 0.998 + \ ...$$

Summary I

- We described the formalism that allows to study form factors of vector mesons in the AdS/QCD model with hard-wall cutoff *(HRG, Radyushkin, PLB650, 2007)*.
- Analogous formalism for the soft-wall model *(Karch, Katz, Son and Stephanov, PRD74, 2006)* with dilatonic field $\chi(z) = z^2$ was developed in *(HRG, A. Radyushkin, PRD76, 2007)*.
- We introduced "ϕ wave functions" that have the properties necessary for the light-cone interpretation proposed in *(Brodsky, PRL96, 2006)* and discussed in *(Radyushkin, PLB642, 2006)*.
- The calculated electric radius is: $\langle r_\rho^2 \rangle_C = 0.53\,\text{fm}^2$.
- AdS/QCD model predicts a very specific VMD pattern. Form factors are given by contributions due to the first two bound states.
- The addition of dimension six terms gives three different form factors and more realistic values for μ and D *(HRG, 2007)*.

Pion in AdS/QCD

- We describe a formalism to calculate the pion form factor in the chiral limit of QCD with $N_f = 2$.
- We also study the behavior of the f_π and $\langle r_\pi^2 \rangle_C$ in various regions of the holographic parameters space (z_0, σ).
- The holographic dictionary in the axial gauge ($A_z = 0$) is:

$$J^a_{A\mu}(x) = \bar{q}(x)\gamma_\mu\gamma_5 t^a q(x) \longrightarrow A^a_\mu(x, z)$$

$$\Sigma^{\alpha\beta}(x) = \langle \bar{q}^\alpha_L(x) q^\beta_R(x) \rangle \longrightarrow \frac{2}{z} X^{\alpha\beta}(x, z)$$

Initial Setup

Axial-vector and pseudoscalar sectors of the hard-wall model (EKSS) are described by the action

$$S^{A}_{\mathrm{AdS}} = \mathrm{Tr}\int d^4x\, dz\, \sqrt{g}\left[|DX|^2 + 3|X|^2 - \frac{1}{4g_5^2}F_A^2\right] ,$$

where $DX = \partial X - iA_L X + iXA_R$, ($A_{L(R)} = V \pm A$) and

$$X(x,z) = \frac{1}{2}v(z)U(x,z) ,$$

$$U(x,z) = \exp\left(2it^a\pi^a(x,z)\right) ,$$

$$v(z) = m_q z + \sigma z^3 ,$$

m_q– quark mass and σ– quark condensate.

Initial Setup

Expanding $U(x,z) \Rightarrow$

$$S^{A\,(2)}_{\mathrm{AdS}} = \mathrm{Tr} \int d^4x\, dz \left[-\frac{1}{4g_5^2 z} A^{MN} A_{MN} + \frac{v^2(z)}{2z^3} (A^a_M - \partial_M \pi^a)^2 \right]$$

In general $A = A_\perp + A_{||}$, where $A_\perp$ and $A_{||}$ are transverse and longitudinal components of the axial-vector field.

SSB causes $A_{||}$ to be physical and associated with the GB – pion.

The $||$ component may be written as

$$A^a_{M||}(x,z) = \partial_M \psi^a(x,z) \Rightarrow \psi^a(x,z) \leftrightarrow \text{pion field}$$

Equations of Motion

Varying the action with respect to $A^a_{\perp\mu}(x,z)$ and representing the Fourier image of $A^a_{\perp\mu}(x,z)$ as $\tilde{A}^a_{\perp\mu}(p,z) = \mathcal{A}(p,z)A^a_\mu(p)$ we will get

$$\left[z^3\partial_z\left(\frac{1}{z}\partial_z\mathcal{A}\right) + p^2z^2\mathcal{A} - g_5^2v^2\mathcal{A}\right] = 0\ ,$$

with b.c. $\mathcal{A}(p,0) = 1$ and $\mathcal{A}'(p,z_0) = 0$. Remember that $v(z) = \sigma z^3$. Variation with respect to the longitudinal component $\partial_\mu\psi^a$ gives

$$z^3\partial_z\left(\frac{1}{z}\,\partial_z\psi^a\right) - g_5^2v^2\left(\psi^a - \pi^a\right) = 0\ .$$

Finally, varying with respect to A_z produces

$$p^2z^2\partial_z\psi^a - g_5^2v^2\partial_z\pi^a = 0\ ,$$

with b.c. $\partial_z\psi(z_0) = 0$, $\psi(\epsilon) = 0$ and $\pi(\epsilon) = 0$.

Equations of Motion

In the chiral limit, the equation for ψ becomes

$$z^3 \partial_z \left(\frac{1}{z} \partial_z \Psi \right) - g_5^2 v^2 \Psi = 0$$

where $\Psi \equiv \psi - \pi$ and since $\pi = -1 \Rightarrow \Psi(\epsilon) = 1$ and $\Psi'(z_0) = 0 \Rightarrow$

$$\Psi(z) = \mathcal{A}(0, z) \, .$$

It is useful to define the conjugate w.f. $\Phi(z)$ as

$$\Phi(z) = -\frac{1}{g_5^2 f_\pi^2} \left(\frac{1}{z} \partial_z \Psi(z) \right) \, ,$$

then $\Phi(0) = 1$ and $\Phi(z_0) = 0$.

Definitions

The spectrum in the axial-current channel consists of the pion

$$\langle 0|J_A^\alpha|\pi(p)\rangle = if_\pi p^\alpha$$

and axial-vector mesons

$$\langle 0|J_A^\alpha|A_n(p,\sigma)\rangle = F_{A,n}\epsilon_n^\alpha(p,\sigma)\ ,$$

where $F_{A,n}$ correspond to the n^{th} axial-vector meson decay constant.

Two-point Function

The two-point function for the axial-vector currents can be written as

$$\langle J_A^\alpha(p) J_A^\beta(-p) \rangle = p^\alpha p^\beta \frac{f_\pi^2}{p^2} + \left(-\eta^{\alpha\beta} + \frac{p^\alpha p^\beta}{p^2} \right) \sum_n \frac{F_{A,n}^2}{p^2 - M_{A,n}^2} + \cdots ,$$

in which the second term on the rhs is explicitly transverse to p.

