Baryon as skyrmion-like soliton from the holographic dual model of QCD

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Overview

- We start by discussing holographic model with *hard-wall* cutoff, which is constructed to be dual to QCD with $N_f = 2$ at low energies and in the chiral limit.
- Then we show how to reproduce the Skyrme model and its generalization including vector mesons.
- We extend this model to incorporate isosinglet vector mesons and CS term, required to generate appropriate QCD anomaly and coupling between the isosinglet (ω_μ) and (topological) baryon (B_μ) currents.
- Although this approach is *bottom-up*, it reproduces results which are very similar to ones applying *top-down* "stringy" setups.
- We also discuss that our soliton is only localized in 4D and can'd be viewed as a 5D localized instanton.

Introduction

- The AdS/CFT correspondence conjectures the equivalence of weakly coupled gravity theory (Type IIB string theory) on $AdS_5 \times S^5$, and strongly coupled ($\mathcal{N} = 4$ SYM) CFT_4 *(Maldacena '97)*.
- AdS/CFT states that for $\forall \mathcal{O}(x) \in \{CFT_4\ operator\}$, $\exists! \phi(x,z) \in \{5D\ bulk\ field\}$ s.t. $\phi(x,0) = \phi_0(x),\ x \in \partial AdS_5$.
- If $S_5[\phi_0(x)]$ is the gravity or string action of $\phi(x,z)$ with $\phi(x,0) = \phi_0(x)$, then the correspondence takes the form

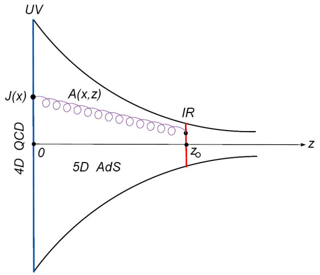
$$\langle \exp(i \int d^4x \phi_0(x) \mathcal{O}(x)) \rangle_{CFT} = \exp(iS_5[\phi_0(x)]) \ ,$$

(Witten '98; Gubser, Klebanov & Polyakov '98)

- For small z, the solution of EOM is: $\phi(x,z) \sim z^{4-\Delta}\phi_0 + \frac{1}{2\Delta - 4} z^{\Delta} \langle O \rangle$, where Δ – conformal dimension of $O(x)$ and $m_\phi^2 = \Delta(\Delta - 4)$.

Addition of IR Brane

- Since QCD is not CFT, direct application of AdS/CFT is meaningless.
- To fix the problem the IR brane is introduced which breaks the conformal symmetry in the 5D bulk, allowing to have both particles and S-matrix elements.
- 5D KK modes are interpreted as 4D QCD resonances at large N_c.



- AdS/QCD suggests that 5D theory with IR brane and certain field content is dual to 4D QCD at low energies.

We work with QCD in the limit, where both N_c and λ are large.

Holographic Dictionary

- The 5D field content is specified by *holographic dictionary*:

$$J^a_{L\mu}(x) = \bar{q}_L \gamma_\mu t^a q(x)_L \leftrightarrow L^a_M(x,z)$$

$$J^a_{R\mu}(x) = \bar{q}_R \gamma_\mu t^a q(x)_R \leftrightarrow R^a_M(x,z)$$

so that $L^a_\mu(x,0)$ is the source for $J^a_{L\mu}(x)$ (same for $L \leftrightarrow R$).

- The ChSB occurs due to IR BC: $L_\mu(x,z_0) - R_\mu(x,z_0) = 0$.
- Chiral fields are expressed through the Wilson lines.
- This holographic construction is discussed in (*Hirn & Sanz '05*) and is similar to models in (*Erlich, Katz, Son & Stephanov '05; Da Rold & Pomarol '05*).

5D Action

- The 5D action corresponds to $SU(2)_L \times SU(2)_R$ YM theory in the bulk of the sliced AdS space:

$$S_{YM} = -\frac{1}{4g_5^2} \int d^5x \sqrt{g}\ \mathrm{Tr}\left[L_{MN} L^{MN} + R_{MN} R^{MN} \right] ,$$

$L_{MN} = \partial_M L_N - \partial_N L_M - i[L_M, L_N],\ L = L^a t^a, t^a \in SU(2),\ a = 1, 2, 3$ and $M, N = 0, 1, 2, 3, z\ (L \leftrightarrow R)$.

- The local gauge invariance in 5D will produce 4D global chiral symmetry of QCD.

- The sliced AdS metric is defined by:

$$ds^2 = \frac{1}{z^2} \left(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2 \right) , \qquad 0 < z \leq z_0 ,$$

$\eta_{\mu\nu} = \mathrm{Diag}\,(1, -1, -1, -1)$ and $z_0 \sim 1/\Lambda_{QCD}$ is the IR scale.

Two-point Function

AdS/QCD predicts that the 2-point function is

$$\int d^4x\, e^{iq\cdot x}\langle J_\mu(x)J_\nu(0)\rangle = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Sigma(q^2)$$

$$\Sigma(q^2) = \sum_{n=1}^{\infty} \frac{f_n^2}{q^2 - M_n^2}\ , \quad M_n = \frac{\gamma_{0,n}}{z_0}\ , \quad f_n^2 = \frac{2M_n^2}{g_5^2 z_0^2 J_1^2(\gamma_{0,n})}$$

$$\Sigma(q^2 \gg 1/z_0^2) \sim \frac{1}{2g_5^2} q^2 \ln(q^2\epsilon^2)\ ,$$

by matching with QCD (*Erlich et al '05*): $g_5^2 = 12\pi^2/N_c$.

