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THE FREE OSCILLATIONS OF LAKE ST. CLAIR -- AN APPLICATION OF LANCZOS' PROCEDURE

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THE FREE OSCILLATIONS OF LAKE ST. CLAIR--AN APPLICATION OF LANCZOS' PROCEDURE[1]

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The frequencies and structures of the five lowest free oscillations of Lake St. Clair are determined by a Lanczos procedure. With the Lanczos procedure, a high resolution numerical grid can be used to resolve the detailed structures of the modes. The lowest mode has a calculated period of 4.06 h.

1. INTRODUCTION

The purpose of this paper is to present a method of determining the frequencies of free oscillation of an enclosed basin and, more importantly, a high resolution representation of the modal structures. The method used is a Lanczos procedure applied to the linear shallow water equations. The advantage of this procedure is that the governing differential operator can be discretized on a high resolution numerical grid without taxing the memory capacity of the computer.

2. GOVERNING EQUATIONS

The linearized, inviscid shallow water equations in the absence of rotation and forcing can be written as

$$\frac{\partial \vec{v}}{\partial t} = -g\nabla\zeta \tag{1}$$

and

$$\frac{\partial \zeta}{\partial t} + \nabla\cdot(h\vec{v}) = 0 \; , \tag{2}$$

where $\vec{v}$ is the horizontal velocity vector, ζ is the free surface fluctuation, h is the water depth, t is the time, and ∇ is the horizontal gradient operator. The boundary condition is that of perfect reflection, that is,

$$\vec{v}\cdot\vec{n} = 0 \tag{3}$$

at the shoreline. The term $\vec{n}$ is a vector normal to the shoreline.

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Eliminating $\vec{v}$ from (1) and (2) to find a single equation for ζ, we find

$$\frac{\partial^2 \zeta}{\partial t^2} - g\nabla\cdot h\nabla\zeta = 0 \ . \tag{4}$$

The boundary condition for ζ is, from (1) and (3)

$$\frac{\partial \zeta}{\partial n} = 0 \ . \tag{5}$$

The free solutions of (4) are

$$\zeta_n(x,y,t) = \eta_n(x,y)e^{i\sigma_n t} . \tag{6}$$

Substituting into (4) and eliminating the time dependent part gives

$$-\sigma_n^2\eta_n - g\nabla\cdot h\nabla\eta_n = 0$$

or

$$\nabla\cdot h\nabla\eta_n = \lambda_n\eta_n , \tag{7}$$

where the $n^{\underline{th}}$ characteristic value, λ_n, is defined as

$$\lambda_n = \frac{-\sigma_n^2}{g} \ . \tag{8}$$

To show that λ_n is real and negative, and therefore that σ_n is real so the free solutions are purely harmonic, multiply (7) by η_n^* and integrate over the area of the basin.

$$\int\eta_n^*\nabla\cdot h\nabla\eta_n dA = \lambda_n\int\eta_n^*\eta dA \ .$$

Partial integration and use of the boundary condition (5) gives

$$\lambda_n = -\int h\nabla\eta_n^*\cdot\nabla\eta_n dA/\int\eta_n^*\eta_n dA \ .$$

This shows λ_n to be real and negative.

One can also show the η_n to be orthogonal. Since λ_n is real, η_n is real.

One can write (7) for a different mode m as

$$\nabla\cdot h\nabla\eta_m = \lambda_m\eta_m. \tag{9}$$

Now multiply (7) by η_m, (9) by η_n, and subtract to obtain

$$\eta_m\nabla\cdot h\nabla\eta_n - \eta_n\nabla\cdot h\nabla\eta_m = (\lambda_n - \lambda_m)\eta_n\eta_m. \tag{10}$$

Integrate (10) over the area of the basin.

$$\int(\eta_m\nabla\cdot h\nabla\eta_n - \eta_n\nabla\cdot h\nabla\eta_m)dA = (\lambda_n - \lambda_m)\int\eta_n\eta_m dA$$

Partial integration of the left side and use of the boundary condition (5) gives

$$(\lambda_n - \lambda_m)\int\eta_n\eta_m dA = 0. \tag{11}$$

If $m = n$, this equation is satisfied trivially. For two different characteristic values, $(\lambda_n - \lambda_m) \neq 0$ so

$$\int\eta_n\eta_m dA = 0. \tag{12}$$

The eigenfunctions are orthogonal.