As noted in AdS/QCD model (EKSS):

$$f_\pi^2 = -\frac{1}{g_5^2} \left(\frac{1}{z} \partial_z \mathcal{A}(0,z) \right)_{z=\epsilon\to 0} .$$

Three-Point Function

To obtain the pion form factor, we need to calculate three-point correlation function.

Correlator should include the external EM current $J_\mu^{el}(0)$ and currents $J^a_{5\alpha}(x_1), J^{a\dagger}_{5\beta}(x_2)$ having nonzero projection onto the pion states $\Rightarrow$

$$\mathcal{T}_{\mu\alpha\beta}(p_1,p_2) = \int d^4x_1 \int d^4x_2 \, e^{ip_1x_1 - ip_2x_2} \, \langle 0|\mathcal{T} J^\dagger_{5\beta}(x_2) J^{\mathrm{el}}_\mu(0) J_{5\alpha}(x_1)|0\rangle \, ,$$

The momentum transfer carried by the EM source is $q = p_2 - p_1$ ($q^2 = -Q^2 \leq 0$).

Three-Point Function

The spectral representation for the three-point function is

$$\mathcal{T}^{\mu\alpha\beta}(p_1,p_2) = p_1^\alpha p_2^\beta (p_1+p_2)^\mu \frac{f_\pi^2 F_\pi(Q^2)}{p_1^2 p_2^2} + \sum_{n,m} (\perp \text{ terms}) + \cdots ,$$

1^{st} term is longitudinal to p_1^α and p_2^β and contains pion form factor

$$\langle \pi(p_1)|J_\mu^{el}(0)|\pi(p_2)\rangle = F_\pi(q^2)(p_1+p_2)_\mu .$$

Other pole terms are transverse to p_1^α or $p_2^\beta \Rightarrow$

$$p_{1\alpha} p_{2\beta} \mathcal{T}^{\mu\alpha\beta}(p_1,p_2)|_{p_1^2=0, p_2^2=0} = (p_1+p_2)^\mu f_\pi^2 F_\pi(Q^2) .$$

Three-Point Function

To obtain form factor from the holographic model, we need the action at the third order in the fields.

There are two types of terms contributing to the pion electromagnetic form factor: $|DX|^2$ term and F^2 terms.

The part of F^2 term which contributes to $\langle J_{5\alpha} J_\mu J_{5\beta} \rangle$ is:

$$W_3 = \frac{i}{g_5^2} \,\mathrm{Tr} \int d^4x\, dz\, \frac{1}{z} \left(V_{\mu\nu} [A^\mu, A^\nu] + A_{\mu\nu} [V^\mu, A^\nu] \right).$$

Similarly, the relevant part from $|DX|^2$ has the form:

$$U_3 = \epsilon_{abc} \int d^4x\, dz\, \left[\frac{v^2(z)}{z^3} (A_M^a - \partial_M \pi^a)\, \pi^b\, V^{c\,M} \right],$$

Three-Point Function

Varying W_3 and U_3 terms with respect to the sources and representing

$$\langle J^{\mu}_{V,a}(q) J^{\alpha}_{\|A,b}(p_1) J^{\beta}_{\|A,c}(-p_2)\rangle =$$

$$= i(2\pi)^4 \delta^{(4)}(q + p_1 - p_2)\, \epsilon_{abc} \mathcal{T}^{\mu\alpha\beta}(p_1, p_2) \, ,$$

we get for the total form factor:

$$F_\pi(Q^2) = \frac{1}{g_5^2 f_\pi^2} \int_0^{z_0} dz\, z\, \mathcal{J}(Q, z) \left[\left(\frac{\partial_z \Psi}{z} \right)^2 + \frac{g_5^2 v^2}{z^4} \Psi^2(z) \right] \, .$$

(HRG, Radyushkin, PRD76, 2007)

Form Factor

It can be shown that

$$F_\pi(Q^2) = -\int_0^{z_0} dz\, \mathcal{J}(Q,z)\, \partial_z \Big(\Psi(z)\, \Phi(z) \Big) \Rightarrow$$

$$F_\pi(0) = -\int_0^{z_0} dz\, \partial_z \Big(\Psi(z)\, \Phi(z) \Big) = \Psi(0)\, \Phi(0) = 1 \;,$$

since $\mathcal{J}(0,z) = 1$ and other BC. We can also write $F_\pi(Q^2)$ as

$$\begin{aligned} F_\pi(Q^2) &= \int_0^{z_0} dz\, z\, \mathcal{J}(Q,z) \left[g_5^2 f_\pi^2 \Phi^2(z) + \frac{\sigma^2}{f_\pi^2}\, z^2\, \Psi^2(z) \right] \\ &\equiv \int_0^{z_0} dz\, z\, \mathcal{J}(Q,z)\, \rho(z) \;, \end{aligned}$$

and interpret the function $\rho(z)$ as the radial distribution density.

Form Factor

The function $\phi_\pi(z) \equiv g_5 f_\pi \Phi(z)$ is an analog of $\rho-$meson w.f.

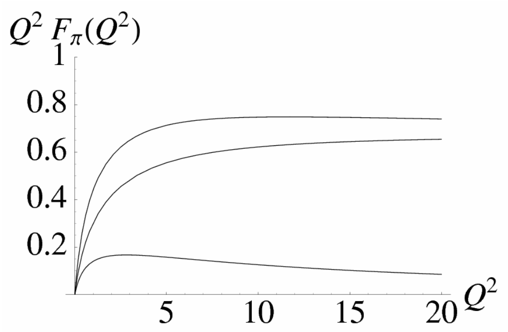


Figure: Contributions to the $Q^2 F_\pi(Q^2)$ from Ψ-term (lower curve), from Φ-term (middle curve) and total contribution (upper curve).