To get $M_1 \equiv M_\rho^{\rm exp} = 775.8$ MeV, for $N_c = 3$, one takes $z_0 = 1/(323\ \text{MeV})$.

As a result: $f_1 \equiv f_\rho = (392\ \text{MeV})^2$, cnf. to $f_\rho^{\rm exp} = (401 \pm 4\ \text{MeV})^2$ (*PDG, 2007*)

The Emergence of Global Chiral Symmetry

- In terms of flat Minkowski indices the 5D action is:

$$S_{YM} = -\frac{1}{4g_5^2}\,\mathrm{Tr}\int d^4x \int_0^{z_0} \frac{dz}{z}\Big[L_{\mu\nu}L^{\mu\nu} + 2L_{\mu z}L^{\mu z} + (L \leftrightarrow R)\Big]$$

- For S_{YM} to be finite at $z = 0 \;\Rightarrow\; L_M$ (and R_M) should satisfy

$$L_M(x, z \to 0) = i\, U_L^\dagger(x)\partial_M U_L(x)\;, \quad U_L(x) \in SU(2)_L$$

- Partially fixing the gauge: $\quad L_M(x, z \to 0) = 0\;,\;\; R_M(x, z \to 0) = 0$

- Under additional gauge transformations $g_L(x, z) \in SU(2)_L$

$$L_M(x, z \to 0) \to i g_L^\dagger \partial_M g_L(x, z \to 0)$$

The Emergence of Global Chiral Symmetry

- $L_M(x, z \to 0) = 0$ remains unaltered under the residual gauge transformations $g_L^{res}(x, z)$ s.t.

$$\partial_M g_L^{res}(x, z \to 0) = 0$$

The same is true when $L \leftrightarrow R \Leftrightarrow g_{L,R}^{res}(x, z)$ become constant matrices $g_{L,R} \in SU(2)_{L,R}$ at $z = 0$.

- In the holographic model $(g_L, g_R) \in SU(N_f)_L \times SU(N_f)_R$ corresponds to global chiral symmetry of QCD, at $z = 0$.

- Defining vector field as $V_M \equiv (L_M + R_M)/2$ and axial-vector field as $A_M \equiv (L_M - R_M)/2$, the UV BC that produce appropriate global symmetry of QCD can be written as:

$$V_\mu(x, 0) = 0 \,, \;\; A_\mu(x, 0) = 0$$

Axial-like Gauge

- Gauge tr. that generates the axial-like gauge $L_z(x,z) = 0$ is

$$W_L(x,z) = P\exp\left\{ i\int_{z_0}^{z} dz' L_z(x,z') \right\}$$

since $L_z(x,z) \to W_L^\dagger L_z W_L + i W_L^\dagger \partial_z W_L = 0$.

- In the axial-like gauge the UV BC $L_\mu(x,0) = 0$ changes to:

$$L_\mu(x,0) = i\xi_L^\dagger(x)\partial_\mu \xi_L(x)$$

where $\xi_L(x) = W_L(x,0)$ (similarly for R_μ).

- This is equivalent to having *sources* in UV:

$$A_\mu, V_\mu(x,0) = \frac{i}{2}\left[\xi_L^\dagger(x)\partial_\mu\xi_L(x) \pm \xi_R^\dagger(x)\partial_\mu\xi_R(x)\right]$$

Separation to Dynamical and Source Fields

Writing $V_M(x,z)$ and $A_M(x,z)$ in the axial-like gauge ($L_z = R_z = 0$) as $\hat{V}_\mu\,(x,z)$ and $\hat{A}_\mu\,(x,z)$, separate these into *dynamical* and *source* parts as:

$$\hat{V}_\mu\,(x,z) \equiv V_\mu\,(x,z) + \hat{V}_\mu\,(x,0)\ ,$$

$$\hat{A}_\mu\,(x,z) \equiv A_\mu\,(x,z) + \alpha\,(z)\,\hat{A}_\mu\,(x,0)\ .$$

and require dynamical fields $V_\mu\,(x,z)$ and $A_\mu\,(x,z)$ to satisfy BC:

- $V_\mu(x,0) = 0\ ,\ \ A_\mu(x,0) = 0 \ \Leftrightarrow\ L_\mu(x,0) = R_\mu(x,0) = 0$
- $\partial_z V_\mu(x,z_0) = 0 \ \Leftrightarrow\ V_{z\mu}(x,z_0) = 0$
- $A_\mu(x,z_0) = 0 \ \Leftrightarrow\ L_\mu(x,z_0) = R_\mu(x,z_0)$

These BC + absence of $a_1\pi$-like mixing give: $\alpha(z) = 1 - z^2/z_0^2$.

N.B. $V_{z\mu}\,(x,z_0) = \hat{V}_{z\mu}\,(x,z_0) = 0$ and $A_\mu\,(x,z_0) = \hat{A}_\mu\,(x,z_0) = 0$.

Source Fields

In the axial-like gauge the sources can be written as:

$$\hat{A}_\mu(x,0) \equiv \frac{1}{2}\alpha_\mu(x) \equiv \frac{i}{2}\left\{\xi_L^\dagger \partial_\mu \xi_L - \xi_R^\dagger \partial_\mu \xi_R\right\} ,$$

$$\hat{V}_\mu(x,0) \equiv \beta_\mu(x) \equiv \frac{i}{2}\left\{\xi_R^\dagger \partial_\mu \xi_R + \xi_L^\dagger \partial_\mu \xi_L\right\} ,$$

where

$$\xi_L(x) = P\exp\left\{-i\int_0^{z_0} dz' L_z(x,z')\right\} ,$$

$$\xi_R(x) = P\exp\left\{-i\int_0^{z_0} dz' R_z(x,z')\right\}$$

are Wilson lines from UV to IR boundaries by left or right fields.

Chiral Symmetry Breaking by BC

- From IR BC: $L_\mu(x, z_0) = R_\mu(x, z_0) \ \Rightarrow$

$$h(x) = g_L^{res}(x, z_0) = g_R^{res}(x, z_0) \in SU(2)_V$$

- Then, the Wilson lines $\xi_{L,R}$ should transform as

$$\xi_{L,R} \longrightarrow g_{L,R}^{res}(x, 0)\ \xi_{L,R}\ h(x)^\dagger$$

- This allows to define a chiral field as:

$$U(x) = \xi_R(x)\xi_L^\dagger(x) \ \longrightarrow \ g_R U(x) g_L^\dagger \ .$$

- This chiral field transforms exactly the same way as in the non-linear sigma model with respect to the global chiral transformations.

Dynamical Fields

- The dynamical vector fields can be written as:

$$V_\mu(x,z) = \sum_{n=1}^{\infty} V_\mu^{(n)}(x)\psi_n(z)$$

where $\psi_n(z)$ satisfy EOM

$$\left[z^2\partial_z^2 - z\partial_z + M_n^2 z^2\right]\psi_n(z) = 0\ ,$$

with BC $\psi_n(0) = \partial_z\psi_n(z_0) = 0$ $(V_\mu(x,0) = \partial_z V_\mu(x,z_0) = 0)$.

- The dynamical axial-vector fields can be written as:

$$A_\mu(x,z) = \sum_{n=1}^{\infty} A_\mu^{(n)}(x)\psi_n^A(z)$$

where $\psi_n^A(z)$ satisfy same EOM as $\psi_n(z)$ but with different BC $\psi_n^A(0) = \psi_n^A(z_0) = 0$ $(A_\mu(x,0) = A_\mu(x,z_0) = 0)$.

Dynamical Fields

- In particular: $V_\mu^{(1)}(x) = g_5 \rho_\mu(x)$ and $A_\mu^{(1)}(x) = g_5 a_{1\mu}(x)$.
- The solution for $\psi_n(z)$ is

$$\psi_n(z) = \frac{\sqrt{2}}{z_0 J_1(\gamma_{0,n})}\, z J_1(M_n z)$$

where M_n is determined from $J_0(M_n z_0) = 0$. It is normalized as

$$\int_0^{z_0} \frac{dz}{z}\, |\psi_n(z)|^2 = 1$$

- The solution for axial-vector sector is

$$\psi_n^A(z) \propto z J_1(M_n^A z)$$

where M_n^A is determined from IR BC: $J_1(M_n^A z_0) \simeq 0$.

In case dynamical fields vanish $A_\mu = V_\mu = 0$, the action after integrating over the z becomes:

$$S_{YM} = \int d^4x \,\mathrm{Tr}\left\{\frac{f_\pi^2}{4}\left(\partial_\mu U^\dagger \partial^\mu U\right) + \frac{1}{32e^2}\left[U^\dagger\partial_\mu U, U^\dagger \partial_\nu U\right]^2\right\} ,$$

The coefficients f_π and e are given by the integrals:

$$f_\pi^2 = \frac{1}{g_5^2}\int_0^{z_0} \frac{dz}{z}\left(\partial_z \alpha\right)^2 = \frac{2}{g_5^2 z_0^2} ,$$

$$\frac{1}{e^2} = \frac{1}{g_5^2}\int_0^{z_0} \frac{dz}{z}\left(1-\alpha^2\right)^2 = \frac{11}{24 g_5^2} .$$

This establishes the relation between the 5D AdS/QCD and the 4D Skyrme model for two flavors.

Baryon as a Skyrmion

- In order for the Skyrmion action to be finite, the chiral fields should satisfy the following conditions at spacial infinity $|\mathbf{x}| \to \infty$: $U(\mathbf{x}) \to \mathbf{1}$.

- These conditions describe topologically non trivial mapping $\mathbf{R}^3 \to S^3 \to SU(2)$ which is characterized by the homotopy group $\pi_3(SU(2)) = \mathbf{Z}$.

- Defining $L_i = U^\dagger \partial_i U$, the topological charge = baryon number B is

$$B = \frac{1}{24\pi^2} \int d^3x \epsilon_{ijk} \operatorname{Tr}\left(L_i L_j L_k\right) \ .$$

- The ansatz for the chiral field with $B = 1$ is:

$$U(\mathbf{x}) = e^{i\tau_a \hat{x}_a F(r)} \ ,$$

where $\hat{x}_a = x_a/r$, $r = \sqrt{|\mathbf{x}|^2}$, $F(0) = \pi$ and $F(\infty) = 0$ (*Skyrme '54*).

