# Small-$x$ evolution in the next-to-leading order

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# Outline

- Regge limit in the coordinate space.
- Light-cone OPE versus OPE in color dipoles.
- High-energy scattering and Wilson lines.
- Factorization in rapidity: Feynman diagrams in a shock-wave background.
- Leading order: BK equation.
- Non-linear evolution equation in the NLO.
- $\mathcal{N}=4$: study of 2-dim conformal invariance at high energies
- NLO BK kernel in $\mathcal{N}=4$.
- NLO BK kernel in QCD
- Conclusions.

# Small-$x$ (Regge) limit in the coordinate space

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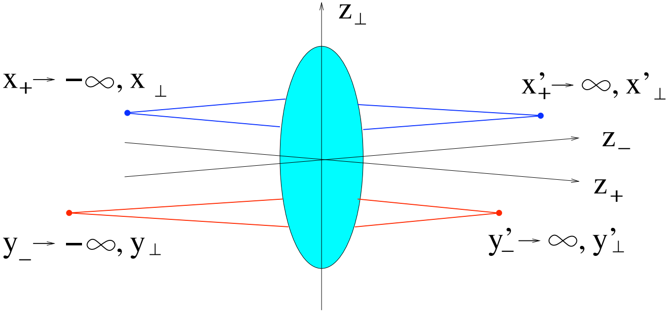


Regge limit symmetry in a conformal theory: 2-dim conformal Möbius group $SL(2, C)$.

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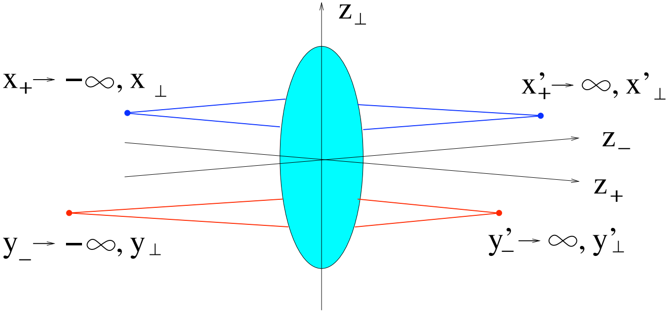


LLA: $\alpha_s \ll 1$, $\alpha_s \ln \rho \sim 1$, $\Rightarrow \sum(\alpha_s \ln \rho)^n \equiv$ BFKL pomeron.
LLA $\Leftrightarrow$ tree diagrams $\Rightarrow$ the BFKL pomeron is Möbius invariant .

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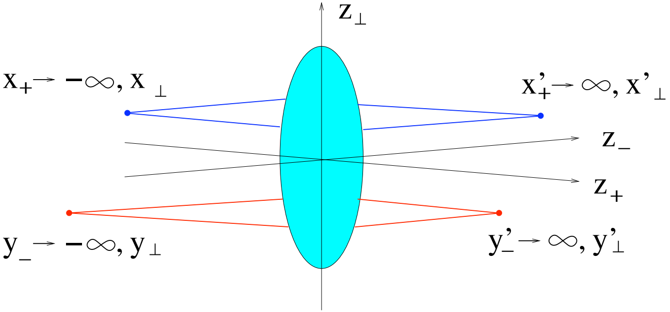


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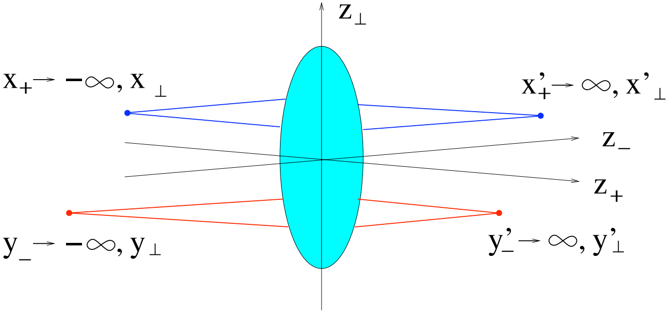


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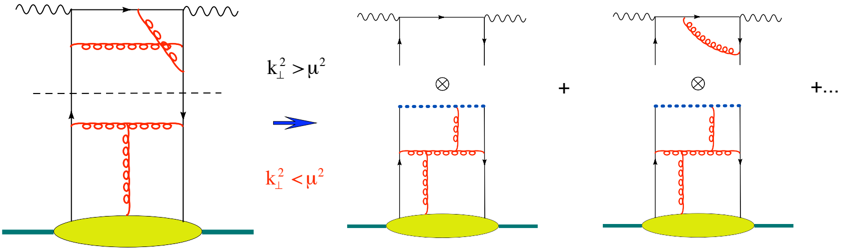
In QCD, we expect to have running coupling part plus conformal part.