Density Function

$$a(z_0, \alpha) \equiv \alpha z_0^3 = \frac{1}{3}\, g_5 \sigma z_0^3, \quad a_0 \equiv a(z_0 = \tfrac{1}{323}, \alpha = 424) = 2.26$$

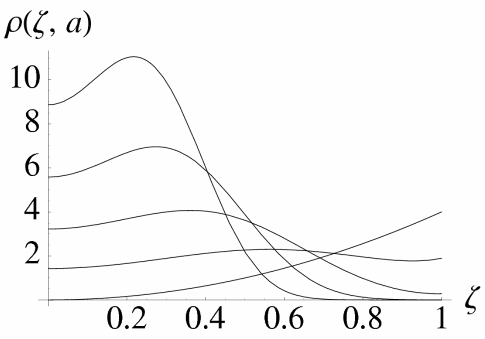


Figure: Function $\rho(\zeta, a)$ for $a = 0$, $a = 1$, $a = 2.26$, $a = 5$ and $a = 10$.

Density Function

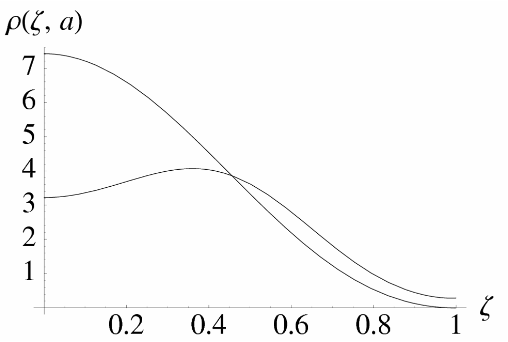


Figure: Densities $\rho(\zeta, 2.26)$ for pion and $\rho_\rho(\zeta)$ for ρ-meson in the hard-wall model.

Decay Constant

Explicitly

$$\Phi(z) = -\frac{1}{g_5^2 f_\pi^2}\left(\frac{1}{z}\,\partial_z \Psi(z)\right)$$

$$= \frac{3\,z^2}{g_5^2 f_\pi^2}\,\Gamma\,[2/3]\left(\frac{\alpha^4}{2}\right)^{1/3}\left[-I_{2/3}\left(\alpha z^3\right) + I_{-2/3}\left(\alpha z^3\right)\frac{I_{2/3}\left(\alpha z_0^3\right)}{I_{-2/3}\left(\alpha z_0^3\right)}\right]\,,$$

where $\alpha = g_5\sigma/3 \approx 1.481\,\sigma$ ($g_5 = \sqrt{2}\pi$).

The formula $\Phi(z_0) = 0$ establishes the relation

$$f_\pi^2 = 3\cdot 2^{1/3}\,\frac{\Gamma[2/3]}{\Gamma[1/3]}\,\frac{I_{2/3}\left(\alpha z_0^3\right)}{I_{-2/3}\left(\alpha z_0^3\right)}\,\frac{\alpha^{2/3}}{g_5^2}$$

Decay Constant

For sufficiently large values of the confinement radius, $z_0 \gtrsim 1/\alpha^{1/3}$,

$$f_\pi|_{z_0\to\infty} \approx \frac{\alpha^{1/3}}{3.21} .$$

Requiring that $f_\pi|_{z_0\to\infty} = f_\pi^{\exp} \approx 131\,\text{MeV} \Rightarrow \alpha^{1/3} \approx 420\,\text{MeV}$.

To get $f_\pi \approx 131\,\text{MeV}$ from the exact formula for $1/z_0 = 323\,\text{MeV}$, we should take $\alpha^{1/3} \approx 424\,\text{MeV} \equiv \alpha_0^{1/3}$.

Define $\;a \equiv \alpha z_0^3 = \frac{1}{3}\, g_5 \sigma z_0^3 \;\;\Rightarrow\;\; a = 2.26 \equiv a_0\;$ for $\alpha_0^{1/3} = 424\,\text{MeV}$ and $1/z_0^\rho = 323\,\text{MeV}$.

Decay Constant

$$f_\pi \Big|_{a \gtrsim 2} \approx 0.311\,\alpha^{1/3}\,, \qquad f_\pi \Big|_{a \lesssim 1} \approx 0.338\,\frac{a}{z_0}\,.$$

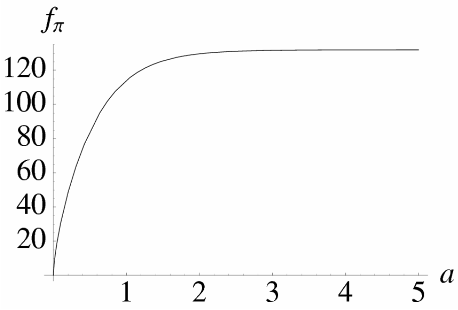


Figure: Pion decay constant f_π as a function of a for fixed $\alpha^{1/3} = 424\,\text{MeV}$.

Electric Radius

The Q^2-expansion of the vector source takes a form

$$\mathcal{J}(Q,\zeta,z_0) = 1 - \frac{Q^2}{4} z_0^2 \zeta^2 \left[1 - 2 \ln \zeta \right] + \ldots$$

Using it, we obtain for the pion charge radius:

$$\langle r_\pi^2 \rangle = \frac{4}{3} z_0^2 \left\{ 1 - \frac{a^2}{4} + \mathcal{O}(a^4) \right\} .$$

Electric Radius

For fixed z_0 and small $a \ll 1$, the radius is determined by z_0 alone.

$$\langle r_\pi^2 \rangle|_{\alpha=0} = \frac{4}{3} z_0^2 \,.$$

For $z_0 = z_0^\rho \approx 1/323\,\mathrm{MeV} = 0.619\,\mathrm{fm} \Rightarrow \langle r_\pi^2 \rangle = 0.51\,\mathrm{fm}^2$.

Increasing α pion becomes smaller. The experimental value of $0.45\,\mathrm{fm}^2$ is reached for $a \sim 0.9$. However, the corresponding value $f_\pi \approx 80\,\mathrm{MeV}$ is too small!

If we take $a = a_0 = 2.26$, then $\langle r_\pi^2 \rangle = 0.34\,\mathrm{fm}^2$. The pion radius is smaller than the experimental value.

