Next-to-leading order evolution of color dipoles

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JLAB & ODU

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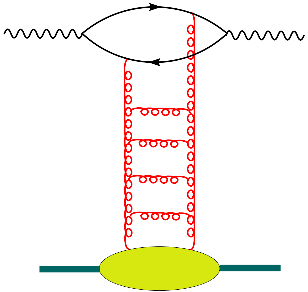
Outline

- DIS from nucleus at high energy and Wilson lines.
- Evolution equation for color dipoles.
- Leading order: BK equation.
- Non linear evolution equation in the NLO.
- Argument of coupling constant in the BK equation.
- NLO kernel.
- $\mathcal{N} = 4$: study of 2-dim conformal invariance at high energies
- NLO kernel in $\mathcal{N} = 4$.
- Conclusions and outlook.

DIS at high energy

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



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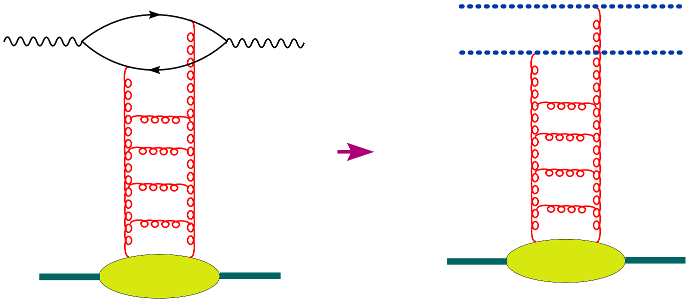

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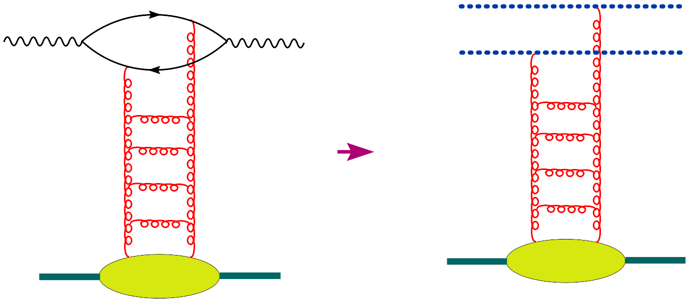

$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp)\langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

$$U(x_\perp) = Pe^{ig\int_{-\infty}^{\infty} du\ n^\mu A_\mu(un+x_\perp)} \qquad \text{Wilson line}$$

Evolution Equation

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

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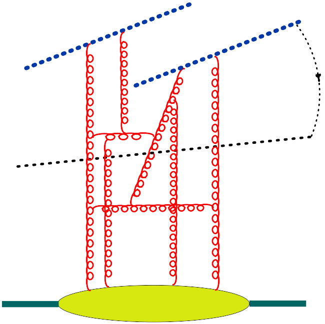

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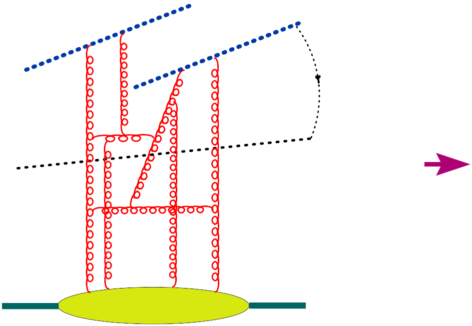

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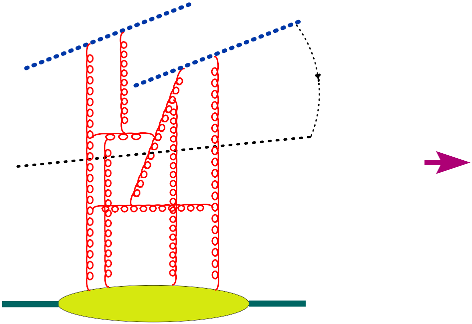
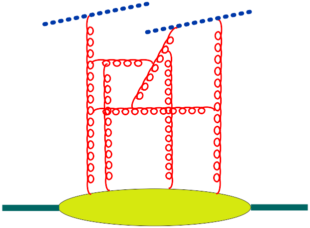


$$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \otimes$$



Spectator frame: propagation in the shock-wave background.

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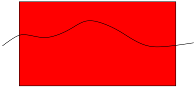

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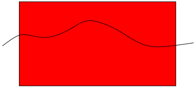

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Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

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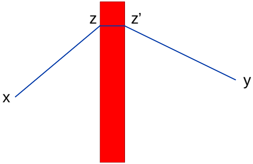

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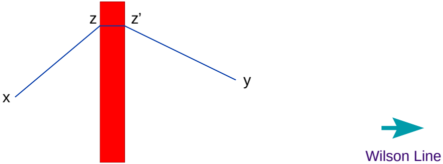

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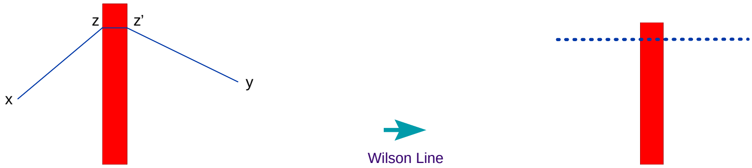

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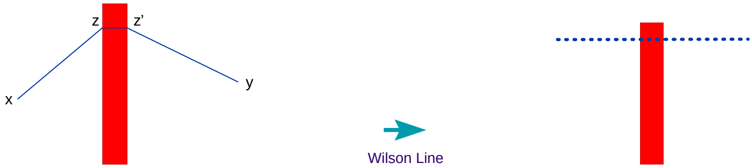

[$x \to z$: free propagation]×

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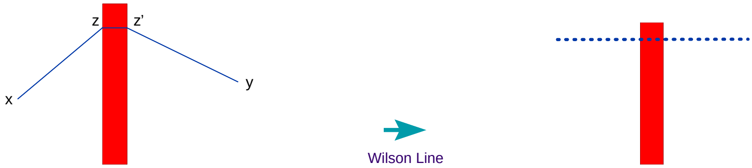

[$x \rightarrow z$: free propagation]$\times$
$[U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$]\times$

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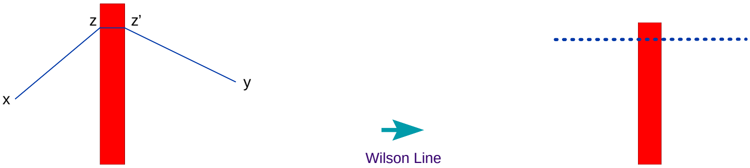
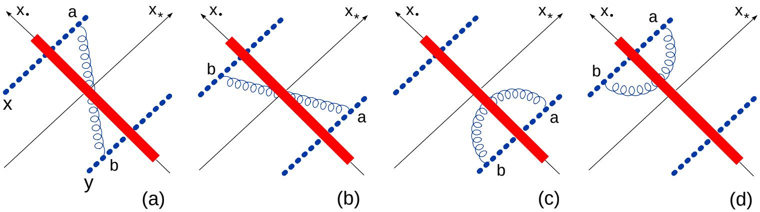


[$x \to z$: free propagation]$\times$
$[U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$]\times$
[$z \to y$: free propagation]

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$$U^{ab}_z = \mathrm{Tr}\{t^a U_z t^b U^\dagger_z\} \quad \Rightarrow (U_x U^\dagger_y)^{\eta_1} \to (U_x U^\dagger_y)^{\eta_1} + \alpha_s(\eta_1-\eta_2)(U_x U^\dagger_z U_z U^\dagger_y)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

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I. B. (1996), Yu. Kovchegov (1999)

Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1,\ \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Why NLO correction?

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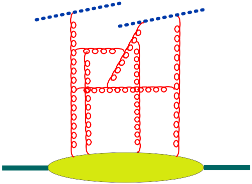
- To check conformal invariance (in $\mathcal{N}$=4 SYM)

Non-linear evolution equation in the NLO

$$\frac{d}{d\eta}Tr\{U_xU^\dagger_y\} =$$
$$\int\frac{d^2z}{2\pi^2}\left(\alpha_s\frac{(x-y)^2}{(x-z)^2(z-y)^2}+\alpha_s^2K_{NLO}(x,y,z)\right)[Tr\{U_xU^\dagger_z\}Tr\{U_zU^\dagger_y\}-N_cTr\{U_zU^\dagger_y\}]+$$
$$\alpha_s^2\int d^2zd^2z'\left(K_4(x,y,z,z')\{U_x,U^\dagger_{z'},U_z,U^\dagger_y\}+K_6(x,y,z,z')\{U_x,U^\dagger_{z'},U_{z'},U_z,U^\dagger_z,U^\dagger_y\}\right)$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K_4 and K_6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Regularizing the rapidity divergence



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

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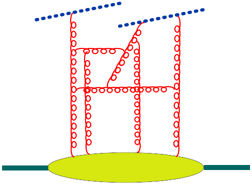

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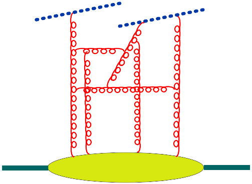
Regularization by: slope $\parallel$ velocity

$$U^\eta(x_\perp) = \text{Pexp}\Big\{ig\int_{-\infty}^{\infty} du\, n_\mu\, A^\mu(un + x_\perp)\Big\} \qquad n = p_1 + e^{-\eta}p_2$$



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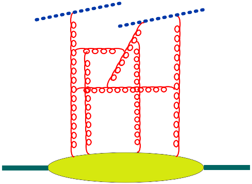
Regularization by: rigid cut-off

$$U_x^\eta = \mathrm{Pexp}\Big[ig\int_{-\infty}^\infty du\, p_1^\mu A_\mu^\eta(up_1 + x_\perp)\Big]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$



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Leads to (almost) conformal NLO kernel

Definition of the NLO kernel

In general

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

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Subtraction of the (LO) contribution (with the rigid rapidity cutoff)
$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

Typical integral

$$\int_0^1 dv\, \frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) =$$

$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_\perp \frac{(\vec{x}-\vec{y})^2_\perp}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{z}_\perp-\vec{y}_\perp)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

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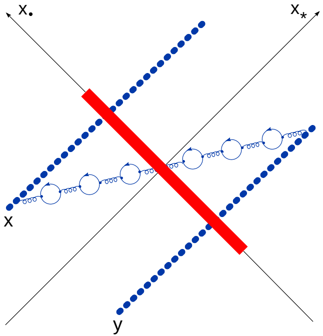
Renormalon-based approach: summation of quark bubbles



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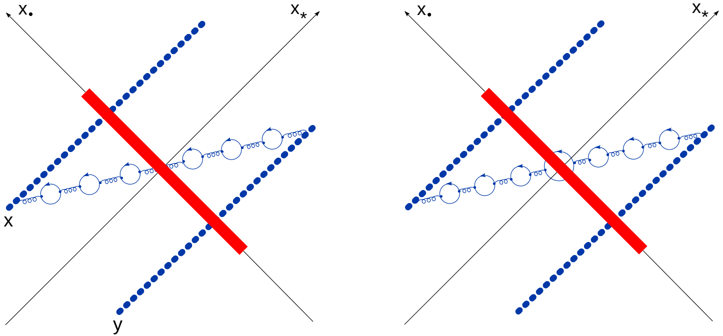

$$-\frac{2}{3}n_f \to b = \frac{11}{3}N_c - \frac{2}{3}n_f$$

Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} \;=\; \frac{\alpha_s((x-y)^2)}{2\pi^2}\int d^2z\,\left[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\right]$$

$$\times\left[\frac{(x-y)^2}{X^2Y^2} + \frac{1}{X^2}\left(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1\right) + \frac{1}{Y^2}\left(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1\right)\right] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

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$$\times\Big[\frac{(x-y)^2}{X^2Y^2} + \frac{1}{X^2}\Big(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1\Big) + \frac{1}{Y^2}\Big(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1\Big)\Big] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$\frac{\alpha_s((x-y)^2)}{2\pi^2}\frac{(x-y)^2}{X^2Y^2}$	$\lvert x-y\rvert \ll \lvert x-z\rvert, \lvert y-z\rvert$
$\frac{\alpha_s(X)^2)}{2\pi^2X^2}$	$\lvert x-z\rvert \ll \lvert x-y\rvert, \lvert y-z\rvert$
$\frac{\alpha_s(Y)^2)}{2\pi^2Y^2}$	$\lvert y-z\rvert \ll \lvert x-y\rvert, \lvert x-z\rvert$

Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{\alpha_s((x-y)^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}]$$
$$\times\Big[\frac{(x-y)^2}{X^2Y^2} + \frac{1}{X^2}\Big(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1\Big) + \frac{1}{Y^2}\Big(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1\Big)\Big] + \ldots$$

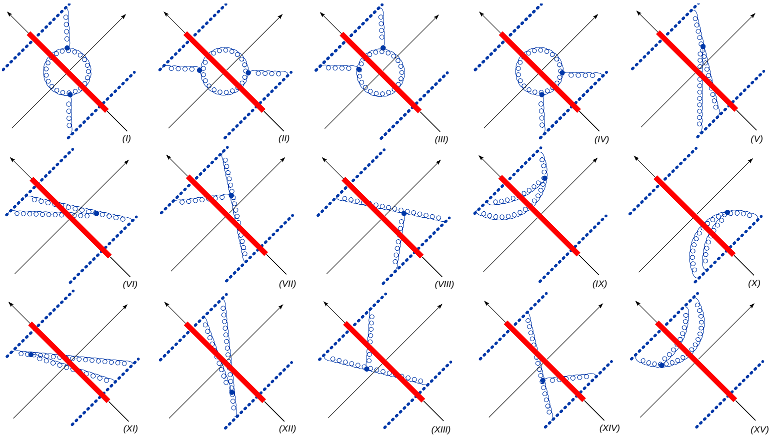
I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

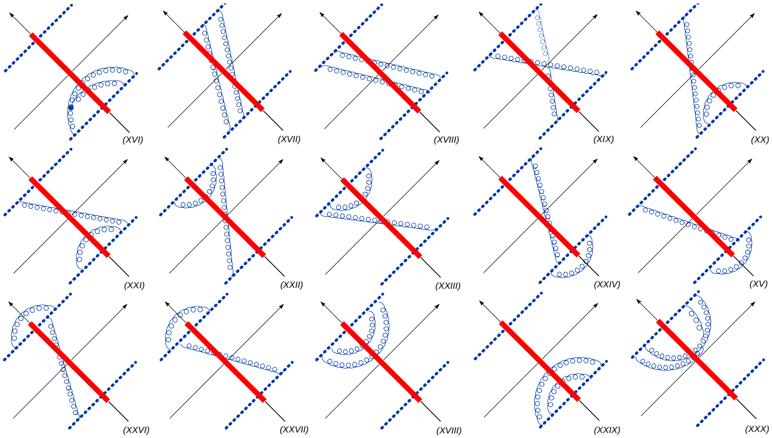
$$\frac{\alpha_s((x-y)^2)}{2\pi^2}\frac{(x-y)^2}{X^2Y^2} \qquad |x-y| \ll |x-z|, |y-z|$$
$$\frac{\alpha_s(X)^2)}{2\pi^2X^2} \qquad |x-z| \ll |x-y|, |y-z|$$
$$\frac{\alpha_s(Y)^2)}{2\pi^2Y^2} \qquad |y-z| \ll |x-y|, |x-z|$$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.

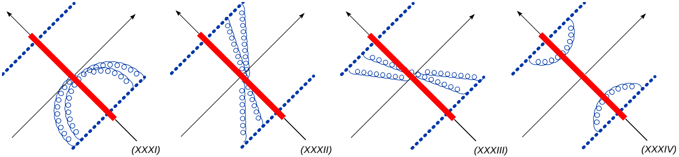
Gluon part of the NLO BK kernel: diagrams



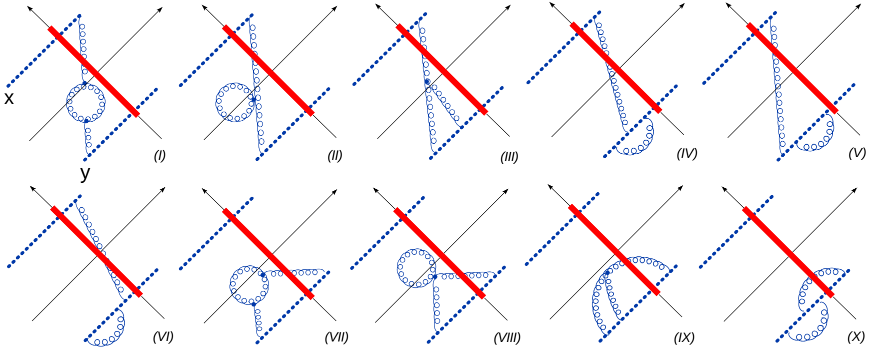
Diagrams with 2 gluons interaction



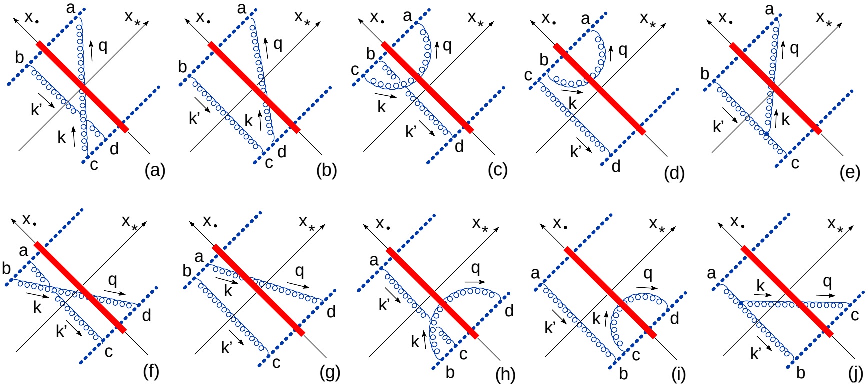
Diagrams with 2 gluons interaction



"Running coupling" diagrams



$1 \rightarrow 2$ dipole transition diagrams



$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}]$$

$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$

$$+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}$$

$$-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]$$

$$+[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\} - (z'\to z)]$$

$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)$$

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} ~=~ \frac{\alpha_s}{2\pi^2}\int d^2z~\Big([\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-N_c\mathrm{Tr}\{U_xU^\dagger_y\}]\\
&\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big]\\
&-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}-~\frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
&+\frac{\alpha_s}{4\pi^2}\int d^2z'~\Big\{[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}\\
&-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]\\
&+[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\}-(z'\to z)]\\
&\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)=
\end{aligned}
$$

Running coupling part

Gluon contribution to the NLO kernel

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} \;=\; &\frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}]\\
&\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]\\
&-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \color{red}{\frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}}\Big\}\\
&+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}\\
&-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]\\
&+[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\} - (z'\to z)]\\
&\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =
\end{aligned}
$$

Running coupling part + Non-conformal part

Gluon contribution to the NLO kernel

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$

$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3})\Big]$$

$$-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}-\frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$

$$+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$

$$-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]$$

$$+[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\}-(z'\to z)]$$

$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)=$$

Running coupling part $+$ Non-conformal part $+$ Conformal "non-analytic" part

Gluon contribution to the NLO kernel

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} \;=\; &\frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}]\\
&\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big]\\
&-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}-\frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
&+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}]\\
&-(z'\to z)\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]\\
&+[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\} - (z'\to z)]\\
&\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =
\end{aligned}
$$

Running coupling part + Non-conformal part + Conformal "non-analytic" part + "conformal-analytic" ($\mathcal{N}=4$) part

Gluon contribution to the NLO kernel

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}]\\
&\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3})\Big]\\
&-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
&+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}\\
&-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]\\
+\;&[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\} - (z'\to z)]\\
&\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) + \frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\{U_xU^\dagger_y\} =
\end{aligned}
$$

Our result + Extra term $\Rightarrow$ Agrees with NLO BFKL

(Comparing the eigenvalue of the forward kernel)

Gluon contribution to the NLO kernel

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} \;=\;& \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-N_c\mathrm{Tr}\{U_xU_y^\dagger\}]\\
&\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]\\
&-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}-\frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
&+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}\\
&-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]\\
&+[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\}-(z'\to z)]\\
&\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)+\frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\{U_xU_y^\dagger\}
\end{aligned}
$$

However, the term $\frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}U_xU_y^\dagger$ contradicts the requirement $\frac{d}{d\eta}U_xU_y^\dagger=0$ at $U=1$ (or at $x=y$).

$\mathcal{N}$=4 SYM:
study of 2-dim conformal invariance in the $\perp$ plane at high energies

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+\, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^+ \to x^+/x_\perp^2$

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$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2} \, A_+\Big(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\Big)\right\} \;=\; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

Conformal invariance of the BK equation

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Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^+ \to x^+/x_\perp^2 \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2}\, A_+\left(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\right)\right\} \;=\; \left[\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}\right]$$

$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2z}{\not{z}^4}\, \frac{(x-y)^2\ \not{z}^4}{(x-z)^2(z-y)^2}\left[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}\right]$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

Conformal invariance of the BK equation

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$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}\mathrm{Tr}\{U_x U_y^\dagger\}]$$

$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

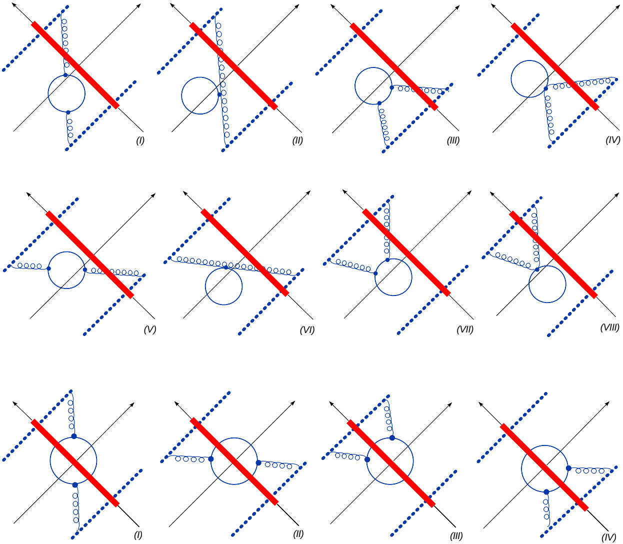
Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}\mathrm{Tr}\{U_x U_y^\dagger\}]$$

$$\Rightarrow \Big[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\Big]K(x,y,z) = 0$$

In the leading order - OK. In the NLO - ?

$\mathcal{N} = 4$ diagrams (scalar and gluino loops)



Evolution equation in $\mathcal{N}=4$ (I.B. and G. Chirilli)

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z\ \Big([\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}]$$
$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1-\frac{\alpha_s\pi N_c}{12}\Big] - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$
$$+\frac{\alpha_s}{16\pi^2}\int d^2z'\ \ [\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}$$
$$-\mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\} - (z'\to z)]$$
$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)$$

Evolution equation in $\mathcal{N} = 4$

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U^\dagger_y\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_y\} - N_c\mathrm{Tr}\{U_x U^\dagger_y\}]\\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1-\frac{\alpha_s\pi N_c}{12}\Big] - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
& + \frac{\alpha_s}{16\pi^2}\int d^2z'\;\;[\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_{z'}\}\mathrm{Tr}\{U_{z'} U^\dagger_y\} - \mathrm{Tr}\{U_x U^\dagger_z U_{z'} U^\dagger_y U_z U^\dagger_{z'}\}\\
& \qquad - \mathrm{Tr}\{U_x U^\dagger_{z'} U_z U^\dagger_y U_{z'} U^\dagger_z\} - (z' \to z)]\\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2 - X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)
\end{aligned}
$$

Non-conformal part

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\, \Big([\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}] \\
& \times \Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1-\frac{\alpha_s\pi N_c}{12}\Big] - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
+ \frac{\alpha_s}{16\pi^2}\int d^2z' \; & [\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\} \\
& -\mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\} - (z'\to z)] \\
& \times \Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)
\end{aligned}
$$

Non-conformal part $+$ Conformal analytic part

Evolution equation in $\mathcal{N}=4$

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{12\pi}(1-\pi^2)\Big] - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$
$$+\frac{\alpha_s}{16\pi^2}\int d^2z'\ \Big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$
$$-\mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)\Big]$$
$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big) + \frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\{U_xU_y^\dagger\} =$$

Our result $+$ Extra term $\Rightarrow$ Agrees with NLO BFKL in $\mathcal{N}=4$
(Lipatov and Kotikov, 2004)

(Comparing the forward kernel)

Linearized forward evolution kernel

$$\frac{d}{d\eta}\langle\hat{\mathcal{U}}(x)\rangle = \frac{\alpha_s N_c}{\pi^2}\int d^2z\,\frac{x^2}{z^2}\Big\{\frac{1}{(x-z)^2}\Big(1+\frac{\alpha_s N_c}{4\pi}\Big[-\frac{\pi^2}{3} - 2\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}\Big]\Big)[\langle\hat{\mathcal{U}}(z)\rangle - \frac{1}{2}\langle\hat{\mathcal{U}}(x)\rangle] + \frac{\alpha_s N_c}{4\pi}F(x,z)\,\langle\hat{\mathcal{U}}(z)\rangle\Big\}$$

$\langle\mathcal{U}(x)\rangle \equiv \langle(1-\frac{1}{N_c}\mathrm{Tr}\{U_x U_0^\dagger\}\rangle$

$$F(x,z) = \frac{(x^2-z^2)}{(x-z)^2(x+z)^2}\Big[\ln\frac{x^2}{z^2}\ln\frac{x^2z^2(x-z)^4}{(x^2+z^2)^4} + 2\mathrm{Li}_2\Big(-\frac{z^2}{x^2}\Big) - 2\mathrm{Li}_2\Big(-\frac{x^2}{z^2}\Big)\Big] - \Big(1-\frac{(x^2-z^2)^2}{(x-z)^2(x+z)^2}\Big)\Big[\int_0^1 - \int_1^\infty\Big]\frac{du}{(x-zu)^2}\ln\frac{u^2z^2}{x^2}$$

symmetric under $x \leftrightarrow z$

Linearized forward evolution kernel

$$\frac{d}{d\eta}\langle\hat{\mathcal{U}}(x)\rangle = \frac{\alpha_s N_c}{\pi^2}\int d^2z\,\frac{x^2}{z^2}\Big\{\frac{1}{(x-z)^2}\Big(1+\frac{\alpha_s N_c}{4\pi}\Big[-\frac{\pi^2}{3} - 2\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}\Big]\Big)[\langle\hat{\mathcal{U}}(z)\rangle - \frac{1}{2}\langle\hat{\mathcal{U}}(x)\rangle] + \frac{\alpha_s N_c}{4\pi}F(x,z)\,\langle\hat{\mathcal{U}}(z)\rangle\Big\}$$

$\langle\mathcal{U}(x)\rangle \equiv \langle(1-\frac{1}{N_c}\mathrm{Tr}\{U_xU_0^\dagger\}\rangle$

$$F(x,z) = \frac{(x^2-z^2)}{(x-z)^2(x+z)^2}\Big[\ln\frac{x^2}{z^2}\ln\frac{x^2z^2(x-z)^4}{(x^2+z^2)^4} + 2\mathrm{Li}_2\Big(-\frac{z^2}{x^2}\Big) - 2\mathrm{Li}_2\Big(-\frac{x^2}{z^2}\Big)\Big] - \Big(1-\frac{(x^2-z^2)^2}{(x-z)^2(x+z)^2}\Big)\Big[\int_0^1 - \int_1^\infty\Big]\frac{du}{(x-zu)^2}\ln\frac{u^2z^2}{x^2}$$

symmetric under $x \leftrightarrow z$

To compare with NLO BFKL we rewrite the evol. eqn. in terms of $\mathcal{V}(x) = \partial^2\mathcal{U}(x)$

$$\frac{d}{d\eta}\langle\hat{\mathcal{V}}(x)\rangle = \frac{\alpha_s N_c}{\pi^2}\int d^2z\,\Big\{\frac{1}{(x-z)^2}\Big(1+\frac{\alpha_s N_c}{4\pi}\Big[-\frac{\pi^2}{3} - 2\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}\Big]\Big)[\langle\hat{\mathcal{V}}(z)\rangle - \frac{x^2}{2z^2}\langle\hat{\mathcal{V}}(x)\rangle] + \frac{\alpha_s N_c}{4\pi}F(x,z)\,\langle\hat{\mathcal{V}}(z)\rangle\Big\}$$

Forward NLO BFKL kernel

$$A(s,t) = \frac{1}{4\pi^2}\int \frac{d^2q}{q^2}\frac{d^2q'}{q'^2}\Phi_A(q)\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\left(\frac{s}{qq'}\right)^{\omega} G_\omega(q,q')$$

$\Phi_A(q)$, $\Phi_B(q')$): impact factors
$G_\omega(q,q')$: the partial wave of the forward reggeized gluon scattering amplitude

$$\omega G_\omega(q,q') = \delta^{(2)}(q-q') + \int d\!\!{}^-{}^2p K(q,p) G_\omega(p,q')$$

$$\int d\!\!{}^-{}^2p K(q,p) f(p) = 4\alpha_s N_c \int d\!\!{}^-{}^2p \left\{\frac{1}{(q-p)^2}\left(1-\frac{\alpha_s N_c \pi}{12}\right)[f(p) - \frac{q^2}{p^2+(q-p)^2} f(q)] + \frac{\alpha_s N_c}{4\pi}\left(-\frac{\ln^2 q^2/p^2}{(q-p)^2} + F(q,p)\right)\right\} f(p) + \frac{3\alpha_s^2 N_c^2}{2\pi^2}\zeta(3) f(p)$$

To write the evolution equation we need $\left(\frac{s}{qq'}\right)^{\omega} \to \left(\frac{s}{q'^2}\right)^{\omega}$

$$K^{\rm evol}(q,q') = K(q,q') - \frac{1}{2}\int d\!\!{}^-{}^2p\, K(q,p)\ln\frac{q^2}{p^2} K(p,q') + O(\alpha_s^2)$$

Comparison of the dipole evolution and NLO BFKL

$$\int đ^2 p K^{\rm evol}(q,p) f(p) \;=\; 4\alpha_s N_c \int đ^2 p \left\{ \frac{1}{(q-p)^2}\Big(1 - \frac{\alpha_s N_c \pi}{12}\Big)[f(p) - \frac{q^2}{2p^2} f(q)] \right.$$
$$\left. + \frac{\alpha_s N_c}{4\pi}\Big(- \frac{2\ln\frac{q^2}{p^2}\ln\frac{(q-p)^2}{p^2}}{(q-p)^2} + F(q,p)\Big)\right\} f(p) + \frac{3\alpha_s^2 N_c^2}{2\pi^2}\zeta(3) f(p)$$

Comparison of the dipole evolution and NLO BFKL

$$\int d\!\!{}^-{}^2pK^{\rm evol}(q,p)f(p) \;=\; 4\alpha_sN_c\int d\!\!{}^-{}^2p\,\Big\{\frac{1}{(q-p)^2}\Big(1-\frac{\alpha_sN_c\pi}{12}\Big)[f(p)-\frac{q^2}{2p^2}f(q)]$$
$$+\frac{\alpha_sN_c}{4\pi}\Big(-\frac{2\ln\frac{q^2}{p^2}\ln\frac{(q-p)^2}{p^2}}{(q-p)^2}+F(q,p)\Big)\Big\}\,f(p)+\frac{3\alpha_s^2N_c^2}{2\pi^2}\zeta(3)f(p)$$

In the coordinate space

$$\int d^2zK^{\rm evol}(x,z)f(z) = \frac{\alpha_sN_c}{\pi^2}\int d^2z\,\Big\{\frac{1}{(x-z)^2}\Big(1+\frac{\alpha_sN_c}{4\pi}\Big[-\frac{\pi^2}{3}-$$
$$2\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}\Big]\Big)[f(z)-\frac{x^2}{2z^2}f(x)]+\frac{\alpha_sN_c}{4\pi}F(x,z)\Big\}\,f(z)+\frac{3\alpha_s^2N_c^2}{2\pi^2}\zeta(3)f(x)$$

Comparison of the dipole evolution and NLO BFKL

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$$+\frac{\alpha_s N_c}{4\pi}\Big(-\frac{2\ln\frac{q^2}{p^2}\ln\frac{(q-p)^2}{p^2}}{(q-p)^2}+F(q,p)\Big)\Big\}\,f(p)+\frac{3\alpha_s^2N_c^2}{2\pi^2}\zeta(3)f(p)$$

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$$2\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}\Big]\Big)[f(z)-\frac{x^2}{2z^2}f(x)]+\frac{\alpha_s N_c}{4\pi}F(x,z)\Big\}\,f(z)+\frac{3\alpha_s^2N_c^2}{2\pi^2}\zeta(3)f(x)$$

while we have

$$\frac{d}{d\eta}\langle\hat{\mathcal{V}}(x)\rangle \;=\; \frac{\alpha_s N_c}{\pi^2}\int d^2 z\,\Big\{\frac{1}{(x-z)^2}\Big(1+\frac{\alpha_s N_c}{4\pi}\Big[-\frac{\pi^2}{3}$$

$$-2\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}\Big]\Big)[\langle\hat{\mathcal{V}}(z)\rangle-\frac{x^2}{2z^2}\langle\hat{\mathcal{V}}(x)\rangle]+\frac{\alpha_s N_c}{4\pi}F(x,z)\;\langle\hat{\mathcal{V}}(z)\rangle\Big\}$$

Comparison of the dipole evolution and NLO BFKL

$$\int d\!\!\!{}^-{}^2 p K^{\rm evol}(q,p)f(p) \;=\; 4\alpha_s N_c \int d\!\!\!{}^-{}^2 p \left\{\frac{1}{(q-p)^2}\Big(1-\frac{\alpha_s N_c \pi}{12}\Big)[f(p)-\frac{q^2}{2p^2}f(q)] + \frac{\alpha_s N_c}{4\pi}\Big(-\frac{2\ln\frac{q^2}{p^2}\ln\frac{(q-p)^2}{p^2}}{(q-p)^2}+F(q,p)\Big)\right\} f(p) + \frac{3\alpha_s^2 N_c^2}{2\pi^2}\zeta(3)f(p)$$

In the coordinate space

$$\int d^2z K^{\rm evol}(x,z)f(z) = \frac{\alpha_s N_c}{\pi^2}\int d^2z \left\{\frac{1}{(x-z)^2}\Big(1+\frac{\alpha_s N_c}{4\pi}\Big[-\frac{\pi^2}{3} - 2\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}\Big]\Big)[f(z)-\frac{x^2}{2z^2}f(x)] + \frac{\alpha_s N_c}{4\pi}F(x,z)\right\} f(z) + \frac{3\alpha_s^2 N_c^2}{2\pi^2}\zeta(3)f(x)$$

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$\Rightarrow$ Same kernel up to $\frac{\alpha_s^2 N_c^2}{2\pi^2}\zeta(3)f(x)$ term

Conclusions

- The NLO kernel for the evolution of the color dipole consists of three parts: the running-coupling part proportional to β-function, the conformal part describing 1 $\rightarrow$ 3 dipoles transition and the non-conformal term.
- The result agrees with the forward NLO BFKL kernel up to a term proportional $\alpha_s^2\zeta(3)$ times the original dipole.
- For the creation of dipoles in the small-x evolution, the coupling constant is determined by the size of the smallest dipole.
- With rigid $|\alpha| < \sigma$ cutoff, the NLO-BK and the NLO-BFKL for $\mathcal{N} = 4$ is (almost) conformally invariant in the transverse plane.

Outlook: two problems

Problem #1: $\alpha_s^2\zeta(3)U_xU_y^\dagger$

$$\langle\mathrm{Tr}\{U_xU_y^\dagger\}\rangle_{\rm shockwave} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big(\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}$$

$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1-\frac{\alpha_s\pi N_c}{12}\Big]-\frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$

$$+\frac{\alpha_s}{16\pi^2}\int d^2z'\,[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$

$$-\mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\}-(z'\to z)]$$

$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}+c\times\mathrm{Tr}\{U_xU_y^\dagger\}\Big)$$

Problem #1: $\alpha_s^2\zeta(3)U_xU_y^\dagger$

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$$-\mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\}-(z'\to z)\big]$$
$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}+c\times\mathrm{Tr}\{U_xU_y^\dagger\}\Big)$$

To get c we set $U_z=1$ (no shock wave) then

$$\langle\mathrm{Tr}\{U_xU_y^\dagger\}\rangle=1$$

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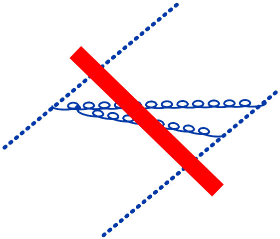
To get c we set $U_z=1$ (no shock wave) then

$$\langle\mathrm{Tr}\{U_xU_y^\dagger\}\rangle=1$$

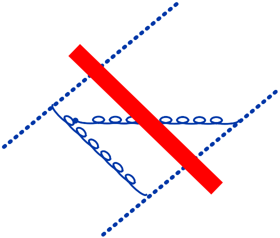
We checked this with gauge/scalar links at infinity

$$\mathrm{Tr}\{U_xU_y^\dagger\} = \lim_{L\to\infty}\mathrm{Tr}\{[Lp_1+x_\perp,-Lp_1+x_\perp][-Lp_1+x_\perp,-Lp_1+y_\perp]$$
$$\times[-Lp_1+y_\perp,Lp_1+y_\perp][Lp_1+y_\perp,Lp_1+x_\perp]\}$$

Problem #2: Conformal invariance of the $\mathcal{N} = 4$ evolution kernel



Conformal



Non−Conformal

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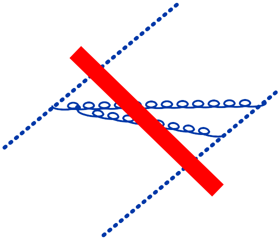
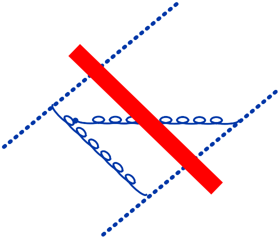


Conformal



Non−Conformal

The high-energy evolution may still be conformal in the proper (effective action?) language symmetric with respect to projectile↔target

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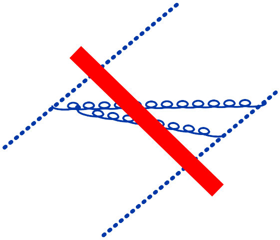

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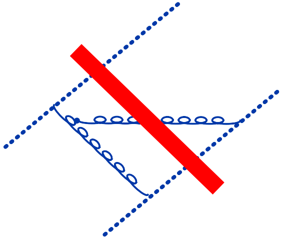


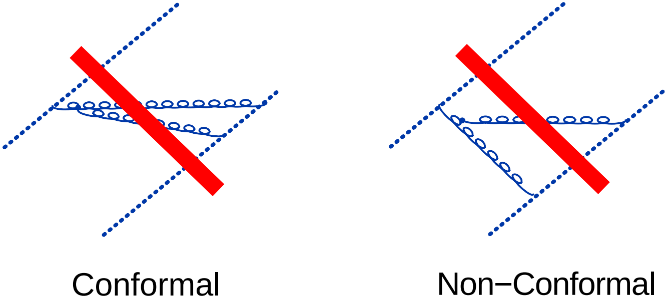
Non−Conformal

The high-energy evolution may still be conformal in the proper (effective action?) language symmetric with respect to projectile↔target

The projectile↔target symmetric forward NLO BFKL kernel is “conformal” under the inversion $x \to x/x^2$.

Outlook: two problems

Problem #2: Conformal invariance of the $\mathcal{N} = 4$ evolution kernel



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The projectile↔target symmetric forward NLO BFKL kernel is "conformal" under the inversion $x \to x/x^2$.

This indicates that the non-forward NLO BFKL, properly symmetrized, may be conformally invariant.