Masses of Nucleon and Delta

- Substituting the ansatz into the energy functional we get:

$$E[F] = \frac{2\pi f_\pi}{e} \int_0^\infty dx \left\{ \left(\frac{\partial F}{\partial x}\right)^2 \left[\frac{x^2}{2} + 4\sin^2 F\right] + \sin^2 F + \frac{2\sin^4 F}{x^2} \right\}$$

Let $F^*(x)$ minimizes $E[F]$, then $M_{cl} = E[F^*]$.

- Quantizing the skyrmion, we get masses of nucleon and delta

$$M_N = 73\frac{f_\pi}{e} + \frac{f_\pi e^3}{142.3}, \quad M_\Delta = 73\frac{f_\pi}{e} + \frac{f_\pi e^3}{28.5}.$$

- It is also convenient to use the mass difference as a parameter

$$\Delta M \equiv M_\Delta - M_N = \frac{f_\pi e^3}{35.6}.$$

These are the results from ANW model *(Adkins, Nappi, Witten '83)*

Results for Skyrmion

We take $z_0 = 1/(323\text{ MeV})$ and $g_5 = 2\pi \;\; (N_c = 3)$ from Model A of (Erlich *et al* '05).

Quantity	Prediction	ANW	Experiment
M_ρ (MeV)	776 (input)	–	776
f_π (MeV)	72.7	64.5	92.4
e	9.3	5.44	–
$E_{ANW} \equiv \frac{f_\pi}{2\, e}$ (MeV)	3.92	5.93	–
$r_{ANW} \equiv \frac{1}{e\, f_\pi}$ (fm)	0.29	0.56	–
M_{cl} (MeV)	572	864.3	–
$\sqrt{\langle r^2 \rangle} = \sqrt{2} r_{ANW}$ (fm)	0.41	0.8	0.6-0.8
M_N (MeV)	980	938.9 (input)	938.9
$M_\Delta - M_N$ (MeV)	1632 ?!	293.1 (input)	293.1

N.B. If we take $g_5 = 1.17\pi$, $z_0 = 1/(168\text{ MeV}) \Rightarrow$ exactly ANW results.

Turning on the ρ-meson

In case $A_\mu = 0$ and $V_\mu(x,z) = g_5 \rho_\mu(x)\psi_1(z)$, the YM lagrangian becomes:

$$
\begin{aligned}
-\mathcal{L}_{YM} = {} & \frac{f_\pi^2}{4}\,\mathrm{Tr}\left(\partial^\mu U^\dagger \partial_\mu U\right) - \frac{1}{32e^2}\,\mathrm{Tr}\left(\left[U^\dagger\partial_\mu U, U^\dagger \partial_\nu U\right]^2\right) \\
& + \frac{1}{2}\,\mathrm{Tr}\left(\rho_{\mu\nu}^2\right) + M_\rho^2\,\mathrm{Tr}\left(\rho_\mu^2\right) + ik_{3\rho}\,\mathrm{Tr}\left(\rho_{\mu\nu}[\rho_\mu,\rho_\nu]\right) - \frac{1}{2}k_{4\rho}\,\mathrm{Tr}\left([\rho_\mu,\rho_\nu]^2\right) \\
& - ik_1\,\mathrm{Tr}\left(\rho_{\mu\nu}[\alpha_\mu,\alpha_\nu]\right) + k_2\,\mathrm{Tr}\left\{[\alpha_\mu,\alpha_\nu][\rho_\mu,\rho_\nu]\right\} \\
& + k_1\,\mathrm{Tr}\left\{[\alpha_\mu,\alpha_\nu]([\beta_\mu,\rho_\nu]+[\rho_\mu,\beta_\nu])\right\} \\
& + i\,\mathrm{Tr}\left\{\rho_{\mu\nu}([\beta_\mu,\rho_\nu]+[\rho_\mu,\beta_\nu])\right\} - k_{3\rho}\,\mathrm{Tr}\left\{[\rho_\mu,\rho_\nu]([\beta_\mu,\rho_\nu]+[\rho_\mu,\beta_\nu])\right\} \\
& - \frac{1}{2}k_3\,\mathrm{Tr}\left([\alpha_\mu,\rho_\nu]+[\rho_\mu,\alpha_\nu]\right)^2 - \frac{1}{2}\,\mathrm{Tr}\left([\beta_\mu,\rho_\nu]+[\rho_\mu,\beta_\nu]\right)^2
\end{aligned}
$$

This lagrangian, is identical in form to the lagrangian obtained in a Sakai-Sugimoto like setup by (*Nawa et al '06*).

N.B. To agree with their conventions we made the following substitutions: $g_5 \leftrightarrow -g_5$, $\beta \leftrightarrow -\beta$ and $\eta_{\mu\nu} \leftrightarrow -\eta_{\mu\nu}$.

Turning on the ρ-meson

The 4 independent couplings k_i in the lagrangian are defined as:

$$k_{3\rho} = g_5 \int_0^{z_0} \frac{dz}{z}\psi_1^3(z) \simeq g_5 \ , \quad k_{4\rho} = g_5^2 \int_0^{z_0} \frac{dz}{z}\psi_1^4(z) \simeq 1.3 g_5^2 \ ,$$

$$k_1 = \frac{1}{4g_5} \int_0^{z_0} \frac{dz}{z}(1-\alpha^2)\psi_1 \simeq \frac{1}{6g_5} \ , \quad k_2 = \frac{1}{4} \int_0^{z_0} \frac{dz}{z}(1-\alpha^2)\psi_1^2 \simeq \frac{3}{16} \ .$$

The table comparing the corresponding couplings in two different models is presented below:

Coefficients k_i

•	$k_{3\rho}$	$k_{4\rho}$	k_1	k_2
Our Model	6.28	51	0.03	0.19
Nawa et al	5.17	29.7	0.03	0.18

Ansatz for the Solution.