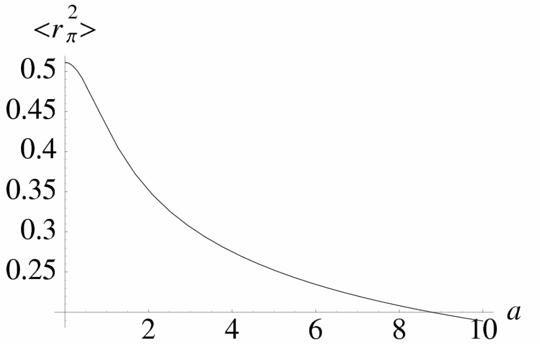


Figure: $\langle r_\pi^2 \rangle$ in fm^2 as a function of a for $z_0 = z_0^\rho \approx 0.619\,\mathrm{fm}$.

Electric Radius

For large a, it follows

$$\langle r_\pi^2 \rangle \Big|_{a \gtrsim 2} = \frac{3}{2\pi^2 f_\pi^2} \left[1 + \frac{1}{3} \ln \left(\frac{\alpha z_0^3}{2.54} \right) \right] \; .$$

In the limit $z_0 \to \infty$, we have $\langle r_\pi^2 \rangle \to \infty \Leftarrow m_\pi = 0$.

In $SU(2)$ ChPT, the radius is

$$\langle r_\pi^2 \rangle_{\rm ChPT} = \frac{1}{8\pi^2 f_\pi^2} \left(\bar{\ell}_6 - 1 \right) + \mathcal{O}(m_\pi^2) \; ,$$

where $\bar{\ell}_6 \simeq 16.5 \pm 1.1$ (from $\langle r_\pi^2 \rangle_{\rm exp}$) and $\bar{\ell}_6 \simeq 13.0$ (NJL).

(Tarrach, Z.Phys.C2, 1979; Hippe, PRC52, 1995.)

Form Factor at Large Q^2

The large-Q^2 behavior of $F_\pi(Q^2)$ is determined by

$$F_\pi(Q^2) \to \frac{2\,\rho(0)}{Q^2} = \frac{4\pi^2 f_\pi^2}{Q^2}\ .$$

AdS/QCD predicts $Q^2 F_\pi(Q^2) \to 4\pi^2 f_\pi^2 \approx 0.68\,\mathrm{GeV}^2$.

However, JLab experiment gives $Q^2 F_\pi^{\exp}(Q^2) \approx 0.4\,\mathrm{GeV}^2$!

This may be a signal that we are reaching a region where AdS/QCD models should not be expected to work.

$Q^2 F_\pi(Q^2) \to 4\pi^2 f_\pi^2 \approx 0.68\,\mathrm{GeV}^2$, $(0.4\,\mathrm{GeV}^2$ if $f_\pi \approx 101$ MeV$) \Rightarrow$
$a = 1.26 \Rightarrow \langle r_\pi^2 \rangle = 0.43\ \mathrm{fm}^2$.

Experimental Data

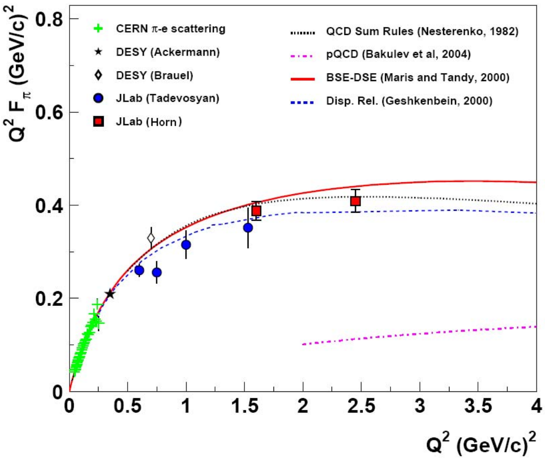


Figure: The plot is taken from T. Horn *et al.* [Jefferson Lab F(pi)-2 Collaboration], "Determination of the Charged Pion Form Factor at $Q^2 = 1.60$ and $2.45\ (\mathrm{GeV}/c)^2$," Phys. Rev. Lett. **97**, 192001 (2006).

Summary II

- We described a formalism that allows to extract pion form factor in the framework of the AdS/QCD model *(HRG, Radyushkin, PRD76, 2007)*.
- $a = g_5 \sigma z_0^3/3$ determines regions where pion is either governed by confinement or by ChSB ($a_0 = 2.26$).
- E.g. for $a > 2 \Rightarrow f_\pi = f_\pi(\sigma)$ and $f_\pi^2 \langle r_\pi^2 \rangle$ depends only on $\ln a/a_0$
- For $a < 1 \Rightarrow f_\pi \sim a/z_0$ and $\langle r_\pi^2 \rangle = \frac{4}{3} z_0^2$.
- In AdS/QCD $\langle r_\pi^2 \rangle = 0.34\,\mathrm{fm}^2 < 0.45\,\mathrm{fm}^2$ (experimental). "Softening" the IR wall can solve the problem.
- The model is not good for Q^2 above $1\,\mathrm{GeV}^2$. We think this is due to the possibility that for $Q^2 > 1\ \mathrm{GeV}^2$, the quark substructure of the pion may be resolved by the electromagnetic probe.

Anomalous Amplitude

- It is known from *(Witten, 1983)* that $\pi \to \gamma\gamma$ anomalous decay can be incorporated into the low energy theory by gauging the WZW term $\Gamma(U, A_L, A_R)$ with $A_L = A_R = QA$, where $Q = \mathrm{diag}\{2/3,\ -1/3\}$.
- On the other hand the WZW term naturally arises from the suitably compactified 5D theory with CS term built of YM gauge fields *(Hill, PRD73, 2006)*.
- Then it is expected that 5D holographic dual model of QCD with CS term, should naturally reproduce the anomaly.
- Since the CS (WZW) term for $SU(2)$ gauge (global) group is vanishing, the flavor symmetry is extended to $U(2)_L \times U(2)_R$. Therefore, we can write fields as: $\mathcal{B}_\mu = t^a B^a_\mu + \frac{1}{2}\hat{B}_\mu$.

Initial Setup

The 4D isovector $J_\mu^{\{I=1\},a}(x)$ and isosinglet vector $J_\mu^{\{I=0\}}(x)$ currents correspond to:

$$J_\mu^{\{I=1\},3} = \frac{1}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right) = \bar{q}\gamma_\mu \frac{\sigma^3}{2} q \to V_\mu^3(x,z) \; ,$$
$$J_\mu^{\{I=0\}} = \frac{1}{2}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d\right) = \frac{1}{2}\bar{q}\gamma_\mu \mathbf{1} q \to \hat{V}_\mu(x,z) \; ,$$

and the electromagnetic current

$$J_\mu^{\mathrm{EM}} = J_\mu^{\{I=1\},3} + \frac{1}{3} J_\mu^{\{I=0\}}$$

has both isovector ("ρ-type") and isosinglet ("ω-type") terms. The matrix element $\langle 0|J^{\mathrm{EM}} J^{\mathrm{EM}}|\pi^0\rangle$ is nonzero since it contains $\langle 0|J^{\{I=1\},3} J^{\{I=0\}}|\pi^0\rangle \leftrightarrow \langle 0|J^{\{I=1\},3} J^{\{I=0\}} J_A^3|0\rangle \sim \mathrm{Tr}(\sigma^3\sigma^3)$ part.

Anomalous Form Factor

The $\pi^0\gamma^*\gamma^*$ form factor is defined by

$$\int \langle \pi, p | T \left\{ J^\mu_{\mathrm{EM}}(x)\, J^\nu_{\mathrm{EM}}(0) \right\} |0\rangle e^{-iq_1 x} d^4x$$

$$= \epsilon^{\mu\nu\alpha\beta} q_{1\,\alpha} q_{2\,\beta}\, F_{\gamma^*\gamma^*\pi^0}\left(Q_1^2, Q_2^2\right) \ .$$

Its value for real photons

$$F_{\gamma^*\gamma^*\pi^0}(0,0) = \frac{N_c}{12\pi^2 f_\pi}$$

is related in QCD to the axial anomaly.

Holographic Action with CS Term

The $\mathcal{O}(B^3)$ part of the 5D CS action, in the axial gauge $B_z = 0$ can be written as

$$S_{\text{CS}}^{(3)}[\mathcal{B}] = k\frac{N_c}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\text{Tr}\int d^4x\,dz\,(\partial_z\mathcal{B}_\mu)\left[\mathcal{F}_{\nu\rho}\mathcal{B}_\sigma + \mathcal{B}_\nu\mathcal{F}_{\rho\sigma}\right],$$

where $k = 2$ to reproduce the QCD anomaly result.
Then, in the AdS/QCD model (cnf. *Domokos and Harvey, PRL99, 2007*) the CS term is:

$$S_{\text{CS}}^{\text{AdS}}[\mathcal{B}_L, \mathcal{B}_R] = S_{\text{CS}}^{(3)}[\mathcal{B}_L] - S_{\text{CS}}^{(3)}[\mathcal{B}_R]\,.$$

Holographic Action with CS Term

Taking into account that $\mathcal{B}_{L,R} = \mathcal{V} \pm \mathcal{A}$, and keeping only the longitudinal component of the axial-vector field $A = A_{\|}$ (that brings in the pion), for which $F^{A}_{\mu\nu} = 0$, we will have

$$\begin{aligned} S^{\rm AdS}_{\rm CS} &= \frac{N_c}{12\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4x \int_0^{z_0} dz \\ &\times \left[\left(\partial_\rho V^a_\mu\right) \left(A^a_{\|\sigma} \overleftrightarrow{\partial_z} \hat{V}_\nu \right) + \left(\partial_\rho \hat{V}_\mu\right) \left(A^a_{\|\sigma} \overleftrightarrow{\partial_z} V^a_\nu \right) \right] , \end{aligned}$$

where $\overleftrightarrow{\partial_z} \equiv \overrightarrow{\partial_z} - \overleftarrow{\partial_z}$. After integration by parts, taking appropriate care on the IR boundary, we get:

$$S = -\frac{N_c}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4x \int_0^{z_0} dz \left(\partial_z A^a_{\|\sigma}\right) \left(\partial_\rho V^a_\mu\right) \hat{V}_\nu \,.$$

Anomalous Holographic Form Factor

Varying S we get the 3-point function:

$$\langle J_\alpha^{A,3}(-p) J_\mu^{\mathrm{EM}}(q_1) J_\nu^{\mathrm{EM}}(q_2)\rangle = T_{\alpha\mu\nu}(p,q_1,q_2)\, i(2\pi)^4 \delta^{(4)}(q_1+q_2-p)\ ,$$

where

$$T_{\alpha\mu\nu}(p,q_1,q_2) = \frac{N_c}{12\pi^2}\frac{p_\alpha}{p^2}\,\epsilon_{\mu\nu\rho\sigma}\, q_1^\rho q_2^\sigma K(Q_1^2,Q_2^2)$$

and

$$K(Q_1^2,Q_2^2) = -\int_0^{z_0} \mathcal{J}(Q_1,z)\mathcal{J}(Q_2,z)\,\partial_z\psi(z)\,dz\ .$$

Here, p is the momentum of the pion and q_1, q_2 are the momenta of photons ($\psi(z)$ is the pion wave function).

(HRG, Radyushkin, in preparation)

Conforming to Anomaly

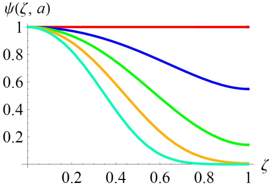
From QCD, we expect $K^{\text{QCD}}(0,0) = 1$. However, AdS/QCD gives:

$$K_b(0,0) = -\int_0^{z_0} \partial_z \psi(z)\, dz = -\psi(z_0) = \Big[1 - \Psi(z_0)\Big] \,,$$

$$\Psi(z_0) = \frac{\sqrt{3}\,\Gamma\left(2/3\right)}{\pi I_{-2/3}(a)} \left(\frac{1}{2a^2}\right)^{1/3} \,.$$

For $a = a_0 = 2.26$, we have $\Psi(z_0) = 0.14$ (e.g., $\Psi(z_0)|_{a=4} \approx 0.02$).

N.B. Plot of $\Psi(\zeta = z/z_0, a)$ for: $a = 0$ (uppermost line), $a = 1$, $a = 2.26$, $a = 5$, $a = 10$ (lowermost line) is given below.



Conforming to Anomaly

Since $\Psi(z_0) \neq 0$, for finite a, it seems to be impossible to reproduce exactly the anomaly of QCD.

To fix this problem, we add an IR surface term, such that,

$$K(Q_1^2, Q_2^2) = \Psi(z_0)\mathcal{J}(Q_1, z_0)\mathcal{J}(Q_2, z_0) - \int_0^{z_0} \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\, \partial_z \Psi(z)\, dz\,.$$

In this case, we have $K(0,0) = 1$!

Notice, that for large $Qz_0 \gg 1$, we have

$$\mathcal{J}(Q, z_0) = \frac{1}{I_0(Qz_0)} \sim e^{-Qz_0}\,.$$

Small Virtualities

For $Q_1^2 = 0$ and $Q_2^2 = Q^2 \ll 1/z_0^2$

$$K(0, Q^2) \simeq 1 - 0.66\,\frac{Q^2 z_0^2}{4} \simeq 1 - 0.96\,\frac{Q^2}{m_\rho^2}\ .$$

The predicted slope is $\simeq 1/m_\rho^2$ which is expected from naive VMD.

Experimentally, the slope for small timelike Q^2 is measured through the Dalitz decay $\pi^0 \to e^+e^-\gamma$.

Predictions

The usual representation of the results is

$$a_\pi \equiv -m_\pi^2 \left(\frac{dK(0,Q^2)}{dQ^2} \right)_{Q^2 \to 0} .$$

For the Q^2-slope AdS/QCD model predicts: $a_\pi \approx 0.031$.

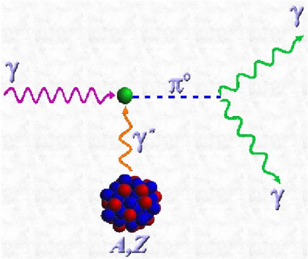
This number is not very far from the central values of two last experiments, $a_\pi = 0.026 \pm 0.024 \pm 0.0048$ (*Farzanpay, 1992*), $a_\pi = 0.025 \pm 0.014 \pm 0.026$ (*Meijer, Drees, 1992*), but the experimental errors are rather large.

The CELLO collaboration (*Behrend, 1990*) gives the value $a_\pi = 0.0326 \pm 0.0026$ that is very close to our result.

In the spacelike region, the data are available only for the values $Q^2 \gtrsim 0.5\ \mathrm{GeV}^2$ (*CELLO, Behrend, 1990*) and $Q^2 \gtrsim 1.5\ \mathrm{GeV}^2$ (*CLEO, Gronberg, 1997*) which cannot be treated as very small.

PRIMEX experiment at JLab

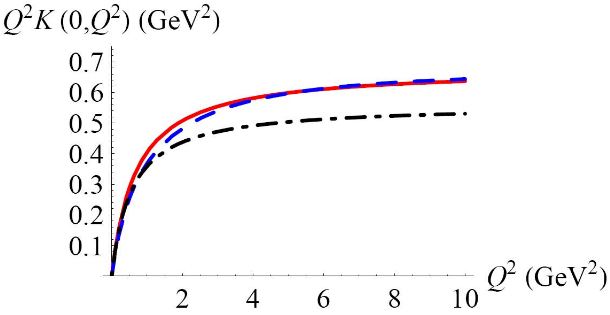
It would be interesting to have data on the slope from the spacelike region of very small Q^2, which may be obtained by modification of the PRIMEX experiment at JLab.



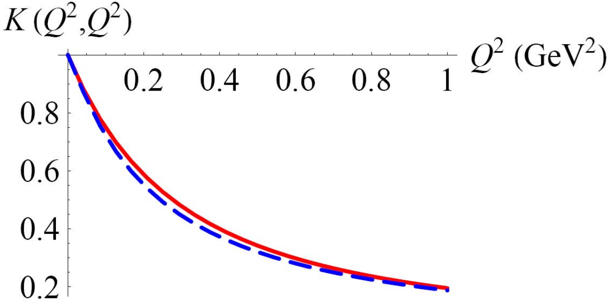
The aim of the Primakoff Experiment (PRIMEX) is to perform a precise measurement of π^0 lifetime from the Primakoff effect (using the small angle coherent photoproduction of the π^0 in the Coulomb field of a nucleus). The figure is taken from www.jlab.org/primex/.

From Small to Large Virtualities

Function $Q^2 K(0, Q^2)$ in AdS/QCD model (solid curve, red online) and in local quark hadron duality model, coinciding with Brodsky-Lepage interpolation formula $1/(1 + Q^2/s_0)$, where $s_0 = 0.68$ GeV^2 (dashed curve, blue online). The monopole fit of CLEO data is shown by dash-dotted curve (black online).



Form factor $K(Q^2, Q^2)$ in AdS/QCD model (solid curve, red online) compared to the local quark-hadron duality model prediction (dashed curve, blue online).



Summary III

- We showed *(HRG, Radyushkin '08)* that by including the 5D CS term into the holographic action and extending the symmetry group with appropriately defined dictionary the anomalous decay amplitude of the pion can be incorporated.
- Although we get non zero result for the anomalous amplitude, the existence of the IR wall at finite z spoils the normalization of form factor. However, when the wall is taken to infinity, the normalization is recovered.
- The slope predicted from the AdS/QCD model is in a good agreement with experiment.
- Surprisingly, it appears that the predictions of the AdS/QCD model are almost perfect in the domain of pQCD.

