# High-energy amplitudes in $\mathcal{N} = 4$ SYM at the next-to-leading order

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# Conformal four-point amplitude

$$A(x,y,x',y') = (x-y)^4(x'-y')^4 N_c^2 \langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$

$\mathcal{O} = \mathrm{Tr}\{Z^2\}$ ($Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$) - chiral primary operator

In a conformal theory the amplitude is a function of two conformal ratios

$$A = F(R,R')$$
$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \quad R' = \frac{(x-y)^2(x'-y')^2}{(x-y')^2(x'-y)^2}$$

At large $N_c$

$$A(x,y,x',y') = A(g^2N_c) \qquad\qquad g^2N_c = \lambda \; - \; \text{'t Hooft coupling}$$

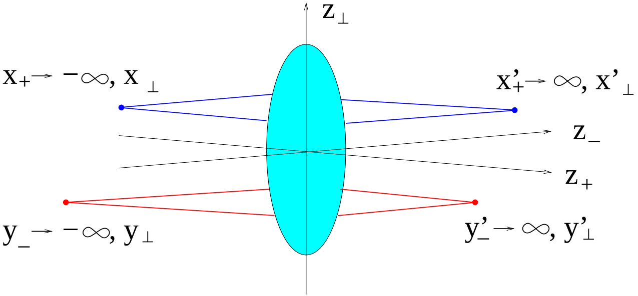
AdS/CFT gives predictions at large $\lambda \to \infty$.

Our goal is perturbative expansion and resummation of $(\lambda \ln s)^n$ at large energies in the next-to-leading approximation

$$(\lambda \ln s)^n (c_n^{\mathrm{LO}} + c_n^{\mathrm{NLO}}\lambda)$$

# Regge limit in the coordinate space

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+,\ y_- \to \rho' y_-,\ y'_- \to \rho' y{-}_-$ $\qquad \rho, \rho' \to \infty$



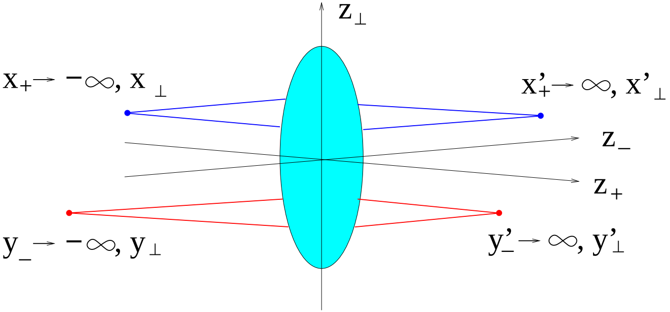

Full 4-dim conformal group: $A = F(R_1, R_2)$

$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \to \frac{\rho^2\rho'^2 x_+ x'_+ y_- y'_-}{(x-x')^2_\perp (y-y')^2_\perp}$$

$$r = \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y')^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2}$$

$$\to \frac{[(x'-y')^2_\perp x_+ y_- + x'_+ y'_-(x-y)^2_\perp + x_+ y'_-(x'-y)^2_\perp + x'_+ y_-(x-y')^2_\perp]^2}{(x-x')^2_\perp (y-y')^2_\perp x_+ x'_+ y_- y'_-}$$

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+, y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\quad \rho,\rho' \to \infty$




Regge limit symmetry: 2-dim conformal group $SL(2,C)$ formed from $P_1, P_2, M^{12}, D, K_1$ and $K_2$ which leave the plane $(0,0,z_\perp)$ invariant.

Inversion: $x_\perp \to \frac{x_\perp}{x_\perp^2},\ x_+ \to \frac{x_+}{x_\perp^2},\ y_- \to \frac{y_-}{y_\perp^2}$.

All the ratios of the type

$$\frac{(x-y)_\perp^2 (x'-y')_\perp^2}{(x-x')_\perp^2 (y-y')_\perp^2} \quad \text{or} \quad \frac{x_+ y'_-}{(x-y')_\perp^2} \qquad \text{are invariant}$$

$\Rightarrow$ much less restrictive

$$A(x,y;x',y') \overset{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\omega(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\omega(\lambda,\nu)/2}$$