- For the chiral field we choose the following Skyrme ansatz:

$$U(\mathbf{x}) = e^{i\tau_a \hat{x}_a F(r)} , \quad \text{with} \;\; F(0) = \pi , \; F(\infty) = 0 .$$

- For the ρ-meson we choose the following hedgehog ansatz ($\rho_0(\mathbf{x}) = 0$)

$$\rho_i(\mathbf{x}) = \epsilon_{iab} \tau_a \hat{x}_b \frac{G(r)}{r} ,$$

  where $G(r)$ is a profile function.

- Substituting these into the energy functional, we get

$$E[F(r), G(r)] \equiv 4\pi \int_0^\infty dr \, r^2 \, \mathcal{E}[F(r), G(r)] .$$

- Minimizing the energy E with given BC, we obtain F and G describing Skyrmion-like soliton.

Skyrmion with ω-mesons

- The non-linear σ model contains topologically stable but not energetically stable soliton-like solutions.
- To stabilize the solitons the higher derivative terms should be included to overcome Hobart-Derrick's theorem. One such term is Skyrme term.
- However, as is shown in (*Adkins & Nappi '84*) the vector mesons are sufficient to stabilize the soliton, even without the Skyrme term. This is demonstrated on the example of isoscalar vector mesons.
- In order to incorporate ω-meson the symmetry group has to be extended and CS term has to be added.
- This is also important in order to reproduce correct QCD anomaly.

Extending Dictionary

- Since the CS (WZW) term for $SU(2)$ gauge (global) group is vanishing, the flavor symmetry is extended to $U(2)_L \times U(2)_R$. Therefore, we can write fields as: $\mathcal{B}_\mu = t^a B^a_\mu + \frac{1}{2}\hat{B}_\mu$.
- The 4D isovector $J^{\{I=1\},a}_\mu(x)$ and isosinglet vector $J^{\{I=0\}}_\mu(x)$ currents correspond to:

$$J^{\{I=1\},3}_\mu = \frac{1}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right) = \bar{q}\gamma_\mu \frac{\sigma^3}{2} q \rightarrow V^3_\mu(x,z)$$

$$J^{\{I=0\}}_\mu = \frac{1}{2}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d\right) = \frac{1}{2}\bar{q}\gamma_\mu \mathbf{1} q \rightarrow \hat{V}_\mu(x,z)$$

- The EM current

$$J^{\text{EM}}_\mu = J^{\{I=1\},3}_\mu + \frac{1}{3} J^{\{I=0\}}_\mu$$

has both isovector ("ρ-type") and isosinglet ("ω-type") terms.

Holographic Action with CS Term

The $\mathcal{O}(B^3)$ part of the 5D CS action, in the axial gauge $B_z = 0$ is

$$S_{\text{CS}}^{(3)}[\mathcal{B}] = \frac{N_c}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\text{Tr}\int d^4x\,dz\,(\partial_z\mathcal{B}_\mu)\left[\mathcal{F}_{\nu\rho}\mathcal{B}_\sigma + \mathcal{B}_\nu\mathcal{F}_{\rho\sigma}\right]$$

In holographic model (cnf. *Domokos & Harvey '07*) the CS term is:

$$S_{\text{CS}}^{\text{AdS}}[\mathcal{B}_L,\mathcal{B}_R] = S_{\text{CS}}^{(3)}[\mathcal{B}_L] - S_{\text{CS}}^{(3)}[\mathcal{B}_R]$$

In the absence of dynamical axial-vector fields:

$$A_\mu^a(x,z) = \alpha(z)\partial_\mu\int_0^{z_0} dz'\,A_z^a(x,z') \equiv \alpha(z)(\partial_\mu\pi^a)$$

Substituting $\mathcal{B}_{L,R} = \mathcal{V} \pm \mathcal{A}$ and taking appropriate care on IR:

$$S^{\text{anom}} = \frac{N_c}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\int_0^{z_0} dz\,(\partial_z\alpha)\int d^4x\,\pi^a\left(\partial_\rho V_\mu^a\right)\left(\partial_\sigma\hat{V}_\nu\right)$$

N.B. $\alpha(z) = 1 - z^2/z_0^2$ corresponds to $\psi_0^A(z)$ for $m_0 = 0$.

Correct Anomaly

- In QCD the $\pi^0\gamma^*\gamma^*$ form factor is defined by

$$\int d^4x\, e^{-iq_1x}\langle\pi, p|T\{J^\mu_{\rm EM}(x)\, J^\nu_{\rm EM}(0)\}|0\rangle = \epsilon^{\mu\nu\alpha\beta} q_{1\,\alpha} q_{2\,\beta}\, F_{\gamma^*\gamma^*\pi^0}\left(Q_1^2, Q_2^2\right)$$

where $p = q_1 + q_2$ and $q^2_{1,2} = -Q^2_{1,2}$.

- Varying $S^{\rm anom}$ we get the 3-point function:

$$T_{\alpha\mu\nu}(p, q_1, q_2) = \frac{N_c}{12\pi^2}\frac{p_\alpha}{p^2}\,\epsilon_{\mu\nu\rho\sigma}\, q_1^\rho q_2^\sigma K(Q_1^2, Q_2^2)$$

- QCD anomaly requires that $K^{QCD}(0,0) = 1$.
- Indeed, in this extended holographic model, we get:

$$K(0,0) = -\int_0^{z_0} \mathcal{J}^2(0,z)\partial_z\alpha(z)dz = -\int_0^{z_0}\partial_z\alpha(z)\,dz = \alpha(0) = 1\,!$$

N.B. $K(Q_1^2, Q_2^2) = -\int_0^{z_0} dz \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\partial_z\alpha$, where $\mathcal{J}(Q, z)$ is NN-mode s.t. $\mathcal{J}(0, z) = 1$.

Turning only ω-meson

In case $A_\mu = V_\mu = 0$ and $\hat{V}_\mu(x,z) = g_5 \omega_\mu(x)\psi_1(z)$, we have

$$\mathcal{L}_{YM} = \frac{f_\pi^2}{4} \operatorname{Tr}\left(\partial^\mu U^\dagger \partial_\mu U\right) + \frac{1}{32e^2} \operatorname{Tr}\left(\left[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U\right]^2\right)$$

$$- \frac{1}{4}\omega_{\mu\nu}^2 + \frac{1}{2}M_\omega^2 \operatorname{Tr}\left(\omega_\mu^2\right) - \frac{i}{2}k_1 \omega_{\mu\nu} \operatorname{Tr}\left[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U\right] + \kappa \omega_\mu B^\mu$$

where

$$B^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\alpha\beta} \operatorname{Tr}\left[(U^\dagger \partial_\nu U)(U^\dagger \partial_\alpha U)(U^\dagger \partial_\beta U)\right]$$

is the conserved, normalized CS current (κ is determined from z_0 and g_5).

N.B. This lagrangian is similar to one in (*Adkins & Nappi '84*) in the chiral limit but with Skyrme term and with term $\sim k_1$.

Instanton Number

- From 5D point of view, the Chern-Pontryagin index is

$$Q_{L,R} = \frac{1}{32\pi^2} \int d^3x dz \, \epsilon_{MNPQ} \, \mathrm{Tr}(F_{L,R}^{MN} F_{L,R}^{PQ}) \; .$$

where z is very much like compact 4D Euclidean time.

- This topological characteristic is diffeomorphism invariant and is not sensitive to the local small perturbations around current value of fields.

- The topological charges can be also written as

$$Q_L = \frac{i}{24\pi^2} \oint d\sigma_\rho \epsilon^{\rho\alpha\beta\mu} \, \mathrm{Tr}[\ell_\alpha \ell_\beta \ell_\mu] \; ,$$
$$Q_R = \frac{i}{24\pi^2} \oint d\sigma_\rho \epsilon^{\rho\alpha\beta\mu} \, \mathrm{Tr}[r_\alpha r_\beta r_\mu] = -Q_L \; ,$$

$$\ell_\mu(x) = iU\partial_\mu U^\dagger(x) \; , \quad r_\mu(x) = iU^\dagger \partial_\mu U(x)$$

Baryon and Instanton Numbers

- In $A_z = 0$ gauge, for the solutions which interpolate between the UV degenerate classical vacua $L_i = R_i = 0$ and $L_i = \ell_i$, $R_i = r_i$ we have:

$$Q_L = -\frac{i}{24\pi^2} \int d^3x \epsilon^{ijk} \,\mathrm{Tr}[\ell_i \ell_j \ell_k] + Q_{IR} \; ,$$
$$Q_R = -\frac{i}{24\pi^2} \int d^3x \epsilon^{ijk} \,\mathrm{Tr}[r_i r_j r_k] + Q_{IR} \; ,$$

  where Q_{IR} is some integral over the $h(x)$ fields on the IR brane.

- The instanton number N is defined as

$$N \equiv \frac{1}{2}(Q_L - Q_R) = -\frac{i}{24\pi^2} \int d^3x \epsilon^{ijk} \,\mathrm{Tr}[\ell_i \ell_j \ell_k] = B \; .$$

- $\Rightarrow$ the instanton number N is the same as the baryon number B: Does this mean that the skyrmion is a “preimage” of 5D instanton?

Witten's ansatz

- The fields of minimum action for fixed BC are solutions of $F = \tilde{F}$ (BPST). We will seek the solutions which are invariant under combined isospin and spin symmetries.
- This symmetry is called a cylindrical symmetry, since it determines the dependence of the fields on the 3D polar angles and leaves the unknown only the dependence on the 3D radius r and Euclidean time, which in our case (effectively) is z.
- The most general gauge field with this cylindrical symmetry is given by (*Witten '76*)

$$A_j^a(x,z) = \frac{1+\phi_2(r,z)}{r^2}\epsilon_{jak}x_k + \frac{\phi_1(r,z)}{r^3}\left[\delta_{ja}r^2 - x_j x_a\right] + A_1(r,z)\frac{x_j x_a}{r^2} \ ,$$

$$A_z^a(x,z) = \frac{A_2(r,z)x^a}{r} \ .$$

N.B. This is true for general gauge.

Witten's ansatz for two YM fields

- Witten found general solution of $F_{ab} = \tilde{F}_{ab}$ $(a, b = 1, 2, 3, z)$ that can be written in the above form in case of the unbounded Euclidean space.
- The same problem for space with bounded z was discussed by (*Pomarol & Wulzer '08*), and the solution was obtained using numerical calculations.
- Since we have two fields in the bulk, some simplifications can be made:

$$\phi_1 \equiv -\phi_1^L = \phi_1^R\ , \quad \phi_2 \equiv \phi_2^L = \phi_2^R\ ,$$

$$A_1 \equiv A_1^R = -A_1^L\ , \quad A_2 \equiv A_2^R = -A_2^L$$

Witten's ansatz for two YM fields

As a result: $V_z(x,z) = 0$,

$$V_j^a(x,z) = \frac{1+\phi_2(r,z)}{r^2}\epsilon_{jak}x_k$$

and

$$A_j^a(x,z) = -\frac{\phi_1(r,z)}{r^3}\left[\delta_{ja}r^2 - x_j x_a\right] - A_1(r,z)\frac{x_j x_a}{r^2}$$

$$A_z^a(x,z) = -\frac{A_2(r,z)x^a}{r}$$

In order to satisfy the BC of the original gauge fields, we have to require:

$$\phi_1(r,z_0) = 0\ ,\quad \partial_z\phi_2(r,z_0) = 0\ ,$$

$$A_1(r,z_0) = 0\ ,\quad \partial_z A_2(r,z_0) = 0\ ,$$

$$\phi_1(r,0) = 0\ ,\quad \phi_2(r,0) = -1\ ,$$

$$A_1(r,0) = 0\ ,\quad \partial_z A_2(r,0) = 0\ .$$

Absence of Instanton

- The condition $F_{ab} = \tilde{F}_{ab}$ is equivalent to Witten's duality equations:

$$\partial_z \phi_1 + A_2 \phi_2 = \partial_r \phi_2 - A_1 \phi_1$$

$$\partial_r \phi_1 + A_1 \phi_2 = -\partial_z \phi_2 + A_2 \phi_1$$

$$\partial_z A_1 - \partial_r A_2 = \frac{1}{r^2} \left(1 - \phi_1^2 - \phi_2^2\right)$$

- Expressing our ansatz for vector and chiral fields in terms of $\phi_{1,2}$ and $A_{1,2}$, we find that the letter don't satisfy Witten's equations
- $\Rightarrow$ our soliton can't be a 5D localized instanton
- Notice that in our case the soliton consists of ρ-mesons only, since: $\phi_2(r, z) = g_5 G(r) \psi_1(z) - 1$
- One can show that to have an instanton the whole tower of vector resonances should be incorporated

Naive Soliton-like Solution from Vector Fields

- Let $A_j^a(x,z) = 0 \;\Leftrightarrow\; \phi_1 = A_1 = 0$. This is true e.g. on IR boundary. As a result Witten's duality equations become:

$$\partial_r \phi_2 = A_2 \phi_2 \;,$$

$$\partial_z \phi_2 = 0 \;,$$

$$\partial_r A_2 = \frac{1}{r^2}\left(\phi_2^2 - 1\right) \;.$$

- From 2^{nd} equation $\;\Rightarrow\;$ no z dependence for vector fields.
- Substituting: $\phi_2(r) = r e^{\rho(r)}$, we will get:

$$\partial_r^2 \rho = e^{2\rho} \;,$$

$$A_2(r) = \partial_r \rho + \frac{1}{r} \;.$$

Soliton-like Solution

The equation $\partial_r^2 \rho = e^{2\rho}$ is Liouville's equation, with general solutions:

$$\rho(r) = -\ln\left[-\frac{i}{a}\cosh(ar+b)\right] .$$

Substituting this back to ϕ_2 and A_2, we get

$$\phi_2(r) = \frac{iar}{\cosh(ar+b)} ,$$

$$A_2(r) = \frac{1}{r} - a\tanh(ar+b) .$$

To make the solution for A_2 regular at $r=0$, we demand $b = -i\pi/2$ so that

$$\phi_2(r) = -\frac{ar}{\sinh(ar)} ,$$

$$A_2(r) = \frac{1}{r} - a\coth(ar) .$$

N.B. For $r \to 0$, we have $A_2 \to -a^2 r/3$ and $\phi_2 \to -1 + a^2 r^2/6$.

Topological Charge and Mass

The topological charge in terms of Witten's ansatz is:

$$Q = \frac{1}{2\pi} \int_0^\infty dr \int_0^{z_0} dz \epsilon^{\bar{\mu}\bar{\nu}} \left[\partial_{\bar{\mu}} \left(-i\phi^* D_{\bar{\nu}} \phi + \text{h.c.} \right) + F_{\bar{\mu}\bar{\nu}} \right] ,$$

where $D_{\bar{\mu}} \phi_i = \partial_{\bar{\mu}} + \epsilon_{ij} A_{\bar{\mu}} \phi_j$, $\bar{\mu}, \bar{\nu} = 1, 2$, $\phi = \phi_1 + i\phi_2$ and $F_{\bar{\mu}\bar{\nu}} = \partial_{[\bar{\mu}} A_{\bar{\nu}]}$.