Baryon as a Skyrmion

In the large N_c limit, two flavor QCD only describes weakly interacting pions, with the coupling constant $g \sim 1/\sqrt{N_c}$. In this limit baryons may only appear non-perturbatively as solitonic objects, with masses $M \sim 1/g^2 \sim N_c$.
The leading order non-linear $\sigma-$model lagrangian

$$\mathcal{L}_2 = \frac{f_\pi^2}{4} \mathrm{Tr} \left(\partial_\mu U \partial^\mu U^\dagger \right) \ ,$$

doesn't give stable solitonic solution (Hobart-Derrick theorem). One of the ways to fix the problem is to add the "Skyrme quartic term"

$$\mathcal{L}_{sk} = \frac{1}{32e^2} \mathrm{Tr} \left[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2 \ .$$

Baryon as a Skyrmion

In order for the action to be finite, the chiral fields should satisfy the following conditions at spacial infinity $|\mathbf{x}| \to \infty$: $U(\mathbf{x}) \to \mathbf{1}$.
These conditions describe topologically non trivial mapping $\mathbf{R}^3 \to S^3 \to SU(2)$ which is characterized by the homotopy group $\pi_3(SU(2)) = \mathbf{Z}$.
The ansatz for the chiral field (Skyrme, 54) is:

$$U(\mathbf{x}) = e^{i\tau_a \hat{x}_a F(r)} ,$$

where $\hat{x}_a = x_a/r$, $r = \sqrt{|\mathbf{x}|^2}$, $F(0) = \pi$ and $F(\infty) = 0$.
Defining $L_i = U^\dagger \partial_i U$, the topological charge which identified with the baryon number B is

$$B = \frac{1}{24\pi^2} \int d^3x \epsilon_{ijk} \operatorname{Tr}\left(L_i L_j L_k\right) .$$

Here, we will show how to derive the Skyrme lagrangian from the 5D YM theory. For this consider the action:

$$S_{YM} = -\frac{1}{4g_5^2}\int d^5x\sqrt{g}\ \mathrm{Tr}\left[L_{MN}L^{MN} + R_{MN}R^{MN}\right] \ ,$$

The gauge fields transform according to

$$\begin{aligned} L_M(x,z) &\to g_L L_M g_L^{-1}(x,z) + i g_L \partial_M g_L^{-1}(x,z) \ , \\ R_M(x,z) &\to g_R R_M g_R^{-1}(x,z) + i g_R \partial_M g_R^{-1}(x,z) \ , \end{aligned}$$

where $g_{L,R}(x,z) \in SU(N_f)_{L,R}$.
Define $L_\mu(x,0) = \ell_\mu(x)$ and $R_\mu(x,0) = r_\mu(x)$, where $\ell_\mu(x)$ and $r_\mu(x)$ are the sources for the left- and right- 4D currents.

In order for the action to be finite at $z = 0$,

$$L_M(x, z \to 0) = i\, U_L^\dagger(x) \partial_M U_L(x) \; ,$$

where $U_L(x) \in SU(N_f)$ (the same for R_M). Therefore, we can partially fix the gauge, so that

$$L_M(x, z \to 0) = 0 \; , \;\; R_M(x, z \to 0) = 0 \; .$$

Nothing changes if we perform additional gauge transformations s.t.

$$\partial_M g^{res}(x, z \to 0) = 0 \; .$$

That is: $g^{res}(x, z)$ goes to constant matrix $g_{L,R} \in SU(N_f)_{L,R}$ at $z = 0$. In the holographic model $(g_L, g_R) \in SU(N_f)_L \times SU(N_f)_R$ corresponds to global chiral symmetry of QCD, at $z = 0$.

It is convenient to define the following path ordered Wilson line:

$$\tilde{\xi}_A(x,z) = P\exp\left\{ i\int_{z_0}^{z} dz' A_z(x,z') \right\} ,$$

where by $A = (L, R)$. Two other objects of immediate interest are defined as: $\tilde{\xi}_A(x,0) = \xi_A(x)$, that is:

$$\xi_L(x) = P\exp\left\{ -i\int_{0}^{z_0} dz' L_z(x,z') \right\} ,$$

$$\xi_R(x) = P\exp\left\{ -i\int_{0}^{z_0} dz' R_z(x,z') \right\} .$$

Here $\xi_{L,R}$ transform with the gauge function $g^{res}(x,z)$ as

$$\xi_{L,R} = g_{L,R}\, \xi_{L,R}\, h(x)^\dagger ,$$

where $h(x) \equiv g^{res}(x, z = z_0)$ is the element of $SU(N_f)$ local gauge symmetry on the 4D IR brane.

Notice, that requiring $g_L^{res}(x, z = z_0) = g_R^{res}(x, z = z_0) = h(x)$, we broke the gauge symmetry in the bulk.
This prescription is similar to the one proposed by *(Hirn and Sanz, 2005)*, where the chiral symmetry is broken by the b.c.

$$L_\mu(x, z_0) - R_\mu(x, z_0) = 0 \ ,$$

from which it automatically follows that the gauge transformations should satisfy the condition: $h(x) = g_L(x) = g_R(x)$. The other b.c. condition, of this model we will adopt is:

$$L_{z\mu}(x, z_0) + R_{z\mu}(x, z_0) = 0 \ ,$$

The chiral field can be written as

$$U(x) = \xi_R(x)\xi_L^{-1}(x) \ ,$$

which transforms as: $U(x) \to g_R U(x) g_L^{-1}$, the same way as the chiral field in the non-linear sigma model with respect to the global chiral transformations.
Therefore, the pion field is a product of the Wilson lines extending from one boundary to the other.
If the vacuum corresponds to $U = \mathbf{1}$, then it is invariant under (g_L, g_R) transformations, only if $g_L = g_R$, that is when the chiral symmetry is broken down to its vector subgroup.