L. Cornalba (2007)

$f_+(\omega) = \frac{e^{i\pi\omega}-1}{\sin\pi\omega}$ - signature factor

$\Omega(r,\nu)$ - solution of the eqn $(\Box_{H_3}+\nu^2+1)\Omega(r,\nu)=0$.
Explicit form:

$$\Omega(r,\nu) = \frac{\nu^2}{\pi^3}\int d^2z\Big(\frac{\kappa^2}{(2\kappa\cdot\zeta)^2}\Big)^{\frac{1}{2}+i\nu}\Big(\frac{\kappa'^2}{(2\kappa'\cdot\zeta)^2}\Big)^{\frac{1}{2}-i\nu}$$

$$\zeta = p_1 + \frac{z_\perp^2}{s}p_2 + z_\perp, \qquad p_1^2=p_2^2=0,\ \ 2(p_1,p_2)=s$$

$$\kappa = \frac{1}{2x_+}(p_1-\frac{x^2}{s}p_2+x_\perp) - \frac{1}{2y_+}(p_1-\frac{y^2}{s}p_2+y_\perp), \qquad \kappa^2\kappa'^2 = \frac{1}{R}$$

$$\kappa' = \frac{1}{2x'_-}(p_1-\frac{x'^2}{s}p_2+x'_\perp) - \frac{1}{2y'_-}(p_1-\frac{y'^2}{s}p_2+y'_\perp, \qquad 4(\kappa\cdot\kappa')^2 = \frac{r}{R}$$

The dynamics is described by $\omega(\lambda,\nu)$ and $F(\lambda,\nu)$.

# Pomeron in the conformal theory

$$A(x,y;x',y') \stackrel{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\omega(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\omega(\lambda,\nu)/2}$$

Pomeron intercept $\omega(\nu,\lambda)$ is known in two limits:

$$1. \qquad \lambda\to 0: \qquad \omega(\nu,\lambda) = \frac{\lambda}{\pi}\chi(\nu)+\lambda^2\omega_1(\nu)+...$$

$\chi(\nu)=2\psi(1)-\psi(\frac{1}{2}+i\nu)-\psi(\frac{1}{2}-i\nu)$ - BFKL intercept,

$\omega_1(\nu)$ - NLO BFKL intercept Lipatov, Kotikov (2000)

$$2. \qquad \lambda\to\infty: \qquad AdS/CFT \quad \Rightarrow \quad \omega(\nu,\lambda) = 2-\frac{\nu^2+4}{2\sqrt{\lambda}}+...$$

2 = gravition spin , next term - Brower, Polchinski, Strassler, Tan (2006)
Cornalba, Costa, Penedones (2007)

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The function $F(\nu,\lambda)$ in two limits:

$$1. \qquad \lambda\to 0: \qquad F(\nu,\lambda) \;=\; \lambda^2 F_0(\nu)+\lambda^3 F_1(\nu)+...$$

$F_0(\nu)=\frac{\pi\sinh\pi\nu}{4\nu\cosh^3\pi\nu}$ Cornalba, Costa, Penedones (2007)

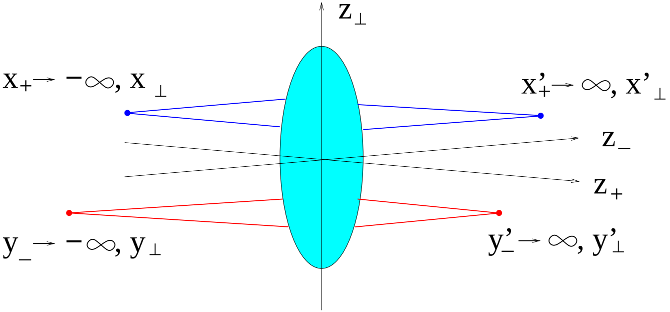
$F_1(\nu)$ = see below G. Chirilli and I.B. (2009)

$$2. \qquad \lambda\to\infty: \qquad AdS/CFT \quad\Rightarrow\quad \omega(\nu,\lambda) \;=\; \pi^3\nu^2\frac{1+\nu^2}{\sinh^2\pi\nu}+...$$

L.Cornalba(2007)

We calculate $F_1(\nu)$ (and confirm $\omega_1(\nu)$) using the expansion of high-energy amplitudes in Wilson lines (color dipoles)

# Conformally invariant evolution equation?




LO BFKL: From the full conformal group we get $A = \sum \alpha_s^n f_n(R_2) \ln^n R_1$

$$R_1 = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \rightarrow \frac{\rho^2\rho'^2 x_+ x'_+ y_- y'_-}{(x-x')_\perp^2 (y-y')_\perp^2}$$

$R_1$ (and $R_2$) are symmetric with respect to projectile $\leftrightarrow$ target $(x \leftrightarrow y,\ x' \leftrightarrow y')$.

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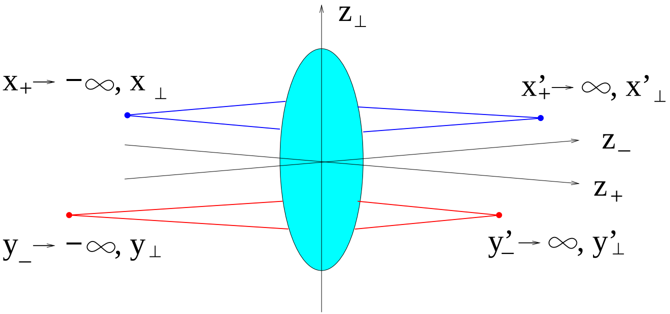


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NLO BFKL: $A = \sum \alpha_s^{n+1} g_n(R_2) \ln^n R_1$

$\Rightarrow$ also should be symmetric with respect to projectile $\leftrightarrow$ target.

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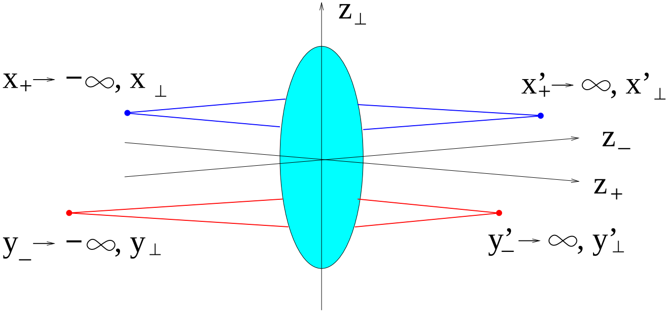


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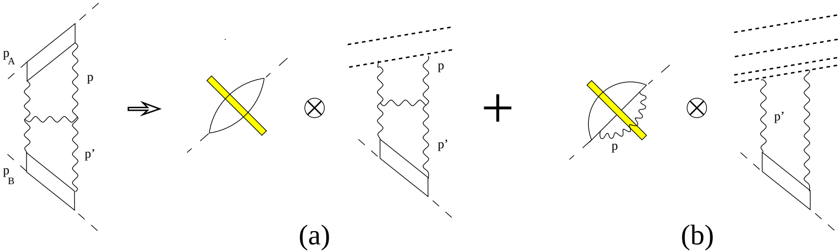
$R_1$ (and $R_2$) are symmetric with respect to projectile $\leftrightarrow$ target ($x \leftrightarrow y,\ x' \leftrightarrow y'$).

NLO BFKL: $A = \sum \alpha_s^{n+1} g_n(R_2) \ln^n R_1$

$\Rightarrow$ also should be symmetric with respect to projectile $\leftrightarrow$ target.

$\Rightarrow$ we need the evolution equation compatible with conformal invariance

# Expansion in Color Dipoles in the next-to-leading order




$$F_2(x_B) \simeq \int d^2z_1 d^2z_2\ I_0(z_1,z_2)\langle \mathrm{tr}\{U_{z_1}U^{\dagger\,\eta}_{z_2}\}\rangle \qquad \eta = \ln\frac{1}{x_B}$$

$$+\ \frac{\alpha_s}{\pi}\int d^2z_1 d^2z_2 d^2z_3\ I_1(z_1,z_2,z_3)\langle \mathrm{tr}\{U_{z_1}U^{\dagger\,\eta}_{z_3}\}\mathrm{tr}\{U_{z_3}U^{\dagger\,\eta}_{z_2}\}\rangle$$

Evolution of the color dipole $\mathrm{tr}\{U_{z_1}U^{\dagger\,\eta}_{z_2}\}$: calculated

$$\frac{d}{d\eta}\{U_{z_1}U^{\dagger\,\eta}_{z_2}\} = K_{\mathrm{BK}}\{U_{z_1}U^{\dagger\,\eta}_{z_2}\} + K_{\mathrm{NLO\ BK}}\{U_{z_1}U^{\dagger\,\eta}_{z_2}\}$$

NLO “Impact factor” $I_1$ - to be calculated

Regge limit in the coordinate space: $x_+, -y_+ \to \infty$, $x'_-, -y'_- \to \infty$