For our case, we have:

$$Q = \frac{1}{2\pi} \int_0^\infty dr \epsilon^{\bar{\mu}\bar{\nu}} F_{\bar{\mu}\bar{\nu}} = -\frac{1}{\pi} A_2(r \to \infty) = \frac{a}{\pi}$$

To get $Q = 1$, we choose $a = \pi \ \Rightarrow$ the classical mass of this object will be

$$M = \frac{8\pi^2}{g_5^2 z_0} = \frac{2}{z_0} = 646 \text{ MeV}$$

N.B. This soliton has to be further quantized in order to find $M_{N,\Delta}$.

Solution with Unit Topological Charge

The solution with $Q = 1$, $V_z^a(x,z) = A_i^a(x,z) = 0$ is given by

$$V_j^a(x,z) = \left[1 - \frac{\pi r}{\sinh(\pi r)}\right]\frac{\epsilon_{jak}x_k}{r^2} \equiv \bar{G}(r)\frac{\epsilon_{jak}x_k}{r}$$

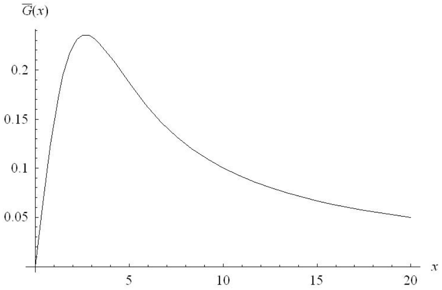
$$A_z^a(x,z) = \left[\pi r \coth(\pi r) - 1\right]\frac{x^a}{r^2} \equiv \bar{F}(r)\frac{x^a}{r}$$

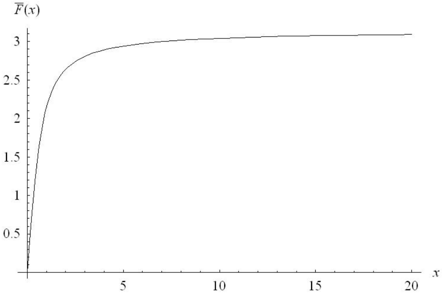
$$U(x) = P\exp\left(i\tau_a \int_0^{z_0} dz' A_z^a(x,z')\right) = \exp\left(i\hat{x}_a\tau_a\, F(r)\right)$$

Near the origin: $\bar{G}/r \to \pi^2/6$ and $\bar{F}/r \to \pi^2/3$

At infinity: $\bar{G}(\infty) \to 0$ and $\bar{F}(\infty) \to \pi$

If the solution is localized at IR $\Rightarrow$ $F(r) = \bar{F}(r)$, and $F(0) = 0$, $F(\infty) = \pi$
$\Rightarrow$ we have anti-skyrmion with $B = -1$.


$\overline{G}(x)$
0.2
0.15
0.1
0.05
5
10
15
20
x


$\overline{F}(x)$
3
2.5
2
1.5
1
0.5
5
10
15
20
x

IR localized soliton

It is easy to see that since, $(F-\tilde{F})^2/2 = F^2 - F\tilde{F} \geq 0$,

$$
\begin{aligned}
E &= \frac{1}{4g_5^2}\int d^3x \int_0^{z_0} dz \frac{1}{z}\,\mathrm{Tr}\left[L_{ab}L^{ab} + R_{ab}R^{ab}\right] \\
&\geq \frac{1}{4g_5^2}\int d^3x \int_0^{z_0} dz \frac{1}{z}\epsilon^{abcd}\,\mathrm{Tr}\left[L_{ab}L_{cd} + R_{ab}R_{cd}\right] \\
&\geq \frac{1}{4g_5^2 z_0}\int d^3x \int_0^{z_0} dz \epsilon^{abcd}\,\mathrm{Tr}\left[L_{ab}L_{cd} + R_{ab}R_{cd}\right] \\
&= \frac{8\pi^2}{g_5^2 z_0}\left(Q_L + Q_R\right) ,
\end{aligned}
$$

where $a, b, c, d \in (1, 2, 3, z)$. If we take $g_5^2 = 4\pi^2$, $Q_L = 1$ and $Q_R = 0$, then:

$$
E \geq \frac{2}{z_0} = 646 \text{ MeV}
$$

IR localized soliton

The bound

$$E \geq \frac{8\pi^2}{g_5^2 z_0} \left(Q_L + Q_R\right)$$

is saturated, when

- $L_{ab}(x,z) = \pm\tilde{L}_{ab}(x,z)$ and $R_{ab}(x,z) = \pm\tilde{R}_{ab}(x,z)$
- the center of the solution is localized at $z = z_0$

This describes 4D instanton localized at IR

It was shown by (*Pomarol & Wulzer '08*) that if ρ is the instanton size:

$$E \geq \frac{8\pi^2}{g_5^2 z_0} \left(1 + \frac{\rho}{2z_0}\right)$$

$\Rightarrow$ instanton will shrink to a point.

N.B. P&W also showed that addition of IR localized gauge kinetic term can stabilize the soliton.

Summary

- We showed how to extract Skyrme model from AdS/QCD.
- We also generalized to include vector meson fields (like ρ-meson), isosinglets (like ω-meson) and $\omega_\mu B^\mu$ interaction.
- In this AdS/QCD model baryon looks like 4D soliton and is not localized in 5D.
- Our ansatz doesn't correspond to instanton in 5D because we have only ρ-mesons that contribute to baryon.
- To have 5D instanton, the whole tower of vector meson resonances should be included.
- We discussed a simple solution, where soliton is made of vector fields and is localized at IR.

THE END