To redefine fields, consider the following combinations of gauge fields

$$\hat{V}_M, \hat{A}_M \equiv \frac{i}{2}\left\{\tilde{\xi}_L^\dagger \left(\partial_M - iL_M\right)\tilde{\xi}_L \pm (L \to R)\right\} ,$$

where $\hat{V}_z = 0, \hat{A}_z = 0$.
In order to separate the *dynamical* fields and external sources, define

$$\begin{aligned} V_\mu(x,z) &\equiv \hat{V}_\mu(x,z) - \hat{V}_\mu(x,0) \ , \\ A_\mu(x,z) &\equiv \hat{A}_\mu(x,z) - \alpha(z)\hat{A}_\mu(x,0) \ . \end{aligned}$$

The b.c. on this redefined field have the following form:

$$\begin{aligned} &V_\mu(x,0) = 0 \ , \ \ \partial_z V_\mu(x,z_0) = 0 \ , \\ &A_\mu(x,0) = 0 \ , \ \ A_\mu(x,z_0) = 0 \ . \end{aligned}$$

To avoid mixing between the pion and the axial resonances:

$$\partial_z \left(\sqrt{g} g^{\mu\nu} g^{zz} \partial_z \alpha(z) \right) = 0 \, .$$

To satisfy b.c. for A, $\alpha(0) = 1$, $\alpha(z_0) = 0$, therefore,

$$\alpha(z) = 1 - \frac{z^2}{z_0^2} \, .$$

Finally, notice, that these redefined fields transform homogeneously under the adjoint representation of $h(x)$

$$\begin{aligned} V_\mu(x,z) &\longmapsto h(x) V_\mu(x,z) h(x)^\dagger \, , \\ A_\mu(x,z) &\longmapsto h(x) A_\mu(x,z) h(x)^\dagger \, . \end{aligned}$$

It is useful to define, the following 4D fields:

$$\begin{aligned} u_\mu(x) &\equiv i\left\{\xi_R^\dagger D_\mu \xi_R - \xi_L^\dagger D_\mu \xi_L\right\} \\ &= i\left\{\xi_R^\dagger(\partial_\mu - ir_\mu)\xi_R - \xi_L^\dagger(\partial_\mu - i\ell_\mu)\xi_L\right\}. \end{aligned}$$

One can show, that this field transforms as: $u_\mu \to h u_\mu h^\dagger$.
The other useful object is

$$\begin{aligned} \Gamma_\mu(x) &\equiv \frac{1}{2}\left\{\xi_R^\dagger(\partial_\mu - ir_\mu)\xi_R + \xi_L^\dagger(\partial_\mu - i\ell_\mu)\xi_L\right\}, \\ \Gamma_\mu &\longmapsto h\Gamma_\mu h^\dagger + ih\partial_\mu h^\dagger, \end{aligned}$$

which transforms as a connection for the 4D transformation $h(x)$.

The $\mathcal{O}(p^4)$ lagrangian is obtained from the terms with two 4D indices in the YM action

$$2\,\mathrm{Tr}\left(R_{\mu\nu}R_{\rho\sigma} + L_{\mu\nu}L_{\rho\sigma}\right) = \mathrm{Tr}\left(F_{+\mu\nu}F_{+\rho\sigma} + F_{-\mu\nu}F_{-\rho\sigma}\right),$$

where, in case, $\ell_\mu = r_\mu = 0$ and $A_\mu = V_\mu = 0$, we have $F_{-\mu\nu} = 0$,

$$u_\mu(x) = i\left\{\xi_R^\dagger \partial_\mu \xi_R - \xi_L^\dagger \partial_\mu \xi_L\right\},$$
$$\Gamma_\mu(x) = \frac{1}{2}\left\{\xi_R^\dagger \partial_\mu \xi_R + \xi_L^\dagger \partial_\mu \xi_L\right\},$$

$$L_{z\mu} = -\frac{1}{2}\left(\partial_z \alpha\right)\xi_L u_\mu \xi_L^\dagger, \qquad R_{z\mu} = \frac{1}{2}\left(\partial_z \alpha\right)\xi_R u_\mu \xi_R^\dagger,$$

$$F_{+\mu\nu} = i\frac{1-\alpha^2}{2}\left[u_\mu, u_\nu\right].$$

Notice, that

$$\xi_R u_\mu \xi_R^\dagger = -iU\partial_\mu U^\dagger \ ,$$
$$\xi_L u_\mu \xi_L^\dagger = -iU^\dagger \partial_\mu U \ .$$

Integrating over the z, we get:

$$S = \int d^4x \, \mathrm{Tr} \left\{ a_1^2 \left(\partial_\mu U^\dagger \partial^\mu U \right) + a_2^2 \left[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2 \right\} \ ,$$

This establishes the relation between the 5D AdS/QCD and the 4D Skyrme models for the two flavors. Note that correspondence of the coefficients should be:

$$a_1^2 = \frac{f_\pi^2}{4} \ , \ \ a_2^2 = \frac{1}{32e^2} \ .$$

As a result, we get:

$$f_\pi^2 = \frac{1}{g_5^2}\int_0^{z_0}\frac{dz}{z}\left(\partial_z\alpha\right)^2 = \frac{2}{g_5^2 z_0^2}\ ,$$

$$\frac{1}{e^2} = \frac{1}{g_5^2}\int_0^{z_0}\frac{dz}{z}\left(1-\alpha^2\right)^2 = \frac{11}{24 g_5^2}\ .$$

Therefore, $f_\pi \simeq 72.7$ MeV and $e \simeq 3.3$ (z_0 is determined from the fit to the ρ-meson physical mass for which $z_0 = 1/(323\text{ MeV})$.
As it is known, for example, from the *(Adkins, Nappi, Witten, 1983)*, the best fit to the hadron masses is for $e = 5.45$ and $f_\pi = 64.5$ MeV (the experimental value is $f_\pi = 92.4$MeV).

Summary IV

- Here we showed how to reproduce the Skyrme lagrangian from the holographic model.
- One can also incorporate vector and axial mesons in the Skyrme lagrangian.
- The physical observables such as magnetic moments and square radii can be calculated similar to (Adkins, Nappi, Witten), expressed in terms of the f_π and e.
  (HRG, Erlich, Carone, in progress)

THE END