In a conformal theory

$$\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle = \int d\nu\ I^A(\alpha_s,\nu)I^B(\alpha_s,\nu)\Omega(r,\nu)R^{\omega(\alpha_s,\nu)}$$

$$= \int d\nu\ [I_0^A(\nu)+\alpha_s I_1^A(\nu)][I_0^B(\nu)+\alpha_s I_1^B(\nu)]\Omega(r,\nu)R^{\omega_{\rm LO}(\nu)+\alpha_s\omega_{\rm NLO}(\nu)}$$

$\omega = j-1$ - pomeron intercept (known in the NLO order and at $\alpha_s N_c \to \infty$)

$$R = \frac{(x-x')(y-y')^2}{(x-y)^2(x'-y')^2} \to \infty,$$

$$r = R\Big[1 - \frac{(x-y')^2(y-x')^2}{(x-x')^2(y-y')^2} + \frac{1}{R}\Big]^2 \simeq O(1)$$

Expansion at $(x-y)^2 \to 0$ should reproduce coeff. functions $c_n$ and anomalous dimensions $\gamma_n$ of local higher-twist operators $G_{+i}D_+^{n-2}G_{+i}$ in all orders in $\alpha_s$ in the limit $n=j\to 1$.

Question: is any part/modification of this structure survives in QCD ?

### Composite dipole with conformally invariant rapidity cutoff

$$[\mathrm{tr}\{U_x U_y^\dagger\}]^{\mathrm{conf}}$$

$$= \mathrm{tr}\{U_x U_y^\dagger\} - \frac{\alpha_s}{4\pi^2}\int d^2z \frac{(x-y)^2}{X^2Y^2} \ln \frac{a(x-y)^2}{X^2Y^2}[\mathrm{tr}\{U_x U_z^\dagger\}\mathrm{tr}\{U_z U_y^\dagger\} - N_c \mathrm{tr}\{U_x U_y^\dagger\}]$$

Evolution equation for conformal dipoles

$$\frac{d}{d\eta}[\mathrm{tr}\{U_x U_y^\dagger\}]^{\mathrm{conf}} = \frac{\alpha_s}{2\pi^2}\int d^2z \,\Big([\mathrm{tr}\{U_x U_z^\dagger\}\mathrm{tr}\{U_z U_y^\dagger\} - N_c \mathrm{tr}\{U_x U_y^\dagger\}]^{\mathrm{conf}}$$

$$\times \frac{(x-y)^2}{X^2Y^2}\Big[1 - \frac{\alpha_s \pi N_c}{12}\Big]$$

$$+ \frac{\alpha_s}{4\pi^2}\int \frac{d^2z'(x-y)^2}{(z-z')^2X^2Y'^2}\,[\mathrm{tr}\{U_x U_z^\dagger\}\mathrm{tr}\{U_z U_{z'}^\dagger\}\mathrm{tr}\{U_{z'} U_y^\dagger\} - (z' \to z)]^{\mathrm{conf}}$$

$$\times \Big[2\ln\frac{(x-y)^2(z-z')^2}{X'^2Y^2} + \Big(1 + \frac{(x-y)^2(z-z')^2}{X^2Y'^2 - X'^2Y^2}\Big)\ln\frac{X^2Y'^2}{X'^2Y^2}\Big]$$

$K_{\mathrm{NLO}}$ is conformally invariant and analytic in complex momentum $j$.

$$\frac{d}{d\eta}\mathrm{tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{tr}\{U_xU_z^\dagger\}\mathrm{tr}\{U_zU_y^\dagger\} - N_c\mathrm{tr}\{U_xU_y^\dagger\}]$$
$$\times\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln(x-y)^2\mu^2 + b\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$
$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z'}{(z-z')^4}\Big\{\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]$$
$$\times[\mathrm{tr}\{U_xU_z^\dagger\}\mathrm{tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\} - (z'\to z)]$$
$$+\frac{(x-y)^2(z-z')^2}{X^2Y'^2}\Big[2\ln\frac{(x-y)^2(z-z')^2}{X'^2Y^2}+\Big(1+\frac{(x-y)^2(z-z')^2}{X^2Y'^2-X'^2Y^2}\Big)\ln\frac{X^2Y'^2}{X'^2Y^2}\Big]$$
$$\times[\mathrm{tr}\{U_xU_z^\dagger\}\mathrm{tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)]\Big\}$$

$$b=\tfrac{11}{3}N_c-\tfrac{2}{3}n_f,\ X\equiv x-z,\ Y\equiv y-z,\ X'\equiv x-z',\ Y\equiv y-z'$$

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part
$+$ Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel.