High-energy scattering at the next-to-leading order

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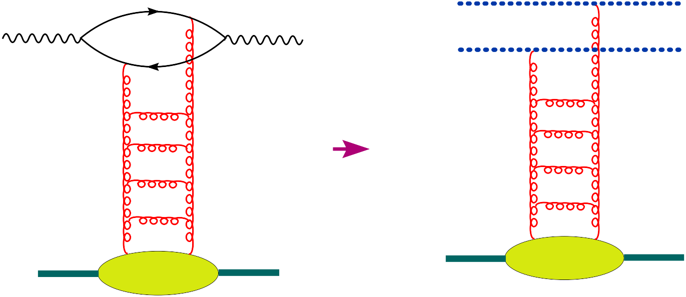
DIS at high energy

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp) \langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

$$U(x_\perp) = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\, n^\mu A_\mu(un + x_\perp)\Big] \qquad \text{Wilson line}$$

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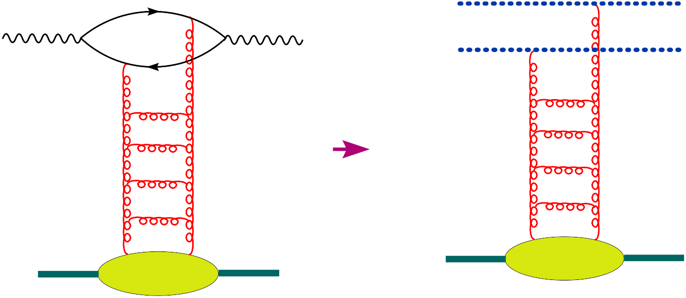

$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp) \langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

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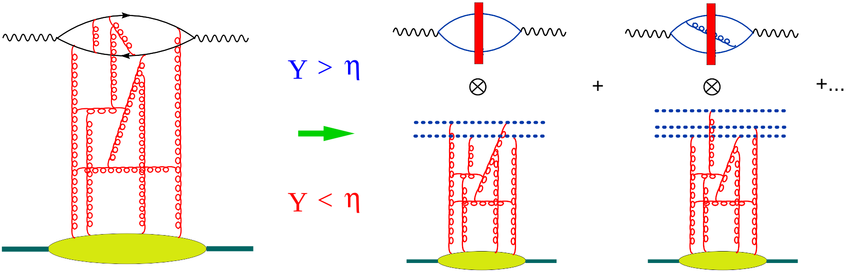
Formally, ➤ means the operator expansion in Wilson lines

Four steps of an OPE

- To factorize an amplitude into a product of coefficient functions and matrix elements of relevant operators.
- To find the evolution equations of the operators with respect to factorization scale.
- To solve these evolution equations.
- To convolute the solution with the initial conditions for the evolution and get the amplitude

High-energy expansion in color dipoles



η - rapidity factorization scale

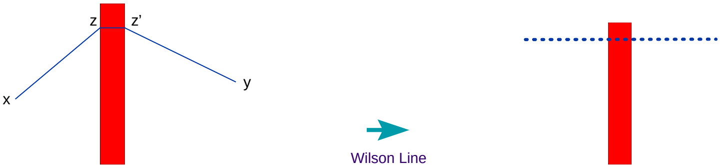
Rapidity Y > η - coefficient function ("impact factor")
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\Big]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

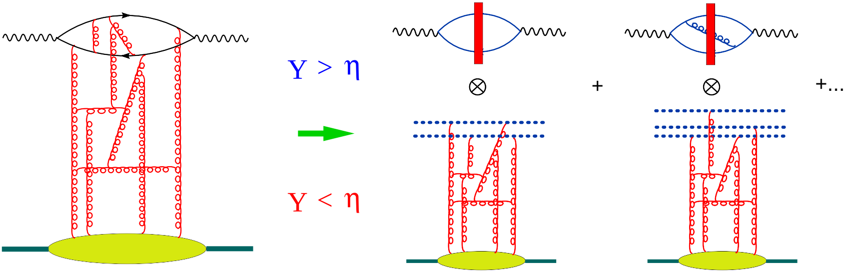
Spectator frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.



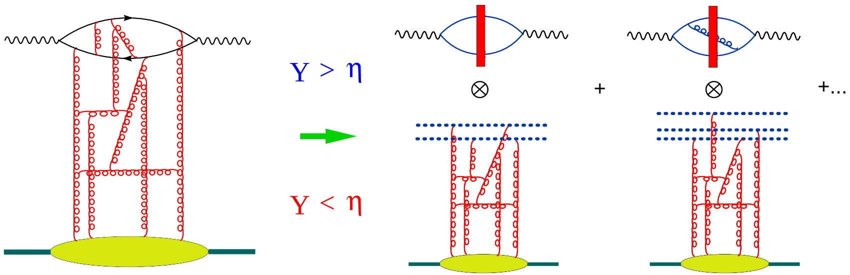
[$x \to z$: free propagation]$\times$
$[U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$]\times$
[$z \to y$: free propagation]

High-energy expansion in color dipoles



The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\ \text{NLO contribution}$$



η - rapidity factorization scale

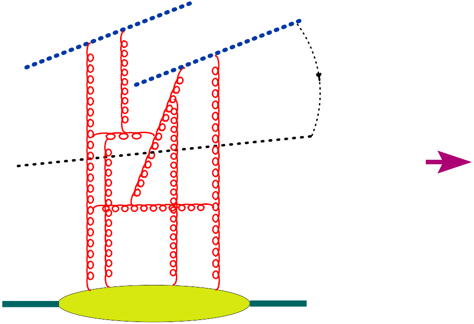
Evolution equation for color dipoles

$$\frac{d}{d\eta}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} - N_c\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} + O(\alpha_s^2)$$

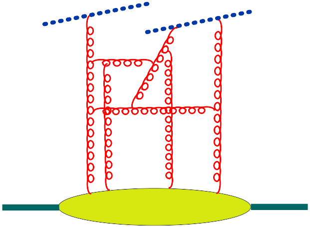
(Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

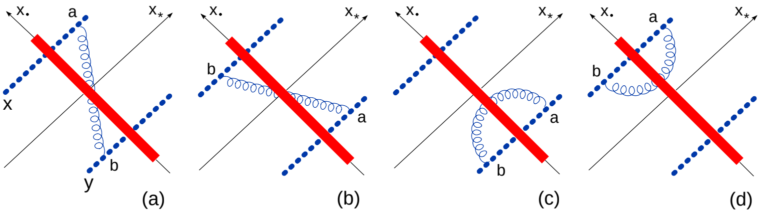


$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \otimes$



$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}$$



$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non-linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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LLA for DIS in pQCD $\Rightarrow$ BFKL eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1,\ \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

- To check that high-energy OPE works at the NLO level.
- To determine the argument of the coupling constant.
- To get the region of application of the leading order evolution equation.
- To check conformal invariance (in $\mathcal{N}$=4 SYM)

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

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Conformal (Möbius) invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}\mathrm{Tr}\{U_x U_y^\dagger\}]$$
$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

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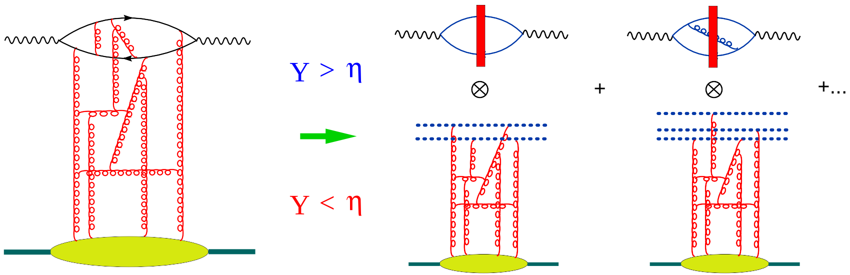
In the leading order - OK. In the NLO - ?



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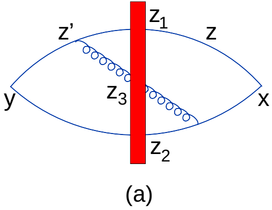
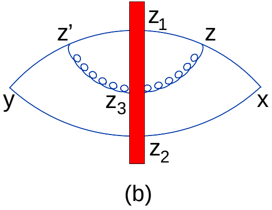
$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order - conf. invariant impact factor

$$I_{\rm LO} = \frac{x_+^{-2}y_+^{-2}}{\pi^2\mathcal{Z}_1^2\mathcal{Z}_2^2}, \qquad \mathcal{Z}_i \equiv \frac{(x-z_i)^2_\perp}{x_+} - \frac{(y-z_i)^2_\perp}{y_+} \qquad CCP, 2007$$

NLO impact factor





$$I^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I^{\rm LO}\times\frac{\lambda}{\pi^2}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\Big[\ln\frac{\sigma s}{4}\mathcal{Z}_3-\frac{i\pi}{2}+C\Big]$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$[{\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} \;=\; {\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\frac{\lambda}{2\pi^2}\int d^2z_3\;\frac{z_{12}^2}{z_{13}^2z_{23}^2}[{\rm Tr}\{T^n\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^n\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c{\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}\;+\;O(\lambda^2)$$

the impact factor becomes conformal in the NLO.

Operator equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

$$\Rightarrow \quad \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

Definition of the NLO evolution kernel

Operator equation

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

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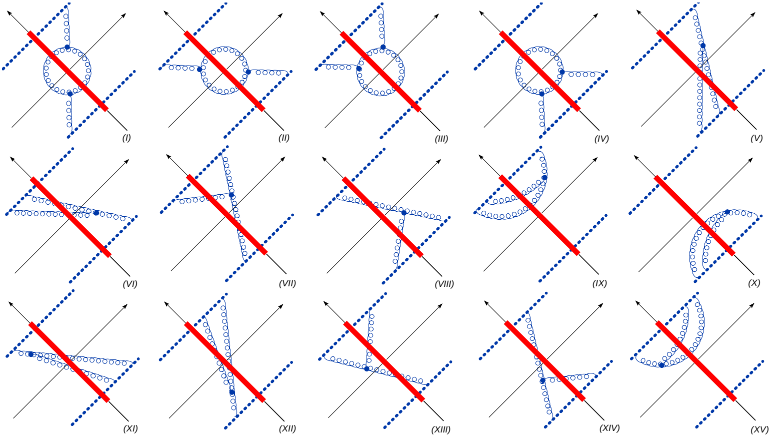
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)

$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

Typical integral

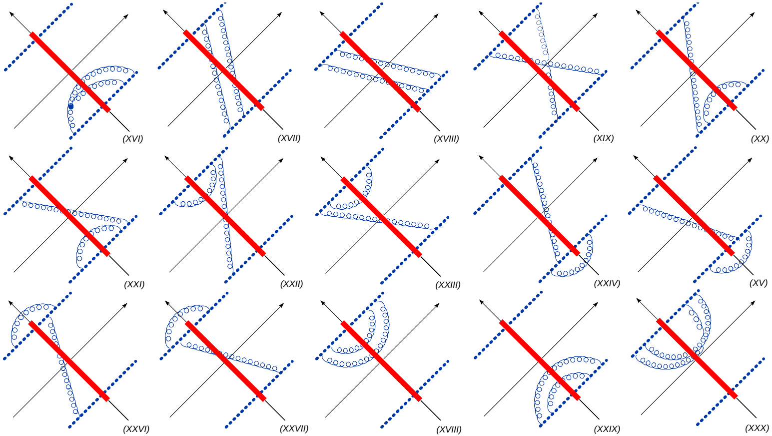
$$\int_0^1 dv\, \frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

Diagrams with 2 gluons interaction



Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction



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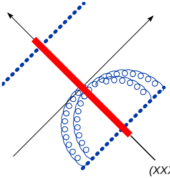
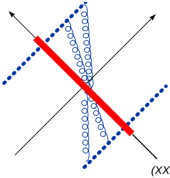

