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The Timing and Magnitude Relationships Between Month-to-Month Changes and Year-to-Year Changes That Make Comparing Them Difficult

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ABSTRACT

When a monthly economic indicator series contracts sharply for a few months and then starts to recover, the published annual and monthly growth rates can give conflicting signals: the annual growth rate can indicate a decrease and the monthly growth rate an increase or vice versa. This is well known to the seasonal adjustment community, see, for example, Shiskin (1957). In this paper, we revisit, illustrate and then explain this potential for conflict more analytically. For example, the annual differences lag the monthly differences by five and a half months because the same-month-year-ago difference is the sum of the current and eleven preceding monthly differences, and the annual sum has a phase shift of five and a half months. Illustrative examples are followed by an elementary formal mathematical derivation using the gain and phase functions of the annual sum.

1 INTRODUCTION

Wyman (2010) notes that there has been a growing interest in understanding the movements in monthly economic time series, most often seasonally adjusted series, as a result of the late 2008 economic downturn and its recovery period. Her paper reviews the basic aspects of seasonal adjustment and how seasonal adjustment helps analysts interpret the economic trend when such a situation occurs. Wyman (2010) can be viewed as an updated version of Shiskin (1957) to publicize the benefits of seasonal adjustment for economic data users. Both papers discuss year-over-year changes as an alternative to seasonal adjustment, its weaknesses and, of most interest to this study, its main limitation that it gives an outdated story. Here we revisit this main limitation, illustrate it with a simple function example and empirically, and then provide an elementary formal mathematical derivation of the delay of the year-over-year comparison with respect to the month-to-month comparison.

Shiskin (1957), pages 230-231, provides a footnote with a reference to Macaulay (1931), pages 134-135,

> Economists have long been critical of same-month-year-ago-comparison. Thus in 1931 Frederick R. Macaulay wrote: "There is a simple and enlightening way to describe the operation of *subtracting* the quotation for the same month last year from the quotation of the present month *[...]*. It amounts to taking a 12-months moving total of the data and using the first differences of this moving total *[...]*. Moreover, as the 12-months moving average does not extend to the end of the data, its first differences do not tell whether, at the present time, the underlying curve of the data is high or low or whether it is rising or falling, but simply *whether it was rising or falling six months ago"*.

Macaulay (1931), pages 135-136, further illustrates this delay of six months with the examples of sine curves of 24-month and 48-month periods.

Rhodes and Elhawary-Rivet (1983) presents the relationship between the monthly and annual rates of change that permits reconciliation of possibly contradictory movements. It provides a graphical example, intuitive arguments and uses the gain and phase shift functions to provide the five and half month delay of the annual growth rate with respect to the monthly growth rate.

The present tutorial paper provides a reproducible and corrected example, simplifies the derivation by using differences instead of rates, and supplies missing details. The most helpful perspective is that an annual difference is the sum of the twelve intervening monthly differences (1). Hence, the phase shift of the annual difference relative to the phase shift of the monthly difference is simply that induced by the annual sum. Equation (D.4) of Findley and Martin (2006) gives the gain and phase function of the annual sum filter without the details of its derivation. We provide a detailed derivation, starting from elementary concepts.

This paper is organized as follows: Section 2 first illustrates the precise problem graphically with an elementary function. Then the New Car Dealer Sales series from the Canadian Monthly Retail Trade Survey serves to provide a real example with sign differences between monthly and annual growth rates. Section 3 develops the relevant business cycle frequency perspective from basic concepts and examples. Section 4 provides the formal derivation of the phase shift induced by the annual sums, leading to the conclusion that annual differences lag monthly differences by five and a half months in a basic way.

2 ILLUSTRATIVE EXAMPLES

2.1 An Artificial Example

It happens occasionally that a monthly time series indicates an annual decrease and at the same time a monthly increase. This can create confusion among users of its data. The following example illustrates the situation. See also the discussions in Shiskin (1957), second column, page 229, and in Macaulay (1931), pages 135-136.

Consider the time series $X_t = \cos(2\pi t/24)$ for $t = 0, 1, 2, \ldots, 24$, displayed in Figure 1. Next consider the monthly differences $X_t - X_{t-1}$ for $t = 1, \ldots, 24$ and the annual difference $X_t - X_{t-12}$ for $t = 12, \ldots, 24$ displayed in Figure 2. The monthly differences, available starting at $t = 1$, are negative from $t = 1$ to $t = 12$ and then positive from $t = 13$ to $t = 24$. They indicate the decrease in X_t from $t = 1$ to $t = 12$ and the subsequent increase in X_t from $t = 13$ to $t = 24$. The continuous line that joins the monthly differences crosses the x-axis at $t = 12.5$. There is no observation at this mid-time. The annual differences only start at $t = 12$. They are negative

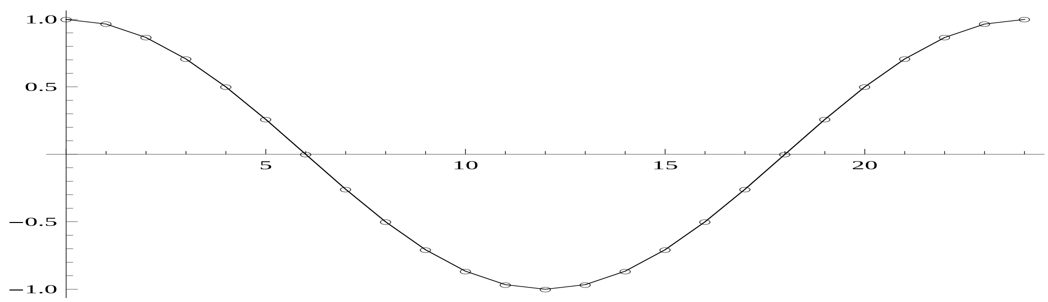

Figure 1: $X_t = \cos(2\pi t/24)$ for $t = 0, 1, 2, \ldots, 24$

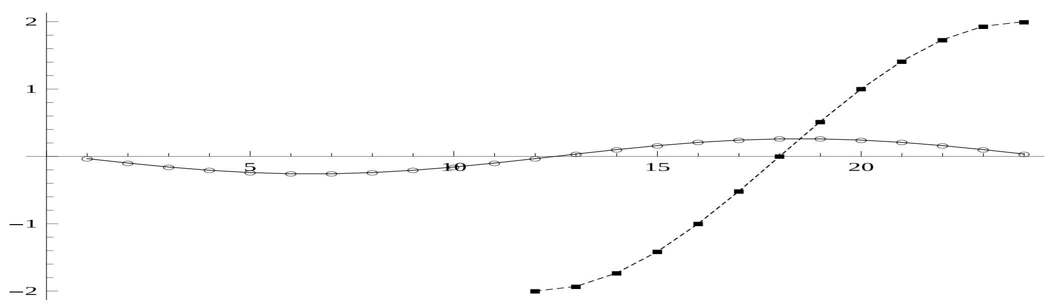

Figure 2: Monthly (circles) and Annual (squares) Differences of X_t

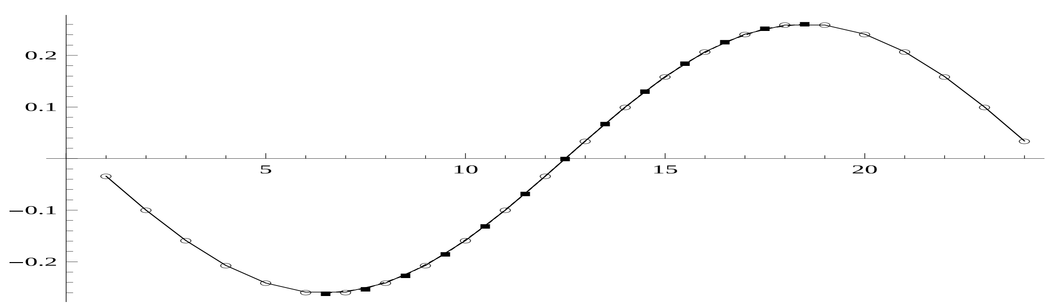

Figure 3: Monthly Differences (circles) and Rescaled and Shifted Annual Differences (squares) of X_t

from $t = 12$ to $t = 17$, zero at time $t = 18$ and then positive from $t = 19$ to $t = 24$. For $t = 13, 14, 15, 16, 17$ and 18, the monthly differences are positive and the annual differences are not. Thus, it seems they provide contradictory information. The time plot of the first differences in Figure 2 clearly indicates that the series started to increase at $t = 13$. The first positive increase from the annual differences occurs six months later at time $t = 19$.

One can observe that the continuous-time curve of the annual differences crosses the x-axis exactly $5.5 = 18 - 12.5$ months after the monthly differences. The explanation will turn out to reside in the fact that each annual difference is the sum of the last twelve monthly differences:

$$X_t - X_{t-12} = (X_t - X_{t-1}) + (X_{t-1} - X_{t-2}) + \ldots + (X_{t-11} - X_{t-12}) \,. \quad (1)$$

As a start, this shows that if series has been decreasing for a few months, it will generally take a few months of positive increase to make the annual differences positive.

The larger scale of the annual differences in Figure 2 is exactly explained by the gain function of annual sums, which will be defined and derived in Section 4. For now, consider the re-scaled annual differences obtained by dividing the annual differences by the ratio of the sine functions in Equation (5) at $\lambda = 1/24$, the frequency of a two-year cycle. Also, shift their graph backward by exactly 5.5 months, corresponding to the phase shift of annual sums at this frequency, derived as (7) below. Now the monthly and re-scaled and time-shifted annual differences, as displayed in Figure 3, tell the same story. In particular, the series was decreasing until $t = 12$ and then started to increase at $t = 13$.

In conclusion for this section, the annual differences were 5.5 months late in identifying the change from decrease to increase relative to the monthly differences. They tell an outdated story. Further information regarding these differences and their phase shifts, or time delays, will be provided below.

2.2 New Car Dealer Sales

This section provides a real example with Statistics Canada New Car Dealer Sales from the Monthly Retail Trade Survey[1]. The estimates are available from Statistics Canada's web site. They are provided in Table 1 for the period January 2007 to July 2010 and are displayed in Figure 4. The seasonally

[1]Older examples are provided in Shiskin (1957).

adjusted series clearly shows the drop in the sales at the end of 2008 and its recovery early 2009.

Before comparing the monthly growth rates and annual growth rates of the seasonally adjusted series, we review the statement in Wyman (2010) that Statistics Canada's main economic data releases use seasonally adjusted series to compare year-over-year measures. This is illustrated in Table 1 and Figure 5 where the annual growth rates of both the raw and seasonally adjusted series are displayed. Figure 5 shows that the growth rate from the seasonally adjusted series is smoother and achieved its lowest value in December 2008, whereas that of the raw series achieved its lowest values two months later in February 2009. The growth rates computed from the raw series are affected by the calendar effects that include trading-day effects, a 2008 leap year February that affected the February 2009 year-over-year comparison, and an April 2009 Easter Sunday combined with a March 2008 Easter Sunday that affected both the March and April comparison. Wyman (2010) discusses these topics in detail. For our further discussion of this example, the annual growth rates computed from the seasonally adjusted series will be used.

The monthly and annual growth rates in the seasonally adjusted series are provided in Table 1 and displayed in Figure 6. The monthly growth rate is positive in January 2009, slightly drops back to a small negative value in February 2009 and then returns to and remains positive through October 2009. The annual growth rates are negative through September 2009. Thus there is a sign contradiction between the monthly and annual growth rates for seven consecutive months. Only in October 2009 do they have the same sign. Publishing only the current monthly and annual growth rates sends a confusing signal because one is positive and the other is negative. Despite the fact that the annual growth rates are negative at the beginning of 2009, Figure 6 shows that the improvement in the annual growth rates also started in January 2009. Most data publications provide neither this graph nor the previous month's annual growth rate.

The Canadian Consumer Price Index publication is a notable exception. In it, both current and previous month annual growth rates are shown and the difference is calculated and commented on. Also publishing the previous month's annual growth rate and commenting on the difference from the current month's annual growth rate would, in general, avoid confusion from disagreements with the signal provided by the recent values of the seasonally adjusted series. More on this topic can be found in Wyman (2010).

Table 1: New Car Dealer Sales from January 2007 to July 2010: raw, seasonally adjusted (SA), annual growth rate in % in the raw (A_GR_Raw), annual growth rate in % in the seasonally adjusted series (A_GR_SA), monthly growth rate in % in the seasonally adjusted series (M_GR_SA)

Date	Raw	SA	A_GR_Raw	A_GR_SA	M_GR_SA
2007Jan	4969575	6256082			
2007Feb	4833879	6174078			-1.31
2007Mar	6833771	6376681			3.28
2007Apr	7268126	6556587			2.82
2007May	8186647	6626927			1.07
2007Jun	7570250	6501631			-1.89
2007Jul	6955440	6464541			-0.57
2007Aug	7338273	6641227			2.73
2007Sep	6130209	6413151			-3.43
2007Oct	6153139	6396495			-0.26
2007Nov	5822440	6428010			0.49
2007Dec	5426751	6653090			3.50
2008Jan	5404653	6689868	8.75	6.93	0.55
2008Feb	5303599	6629311	9.72	7.37	-0.91
2008Mar	6438205	6515989	-5.79	2.18	-1.71
2008Apr	7807596	6406912	7.42	-2.28	-1.67
2008May	7597202	6330020	-7.20	-4.48	-1.2
2008Jun	6928454	6266093	-8.48	-3.62	-1.01
2008Jul	6911068	6105610	-0.64	-5.55	-2.56
2008Aug	6305914	5977386	-14.07	-10.00	-2.10
2008Sep	6245315	6349413	1.88	-0.99	6.22
2008Oct	5950398	6154373	-3.29	-3.79	-3.07
2008Nov	5077802	5827447	-12.79	-9.34	-5.31
2008Dec	4483801	5201588	-17.38	-21.82	-10.74
2009Jan	4232132	5371485	-21.69	-19.71	3.27
2009Feb	4114647	5324453	-22.42	-19.68	-0.88
2009Mar	5852277	5522325	-9.10	-15.25	3.72
2009Apr	6405143	5543801	-17.96	-13.47	0.39
2009May	6684784	5717547	-12.01	-9.68	3.13
2009Jun	6717658	5763867	-3.04	-8.01	0.81
2009Jul	6525669	5873070	-5.58	-3.81	1.89
2009Aug	6240691	5976009	-1.03	-0.02	1.75
2009Sep	6198862	6103334	-0.74	-3.88	2.13
2009Oct	6042329	6219647	1.54	1.06	1.91
2009Nov	5296071	6172845	4.30	5.93	-0.75
2009Dec	5414355	6136234	20.75	17.97	-0.59
2010Jan	4531819	5941702	7.08	10.62	-3.17
2010Feb	4787190	6175739	16.35	15.99	3.94
2010Mar	6991355	6436994	19.46	16.56	4.23
2010Apr	7062390	6086019	10.26	9.78	-5.45
2010May	6991542	6108290	4.59	6.83	0.37
2010Jun	7475499	6234774	11.28	8.17	2.07
2010Jul	6942796	6302895	6.39	7.32	1.09

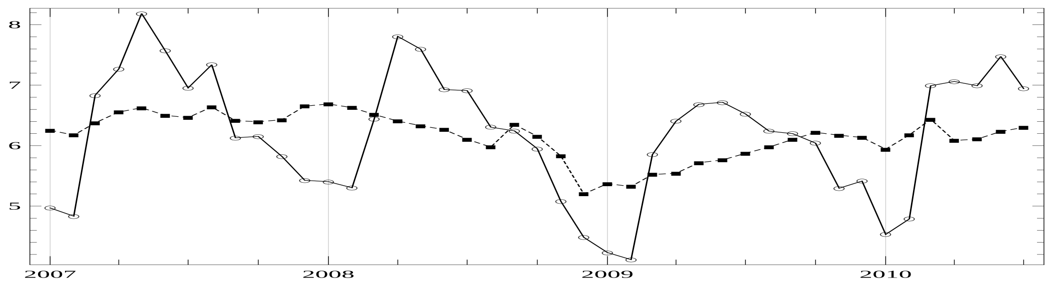

Figure 4: New Car Dealer Sales: Raw (circles) and Seasonally Adjusted Series (squares)

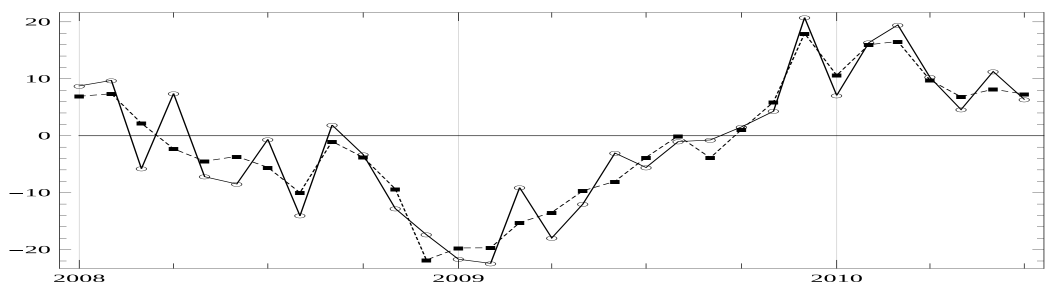

Figure 5: New Car Dealer Sales: Annual Growth Tate in % for the Raw (circles) and Seasonally Adjusted Series (squares)

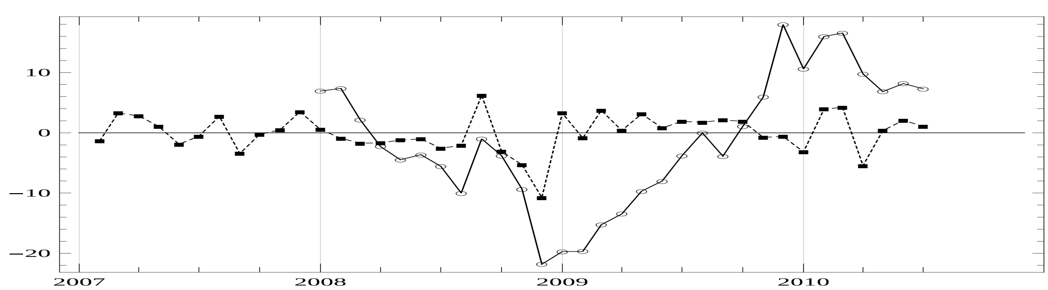

Figure 6: New Car Dealer Sales: Annual (circles) and Monthly (squares) Growth Rates in % for the Seasonally Adjusted Series

3 INTRODUCTORY CONCEPTS

3.1 Frequencies in Time Series

A time series may be considered from two perspectives: time and frequency. In the *time domain*, the series X_t is treated as a succession of T regularly observed values over an interval of months, say, with a time index t varying from 1 to T or some other designation of the months. This is how a time series is generally approached and the time plot of X_t against t shows its evolution over time. Figure 4 provided such a time plot for the raw and seasonally adjusted New Car Dealer Sales' series.

In the *frequency domain*, a time series X_t of length T can be represented by a sum of T periodic functions, specifically sine and cosine functions of typically different amplitudes and possibly different phases. Details of the representation will not be needed in this note. The low frequencies correspond to slowly changing components such as the trend and the business cycle. The high frequencies correspond to the more quickly changing components including seasonal components and more volatile components.

The usual domain of sine and cosine functions is the interval $[0, 2\pi]$, i.e. $0 \leq \omega \leq 2\pi$, or any translation of it in the interval $[-2\pi, 2\pi]$ such as $[-\pi, \pi]$. However it will be seen that, for our purposes, we can focus on positive frequencies in $[0, \pi]$, even on the smaller subinterval of frequencies relevant for business cycle analysis. A given frequency ω within the interval $[0, \pi]$ can be expressed as $\omega = 2\pi\lambda$ with $0 \leq \lambda \leq 1/2$. For example, the graph of $X_t = \cos(2\pi t/24)$ in Figure 1 would represent the cosine function of amplitude 1 with $\lambda = 1/24$. The function $\cos(2\pi\lambda t)$ repeats itself every 24 months since

$$\cos[2\pi(t + 24)/24] = \cos(2\pi t/24 + 2\pi) = \cos(2\pi t/24).$$

For monthly series, the number $1/\lambda$ indicates the number of months it takes for a component of the series with frequency λ to go through a full cycle in the time series, 24 months with $\lambda = 1/24$. The cosine function $\cos(2\pi t/24)$ could provide the fundamental component for modeling a 2-year business cycle in a monthly time series that oscillates around the value zero.

Some frequencies of interest for a monthly economic time series are:

- $\lambda = 1/60$, associated with the five year cycle because $60 = 5 \times 12$.
- $\lambda = 1/24$, associated with the two year cycle because $24 = 2 \times 12$.
- The interval $[1/60, 1/24]$, associated with five down to two year business cycles.
- The interval $[0, 1/60)$, associated with phenomena that take more than 5 years to be fully expressed in the time series. Those with λ close to 0 are related to the long-term trend.
- The values $\lambda = k/12$ with $k = 1, 2, 3, 4, 5, 6$, which are the fundamental seasonal frequency ($k = 1$) and its harmonics. They are associated with phenomena that recur in the time series $1, 2, 3, 4, 5$ or 6 times within a year.

The frequency $\omega = 2\pi\lambda$ with $\lambda = 6/12 = 1/2$ is associated with the 2-month cycle. This is the highest frequency that can be observed in a monthly time series. Hence, in the sequel, λ can be restricted to the interval $[0, 1/2]$, corresponding to $0 \leq \omega \leq \pi$.

3.2 Complex Numbers

The use of complex numbers simplifies the analysis of cycles and phase shifts. A complex number has the form $z = x + iy$ where x and y are real numbers and i is the *imaginary* unit with the property $i^2 = -1$; x is called the real part of the complex number; y is the imaginary part. The complex number $x - iy$ is called the complex conjugate of z and is denoted $\bar{z}$.

A complex number z is graphically represented in the plane by its coordinate pair (x, y). The magnitude of z, also known as the modulus or absolute value, is the distance of (x, y) from the origin $(0, 0)$ and is written $r = |z|$. By Pythagoras' theorem, $r = |z| = |x + iy| = \sqrt{x^2 + y^2} = \sqrt{z \times \bar{z}}$.

For $z \neq 0$, the principal argument of $z = x + iy$, written $\arg(z)$, is the angle which the line from (x, y) to $(0, 0)$ makes with the positive x axis, measured in radians, but with a minus sign if $y < 0$. It is not defined for $z = 0$. The magnitude and argument provide the *polar representation* $z = re^{i\arg(z)}$, with $-\pi < \arg(z) \leq \pi$. A function definition of $\arg(z)$, which we will not explicitly need, can be given with the aid of the atan2 function[2]. See Wikipedia Contributors (2011) for example.

The combination of the magnitude and argument fully specify the position of a point in the plane $(x, y) = (r\cos\varphi, r\sin\varphi)$ different from $(0, 0)$. Hence, a non-zero complex number can be written in various ways: the rectangular form $z = x + iy$, the trigonometric form $z = r(\cos\varphi + i\sin\varphi)$ and the exponential form $z = re^{i\varphi}$ coming from $(x, y) = (r\cos\varphi, r\sin\varphi)$ and $e^{i\varphi} = \cos\varphi + i\sin\varphi$.

A complex number on the unit circle ($r = 1$) can be written as $e^{i\varphi} = \cos\varphi + i\sin\varphi$. These representations provide the following equalities used in this paper: $e^{\pm i2\pi k} = 1,\ k = 0, 1, \ldots$; $i = e^{i2\pi/4}$; and $\sin\varphi = \left(e^{i\varphi} - e^{-i\varphi}\right)/(2i)$.

Multiplication of two complex numbers is simple using the exponential form since $(r_1e^{i\varphi_1}) \cdot (r_2e^{i\varphi_2}) = r_1r_2e^{i(\varphi_1+\varphi_2)}$. When $\varphi_1 + \varphi_2$ falls outside the interval $(-\pi, \pi]$, the principal argument $\varphi_1 + \varphi_2 \pm 2\pi$ in the interval $(-\pi, \pi]$ is usually taken to resolve that ambiguity that $z = re^{i\arg(z)\pm 2\pi k}$ for any $k = 1, 2, \ldots$.

3.3 Moving Averages/Filters

A **moving average** is a weighted sum of a fixed number of time series values that is applied in a sequential manner over a subinterval of the time series data $X_1, \ldots, X_T$, adding and dropping one observation at each step. The value $\hat{X}_t$ of the moving average at time t is given by a formula

$$\hat{X}_t = \sum_{k=-p}^{+f} \theta_k X_{t+k}$$

where the coefficients $\theta_k,\ k, = -p, \ldots, f$ are often called the weights of the moving average. (The weights can have negative values and need not sum to 1.0, so the name can be misleading.) A moving average is also called a **filter**, which is the term we will use. Then the values $\hat{X}_t$ are called the filter

[2]
$$\arg(z) = \operatorname{atan2}(y, x) = \varphi = \begin{cases} \arctan(y/x) & x > 0 \\ \arctan(y/x) + \pi & y \geq 0, x < 0 \\ \arctan(y/x) - \pi & y < 0, x < 0 \\ \pi/2 & y > 0, x = 0 \\ -\pi/2 & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

output and those of X_t the filter input. The output defines a time series in which the value at instant t of the series X_t is replaced by a weighted average of p "past" values of the series, the current value, and f "future" values of the series. Its values cannot be calculated for the first p values and the last f values of the time interval of the X_t values.

We will be concerned with the filter that transforms monthly differences to annual differences. The formula (1) shows that this is the annual sum filter with $p = 11,\ f = 0$, and $\theta_k = 1.0,\ k = -11, \ldots, 0$.

3.4 Gain and Phase Shift Functions

Consider $X_t = Re^{i\omega t} = R\left[\cos(\omega t) + i\sin(\omega t)\right]$, a time series at frequency ω with amplitude R. When a filter is applied to X_t the output is

$$\begin{aligned}\hat{X}_t &= \sum_{k=-p}^{+f} \theta_k R e^{i\omega(t+k)} \\ &= Re^{i\omega t} \sum_{k=-p}^{+f} \theta_k e^{i\omega k} \\ &= X_t \sum_{k=-p}^{+f} \theta_k e^{i\omega k},\end{aligned}$$

which is the initial value X_t multiplied by complex number $\sum_{k=-p}^{+f} \theta_k e^{i\omega k}$.

For ω in the interval $(-\pi, \pi]$, the function

$$G(\omega) = \sum_{k=-p}^{+f} \theta_k e^{i\omega k} = \sum_{k=-p}^{+f} \theta_k \cos(\omega k) + i \sum_{k=-p}^{+f} \theta_k \sin(\omega k)$$

is called the **transfer function** of the filter. It can be expressed as $G(\omega) = |G(\omega)|e^{i\varphi(\omega)}$ using the polar representation of a complex number.

- The function $|G(\omega)| = \left|\sum_{k=-p}^{+f} \theta_k e^{i\omega k}\right|$ is called the **gain function** of the filter. For economic indicator data, usually $\omega = 2\pi\lambda$, with λ in units of cycles per year. The graph of $|G(2\pi\lambda)|$ against $0 \le \lambda \le 1/2$ (see Figure 7 for the annual sum filter) shows the frequencies suppressed, preserved or amplified by the filter. The gain function is graphed only for $0 \le \lambda \le 1/2$ because $|G(-2\pi\lambda)| = |G(2\pi\lambda)|$.

- The function $\varphi(\omega) = \arg[G(\omega)]$, defined only where $G(\omega) \neq 0$, is called the **phase shift function** of the filter. It can be directly calculated for the business cycle frequencies of interest for our example. In general, it is given by

$$\varphi(\omega) = \text{atan2}\left(\sum_{k=-p}^{+f} \theta_k \sin(\omega k), \sum_{k=-p}^{+f} \theta_k \cos(\omega k)\right).$$

  Graphing $\varphi(2\pi\lambda)$ over $0 \leq \lambda \leq 1/2$ or over the business cycle frequencies of interest can show the extent to which the cyclical component at frequency λ is shifted by the filter. For $0 < \lambda \leq 1/2$, the phase shift is commonly graphed as $\varphi(2\pi\lambda)/2\pi\lambda$. This expresses the phase shift as a time shift in units of months (or whatever the sampling interval is). The graphing interval is again restricted positive frequencies because $\varphi(-2\pi\lambda) = -\varphi(2\pi\lambda)$ and $\varphi(-2\pi\lambda)/-2\pi\lambda = \varphi(2\pi\lambda)/2\pi\lambda$. The latter function can be defined at $\lambda = 0$ via $\lim_{\lambda \to 0} \varphi(2\pi\lambda)/2\pi\lambda = \varphi'(0)$ when this limit exists.

4 THE ANNUAL SUM FILTER

The transfer function of the annual sum filter in (1) is

$$G_{AS}(\omega) = 1 + e^{-i\omega} + e^{-2i\omega} + \ldots + e^{-11i\omega}.$$

Using the formula $(1 - z)(1 + z + z^2 + \ldots + z^{11}) = 1 - z^{12}$, we obtain

$$G_{AS}(\omega) = \begin{cases} 12, & \omega = 0 \\ \frac{1-e^{-i12\omega}}{1-e^{-i\omega}}, & \omega \neq 0. \end{cases} \tag{2}$$

Substituting $\omega = 2\pi\lambda$, we can obtain a formula for $G_{AS}(2\pi\lambda)$ that better reveals the gain and phase-shift functions. To do this, we re-express the denominator and numerator in (2) as

$$\begin{aligned} 1 - e^{-i2\pi\lambda} &= \left(e^{i2\pi\lambda/2} - e^{-i2\pi\lambda/2}\right) e^{-i2\pi\lambda/2} \\ &= 2i \sin(2\pi\lambda/2)\, e^{-i2\pi\lambda/2} \\ &= 2\sin(2\pi\lambda/2)\, e^{i2\pi(1/4-\lambda/2)}, \end{aligned} \tag{3}$$

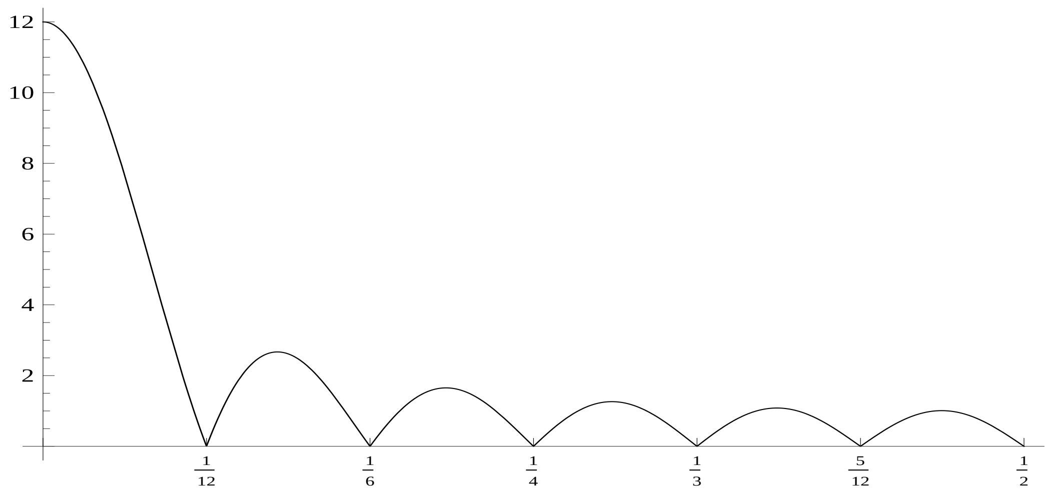

Figure 7: Gain Function of the Annual Sum Filter for $0 \leq \lambda \leq 1/2$

and

$$\begin{aligned} 1 - e^{-i12\times2\pi\lambda} &= \left(e^{i2\pi12\lambda/2} - e^{-i2\pi12\lambda/2}\right) e^{-i2\pi12\lambda/2} \\ &= 2i\sin\left(2\pi12\lambda/2\right) e^{-i2\pi12\lambda/2} \\ &= 2\sin\left(2\pi12\lambda/2\right) e^{i2\pi(1/4-12\lambda/2)}, \end{aligned} \tag{4}$$

respectively. Substitution into (2) yields

$$G_{AS}(2\pi\lambda) = \begin{cases} 12, & \lambda = 0 \\ \frac{\sin(2\pi12\lambda/2)}{\sin(2\pi\lambda/2)} e^{i2\pi(-11\lambda/2)}, & \lambda \neq 0, -1/2 < \lambda \leq 1/2 \end{cases}. \tag{5}$$

The gain function thus has the formula

$$|G_{AS}(2\pi\lambda)| = \begin{cases} 12, & \lambda = 0 \\ \left|\frac{\sin(2\pi12\lambda/2)}{\sin(2\pi\lambda/2)}\right|, & 0 < \lambda \leq 1/2 \end{cases} \tag{6}$$

Its graph in Figure 7 shows that it decreases to 0 at the fundamental seasonal frequency $\lambda = 1/12$ and its harmonics, $\lambda = k/12$, $k = 2, \ldots, 6$. This reveals that annual sums damp seasonal variations.

The formula (5) also immediately reveals the phase shift function of the annual sum filter for $0 < \lambda < 1/12$, which is adequate for cyclical analysis, because it covers all cycles of length greater than one year. Indeed, the sine functions in the formula (5) are both positive for $0 < \lambda < 1/12$, so their ratio is positive and coincides with the gain function over this interval. Consequently, the argument function in the exponential factor coincides with the phase shift function on this interval.

Specifically, for $0 \leq \lambda < 1/12$, (5) shows that the phase shift function for the annual sums filter is

$$\varphi_{AS}(2\pi\lambda) = 2\pi\left(-11\lambda/2\right),$$

in months

$$\frac{\varphi_{AS}(2\pi\lambda)}{2\pi\lambda} = -5.5, \tag{7}$$

(as a limit at $\lambda = 0$). Because this phase shift is constant, it need not be graphed. The frequency $\lambda = 1/12$ is excluded because the phase shift is not defined where the gain function is zero.

This result explains how annual differences reveal cyclical information later than monthly differences. It confirms the annual sum phase shift formula (D.4) of Findley and Martin (2006), stated without a detailed derivation. This reference also provides phase shift graphs of various seasonal adjustment filters that show how seasonal adjustments of recent data can exhibit phase shift.

Remark. The transfer functions of the monthly and annual difference filters are shown in (3) and (4), where they are factorized in a way that is analogous to (5) for the annual sum filter. Only the annual sum filter and its constant phase shift are relevant to the goals of this paper.

5 CONCLUSIONS

The annual difference is the sum of the twelve intervening monthly differences; hence, the phase shift of the annual differences relative to the phase shift of the monthly differences is simply that induced by the annual sum, which we have shown to be -5.5 months quite generally. From a practical point of view, this shows why an analyst who uses both monthly and annual differences may observe contradictory movements, especially right after a turning point. Comparing the current month's annual difference with the annual difference of the previous month may help to resolve such an apparent conflict.

Disclaimer. This article is released to inform interested parties and to encourage discussion of research. All opinions expressed are those of the authors and not necessarily those of Statistics Canada or the U.S. Census Bureau.

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