# Light-cone expansion and DGLAP evolution in the NLO




$\mu^2$ - factorization scale (normalization point)

$k_\perp^2 \ > \ \mu^2$ - coefficient functions

$k_\perp^2 \ < \ \mu^2$ - matrix elements of light-ray operators (normalized at $\mu^2$)

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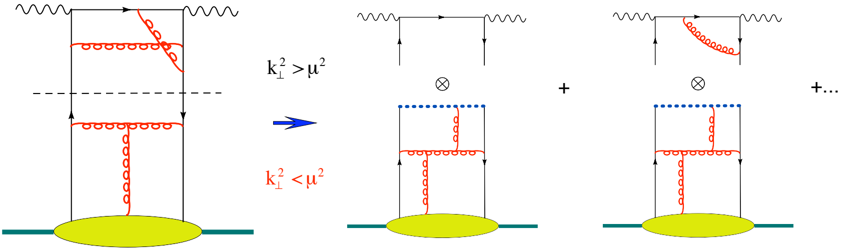


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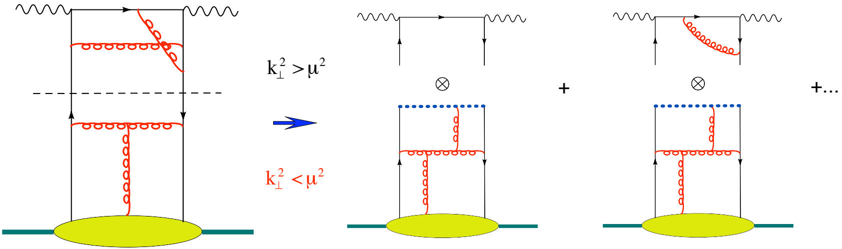
OPE in light-ray operators $(x-y)^2 \to 0$

$$T\{j_\mu(x)j_\nu(y)\} \;=\; \frac{x_\xi}{2\pi^2 x^4}\Big[1+\frac{\alpha_s}{\pi}(\ln x^2\mu^2 + C)\Big]\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y) + O(\frac{1}{x^2})$$

$$[x,y] \;\equiv\; Pe^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)}$$ - gauge link

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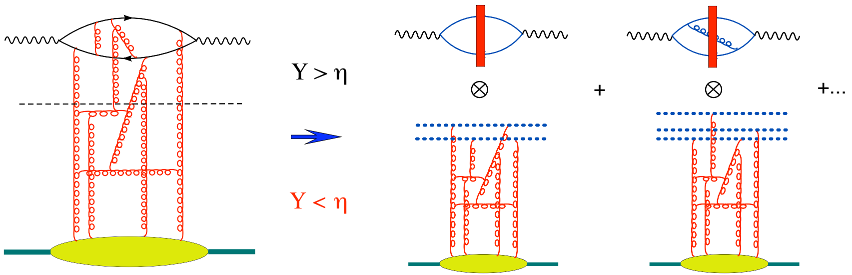
Renorm-group equation for light-ray operators $\Rightarrow$ DGLAP evolution of parton densities $(x-y)^2 = 0$

$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x,y]\psi(y) \ = \ K_{\mathrm{LO}} \bar{\psi}(x)[x,y]\psi(y) + \alpha_s K_{\mathrm{NLO}} \bar{\psi}(x)[x,y]\psi(y)$$

# High-energy expansion in color dipoles in the NLO



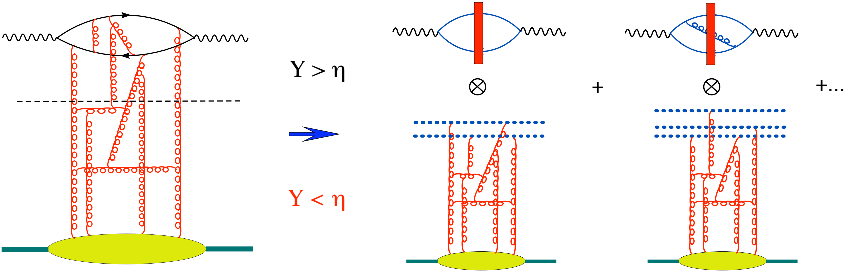

$\eta$ - rapidity factorization scale

Rapidity Y > $\eta$ - coefficient function (“impact factor”)
Rapidity Y < $\eta$ - matrix elements of (light-like) Wilson lines with rapidity divergence cut by $\eta$