(XXXI)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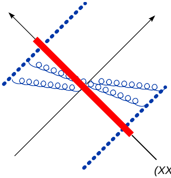
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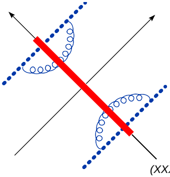
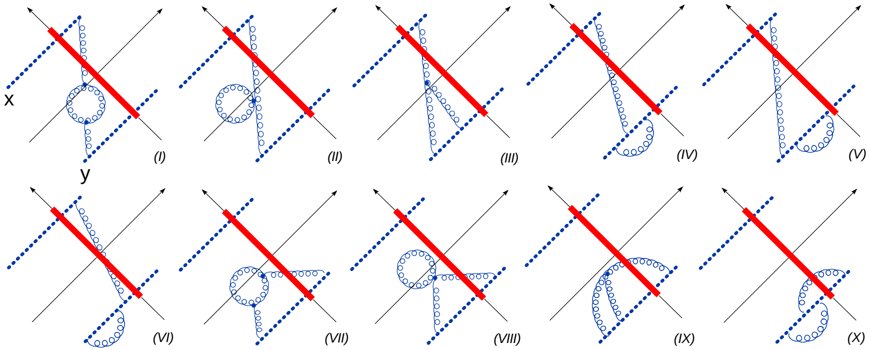
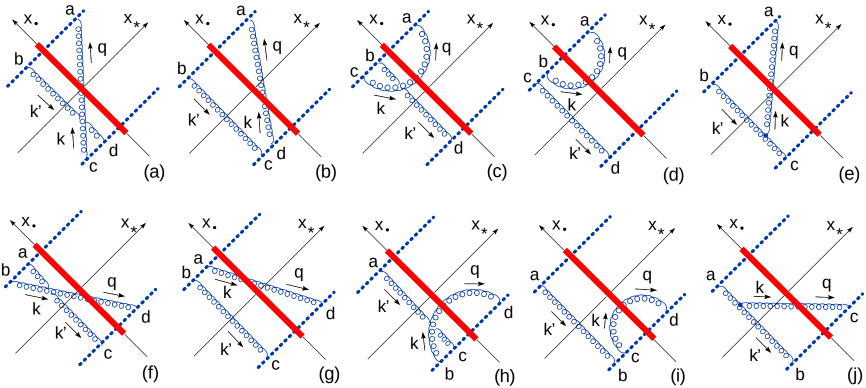

(XXXIV)

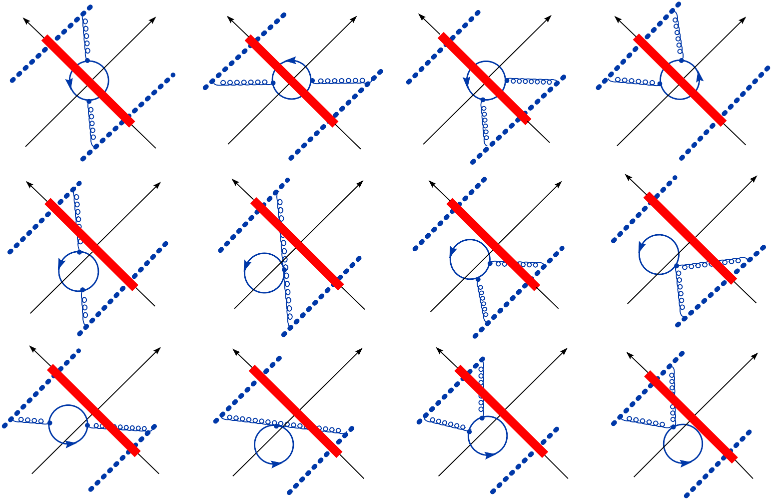
"Running coupling" diagrams



$1 \to 2$ dipole transition diagrams



$\mathcal{N} = 4$ SYM diagrams (scalar and gluino loops)



Evolution equation for color dipole in $\mathcal{N}=4$

(Giovanni A. Chirilli and I.B.)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\,[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\,\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

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(Giovanni A. Chirilli and I.B.)

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$

$$= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}$$

$$\times\,[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$

$$-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}$$

$$\times\,\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N}=4$

$$
\begin{aligned}
&\frac{d}{d\eta}\big[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf}\\
&=\ \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf}\\
&\quad-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}\\
&\quad\times\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}[(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4})^{bb'}-(z_4\to z_3)]
\end{aligned}
$$

Now Möbius invariant!

NLO Amplitude in $\mathcal{N}$=4 SYM theory

The pomeron contribution to a 4-point correlation function in $\mathcal{N}=4$ SYM can be represented as $\lambda \equiv g^2 N_c$

$$(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$
$$= \frac{i}{8\pi^2}\int d\nu\, \tilde{f}_+(\nu)\tanh\pi\nu\frac{\sin\nu\rho}{\sinh\rho}F(\nu,\lambda)R^{\frac{1}{2}\omega(\nu,\lambda)}$$

Cornalba(2007)

$\omega(\nu,\lambda)=\frac{\lambda}{\pi}\chi(\nu)+\lambda^2\omega_1(\nu)+\ldots$ is the pomeron intercept,
$\tilde{f}_+(\omega)=(e^{i\pi\omega}-1)/\sin\pi\omega$ is the signature factor.
$F(\nu,\lambda)=F_0(\nu)+\lambda F_1(\nu)+\ldots$ is the "pomeron residue".

R and r are two conformal ratios:

$$R = \frac{(x-x')(y-y')^2}{(x-y)^2(x'-y')^2}, \quad r = R\Big[1-\frac{(x-y')^2(y-x')^2}{(x-x')^2(y-y')^2}+\frac{1}{R}\Big]^2, \quad \cosh\rho=\frac{\sqrt{r}}{2}$$

In the Regge limit $s\to\infty$ the ratio R scales as s while r does not depend on energy.

The pomeron contribution to a 4-point correlation function in $\mathcal{N} = 4$ SYM can be represented as $\lambda \equiv g^2 N_c$

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$\omega(\nu,\lambda) = \frac{\lambda}{\pi}\chi(\nu) + \lambda^2\omega_1(\nu) + ...$ is the pomeron intercept,
$\tilde{f}_+(\omega) = (e^{i\pi\omega} - 1)/\sin\pi\omega$ is the signature factor.
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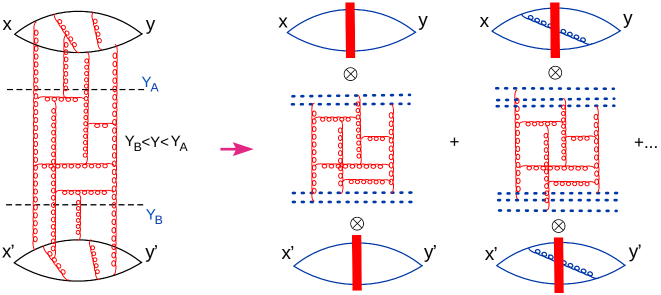
In the Regge limit $s \to \infty$ the ratio R scales as s while r does not depend on energy.

$\omega_0(\nu)$, $\omega_1(\nu)$ and $F_0(\nu)$ were known.

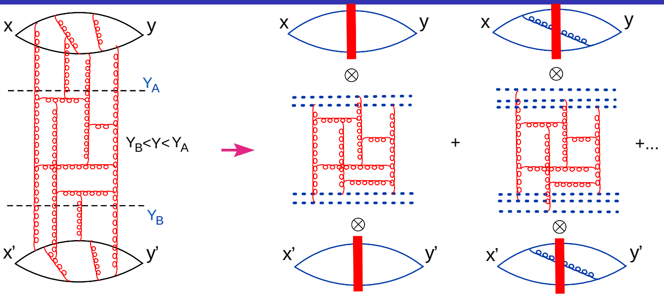
We reproduced $\omega_1(\nu)$ (Lipatov & Kotikov, 2000) and found $F_1(\nu)$

NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

$a_0 = \frac{x_+y_+}{(x-y)^2}$, $b_0 = \frac{x'_-y'_-}{(x'-y')^2}$ $\Leftrightarrow$ impact factors do not scale with energy
$\Rightarrow$ all energy dependence is contained in $[\mathrm{DD}]^{a_0,b_0}$ ($a_0b_0 = R$)



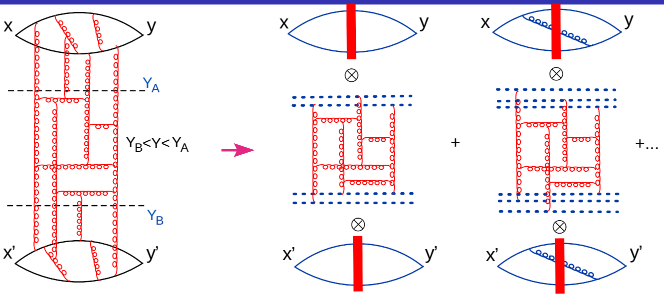
$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

Dipole-dipole scattering

$$\chi(\gamma)\equiv 2C-\psi(\gamma)-\psi(1-\gamma)$$

$$[DD]=\int d\nu\int dz_0\,\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac12+i\nu}\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac12-i\nu}D(\frac12+i\nu;\lambda)R^{\omega(\nu)/2}$$

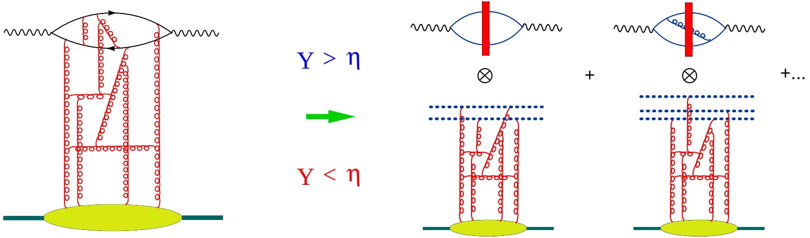
$$D(\gamma;\lambda)\;=\;\frac{\Gamma(-\gamma)\Gamma(\gamma-1)}{\Gamma(1+\gamma)\Gamma(2-\gamma)}\Big\{1-\frac{\lambda}{4\pi^2}\Big[\frac{\chi(\gamma)}{\gamma(1-\gamma)}-\frac{\pi^2}{3}\Big]+O(\lambda^2)\Big\}$$



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

Result : (G.A. Chirilli and I.B.)

$$F(\nu) = \frac{N_c^2}{N_c^2-1}\frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu} \times\Big\{1+\frac{\alpha_s N_c}{\pi}\Big[-2\psi'\big(\frac{1}{2}+i\nu\big)-2\psi'\big(\frac{1}{2}-i\nu\big)+\frac{\pi^2}{2}-\frac{8}{1+4\nu^2}\Big]+O(\alpha_s^2)\Big\}$$



DIS structure function $F_2(x)$: photon impact factor + evolution of color dipoles+ initial conditions for the small-x evolution

Composite “conformal” dipole

$$[\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]_a^{\mathrm{conf}}$$

$$= \mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} - \frac{\alpha_s}{4\pi^2}\int d^2z_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{ae^{2\eta}z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$

Photon impact factor in the LO

$$(x-y)^6 T\{j_\mu(x)j_\nu(y)\} = \frac{1}{\pi^2}\int\frac{d^2z_1d^2z_2}{z_{12}^4}\,\mathcal{R}^3\hat{\mathcal{U}}^{\mathrm{conf}}(z_1,z_2)\frac{\partial^2}{\partial x^\mu\partial y^\nu}\big[-2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)+\kappa^2(\zeta_1\cdot\zeta_2)\big]$$

$$\Delta \equiv (x-y), \qquad x_* = x^+\sqrt{s/2}, \qquad y_* = x^+\sqrt{s/2}, \qquad \mathcal{R} \equiv -\frac{\Delta^2 z_{12\perp}^2}{x_* y_* \mathcal{Z}_1 \mathcal{Z}_2}$$

$$\mathcal{Z}_1 = -\frac{(x-z_1)^2}{x_*} + \frac{(y-z_1)^2}{y_*}, \qquad \mathcal{Z}_2 = -\frac{(x-z_2)^2}{x_*} + \frac{(y-z_2)^2}{y_*}$$

$$\begin{aligned}
I^{NLO}_{\mu\nu}(x,y) = &-\frac{\alpha_s N_c^2}{8\pi^7 x_*^2 y_*^2}\int d^2z_1 d^2z_2\, \mathcal{U}^{\rm conf}(z_1,z_2)\Big\{\Big[\frac{1}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\partial^x_\mu\partial^y_\nu \ln\frac{\Delta^2}{x_*y_*} \\
&+ 2\frac{\big(\partial^x_\mu \mathcal{Z}_1\big)\big(\mathcal{Z}_2\partial^y_\nu\big)}{\mathcal{Z}_1^3\mathcal{Z}_1^3}\Big[\ln\frac{1}{\mathcal{R}} + \frac{1}{2\mathcal{R}} - 2\Big] + \frac{2\big(\partial^x_\mu\mathcal{Z}_1\big)\big(\partial^y_\nu \mathcal{Z}_1\big)}{\mathcal{Z}_1^4\mathcal{Z}_2^2}\Big[\ln\frac{1}{\mathcal{R}} - \frac{1}{2\mathcal{R}}\Big] \\
&- \frac{1}{2}\Big[\frac{\partial^x_\mu\mathcal{Z}_1}{\mathcal{Z}_1^3\mathcal{Z}_2^2}\partial^y_\nu\ln\frac{\Delta^2}{x_*y_*} + \frac{\partial^y_\nu\mathcal{Z}_1}{\mathcal{Z}_1^3\mathcal{Z}_2^2}\partial^x_\mu\ln\frac{\Delta^2}{x_*y_*}\Big]\Big(1-\frac{1}{\mathcal{R}}\Big) - \frac{1}{2\mathcal{Z}_2^2}\Big[\Big(\partial^x_\mu\frac{1}{\mathcal{Z}_1^2}\Big)\partial^y_\nu\mathcal{R} + \Big(\partial^y_\nu\frac{1}{\mathcal{Z}_1^2}\Big)\partial^x_\mu\mathcal{R}\Big]\frac{\ln\mathcal{R}}{1-\mathcal{R}} \\
&- \Big(\partial^x_\mu\partial^y_\nu\frac{\Delta^2}{x_*y_*}\Big)\frac{\mathcal{R}^3}{z_{12}^4}\Big[\frac{1}{\mathcal{R}} + \frac{3}{2\mathcal{R}^2} - 2\Big]\Big(\frac{x_*y_*}{\Delta^2}\Big)^3 + \frac{1}{\mathcal{R}}\Big[\frac{\partial^x_\mu\mathcal{Z}_1}{\mathcal{Z}_1^3\mathcal{Z}_2^2}\Big(\partial^y_\nu\ln\frac{\Delta^2}{x_*y_*}\Big) + \frac{\partial^y_\nu\mathcal{Z}_1}{\mathcal{Z}_1^3\mathcal{Z}_2^2}\Big(\partial^x_\mu\ln\frac{\Delta^2}{x_*y_*}\Big)\Big] \\
&+ 4\frac{\big(\partial^x_\mu\mathcal{Z}_1\big)\big(\partial^y_\nu\mathcal{Z}_2\big)}{\mathcal{Z}_1^3\mathcal{Z}_2^3}\Big[4\mathrm{Li}_2(1-\mathcal{R}) - \frac{2\pi^2}{3} + 2(\ln\mathcal{R}-1)\Big(\ln\mathcal{R} - \frac{1}{\mathcal{R}}\Big)\Big] \\
&+ 2\frac{\big(\partial^x_\mu\mathcal{Z}_1\big)\big(\partial^y_\nu\mathcal{Z}_2\big)}{\mathcal{Z}_1^3\mathcal{Z}_2^3}\Big[\frac{\ln\mathcal{R}}{\mathcal{R}(1-\mathcal{R})} - \frac{1}{\mathcal{R}} + 2\ln\mathcal{R} - 4\Big] + 2\frac{\big(\partial^x_\mu\mathcal{Z}_1\big)\big(\partial^y_\nu\mathcal{Z}_1\big)}{\mathcal{Z}_1^4\mathcal{Z}_2^2}\Big[\frac{\ln\mathcal{R}}{\mathcal{R}(1-\mathcal{R})} - \frac{1}{\mathcal{R}}\Big] \\
&- \Big(\frac{\partial^x_\mu\mathcal{Z}_1}{\mathcal{Z}_1^3\mathcal{Z}_2^2}\partial^y_\nu\ln\frac{\Delta^2}{x_*y_*} + \frac{\partial^y_\nu\mathcal{Z}_2}{\mathcal{Z}_2^3\mathcal{Z}_1^2}\partial^x_\mu\ln\frac{\Delta^2}{x_*y_*}\Big)\Big[\frac{\ln\mathcal{R}}{\mathcal{R}(1-\mathcal{R})} - 2\Big] + (z_1\leftrightarrow z_2)\Big] \\
&- 2\frac{z_{12\perp}^2}{\mathcal{Z}_1^3\mathcal{Z}_2^3}\Big[4\mathrm{Li}_2(1-\mathcal{R}) - \frac{2\pi^2}{3} + 2\Big(\ln\frac{1}{\mathcal{R}} + \frac{1}{\mathcal{R}} + \frac{1}{2\mathcal{R}^2} - 3\Big)\ln\frac{1}{\mathcal{R}} - \Big(6+\frac{1}{\mathcal{R}}\Big)\ln\mathcal{R} + \frac{3}{\mathcal{R}} - 4\Big]\partial^x_\mu\partial^y_\nu\frac{\Delta^2}{x_*y_*}\Big\}
\end{aligned}$$

NLO evolution of composite “conformal” dipoles in QCD

I. B. and G. A. C.

$$a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\rm conf}_a = \frac{\alpha_s}{2\pi^2}\int d^2z_3\ \Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\rm conf}_a$$

$$\times\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$+\ \frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\ \Big\{\Big[-2+\frac{{z_{23}}^2z_{23}^2+z_{24}^2z_{13}^2-4z_{12}^2z_{34}^2}{2({z_{23}}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{{z_{23}}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big]$$

$$\times\ [\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{{z_{23}}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-{z_{23}}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{{z_{23}}^2z_{23}^2}\Big]$$

$$\times\ [\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$$b=\tfrac{11}{3}N_c-\tfrac{2}{3}n_f$$

$K_{\rm NLO\ BK}$ = Running coupling part + Conformal "non-analytic" (in j) part
$+$ Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\rm NLO\ BK}$ reproduces the known result for the forward NLO BFKL kernel.

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$

$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

Argument of coupling constant

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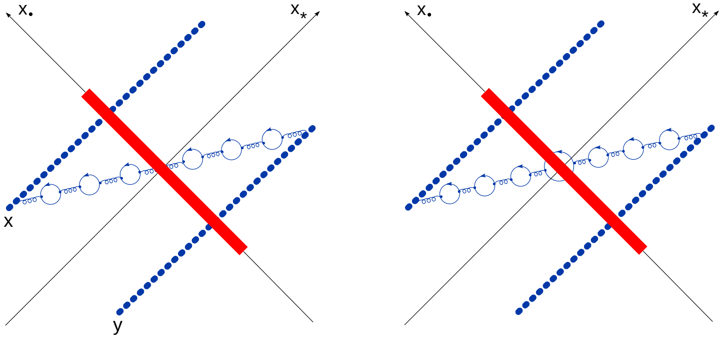
$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

Renormalon-based approach: summation of quark bubbles



$$-\tfrac{2}{3}n_f \rightarrow b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

Argument of coupling constant

$$
\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\ [\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{Tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]
$$

$$
\times\Big[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\Big(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\Big) + \frac{1}{z_{23}^2}\Big(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\Big)\Big] + \ldots
$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$
\frac{\alpha_s(z_{12}^2)}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2} \qquad |z_{12}| \ll |z_{13}|, |z_{23}|
$$

$$
\frac{\alpha_s(z_{13}^2)}{2\pi^2 z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|
$$

$$
\frac{\alpha_s(z_{23}^2)}{2\pi^2 z_{23}^2} \qquad |z_{23}| \ll |z_{12}|, |z_{13}|
$$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N}=4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL eigenvalues.
- The correlation function of four Z^2 operators is calculated at the NLO order.
- The analytic expression for the NLO photon impact factor is calculated (in the coord. space)