The free solutions to equations (1) and (2) are then characterized by pure harmonic time dependence with frequencies σ_n and mathematically orthogonal spatial structures.

3. LANCZOS PROCEDURE

To solve (7) for λ_n and η_n, we use a Lanczos procedure. When applied to the discretized version of a differential operator, the Lanczos procedure (developed by Lanczos, 1950, and explained more fully in Paige, 1972) results in a tridiagonal matrix with the same eigenvalues as the general matrix of the differential operator. To proceed, first expand η_n as follows:

$$\eta_n = \sum_{i=1}^{\infty} C_i^n W_i, \tag{13}$$

where the W_i are orthonormal over the area of the basin and satisfy the same boundary conditions as η, i.e.,

$$\frac{\partial W_i}{\partial n} = 0.$$

The orthonormality condition is

$$\int W_i W_j dA = \delta_{ij}, \tag{14}$$

where δ_{ij} is the Kronecker delta.
Later the W_i will be further specified to simplify the calculation. (7) is then

$$\nabla\cdot h\nabla \sum_{i=1}^{\infty} C_i^n W_i = \lambda_n \sum_{i=1}^{\infty} C_i^n W_i . \tag{15}$$

Multiply by W_j and integrate over the area of the basin.

$$\int W_j \nabla\cdot h\nabla \sum_{i=1}^{\infty} C_i^n W_i dA = \lambda_n \int \sum_{i=1}^{\infty} C_i^n W_i dA \tag{16}$$

Let

$$A_{ij} = \int W_j \nabla\cdot h\nabla W_i dA \tag{17}$$

and use the orthogonality condition (14) to obtain

$$\sum_{i=1}^{\infty} A_{ij} C_i^n = \lambda_n C_j$$

or

$$A\vec{c}_n = \lambda_n \vec{c}_n, \tag{18}$$

which is a standard eigenvalue problem. Note that discretizing the differential operator in (7) would result in the same sort of system as (18). The order of the matrix A would be proportional to the number of grid squares in the digitization of the basin and the structure would be somewhat banded but lacking any symmetric properties. The entire matrix would have to be stored in the computer to obtain eigenvalues and vectors. In the Lanczos procedure the W_i functions are chosen so that the matrix A is symmetric tridiagonal. Special numerical routines described in Smith *et al.* (1974) are used to find the eigenvalues and eigenvectors of the tridiagonal matrix. Only the main diagonal and first off-diagonal of the matrix are stored ***in*** the computer since the rest of the matrix elements are zero. Let

$$\alpha_i = \int W_i \nabla\cdot h\nabla W_i dA \tag{19}$$

$$\hat{W}_{i+1} = (\nabla\cdot h\nabla - \alpha_i)W_i - \beta_i W_{i-1} \qquad i \geq 1 \tag{20}$$

$$\beta_{i+1} = (\int \hat{W}_{i+1}\hat{W}_{i+1} dA)^{1/2} \tag{21}$$

$$W_{i+1} = \hat{W}_{i+1}/\beta_{i+1}. \tag{22}$$

If W_1 is given, (19)-(22) define a recursion relation for W_i, α_i, and β_i. A useful way of writing (20) is

$$\beta_{i+1}W_{i+1} = (\nabla\cdot h\nabla - \alpha_i)W_i - \beta_i W_{i-1}$$

or

$$\nabla\cdot h\nabla W_i = \alpha_i W_i + \beta_{i+1}W_{i+1} + \beta_i W_{i-1}. \tag{23}$$

Multiply (23) by Wj and integrate over the area of the basin.

$$\int W_j \nabla\cdot h\nabla W_i dA = \alpha_i \int W_j W_i dA + \beta_{i+1}\int W_j W_{i+1} dA + \beta_i \int W_j W_{i-1} dA \tag{24}$$

We obtain the diagonal elements of A in (18) by letting j = i in-(24).

$$A_{ii} = \int W_i \nabla\cdot h\nabla W_i dA = a_i \tag{25}$$

The first off-diagonal is obtained by letting j = i-l in (24).

$$A_{i-1\ i} = \int W_{i-1} \nabla\cdot h\nabla W_i dA = \beta_i. \tag{26}$$

It is easy to see that the matrix is symmetric

$$A_{i\ i-1} = \int W_i \nabla\cdot h\nabla W_{i-1} dA = \beta_i \tag{27}$$

and that all other elements are zero.