$$U^\eta_x = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A^\eta_+(x_+, x_\perp)\Big]$$

$$A^\eta_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

# High-energy expansion in color dipoles in the NLO




The high-energy operator expansion is

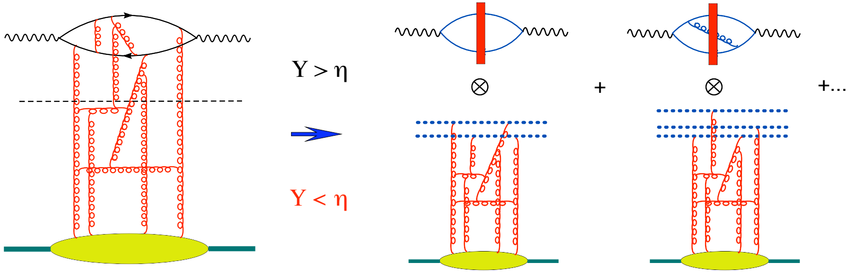
$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \; I^{\rm LO}_{\mu\nu}(z_1,z_2)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+ \int d^2z_1 d^2z_2 d^2z_3 \; I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3)[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c \mathrm{t}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order the impact factor is Mobius invariant

In the NLO one should also expect conf. invariance since $I^{\rm NLO}_{\mu\nu}$ is given by tree diagrams

# High-energy expansion in color dipoles in the NLO




$\eta$ - rapidity factorization scale

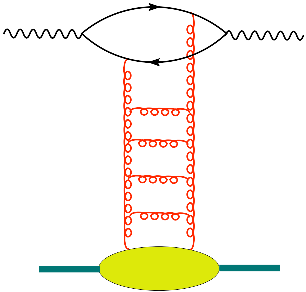
Evolution equation for color dipoles

$$\begin{aligned}\frac{d}{d\eta}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} &= \frac{\alpha_s}{2\pi^2}\int d^2z\frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} \\ &- N_c\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2)\end{aligned}$$

$K_{\mathrm{NLO}}$=? (Linear part of $K_{\mathrm{NLO}}$ = $K_{\mathrm{NLO\ BFKL}}$)

## DIS at high energy

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



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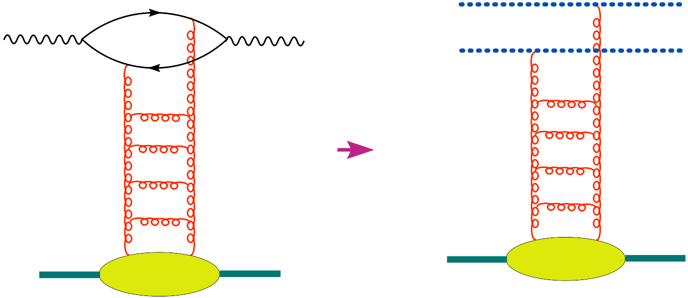

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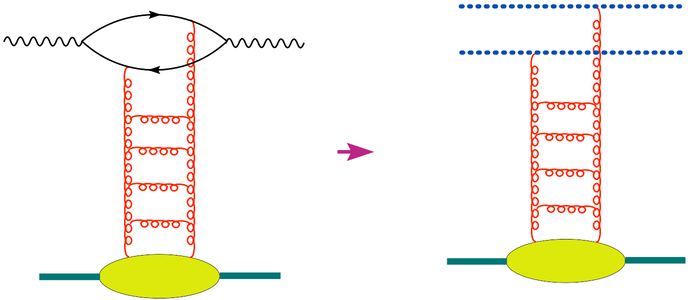

$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp)\langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

$$U(x_\perp) = Pe^{ig\int_{-\infty}^{\infty} du\ n^\mu A_\mu(un+x_\perp)} \qquad \text{Wilson line}$$

# Spectator frame: propagation in the shock-wave background.

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Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

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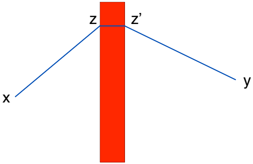

