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Unit Root Properties of Seasonal Adjustment and Related Filters

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Abstract

Linear filters used in seasonal adjustment (model-based or from the X-11 method) contain unit root factors in the form of differencing operators and seasonal summation operators. The extent to which the various filters (seasonal, seasonal adjustment, trend, and irregular) contain these unit root factors determines whether the filters reproduce or annihilate (*i*) fixed seasonal effects, and (*ii*) polynomial functions of time. This paper catalogs which unit root factors are contained by the various filters for the most common approaches to model-based seasonal adjustment, and for X-11 seasonal adjustment with or without forecast extension. Both symmetric and asymmetric filters are considered.

Key Words: time series, ARIMA model, X-11 seasonal adjustment, trend estimation

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1 Introduction

Common approaches to model-based seasonal adjustment are based on linear filters. The same is true of the widely used X-11 method of seasonal adjustment in either the additive or log-additive modes (Ladiray and Quenneville 2001), and Young (1968) asserted this can be regarded as approximately true for X-11's multiplicative mode. These linear filters contain unit root factors in two forms that are of interest here. One is differencing operators, $(1-B)^d$ for some integer $d \geq 1$, where B is the backshift operator ($By_t = y_{t-1}$). The other is the seasonal summation operator, denoted here as $U(B) = 1 + B + \dots + B^{s-1}$, where s is the seasonal period. Filters could also include one or more of the individual polynomial factors of $U(B)$. Those for monthly ($s = 12$) filters are $1 - \sqrt{3}B + B^2$, $1 + \sqrt{3}B + B^2$, $1 - B + B^2$, $1 + B + B^2$, $1 + B^2$, and $1 + B$. Those for quarterly filters are $1 + B^2$ and $1 + B$.

Interest in the presence of unit root factors in filters stems from the fact that this determines whether given filters annihilate or reproduce polynomial functions of time (e.g., $\alpha_0 + \alpha_1 t$) or fixed seasonal effects. For polynomial functions of time, suppose $\omega_S(B) = \sum_i \omega_i B^i$ is a seasonal filter and $\omega_N(B) = 1 - \omega_S(B)$ is the complementary seasonal adjustment filter. If, for example, $\omega_S(B)$ contains $(1-B)(1-F) = -F(1-B)^2$, where $F = B^{-1}$ is the forward shift operator, then $\omega_S(B)$ annihilates a linear time trend (since $(1-B)^2[\alpha_0 + \alpha_1 t] = 0$), while $\omega_N(B)$ reproduces this function. More generally, if $\omega_S(B)$ contains $(1-B)^d$ for $d > 0$, then it will annihilate polynomials of degree up to $d-1$, and $\omega_N(B)$ will reproduce them.

Fixed seasonal effects constitute a deterministic pattern that repeats itself every year and also sums to 0 over any consecutive 12 months (or 4 quarters) of data. For monthly series this pattern can be expressed as $\sum_{i=1}^{11} \beta_i x_{it}$ where the x_{it} are the seasonal contrast variables defined as $x_{1t} = 1$ when t is a January, -1 when t is a December, and 0 otherwise; \dots ; $x_{11t} = 1$ when t is a November, -1 when t is a December, and 0 otherwise. Here $\beta_1, \dots, \beta_{11}$ are the January, \dots , November effects, and $\beta_{12} = -(\beta_1 + \dots + \beta_{11})$ is the December effect, which is defined so that the sum over any 12 months of the seasonal effects is 0. The fixed seasonal pattern can also be defined equivalently as $\sum_{j=1}^5 [\beta_{1j} \cos(2\pi jt/12) + \beta_{2j} \sin(2\pi jt/12)] + \beta_6 \cos(\pi t)$. Both of these parameterizations of fixed seasonal effects are available in the X-12-ARIMA seasonal adjustment program (Findley, et al. 1998; U.S. Census Bureau 2009). Analogous definitions are made for quarterly series.

For fixed seasonal effects the relevant question is whether a seasonal adjustment filter $\omega_N(B)$ annihilates these effects, so that the corresponding seasonal filter $\omega_S(B)$ reproduces them. This will be the case if $\omega_N(B)$ contains $U(B)$. As will be seen, some seasonal adjustment filters contain not just $U(B)$, but $U(B)U(F) = U(B)U(B)F^{11}$, i.e., they, in effect, contain $U(B)$ twice. Such seasonal adjustment filters annihilate not just fixed seasonal effects, but any deterministic function ξ_t such that $U(B)U(B)\xi_t = 0$. The nature of such a ξ_t follows from results on solutions to homogeneous difference equations (Goldberg 1986), and in this case ξ_t can be shown to be a seasonal pattern whose amplitude grows linearly over time. As models to account for such a pattern have been very uncommon in the time series literature (Kitagawa and Gersch (1984) being one exception), we shall not elaborate further

on such patterns, though we will note when seasonal adjustment filters do, in fact, include $U(B)U(F)$ rather than just $U(B)$. There are also seasonal adjustment filters (X-11 symmetric filters) that contain $U(B)$ and an additional factor of $U(F)$, but not the full $U(B)U(F)$.

Trend and irregular filters generally include the $U(B)$ operator and, in some cases, $U(F)$ or some of its factors. Irregular filters generally also include $(1 - B)^d$ for some $d > 0$.

This paper catalogs results on unit root factors in seasonal, seasonal adjustment, trend, and irregular filters used by various proposed model-based approaches to seasonal adjustment, and in such filters used by the X-11 method. Both symmetric and asymmetric filters are considered. The results are presented in a series of tables listing the unit root factors in the various filters along with the deterministic functions (polynomials of what degree or fixed seasonal effects) that are annihilated and that are reproduced by these filters. Some of the results have no doubt been noted before for specific cases, and the results for model-based filters are obvious from their formulas. The general results given here for X-11 filters appear to be new. The focus here, though, is not so much on derivation of the new results, but rather on collecting the complete set of results for all these filters.

Section 2 reviews the general form of seasonal time series models that have been used both for developing model-based seasonal adjustment methods and for forecast extension in the X-11-ARIMA (Dagum 1980) and X-12-ARIMA programs. The key feature of these models that is of interest here is their use of nonseasonal and seasonal differencing operators, yielding differencing of the observed series by $(1 - B)^{d-1}(1 - B^s) = (1 - B)^d U(B)$ for some $d > 0$. Section 2 also provides two theorems on reproduction of deterministic functions (polynomials and fixed seasonal effects) when forecasting from such models. These results are used in the subsequent sections to obtain results on unit root properties of asymmetric filters.

Section 3 presents results on unit root factors in model-based filters, both symmetric and asymmetric, and based on either infinite or finite amounts of data. Section 4 considers symmetric X-11 filters, Section 5 considers asymmetric filters obtained by applying X-11 symmetric filters to a series with full forecast and backcast extension, and Section 6 considers the original X-11 asymmetric filters. Section 7 provides an illustration of the results for X-11 filters, while Section 8 summarizes the general conclusions. Two appendices provide derivations of the results for X-11 filters.

In what follows we usually assume the case of a monthly time series ($s = 12$). Corresponding results for quarterly series are generally either the same as those for monthly series, or follow from the latter with obvious modifications (e.g., replace 12 by 4). When the modifications needed for quarterly series are not obvious, they will be noted.

2 Time Series Models Used in Seasonal Adjustment

The additive decomposition used for seasonal adjustment is

$$y_t = S_t + T_t + I_t \quad (1)$$

where y_t is the observed time series (possibly after transformation, e.g., taking logarithms), and S_t , T_t , and I_t are the seasonal, trend, and irregular components. We also let $N_t = T_t + I_t = y_t - S_t$ denote the nonseasonal component, the estimate of which is known as the seasonally adjusted series. Most of the models that have been proposed for model-based seasonal adjustment use component models that can be written in the following form:

$$\begin{aligned} U(B)S_t &= u_t \\ (1-B)^d T_t &= v_t \\ I_t &\sim i.i.d. N(0, \sigma_I^2) \end{aligned} \quad (2)$$

where u_t and v_t have mean zero for all t and are independent of each other and of I_t . Typically, u_t and v_t follow stationary autoregressive-moving average (ARMA) Gaussian models (Box and Jenkins 1970), though particulars of the models for u_t and v_t are not needed here, for the most part. We require only that u_t and v_t be stationary with autocovariance functions $\gamma_u(k) = \text{Cov}(u_t, u_{t+k})$ and $\gamma_v(k) = \text{Cov}(v_t, v_{t+k})$ that are absolutely summable, i.e., $\sum_{k=-\infty}^{\infty} |\gamma_u(k)| < \infty$ and $\sum_{k=-\infty}^{\infty} |\gamma_v(k)| < \infty$. This summability condition is satisfied by stationary ARMA models. We let $\gamma_u(B) = \sum_{k=-\infty}^{\infty} \gamma_u(k)B^k$ and $\gamma_v(B) = \sum_{k=-\infty}^{\infty} \gamma_v(k)B^k$ denote the autocovariance generating functions (ACGFs) of u_t and v_t . The ACGF of I_t is just σ_I^2 . For results on estimation of N_t , we also define

$$z_t \equiv (1-B)^d N_t = v_t + (1-B)^d I_t,$$

which has ACGF $\gamma_z(B) = \gamma_v(B) + (1-B)^d(1-F)^d \sigma_I^2$.

The Gaussian assumption made above is not essential. Without it, forecasting and signal extraction results given later are interpretable as linear projections, though not as conditional expectations. Also, we could extend (2) to let I_t follow a stationary and invertible ARMA model (instead of requiring it to be white noise, i.e., independent and identically distributed over t). This extension would not alter the results presented here.

With the component models as given in (2), we can write the general model for y_t as

$$\begin{aligned} w_t \equiv (1-B)^{d-1}(1-B^{12})y_t &= (1-B)^d U(B)y_t \\ &= (1-B)^d u_t + U(B)v_t + (1-B)^d U(B)I_t \end{aligned} \quad (3)$$

where w_t is the differenced observed series, with autocovariances $\gamma_w(k) = \text{Cov}(w_j, w_{j+k})$ and ACGF $\gamma_w(B) = \sum_{k=-\infty}^{\infty} \gamma_w(k)B^k$ given by

$$\gamma_w(B) = (1-B)^d(1-F)^d \gamma_u(B) + U(B)U(F)\gamma_v(B) + (1-B)^d U(B)(1-F)^d U(F)\sigma_I^2.$$

The model framework of equations (1)–(3) covers the canonical ARIMA (autoregressive-integrated-moving average) model-based approach to seasonal adjustment as developed in Hillmer and Tiao (1982) and Burman (1980), and implemented in the TRAMO-SEATS software of Gomez and Maravall (1997). (Note that this is the most widely used among model-based procedures used for official seasonal adjustments.) It also covers the structural components models of Harvey (1989), Durbin and Koopman (2001), and Kitagawa and Gersch (1984) (apart from the slight extension in the latter that was mentioned in the Introduction). