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Practical Approximations to a Troposkien by Straight-Line and Circular-Arc Segments

G. E. Reis, B. F. Blackwell

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PRACTICAL APPROXIMATIONS TO A TROPOSKIEN BY STRAIGHT-LINE AND CIRCULAR-ARC SEGMENTS

G. E. Reis
Aeroballistics Division 5625
B. F. Blackwell
Reentry Vehicle Aerothermodynamics Division 5628
Sandia Laboratories, Albuquerque, NM 87115

ABSTRACT

Curves made up of circular arcs and straight lines which approximate troposkiens of interest in the design of crosswind-axis wind turbines have been calculated for a variety of constraints. These curves were fitted to the troposkiens by numerical iteration by using both a least-squares and a least-maximum fit. Curves produced with the least-maximum spacing criteria approximate the troposkiens more closely than curves developed with the least-squares goodness-of-fit criteria.

ACKNOWLEDGMENTS

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ILLUSTRATIONS
(cont)

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LIST OF SYMBOLS

a - Half the distance between the points of attachment of a troposkien on the axis of rotation; normalizing constant

a_ℓ, b_ℓ - Constants in the equation for the straight-line portion of the fitted curve

a_n, b_n - Constants in the equation for the straight line normal to a troposkien

b - Maximum deflection of the troposkien from the axis of rotation

d - Distance between the normalized troposkien and the normalized fitted curve measured normal to the troposkien

$F(\phi,k)$ - Elliptic integral of the first kind with argument ϕ and parameter k

k - Parameter in the elliptic integral (see Eq. (A.34))

m_t - Slope of the normalized troposkien in the first quadrant

r,z - Normalized coordinates used to describe the troposkien (see Figure 2)

r_c, z_c - r and z coordinates of the circular-arc portion of the fitted curve

r_ℓ, z_ℓ - r and z coordinates of the straight-line portion of the fitted curve

r_n, z_n - r and z coordinates of a line normal to the troposkien

r_t, z_t - r and z coordinates of the troposkien

R - r coordinate of the center of the circular-arc portion of the fitted curve

R_i, Z_i - r and z coordinates of the point at which a line normal to the troposkien intersects the fitted curve

R_j, Z_j - r and z coordinates of the point at which the circular-arc and straight-line segments of the fitted curve meet

R_m - r coordinate of the point at which the fitted curve meets the r axis

R_t, Z_t - r and z coordinates of the point at which the troposkien normal intersects the troposkien

Z_m - z coordinate of the point at which the fitted curve intersects the z axis

β - b/a

PRACTICAL APPROXIMATIONS TO A TROPOSKIEN BY STRAIGHT-LINE AND CIRCULAR-ARC SEGMENTS

Introduction

A troposkien,[1,2] the shape assumed by a perfectly flexible uniform cable when attached at two points on a vertical axis and then spun about this axis in the absence of aerodynamic and gravitational forces, has practical application in the design of the Darrieus type[3,4] of crosswind-axis wind turbine. The blades, if fabricated with a troposkien profile, will theoretically be free of bending stress caused by centrifugal loads. Any appreciable deviation from the troposkien shape will induce bending stresses, and practical experience has shown that these bending stresses can be large enough to cause blade yield. An artist's conception of the Darrieus wind turbine is shown in Figure 1.

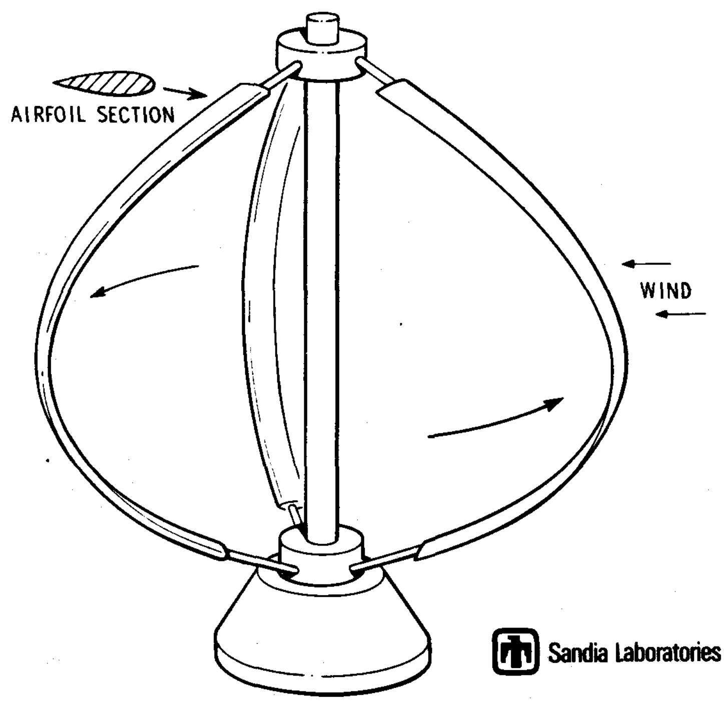

Figure 1. Artist's conception of a Darrieus-type vertical-axis wind turbine

Mathematically, the troposkien shape can be expressed in terms of elliptic integrals.[1] However, fabrication of the troposkien shape may not be economically practical because the use of numerically controlled machining techniques would probably be required. Therefore, it may be advantageous to approximate the troposkien shape with a shape or combination of shapes which are simpler to fabricate. One approximation is the parabola proposed by Templin[3] for an analytic study of the aerodynamic performance of the Darrieus wind turbine. However, the problems connected with the manufacturing of a parabolic wind turbine blade are no simpler than those associated with a troposkien blade. As an alternative it has been suggested[4] that the troposkien be approximated by two straight lines joined to a circular arc, two shapes which are reasonably simple to manufacture. This type of approximation is shown in Figure 2, where all lengths have been normalized by a, half the distance between the attachment points on the axis of rotation.

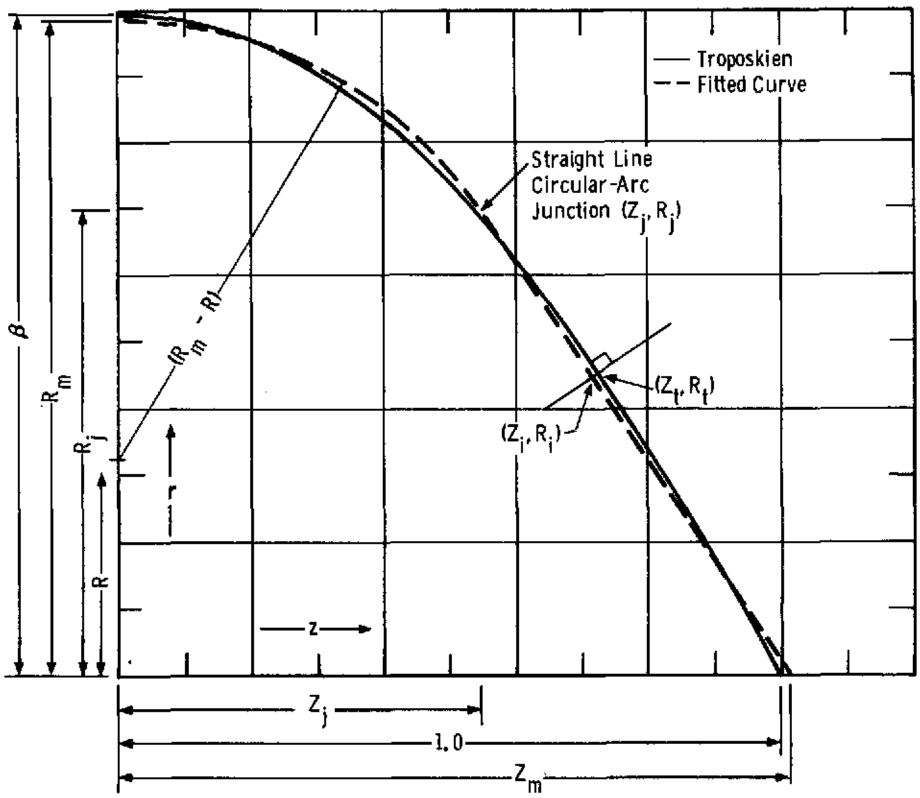

Figure 2. Coordinate system used to describe the troposkien and the fitted curve

The purpose of this report is to explore mathematically several straight-line/circular-arc approximations to troposkiens As was shown in Reference 2, these troposkiens lie in the range of β's between 0.8 and 1.2, where β is the ratio of the maximum blade displacement from the axis of rotation to one-half the spacing between the attachment points on the axis of rotation. The troposkien of maximum swept area* for a given blade length between the points of attachment on the axis of rotation[2] occurs at $\beta = 0.99458568$. This maximum swept area troposkien is used as the reference troposkien in the present study; however, troposkiens with β's of 0.8, 0.9, 1.0, 1.1, and 1.2 are also examined.

Fitting Procedure

Figure 2 presents a schematic of a troposkien and its straight-line/circular-arc approximation. The axis of rotation is the z-axis and, because the troposkien is symmetric about the r axis, the center of the circular portion of the fitted curve lies on the r axis. In this figure, the troposkien is shown as a solid line and the fitted curve as a broken line. Also, in order to keep the present analysis somewhat general, all lengths used in the following discussion have been normalized by a, the distance between the origin in the r,z coordinate system and the point at which the troposkien intersects the z axis. If the maximum displacement of the troposkien from the z axis, which occurs at z=o, is labeled b, it is possible to characterize any particular troposkien by means of the ratio[2]

$$\beta = \frac{b}{a} . \tag{1}$$

With regard to the fitted curve, the center of the circular arc lies on the r axis at a distance R from the origin of the r,z coordinate system, and the intersections of the fitted curve with the r and z axes occur at distances of R_m and Z_m along the respective axes. In addition, it is assumed that the slope of the fitted curve is continuous at the point where the circular arc and the straight line meet.

Several criteria can be used as tests for "goodness-of-fit" between the troposkien and the fitted curve. Two of the more common, the ones considered here, are (1) the least-maximum difference between the two curves (the so-called minimax) and (2) the least root-mean-square of the spacings between the two curves, where the spacings are calculated at selected points along the troposkien. The word "difference"

---

*The swept area is the area common to the volume swept out by a rotating flexible cable (which gives rise to the troposkien shape) and a plane containing the axis of rotation. This area is 4 times the area between the r and z axes and the troposkien in Figure 2.

as used here is not clearly defined and could in fact refer to the differences in either the z coordinates or the r coordinates of the two curves or to some combination of them. The one difference which appears to best describe the closeness of the two curves is the magnitude of the separation between them measured normal to the troposkien; this difference is used in this study. In terms of the notation in Figure 2, this last difference, d, is

$$d = \left[\left(Z_i - Z_t\right)^2 + \left(R_i - R_t\right)^2\right]^{1/2} , \tag{2}$$

where the subscripts t and i refer to the troposkien and to the fitted curve, respectively. The point (Z_i,R_i) is the point of intersection of the fitted curve and the line normal to the troposkien which passes through the point (Z_t,R_t) on the troposkien. The fitted curve intersects the z axis at $(Z_m,0)$ and the r axis at $(0,R_m)$.

Because the center of the circular-arc segment of the fitted curve lies on the r-axis, the two parameters R and R_m uniquely determine the circle which produces the circular portion of the fitted curve. Also, the point $(Z_m,0)$ and one additional point are sufficient for determining the straight-line portion of the fitted curve. The relationship obtained by applying the condition that the slope must be continuous at the circular-arc/straight-line junction permits this second point (specifically, the point of juncture between the circular arc and the straight line) to be expressed in terms of R, Z_m, and R_m. Consequently, the three parameters R, Z_m, and R_m uniquely determine a line made up of a circular arc and a straight line whose slopes are equal at their junction.

In these equations, β is assumed to be given; in the various computer runs, Z_m and R_m are either to be determined by the numerical calculations or are to be constrained so that the fitted curve intersects the troposkien at the point where the troposkien crosses the z and r axes, respectively. In all cases the numerical values of Z_j, R_j, and R are to be determined by the calculations.

The equations used in the computer program for determining the best fit (derived in Appendix A) are listed below for convenience.

Troposkien equations:[1]

$$\beta = \frac{b}{a} = \frac{2k}{\left(1 - k^2\right) F\left(\frac{\pi}{2};k\right)} \tag{3}$$

$$Z_t = 1 - \frac{F\left(\sin^{-1}\left(\frac{R_t}{\beta}\right);k\right)}{F\left(\frac{\pi}{2};k\right)} \tag{4}$$

$$\left.\frac{dr_t}{dz_t}\right]_{R_t} = -\frac{2k}{1-k^2}\left[\left(\frac{R_t^{\,2}}{\beta^2} - 1\right)\left(\frac{R_t^{\,2}k^2}{\beta^2} - 1\right)\right]^{1/2} = m_t \tag{5}$$

Fitted curve:

$$Z_j = \left[\frac{Z_m\left(R_m - R\right)}{R^2 + Z_m^{\,2}}\right]\left\{1 + \left(\frac{\left(R^2 + Z_m^{\,2}\right)^2\left[\left(R_m - R\right)^2 - R^2\right]}{Z_m^{\,2}\left(R_m - R\right)^2}\right)^{1/2}\right\} \tag{6}$$

$$R_j = R + \left[\frac{Z_m Z_j - \left(R_m - R\right)^2}{R}\right] \tag{7}$$

Circular-arc segment:

$$r_c = \left[\left(R_m - R\right)^2 - z_c^{\,2}\right]^{1/2} + R \tag{8}$$

Straight-line segment:

$$r_\ell = \frac{R_j\left(Z_m - z_\ell\right)}{\left(Z_m - Z_j\right)} \tag{9}$$

Troposkien normal-fitted curve intercepts:

When the troposkien normal intercepts the circular arc segment:

$$Z_i = \left[\frac{Z_t + m_t\left(R_t - R\right)}{1 + m_t^{\,2}}\right]\left\{1 + m_t\left(\left(1 + m_t^{\,2}\right)\left[\frac{R_m - R}{Z_t + m_t\left(R_t - R\right)}\right]^2 - 1\right)^{1/2}\right\} \tag{10}$$

$$R_i = R_t - \left(\frac{Z_i - Z_m}{m_t}\right) \tag{11}$$

When the troposkien normal intercepts the straight line segment:

$$Z_i = \left[\frac{\left(Z_m - Z_t - m_t R_t\right)\left(R_j - R\right)}{m_t Z_j - \left(R_j - R\right)}\right] + Z_m \tag{12}$$

$$R_i = \left[\frac{Z_j\left(Z_m - Z_i\right)}{R_j - R}\right] \tag{13}$$

Troposkien-fitted curve spacings:

$$d = \left[\left(Z_i - Z_t\right)^2 + \left(R_i - R_t\right)^2\right]^{1/2} . \tag{14}$$

The details of the logic used to calculate the various quantities are given in Appendix B.

Results and Comments

Values of R, Z_m, and R_m that satisfy the least-squares and/or least-maximum-spacing goodness-of-fit criteria have been determined for several β's between 0.8 and 1.2 and for a variety of constraints on Z_m and R_m. The calculations were performed on a CDC-6600 computer by means of the code MAXDIF.

The numerical results of these calculations are listed in Tables I through III. Table I gives the coordinates for the normalized troposkien of maximum swept area for a fixed blade length at 100 points along the curve. For this case $\beta = 0.99458568$. Table II presents the geometrical characteristics of the straight-line/circular-arc combination which approximates the normalized troposkien of $\beta = 0.99458568$. These data were calculated at either 100 or 1000 points along the curve by using a least-maximum separation or a least-squares goodness-of-fit criterion and various constraints on Z_m and R_m. Parameters calculated for circular-arc/straight-line approximations to troposkiens with β's between 0.8 and 1.2, using a minimax goodness-of-fit criterion, are listed in Table III.

TABLE I

Coordinates of the Normalized Troposkien of Maximum Swept Area
for a Given Arc Length Between the Points of Attachment

z	r	z	r	z	r
1.0000000	0.0000000	.6775324	.5196696	.3270547	.8861823
.9909353	.0156223	.6680150	.5329257	.3164786	.8931654
.9818699	.0312407	.6584729	.5460502	.3058728	.8999280
.9728031	.0468514	.6489053	.5590401	.2952376	.9064686
.9637339	.0624506	.6393119	.5718920	.2845738	.9127856
.9546619	.0780343	.6296920	.5846028	.2738818	.9188774
.9455861	.0935988	.6200451	.5971694	.2631623	.9247424
.9365059	.1091402	.6103708	.6095886	.2524160	.9303792
.9274204	.1246546	.6006686	.6218574	.2416436	.9357865
.9183291	.1401383	.5909380	.6339728	.2308458	.9409629
.9092311	.1555875	.5811786	.6459317	.2200235	.9459072
.9001258	.1709982	.5713899	.6577313	.2091775	.9506180
.8910123	.1863668	.5615716	.6693686	.1983087	.9550943
.8818900	.2016893	.5517233	.6808407	.1874180	.9593350
.8727581	.2169621	.5418446	.6921449	.1765064	.9633389
.8636159	.2321814	.5319353	.7032783	.1655749	.9671052
.8544626	.2473434	.5219949	.7142381	.1546245	.9706328
.8452977	.2624444	.5120233	.7250218	.1436564	.9739210
.8361203	.2774806	.5020202	.7356265	.1326717	.9769688
.8269297	.2924483	.4919853	.7460497	.1216714	.9797756
.8177253	.3073439	.4819185	.7562889	.1106568	.9823407
.8085063	.3221636	.4718195	.7663414	.0996291	.9846634
.7992720	.3369039	.4616884	.7762049	.0885895	.9867431
.7900218	.3515610	.4515249	.7858769	.0775393	.9885793
.7807550	.3661314	.4413291	.7953549	.0664798	.9901717
.7714708	.3806115	.4311008	.8046367	.0554122	.9915197
.7621687	.3949976	.4208401	.8137200	.0443378	.9926231
.7528479	.4092863	.4105470	.8226025	.0332581	.9934816
.7435077	.4234740	.4002216	.8312820	.0221743	.9940949
.7341476	.4375572	.3898641	.8397565	.0110879	.9944630
.7247669	.4515324	.3794745	.8480237	0.0000000	.9945857
.7153650	.4653963	.3690531	.8560817		
.7059411	.4791453	.3586001	.8639285		
.6964948	.4927761	.3481159	.8715621		
.6870254	.5062853	.3376006	.8789806		

TABLE II*

Tabulation of the Calculated Results for a Normalized Troposkien of Maximum Swept Area for a Given Arc Length Between the Points of Attachment

RUN NUMBER	1	2	3	4	5	6	7
NUMBER OF SAMPLING POINTS	100	1000	100	1000	100	1000	100
EQUAL SAMPLING INCREMENTS BASED ON	R	R	R	R	CENTRAL ANGLE	CENTRAL ANGLE	CENTRAL ANGLE
GOODNESS OF FIT CRITERION	MINIMUM SPACING	MINIMUM SPACING	LEAST SQUARES	LEAST SQUARES	MINIMUM SPACING	MINIMUM SPACING	LEAST SQUARES
FIXED POINTS ON FITTED CURVE	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000
	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857
ZJ	0.5369468	0.5369543	0.5539268	0.5539300	0.5369685	0.5369541	0.5439907
RJ	0.7048771	0.7048717	0.6924976	0.6924962	0.7048614	0.7048719	0.6997773
R	0.3521419	0.3521346	0.3356849	0.3356818	0.3521207	0.3521348	0.3452862
RM	0.9945857	0.9945857	0.9945857	0.9945857	0.9945857	0.9945857	0.9945857
ZM	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
VALUE OF GOODNESS OF FIT CRITERION	0.0181518	0.0181536	0.0182762	0.0129371	0.0181473	0.0181536	0.0180391
COORDINATES OF POINTS ON THE TROPOSKIEN AT MAXIMUM SEPARATION	Z=0.7473023 R=0.4177260	Z=0.7505736 R=0.4127531	Z=0.4187293 R=0.8155603	Z=0.4164351 R=0.8175494	Z=0.7528479 R=0.4092863	Z=0.7500479 R=0.4135534	Z=0.4002216 R=0.8312820
	Z=0.3951606 R=0.8354520	Z=0.3963739 R=0.8344574			Z=0.4002216 R=0.8312820	Z=0.3960825 R=0.8346965	
COORDINATES OF POINTS ON THE FITTED CURVE AT MAXIMUM SEPARATION	Z=0.7321699 R=0.4077010	Z=0.7354248 R=0.4027499	Z=0.4331349 R=0.8322173	Z=0.4307953 R=0.8342495	Z=0.7376939 R=0.3993022	Z=0.7349014 R=0.4035465	Z=0.4127709 R=0.8464947
	Z=0.4066198 R=0.8495294	Z=0.4078563 R=0.8485182			Z=0.4117696 R=0.8452809	Z=0.4075596 R=0.8487617	
MAXIMUM SEPARATION	0.0181518	0.0181536	0.0220222	0.0220251	0.0181473	0.0181536	0.0197208
COORDINATES OF THE INTERSECTION OF THE TROPOSKIEN AND THE FITTED CURVE	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000
	Z=0.5559803 R=0.6759037	Z=0.5559885 R=0.6758969	Z=0.5943865 R=0.6296869	Z=0.5943486 R=0.6297479	Z=0.5560892 R=0.6757549	Z=0.5559882 R=0.6758972	Z=0.5713899 R=0.6577313
	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857

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*Machine computer printout; symbols used in the table are those available on the printer.

TABLE II (continued)

RUN NUMBER	8	9	10	11	12	13	14
NUMBER OF SAMPLING POINTS	1000	100	1000	100	1000	100	1000
EQUAL SAMPLING INCREMENTS BASED ON	CENTRAL ANGLE	CENTRAL ANGLE	CENTRAL ANGLE	CENTRAL ANGLE	CENTRAL ANGLE	CENTRAL ANGLE	CENTRAL ANGLE
GOODNESS OF FIT CRITERION	LEAST SQUARES	MINIMUM SPACING	MINIMUM SPACING	MINIMUM SPACING	MINIMUM SPACING	MINIMUM SPACING	MINIMUM SPACING
FIXED POINTS ON FITTED CURVE	Z=1.0000000 R=0.0000000 Z=0.0000000 R=0.9945857	Z=1.0000000 R=0.0000000	Z=1.0000000 R=0.0000000	Z=0.0000000 R=0.9945857	Z=0.0000000 R=0.9945857	(NONE)	(NONE)
ZJ	0.5439895	0.5653837	0.5654058	0.5182856	0.5182969	0.5463539	0.5463654
RJ	0.6997782	0.6721232	0.6721038	0.7230132	0.7230063	0.6920980	0.6920882
R	0.3452874	0.3065280	0.3065025	0.3642354	0.3642230	0.3218586	0.3218413
RM	0.9945857	0.9798174	0.9798138	0.9945857	0.9945857	0.9818433	0.9818397
ZM	1.0000000	1.0000000	1.0000000	1.0187841	1.0187872	1.0153576	1.0153619
VALUE OF GOODNESS OF FIT CRITERION	0.0126804	0.0147683	0.0147719	0.0154910	0.0154936	0.0127424	0.0127460
COORDINATES OF POINTS ON THE TROPOSKIEN AT MAXIMUM SEPARATION	Z=0.4043557 R=0.8278347	Z=0.7621687 R=0.3949976 Z=0.4515249 R=0.7858769 Z=0.0000000 R=0.9945857	Z=0.7668220 R=0.3878165 Z=0.4505068 R=0.7868334 Z=0.0000000 R=0.9945857	Z=1.0000000 R=0.0000000 Z=0.7059411 R=0.4791453 Z=0.3794745 R=0.8480237	Z=1.0000000 R=0.0000000 Z=0.7021653 R=0.4846120 Z=0.3794745 R=0.8480237	Z=1.0000000 R=0.0000000 Z=0.7247669 R=0.4515324 Z=0.4311008 R=0.8046367 Z=0.0000000 R=0.9945857	Z=1.0000000 R=0.0000000 Z=0.7210087 R=0.4570915 Z=0.4331490 R=0.8027962 Z=0.0000000 R=0.9945857
COORDINATES OF POINTS ON THE FITTED CURVE AT MAXIMUM SEPARATION	Z=0.4169914 R=0.8429885	Z=0.7498030 R=0.3869234 Z=0.4616144 R=0.7966614 Z=0.0000000 R=0.9798174	Z=0.7544373 R=0.3797650 Z=0.4605861 R=0.7976324 Z=0.0000000 R=0.9798138	Z=1.0133806 R=0.0078058 Z=0.6932082 R=0.4703255 Z=0.3890038 R=0.8602370	Z=1.0133829 R=0.0078071 Z=0.6894486 R=0.4757612 Z=0.3890054 R=0.8602390	Z=1.0110065 R=0.0064208 Z=0.7142217 R=0.4443794 Z=0.4395804 R=0.8141480 Z=0.0000000 R=0.9818433	Z=1.0110096 R=0.0064226 Z=0.7104743 R=0.4499162 Z=0.4416544 R=0.8122893 Z=0.0000000 R=0.9818397
MAXIMUM SEPARATION	0.0197307	0.0147683	0.0147719	0.0154910	0.0154936	0.0127424	0.0127460
COORDINATES OF THE INTERSECTION OF THE TROPOSKIEN AND THE FITTED CURVE	Z=1.0000000 R=0.0000000 Z=0.5713922 R=0.6577278 Z=0.0000000 R=0.9945857	Z=1.0000000 R=0.0000000 Z=0.5866767 R=0.6391945 Z=0.2493454 R=0.9319205	Z=1.0000000 R=0.0000000 Z=0.5866461 R=0.6392555 Z=0.2493734 R=0.9319310	Z=0.8984140 R=0.1738850 Z=0.5319879 R=0.7032191 Z=0.0000000 R=0.9945857	Z=0.8984086 R=0.1738984 Z=0.5320139 R=0.7031908 Z=0.0000000 R=0.9945857	Z=0.9045226 R=0.1635566 Z=0.5625119 R=0.6682541 Z=0.2392779 R=0.9369206	Z=0.9045233 R=0.1635621 Z=0.5625063 R=0.6682693 Z=0.2393136 R=0.9369239

TABLE III

Tabulation of the Parameters R, R_m, Z_m, Maximum d, R_j and Z_j for Curves Fitted to Normalized Troposkiens With β's of 0.8, 0.9, 1.0, 1.1, 1.2, and 0.99458568 by Use of 100 Points, the Least-Maximum-Spacing Criterion, and Equal Central Angle Increments

β	R	Z_m	R_m	Least-Maximum Spacing	Z_j	R_j
0.8	0.0475462109	1.0124650764	0.7904565415	0.0095434596	0.5676283964	0.5268308475
0.9	0.1937760100	1.0139410840	0.8888200630	0.0111799383	0.5561208918	0.6106882182
0.9945 8568	0.3218585935	1.0153575500	0.9818432850	0.0127424055	0.5463538774	0.6920980209
1.0	0.3289571819	1.0154365990	0.9871677470	0.0128322546	0.5458240196	0.6968121637
1.1	0.4563082750	1.0169705559	1.0855132320	0.0144867693	0.5366786056	0.7847515003
1.2	0.5780035445	1.0184325500	1.1838771380	0.0161228620	0.5285395660	0.8741939198

Locations of the points on the troposkien at which the separation d was calculated were selected in two ways. In the first the sampling points were equally spaced along the r axis. In the second the points were distributed along the troposkien in such a way that all angles between lines drawn from any two adjacent sampling points to the (r,Z) origin were equal.

Figures 3 through 8 have been plotted from the data of Table I and from circular-arc/straight-line approximation curves computed with constants listed in Runs No. 1, 3, 7, 9, 11, and 13 in Table II for 100 points along the troposkien arc. The corresponding runs, Nos. 2, 4, 8, 10, 12, and 14, respectively, which used 1000 calculated points along the troposkien arc, produced plots which could not be distinguished from the plots made from 100 points and are therefore not presented. Also, the plots for Runs No. 5 and 6 could not be distinguished from the plot of Run No. 1; consequently, these two runs are not plotted.

Figure 9 is a plot of the troposkiens and fitted curves for β's of 0.8, 0.9, 1.0, 1.1, and 1.2 with the end points of the fitted curves not constrained. Figure 10 is a plot of the parameters listed in Table III as a function of β.

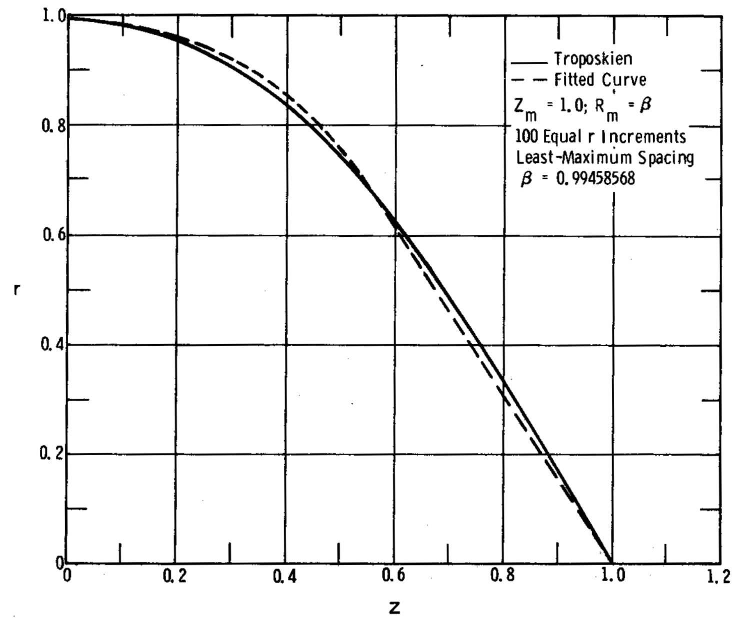

Figure 3. Comparison of troposkien and fitted curve: $\beta = 0.99458568$, $Z_m = 1$, $R_m = \beta$, equal r increments, minimax spacing

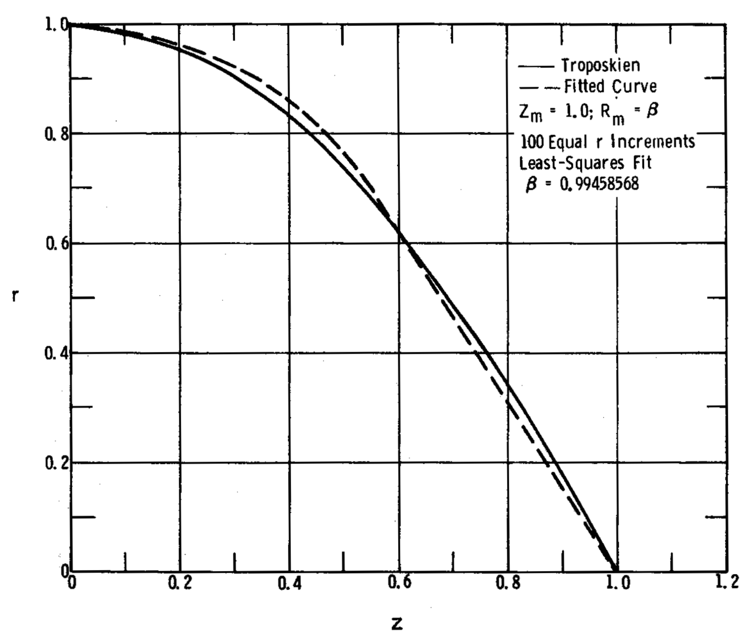

Figure 4. Comparison of troposkien and fitted curve: $\beta = 0.99458568$, $Z_m = 1$, $R_m = \beta$, equal r increments, least-square spacing

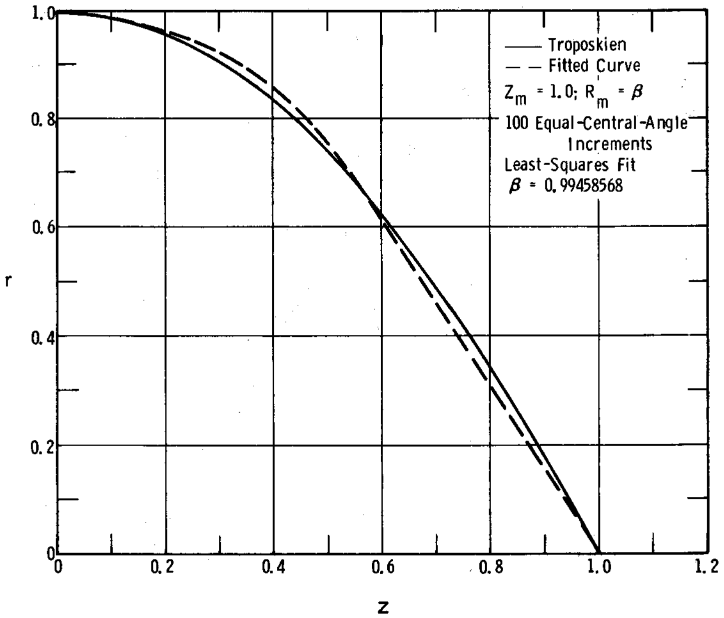

Figure 5. Comparison of troposkien and fitted curve: $\beta = 0.99458568$, $Z_m = 1$, $R_m = \beta$, equal-angle increments, least-square spacing

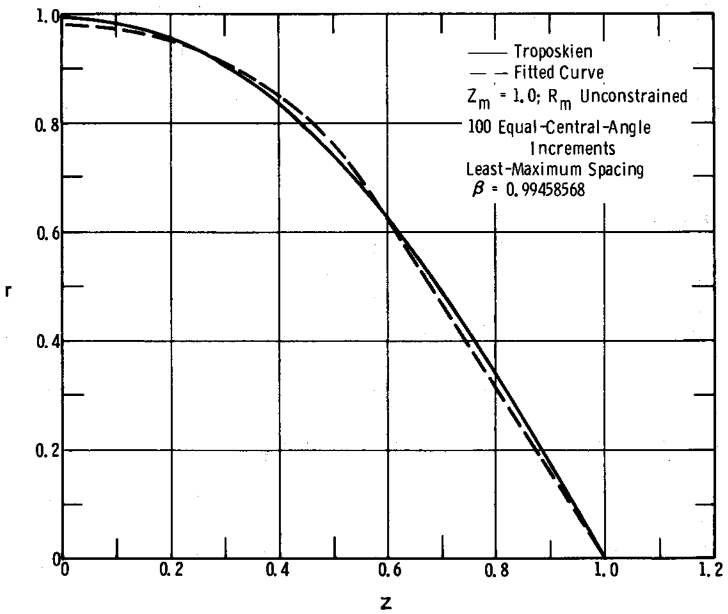

Figure 6. Comparison of troposkien and fitted curve: $\beta = 0.99458568$, $Z_m = 1$, R_m unconstrained, equal-angle increments, minimax spacing

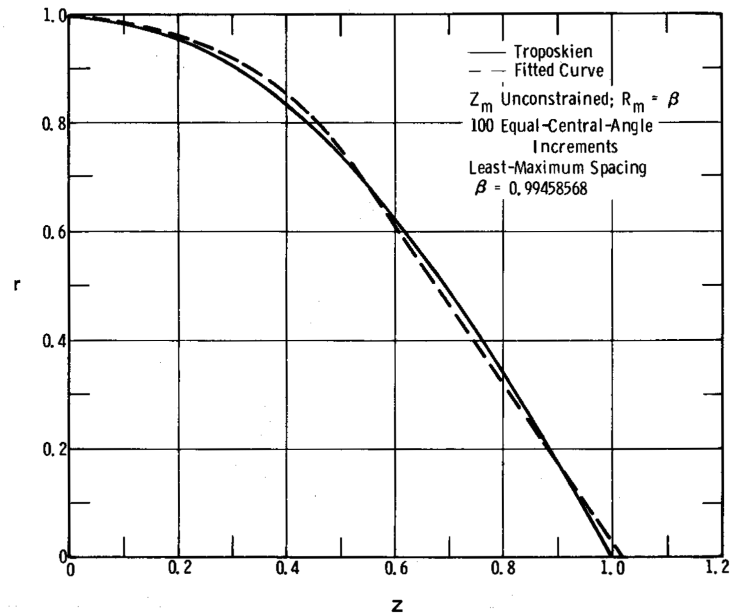

Figure 7. Comparison of troposkien and fitted curve: $\beta = 0.99458568$, Z_m unconstrained, $R_m = \beta$, equal-angle increments, minimax spacing

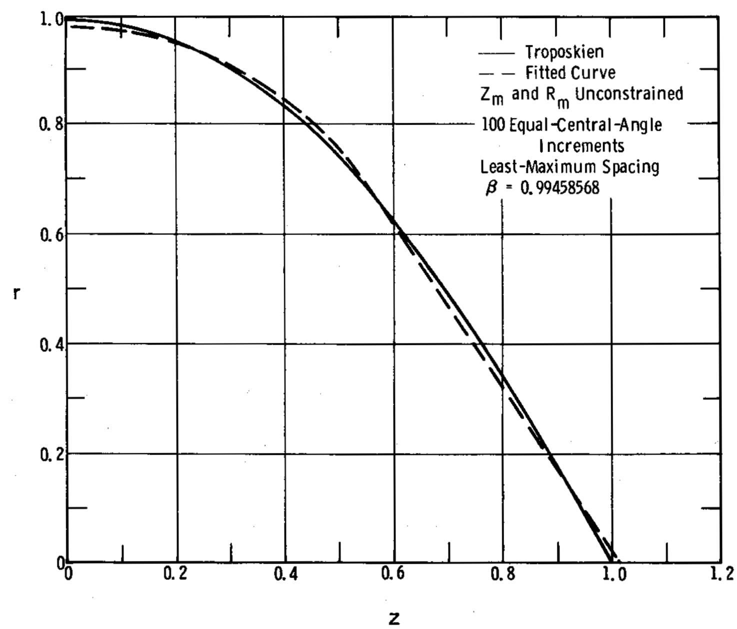

Figure 8. Comparison of troposkien and fitted curve: $\beta = 0.99458568$, Z_m unconstrained, R_m unconstrained, equal-angle increments, minimax-spacing

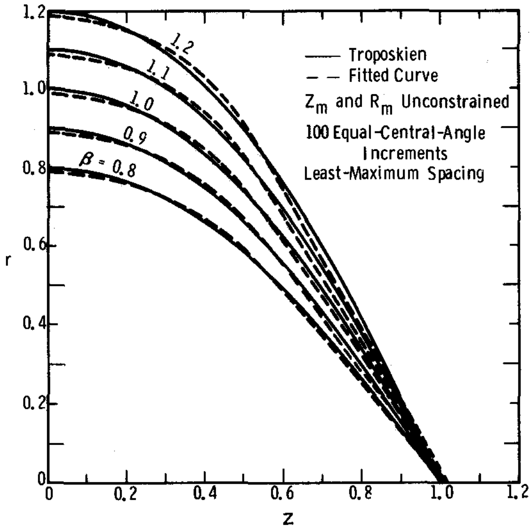

Figure 9. Comparison of troposkien and fitted curve: $\beta = 0.8 - 1.2$, Z_m unconstrained, R_m unconstrained, equal-angle increments, minimax spacing

In the calculations whose results are reported here, all input data have been entered to no less than eight significant figures, and the convergence criterion in the MAXDIF code was set so that the differences in the final two iterated values of each of the quantities R, R_m, and Z_m, when not fixed by the constraints of the particular case under consideration, were less than 10^{-10}. This was done in order to ensure that the results reported here are reliable to the listed seven decimal places. The maximum difference between the final two iterated values of the quantity used as a test for convergence was always $\leq 3 \times 10^{-10}$. The variable used to test for convergence was either the maximum separation between the troposkien and the fitted curve or the least sum of the squares of the calculated distances between the troposkien and the fitted curve.

The results of the present investigation indicate that, when the least-maximum difference is used as the goodness-of-fit criterion, the troposkien of maximum area for a given arc length between the points of attachment ($\beta = 0.99458568$) can be ap approximated by means of a curve made up of a circular arc and a straight line so that the separation between the curves is less than 1.3 percent of half the distance between the troposkien attachment points if the end points of the fitted curve are not constrained. If the fitted curve is constrained to be coincident with the troposkien only at the r-axis intercept, the separation can be held to less than 1.55 percent; if constrained to be coincident only at the z-axis intercept, this separation can be held to less than 1.48 percent. If the ends of the fitted curve are constrained to be coincident with the troposkien at both the r and the z axis intercepts, this separation can be held to less than 1.82 percent.

If the least-squares fit is used as the criterion for goodness-of-fit, these spacings increase by less than 0.5 percent for the four cases investigated.

After examining the calculated results for the cases in which β = 0.99458568, it was felt that 100 equal-angle sampling points were adequate and that the least-maximum spacing criterion was the better of the two goodness-of-fit criteria considered.

Figure 10 shows the dependence of the various parameters on β in the vicinity of β = 1.0 when using 100 sampling points with equal central angle spacings, when using the minimax goodness-of-fit criterion and when neither R_m nor Z_m is constrained. These data indicate that Z_m, R_m, R_j, R, and the least-maximum spacing increase with increasing β and that Z_j decreases with increasing β.

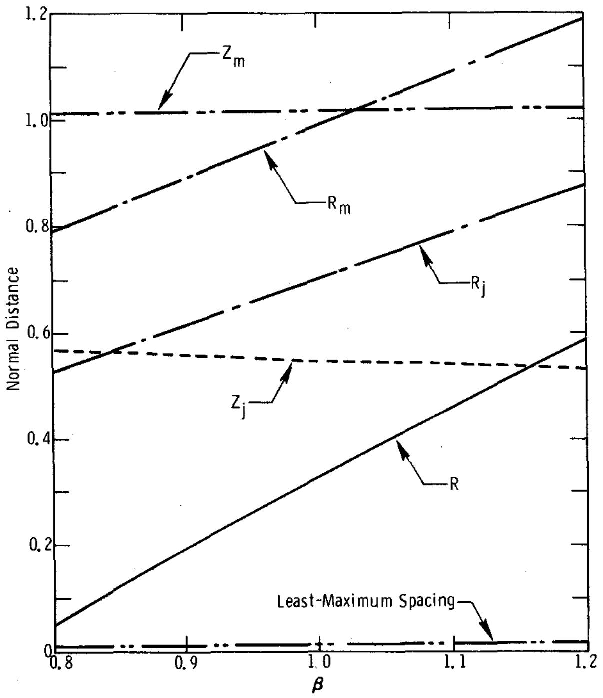

Figure 10. Plot of the calculated values of Z_m, R_m, Z_j, R_j, R, and the minimax spacing between the normalized troposkiens and the fitted curves as a function of β for $0.8 \le \beta \le 1.2$

From this investigation, it appears that troposkien shapes of interest in the design of crosswind-axis wind turbines can be approximated closely with a line made up of a circular arc and a straight line. However, before any decision is made as to which approximation should be used to define the shape of the blades in a Darrieus-type of wind turbine, the magnitude of the bending stresses introduced by the various approximations should be determined.

APPENDIX A

DERIVATION OF THE EQUATIONS

In the following analysis the coordinate system and notation shown in Figure 2 are used. Where normalized quantities are used, the normalizing constant is a.

Let the subscript t refer to the troposkien, c to the circular-arc portion of the fitted curve, ℓ to the straight-line portion of the fitted curve, n to the line normal to the troposkien, and j to the junction between the circular arc and the straight line. Also, let R be the normalized distance along the r-axis between the center of the circular-arc segment of the fitted curve and the origin of the (r,z) coordinate system, Z_m the normalized value of z at the point where the fitted curve meets the z-axis, and R_m the normalized value of r at the point where the fitted curve meets the r-axis. The normalized coordinates of the junction between the fitted curve's circular-arc and straight-line segments are designated Z_j and R_j.

From this notation and the coordinate system of Figure 2, the equation for the fitted-curve circular arc can be written as

$$\left(r_c - R\right)^2 = \left(R_m - R\right)^2 - z_c^{\,2} , \tag{A.1}$$

which has the slope

$$\frac{dr_c}{dz_c} = -\left(\frac{z_c}{r_c - R}\right) . \tag{A.2}$$

The minus sign in front of the right-hand side of Eq. (A.2) is necessary because the slope of the troposkien, the curve to be approximated, is negative in the first quadrant.

The equation for the straight-line section of the fitted curve is

$$r_\ell = a_\ell + b_\ell z_\ell , \tag{A.3}$$

whose slope is

$$\frac{dr_\ell}{dz_\ell} = b_\ell \ . \tag{A.4}$$

At the point where the straight-line section intersects the z-axis, $r_\ell = 0$ and $z_\ell = Z_m$. Substituting Z_m for those values in Eq. (A.3) gives

$$a_\ell = - b_\ell Z_m \ . \tag{A.5}$$

The term a_ℓ in Eq. (A.3) can be eliminated by use of Eq. (A.5); substitution yields

$$r_\ell = - b_\ell \left(Z_m - z_\ell\right) \ . \tag{A.6}$$

In that the coordinates of the circular arc and the straight line are equal at the point (Z_j, R_j) where they meet,

$$z_c = z_\ell = Z_j \tag{A.7}$$

and

$$r_c = r_\ell = R_j \ . \tag{A.8}$$

Substituting Eqs. (A.7) and (A.8) into Eqs. (A.1) and (A.6) gives

$$\left(R_j - R\right)^2 = \left(R_m - R\right)^2 - Z_j{}^2 \tag{A.9}$$

and

$$R_j = - b_\ell \left(Z_m - Z_j\right) \ . \tag{A.10}$$

The slope b_ℓ in Eq. (A.10) can be eliminated by applying the condition that the slope at the junction is continuous. Equating the slopes as given by Eqs. (A.2) and (A.4) and substituting from (A.7) and (A.8) gives

$$\left.\frac{dr_c}{dz_c}\right|_{Z_j, R_j} = \left.\frac{dr_\ell}{dz_\ell}\right|_{Z_j, R_j} = b_\ell = - \frac{Z_j}{R_j - R} \ . \tag{A.11}$$

Eliminating the term b_ℓ in Eq. (A.10) by use of Eq. (A.11) yields

$$R_j = \frac{Z_j}{R_j - R}\left(Z_m - Z_j\right) . \tag{A.12}$$

Equations (A.9) and (A.12) give two independent equations in Z_j and R_j, whose solution can be found as follows. Subtract R from both sides of Eq. (A.12) and multiply the result by $(R_j - R)$ to get

$$\left(R_j - R\right)^2 = Z_j\left(Z_m - Z_j\right) - R\left(R_j - R\right) . \tag{A.13}$$

Substitute for $(R_j - R)^2$ from Eq. (A.9) into Eq. (A.13). This yields

$$R_j - R = \frac{1}{R}\left[Z_m Z_j - \left(R_m - R\right)^2\right] . \tag{A.14}$$

By squaring both sides of this equation and substituting from Eq. (A.9) for the resulting term $(R_j - R)^2$, a quadratic equation for Z_j can be obtained:

$$Z_j^{\,2} - 2\left[\frac{Z_m\left(R_m - R\right)^2}{R^2 + Z_m^{\,2}}\right]Z_j + \left[\frac{\left(R_m - R\right)^4 - R^2\left(R_m - R\right)^2}{R^2 + Z_m^{\,2}}\right] = 0 . \tag{A.15}$$

The two solutions to the above equation are

$$Z_j = \left[\frac{Z_m\left(R_m - R\right)^2}{R^2 + Z_m^{\,2}}\right]\left\{1 \pm \left(\frac{\left(R^2 + Z_m^{\,2}\right)^2\left[\left(R_m - R\right)^2 - R^2\right]}{Z_m^{\,2}\left(R_m - R\right)^2}\right)^{1/2}\right\} . \tag{A.16}$$

The sign to be used in front of the radical in Eq. (A.16) can be determined to be positive by incorporating known numbers into Eq. (A.16). It is known[2] that, when $Z_m = 1.0$ and $R_m = 0.99458568$ and if $R = 0.3$, there will be two points at which the straight line will be tangent to the circular part of the fitted curve, one in the first and another in the fourth quadrant. When the above values of Z_m, R_m, and R are substituted into Eqs. (A.14) and (A.16), the plus sign yields

$$Z_j = 0.5916, \qquad R_j = 0.6639 , \tag{A.17}$$

whereas the minus sign yields

$$Z_j = 0.2936, \qquad R_j = -\,0.3295\ . \tag{A.18}$$

Therefore, because the present analysis is restricted to the first quadrant, both Z_j and R_j must be positive; consequently, the plus sign must be used in Eq. (A.16).

Before the separation d between the troposkien and the fitted curve as given by Eq. (2) in the body of the paper can be calculated, it is necessary to determine Z_i and R_i. These are the coordinates of the point of intersection of the fitted curve and the normal to the troposkien that passes through the point (Z_t,R_t) on the troposkien. The point (Z_i,R_i) depends not only upon Z_t, R_t, R, Z_m, and R_m but also upon whether the troposkien normal intersects the circular-arc or straight-line portion of the fitted curve. First, consider the case where the troposkien normal intersects the straight-line segment of the fitted curve. In this case the equation for the troposkien normal can be written as

$$r_n = a_n + b_n z_n\ , \tag{A.19}$$

where the constants a_n and b_n depend upon the location of the point of interest (Z_t,R_t) along the troposkien. The slope of this line is the negative reciprocal of the slope of the troposkien at the point (Z_t,R_t) where the normal intersects the troposkien. Consequently,

$$\frac{dr_n}{dz_n} = b_n = -\frac{1}{\left[\dfrac{dr_t}{dz_t}\right]_{Z_t,R_t}} = -\frac{1}{m_t}\ . \tag{A.20}$$

Elimination of the term b_n in Eq. (A.19) by use of Eq. (A.20) yields

$$r_n = a_n - \frac{z_n}{m_t}\ . \tag{A.21}$$

At the point where the normal meets the troposkien, their coordinates must be equal, which requires that if (Z_t,R_t) is the point on the troposkien under consideration,

$$z_n = Z_t \quad \text{and} \quad r_n = R_t\ . \tag{A.22}$$

Substituting these values into Eq. (A.21) gives

$$a_n = R_t + \frac{Z_t}{m_t} . \tag{A.23}$$

Eliminating a_n between Eqs. (A.21) and (A.23),

$$r_n = R_t - \left[\frac{z_n - Z_t}{m_t}\right] . \tag{A.24}$$

An equation for the linear portion of the fitted curve can be obtained by substituting for b_ℓ from Eq. (A.11) into Eq. (A.6), so that

$$r_\ell = \left(\frac{Z_j}{R_j - R}\right)\left(z_\ell - Z_m\right) . \tag{A.25}$$

Equations (A.24) and (A.25) may both be evaluated at the point (Z_i, R_i) to yield

$$R_i = R_t - \left(\frac{Z_i - Z_m}{m_t}\right) \tag{A.26}$$

and

$$R_i = - \left(\frac{Z_j}{R_j - R}\right)\left(Z_i - Z_m\right) . \tag{A.27}$$

Equating these last two equations produces an equation for Z_i,

$$Z_i - Z_m = \frac{\left(Z_m - Z_t - m_t R_t\right)\left(R_j - R\right)}{m_t Z_j - \left(R_j - R\right)} . \tag{A.28}$$

Once Z_i is known, R_i can be determined from either Eq. (A.26) or (A.27).

Consider now the case in which the troposkien normal intersects the circular-arc portion of the fitted curve. The equation describing the circular-arc segment is Eq. (A.1), which when evaluated at the point (Z_i, R_i) is

$$\left(R_i - R\right)^2 = \left(R_m - R\right)^2 - Z_i^{\,2} . \tag{A.29}$$

Equation (A.26) is valid at all points on the troposkien and when combined with Eq. (A.29) can be used to determine Z_i for the circular segment as follows:

Subtract R from both sides of Eq. (A.26) and square the result to get

$$\left(R_i - R\right)^2 = \left(R_t - R\right)^2 - \frac{2}{m_t}\left(R_t - R\right)\left(Z_i - Z_t\right) + \frac{1}{m_t^2}\left(Z_i - Z_t\right)^2 . \tag{A.30}$$

Equating the right-hand side of Eqs. (A.29) and (A.30) yields the following equation:

$$Z_i^2 - \left(\frac{2}{1 + m_t^2}\right)\left[Z_t + m_t\left(R_t - R\right)\right]Z_i$$

$$+ \frac{\left[Z_t + m_t\left(R_t - R\right)\right]^2 - m_t^2\left(R_m - R\right)^2}{\left(1 + m_t^2\right)} = 0 . \tag{A.31}$$

This is a quadratic equation in Z_i whose two solutions are

$$Z_i = \left[\frac{Z_t + m_t\left(R_t - R\right)}{1 + m_t^2}\right]\left\{1 \pm m_t\left(\left(1 + m_t^2\right)\left[\frac{R_m - R}{Z_t + m_t\left(R_t - R\right)}\right]^2 - 1\right)^{1/2}\right\} . \tag{A.32}$$

By using the same procedure as was used in connection with the sign in Eq. (A.16), it can be shown that the plus sign is applicable in Eq. (A.32). Once Z_i is known, R_i can be calculated from Eq. (A.26).

In these derivations, the slope m_t of the troposkien as given by Eq. (A.20) is assumed to be known. An expression for m_t, derived in Reference 2, is

$$\frac{dr_t}{dz_t} = - \frac{2k}{1 - k^2}\left[\left(\frac{r_t^2}{\beta^2} - 1\right)\left(\frac{r_t^2 k^2}{\beta^2} - 1\right)\right]^{1/2} , \tag{A.33}$$

where β is the ratio of the maximum blade deflection of the troposkien to one-half the spacing between the attachment points on the axis of rotation. The two parameters β and k are not independent but, rather, are connected by the equation

$$\beta = \frac{2k}{\left(1 - k^2\right) F\left(\frac{\pi}{2};k\right)} , \tag{A.34}$$

where $F(\pi/2;k)$ is the complete elliptic integral of the first kind with parameter k.

APPENDIX B

SEQUENCE OF CALCULATIONS USED IN THE COMPUTER CODE MAXDIF

The actual fitting procedure was started by assuming a value for β and then computing values of Z_t and m_t corresponding to a series of values of R_t by means of Eqs. (3) - (5). Two methods were used to select the values of R_t. In the first method, the various R_t's were assumed to be equally spaced along the r-axis between the origin and the point at which the troposkien intercepts the r-axis. In the second method, the R_t's were distributed along the r-axis in such a way that angles formed by lines drawn from the origin to points on the troposkien defined by any two adjacent values of R_t were equal.

Values of R, R_m, and Z_m were then assumed, and values of Z_j and R_j were calculated by means of Eqs. (6) and (7).

Equations (10) and (12) were then used to compute Z_i and R_i for each value of R_t. For cases in which $R_i > R_j$, d was calculated from Eq. (14) for each value of R_i greater than R_j; Z_i and R_i, given by Eqs. (10) and (11), were used in these calculations. For cases in which $R_i < R_j$, d was calculated from Eq. (14) for each value of R_i less than R_j; Z_i and R_i given by Eqs. (12) and (13) were used in these calculations. It should be noted that Eq. (14) gives the magnitude of the separation between the troposkien and the fitted curve.

At this point, a choice of the goodness-of-fit criterion was made. When the least-maximum-difference criterion was used, the maximum value of d was stored as the maximum error; when the root-mean-square was used, the sum of the squares of the d's computed at each R_t was stored as the maximum error.

The process was continued by assuming a second value for R, with R_m and Z_m fixed, and by repeating the above sequence of calculations to obtain a second value of the maximum error. These two values of the error were compared, and from them an estimate of a new value of R which gives the minimum maximum error was calculated and used to calculate a third estimate of R. This process was continued until the difference between two succeeding values of the maximum error was less than 10^{-10}.

After convergence was obtained for R, then R_m and Z_m (if not fixed by constraints) were varied in the same manner. This operation produced the values of R, R_m, and Z_m listed in the data presented here.

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DISTRIBUTION:

J. Templin
Low Speed Aerodynamics Section
NRC-National Aeronautical Establishment
Ottawa 7, Ontario, Canada K1AOR6

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Faculty of Engineering & Applied Sciences
St. John's, Newfoundland
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