In summary, the Lanczos procedure for the solution to (7) consists of: 1) selecting a starting function W_1; 2) applying (19)-(22) recursively to obtain α_i, β_i and Wi; 3) solving (18) for eigenvalues σ_n and eigenvectors $\vec{C}_n$; and 4) constructing η_n from (13).

4. APPLICATION TO LAKE ST. CLAIR

The outline and bathymetry of Lake St. Clair are digitized on a 1200-m grid yielding 731 square elements as shown in Fig. 1. The depth plotted in this figure includes a 0.8-m stage above low water datum (an average value for recent years). The recursion procedure (19)-(22) is applied with centered difference formulas for the gradient operator and simple summation for the integrations. Analytically, all the W_i are orthogonal according to (14). After applying (19)-(22) to the numerical grid 731 times, we should find that W732 = 0. In practice, however, truncation error sets in after about 50 iterations and W_{50} is not orthogonal to W_1. The Wi do remain orthogonal in a "local" sense in that W_{50} is orthogonal to W25 - W_{75}. Platzman (1975) has shown that the "local" orthogonality is adequate for determining periods and structures of at least the lowest modes.

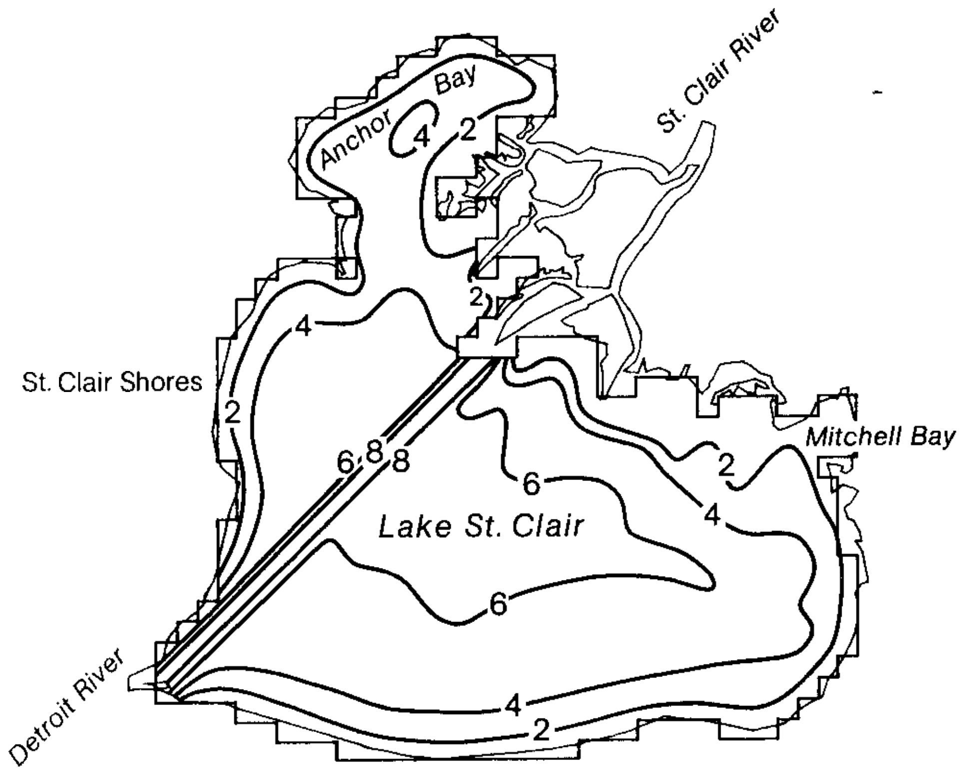

Figure 1.--*Numerical* ***grid for Lake St.*** *Clair*. ***Depth contours at 2-m*** *intervals*.

In the case of Lake se. Clair, the procedure was truncated at 244, 488, and 731 W_i. A large number of Wi are required to resolve the modal structures, but the eigenfrequencies are determined quite well even at low truncation. At higher truncations the loss of orthogonality tends to cause spurious values of λ_i to appear. These can be recognized in two ways. First, the lowest eigenvalues are well determined at low truncation, so that if a new eigenvalue appears at a higher truncation limit, it is probably spurious. Second, we can test the structure of the free mode as constructed by (13) with the parameter ε defined as follows:

$$\varepsilon = \int(\nabla\cdot h\nabla\eta_n - \lambda_n\eta_n)^2 dA/\int\eta_n{}^2 dA. \tag{28}$$

Analytically ε is zero, but when the discretized version of (28) is calculated numerically, ε is some small number and tends to be higher for spurious eigenvalues.

Table 1 lists the lowest 10 frequencies determined by the Lanczos procedure at the three different truncations. The value of ε at truncation 731 is also show". Spurious eigenvalues are marked by asterisks. The lowest four frequencies remain constant to five significant figures for the three different truncations. The fifth mode frequency changes in the second significant figure.

The spatial structures of the five lowest modes are shown in Figs. 2-6. The contour interval is 10 percent of the maximum value. The lowest mode, in Fig. 2, with a period of 4.06 h shows maximum amplitude in the Anchor Bay region north of the main lake basin. The nodal line runs across the lake from the St. Clair giver inflow to the St. Clair Shores region. On the eastern end of the lake, the amplitude attains only 20 percent of its maximum value.

The second mode, shown in Fig. 3, has a period of 3.12 h. There are two nodal lines so that the oscillation in the northern reaches of Anchor Bay is in phase with the eastern end of the main basin where maximum amplitude occurs. This mode is probably more important than the lowest mode in the generation of storm surges since the predominant westerly wind would pile water up at the eastern shore, conforming to the structure of this mode.

The 2.17-h mode in Fig. 4 also has two nodal lines, but the axis of oscillation is oriented north-to-south. The 1.89-h mode in Fig. 5 is the fundamental oscillation of Mitchell Bay. Note that this mode has a very small space scale and requires a fine grid mesh for sufficient resolution. The fifth mode in Fig. 6 at 1.74h involves the whole Lake St. Clair Basin.

5. CONCLUSION

The Lanczos procedures is able to provide more detailed normal mode structures than traditional methods because it is more efficient in terms of computer storage. This was demonstrated on a 1200-m grid representation of Lake St. Clair involving 731 grid squares. The structures of the five lowest modes show detail that could not be resolved on a coarser grid.

Table 1.--Frequencies of oscillation of Lake St. Clair as determined by the Lanczos procedure at various truncations. Frequencies are in cycles per hour. Asterisks denote spurious eigenvalues as explained in the text.

Mode	Truncation 244	488	731	ε
1	0.24619	0.24619	0.24619	7.99×10^{-31}
2	0.32000	0.32000	0.32000	1.95×10^{-31}
3	0.45985	0.45985	0.45985	2.26×10^{-31}
4	0.52949	0.52949	0.46388*	3.54×10^{-15}
5	0.59269	0.57374	0.52949	5.53×10^{-31}
6	0.60725	0.59464*	0.55362*	3.18×10^{-14}
7	0.63336	0.60640	0.57374	8.16×10^{-28}
8	0.68462	0.61965	0.60640	5.39×10^{-27}
9	0.74091	0.68302*	0.61965	3.84×10^{-26}
10	0.77816	0.68385	0.68385	8.11×10^{-29}

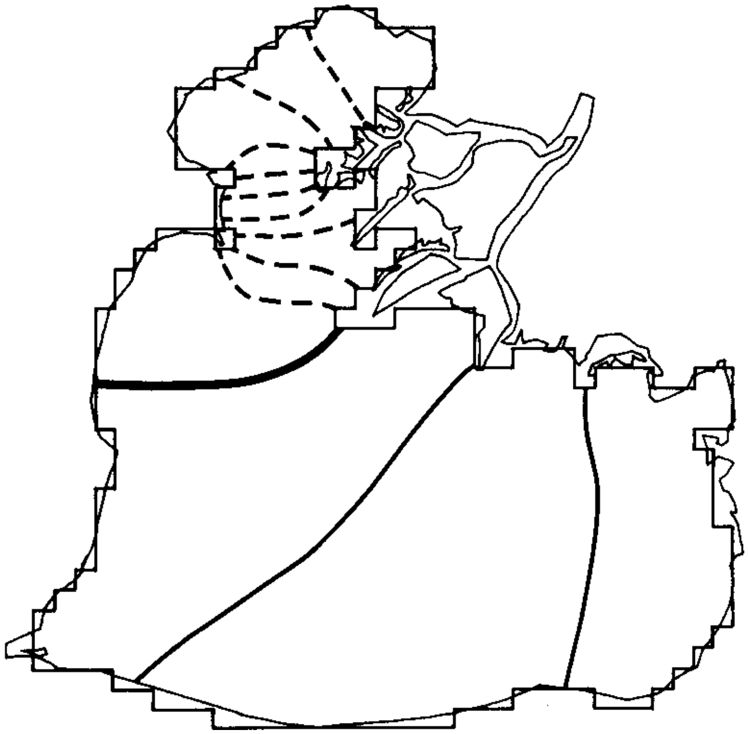

Figure *Z.--Amplitude distribution for the first normal mode* of *Lake St. Clair, with a period of 4.06 h.*

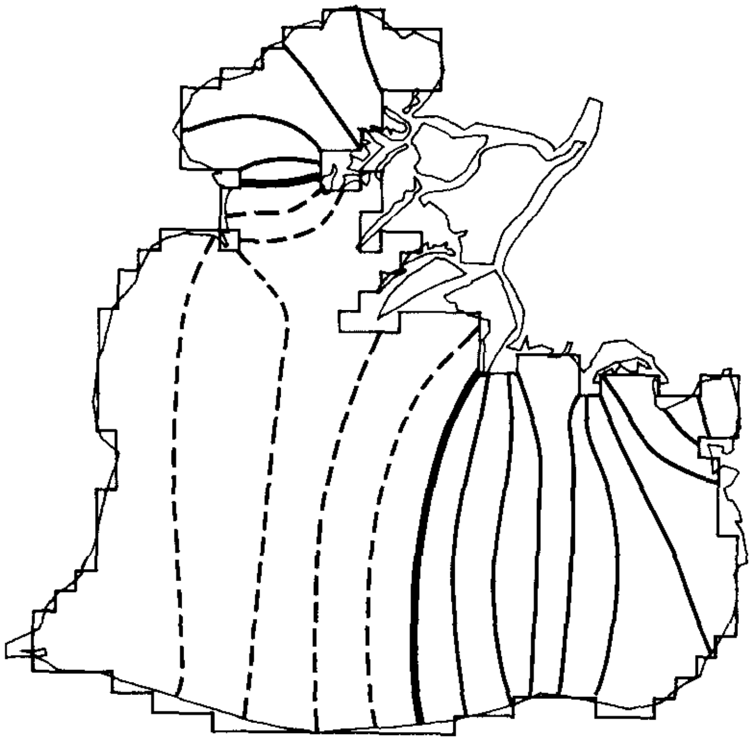

Figure 3.--*Amplitude* ***distribution*** for ***the second normal mode of Lake St. Clair, with*** a period of 3.12 ***h.***

Figure 4.--*Amplitude* ***distribution for the third normal mode of Lake St. Clair, with a*** period of 2.17 ***h.***

Figure 5.--Amplitude distribution for the fourth normal mode of Lake St. Clair, with a period of 1.89 h.

Figure *6.--Amplitude distribution for the fifth normal mode of St. Clair, with a period* of 1.74 *h.*

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