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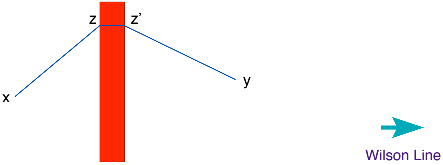

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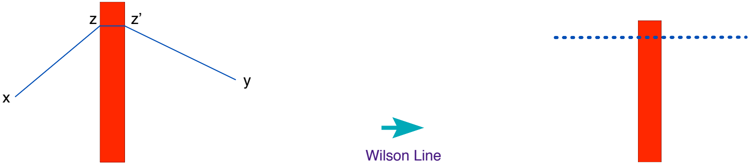


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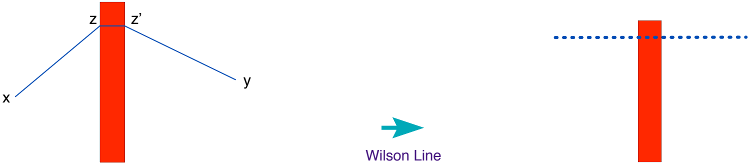


[ $x \to z$: free propagation]$\times$

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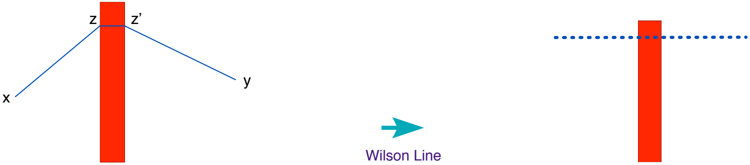


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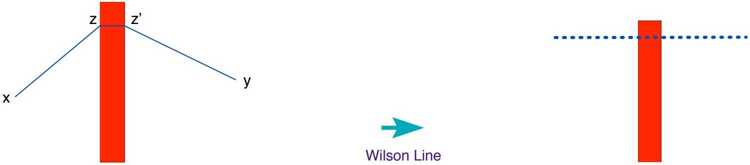


[ $x \to z$: free propagation]$\times$
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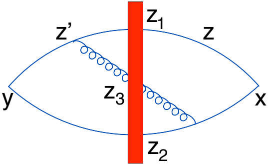
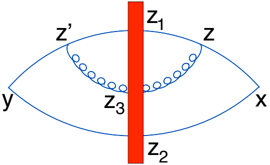
[$z \to y$: free propagation]

# NLO impact factor



(a)



(b)

$$\mathcal{Z}_3 \equiv \frac{(x-z_3)_\perp^2}{x^+} - \frac{(y-z_3)_\perp^2}{y^+}$$

$$I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I^{\rm LO}_{\mu\nu}\times\frac{\alpha_s}{2\pi}\frac{z_{13}^2}{z_{12}^2z_{23}^2}\ln\frac{\sigma s}{4}\mathcal{Z}_3 \;+\; {\rm conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant

However, if we define a composite operator ($a$ - analog of $\mu^{-2}$ for usual OPE)

$$\begin{aligned}
&[{\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf} \;=\; {\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\} \\
&+\frac{\alpha_s}{4\pi}\int d^2z_3\;\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2} \;+\; O(\alpha_s^2)
\end{aligned}$$

the impact factor becomes conformal in the NLO.

# Operator expansion in conformal dipoles

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2){\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO}_{\mu\nu} = \ -I^{\rm LO}_{\mu\nu}\ \frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{12}^2 e^{2\eta}as^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 + {\rm conf.}$$

The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N}=4$ SYM theory (where the $\beta$-function vanishes) one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbaton theory.

# Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to $\eta_1$ and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to $\eta_2$).

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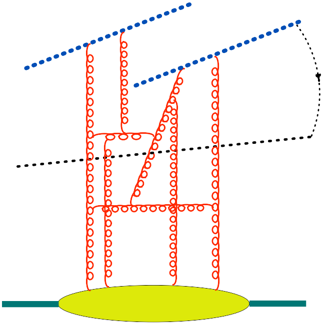

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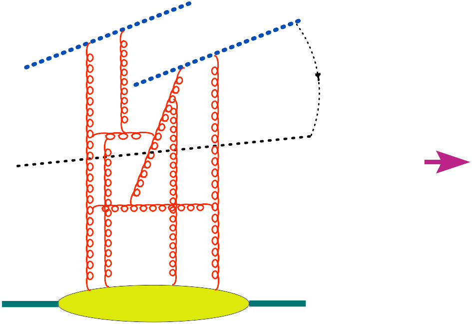

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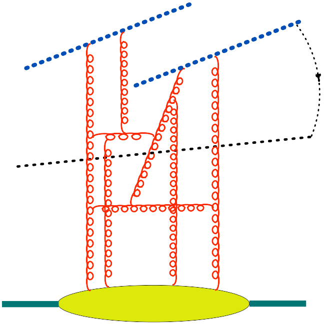
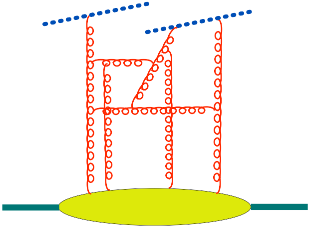
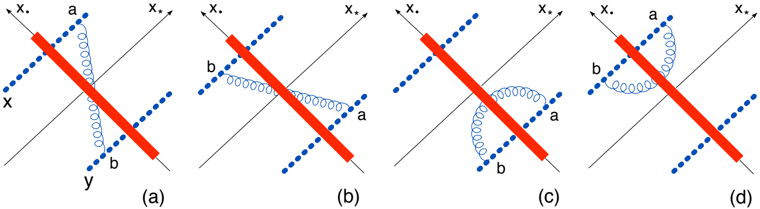


$$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \quad \otimes$$



$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{shockwave}}$$




$$U^{ab}_z = \mathrm{Tr}\{t^a U_z t^b U^\dagger_z\} \quad \Rightarrow (U_x U^\dagger_y)^{\eta_1} \to (U_x U^\dagger_y)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U^\dagger_z U_z U^\dagger_y)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

# Non linear evolution equation

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## BK equation

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I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1,\ \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

# Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty e_+ + x_\perp, -\infty e_+ + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+\, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

# Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty e_+ + x_\perp, -\infty e_+ + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+\, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^+ \to x^+/x_\perp^2 \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2}\, A_+\Big(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\Big)\right\} \;=\; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

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$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2 z}{\cancel{z^4}} \frac{(x-y)^2 \;\cancel{z^4}}{(x-z)^2(z-y)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

# Non-linear evolution equation in the NLO

$$\frac{d}{d\eta}Tr\{U_xU^\dagger_{z_2}\} =$$
$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s\frac{(x-y)^2}{(x-z)^2(z-y)^2}+\alpha_s^2K_{NLO}(x,y,z)\right)[Tr\{U_xU^\dagger_z\}Tr\{U_zU^\dagger_{z_2}\}-N_cTr\{U_zU^\dagger_{z_2}\}]+$$
$$\alpha_s^2\int d^2zd^2z'\left(K_4(x,y,z,z')\{U_x,U^\dagger_{z'},U_z,U^\dagger_{z_2}\}+K_6(x,y,z,z')\{U_x,U^\dagger_{z'},U_{z'},U_z,U^\dagger_z,U^\dagger_{z_2}\}\right)$$

$K_{NLO}$ is the next-to-leading order correction to the dipole kernel and $K_4$ and $K_6$ are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

# Definition of the NLO kernel

The NLO kernel is obtained in the same way as the NLO DGLAP kernel:

1. Write down the general form of the operator equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

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2. Calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle + O(\alpha_s^3)$$

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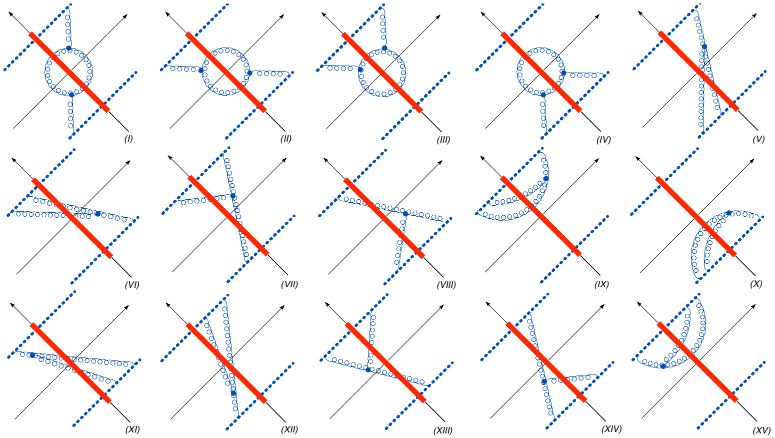
3. Subtract the LO contribution

$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter $v$

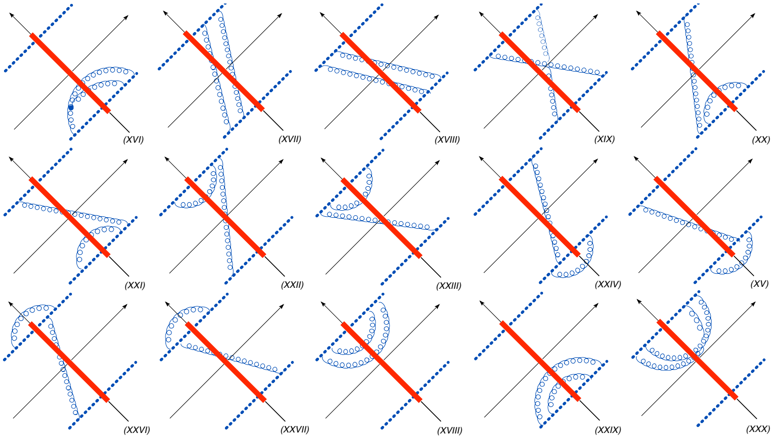
Typical integral

$$\int_0^1 dv\ \frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

# Gluon part of the NLO BK kernel: diagrams



# Diagrams for 1→3 dipoles transition



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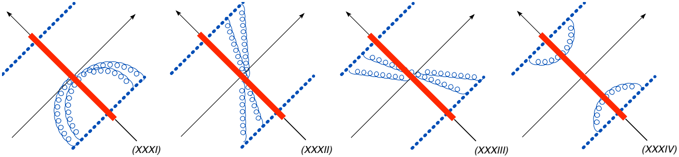
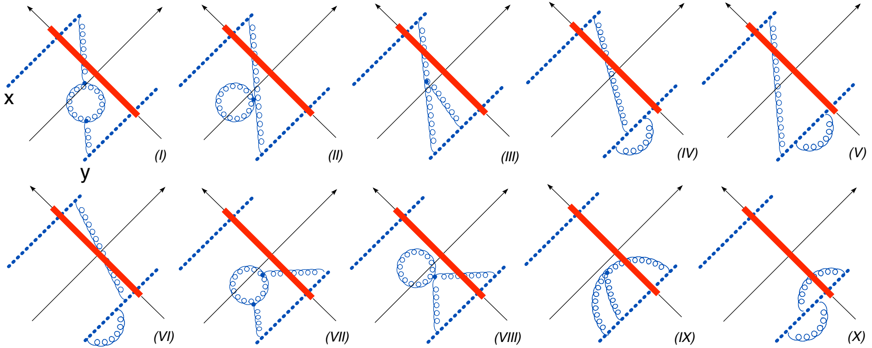
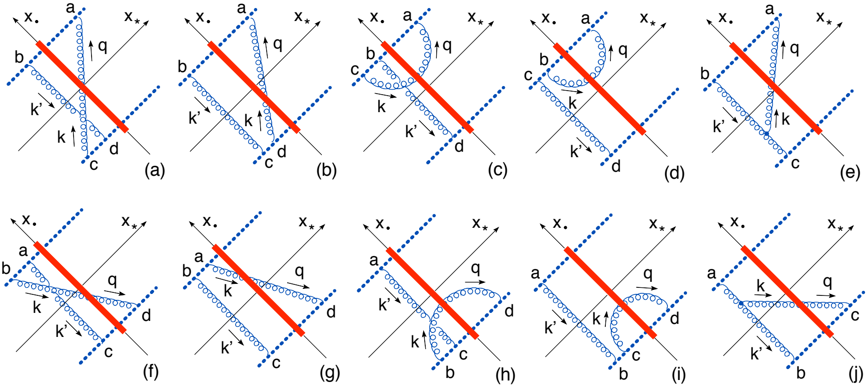


# "Running coupling" diagrams

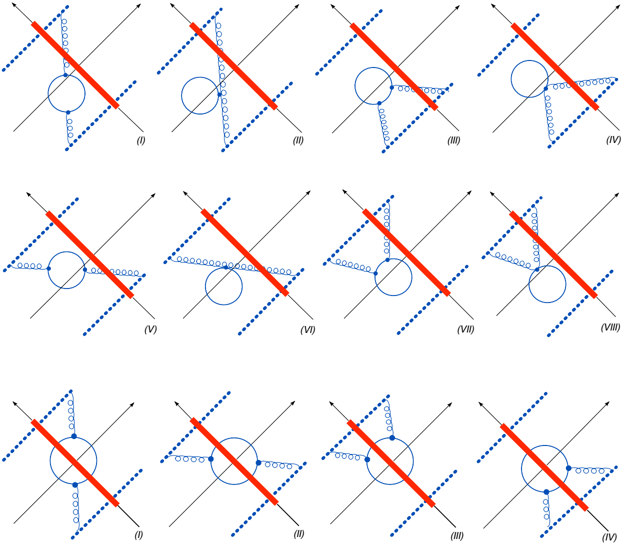


# 1 → 2 dipole transition diagrams



$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{Tr}\{U_{z_3}U^\dagger_{z_2}\}-N_c\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}]\\
\times & \Big\{\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(\frac{11}{3}\ln z_{12}^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big]\\
- & \frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}-\frac{\alpha_sN_c}{2\pi}\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big\}\\
+ & \frac{\alpha_s}{4\pi^2}\int d^2z_4\,\Big\{[\mathrm{Tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{Tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\}-\mathrm{Tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\}\\
- & (z_4\to z_3)]\frac{1}{z_{34}^4}\Big[-2+\frac{z_{14}^2z_{23}^2+z_{24}^2z_{13}^2-4z_{12}^2z_{34}^2}{2(z_{14}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{z_{14}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big]\\
+ & [\mathrm{Tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{Tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\}-\mathrm{Tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z}\}-(z_4\to z_3)]\\
\times & \Big[\frac{z_{12}^4}{z_{13}^2{z_{24}}^2(z_{13}^2{z_{24}}^2-{z_{14}}^2z_{23}^2)}+\frac{z_{12}^2}{z_{34}^2z_{13}^2{z_{24}}^2}\Big]\ln\frac{z_{13}^2{z_{24}}^2}{{z_{14}}^2z_{23}^2}\Big\}\Big)
\end{aligned}
$$

# $\mathcal{N} = 4$ diagrams (scalar and gluino loops)



$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\,[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\,\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\\
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&\times\,[\mathrm{Tr}\{T^a\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
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For the conformal composite dipole the result is Möbius invariant

# Evolution equation for composite conformal dipoles in $\mathcal{N}=4$ SYM

$$
\begin{aligned}
&\frac{d}{d\eta}\big[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\mathrm{conf}}\\
=\ &\frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\mathrm{conf}}\\
&-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}\\
&\times\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}[(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4})^{bb'}-(z_4\to z_3)]
\end{aligned}
$$

Now Möbius invariant!

# NLO BFKL equation in $\mathcal{N} = 4$ SYM

To find $A(x,y;x',y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{\mathcal{U}}^\eta(x,y) \;=\; 1 - \frac{1}{N_c^2-1}\mathrm{Tr}\{\hat{U}_x^\eta \hat{U}_y^{\dagger\eta}\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{\mathcal{U}}^\eta_{\rm conf}(z_1,z_2) \;=\; \hat{\mathcal{U}}^\eta(z_1,z_2)+\frac{\alpha_s N_c}{4\pi^2}\int d^2z\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1,z_3)+\hat{\mathcal{U}}^\eta(z_2,z_3)-\hat{\mathcal{U}}^\eta(z_1,z_2)]$$

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$$\hat{\mathcal{U}}^\eta_{\mathrm{conf}}(z_1, z_2) \;=\; \hat{\mathcal{U}}^\eta(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2}\int d^2z \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{a z_{12}^2}{z_{13}^2 z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1, z_3) + \hat{\mathcal{U}}^\eta(z_2, z_3) - \hat{\mathcal{U}}^\eta(z_1, z_2)]$$

NLO BFKL

$$\begin{aligned}
&\frac{d}{d\eta}\hat{\mathcal{U}}^\eta_{\mathrm{conf}}(z_1, z_2) \\
&= \frac{\alpha_s N_c}{2\pi^2}\int d^2z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1 - \frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}\Big][\hat{\mathcal{U}}^\eta_{\mathrm{conf}}(z_1, z_3) + \hat{\mathcal{U}}^\eta_{\mathrm{conf}}(z_2, z_3) - \hat{\mathcal{U}}^\eta_{\mathrm{conf}}(z_1, z_2)] \\
&+ \frac{\alpha_s^2 N_c^2}{8\pi^4}\int \frac{d^2z_3 d^2z_4}{z_{34}^4}\, \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}\Big\{2\ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \Big[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big\}\hat{\mathcal{U}}^\eta_{\mathrm{conf}}(z_3, z_4) \\
&\qquad + \frac{3\alpha_s^2 N_c^2}{2\pi^3}\zeta(3)\hat{\mathcal{U}}^\eta_{\mathrm{conf}}(z_1, z_2)
\end{aligned}$$

Eigenvalues agree with Kotikov and Lipatov (2000)

# NLO evolution of composite "conformal" dipoles in QCD

$$\frac{d}{d\eta}[\mathrm{tr}\{\hat U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_2}\} - N_c\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\mathrm{conf}}$$

$$\times\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{z_{14}{}^2z_{23}^2+z_{24}{}^2z_{13}^2-4z_{12}^2z_{34}^2}{2(z_{14}{}^2z_{23}^2-z_{24}{}^2z_{13}^2)}\ln\frac{z_{14}{}^2z_{23}^2}{z_{24}{}^2z_{13}^2}\Big]$$

$$\times[\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_4}\}\{\hat U_{z_4}\hat U^\dagger_{z_2}\} - \mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\hat U_{z_4}\hat U^\dagger_{z_2}\hat U_{z_3}\hat U^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}{}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}{}^2}{z_{14}{}^2z_{23}^2}\Big]$$

$$\times[\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_4}\}\mathrm{tr}\{\hat U_{z_4}\hat U^\dagger_{z_2}\} - \mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_4}\hat U_{z_3}\hat U^\dagger_{z_2}\hat U_{z_4}\hat U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$$b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel.

# Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$
$$\frac{\alpha_s(?)N_c}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3) + \hat{\mathcal{U}}(z_3,z_2)) - \hat{\mathcal{U}}(z_1,z_2) - \hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

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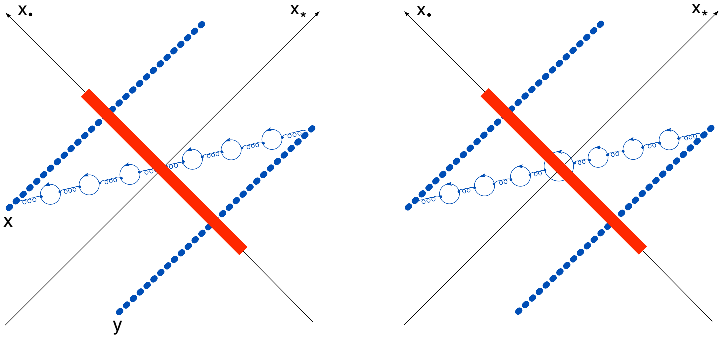
Renormalon-based approach: summation of quark bubbles



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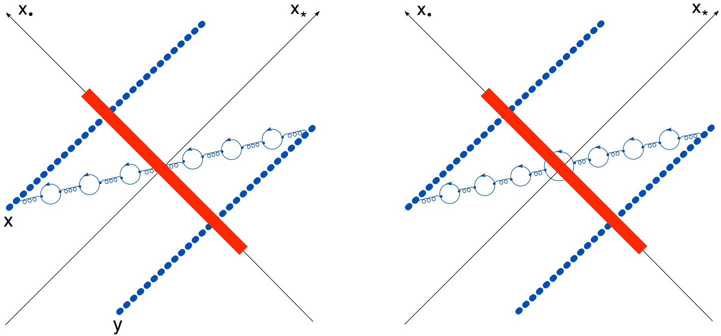


$$-\tfrac{2}{3}n_f \rightarrow b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

# Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\,\left[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{Tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}\right]$$

$$\times\left[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\left(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\right) + \frac{1}{z_{23}^2}\left(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\right)\right] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

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When the sizes of the dipoles are very different the kernel reduces to:

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$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.

# Conclusions

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- The NLO BK kernel in QCD and $\mathcal{N}=4$ SYM agrees with NLO BFKL eigenvalues.
- The NLO BK kernel for the evolution of conformal composite dipoles in $\mathcal{N}=4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel in QCD is a sum of the running-coupling part and conformal part.
- The coupling constant in the BK equation is determined by the size of smallest dipole.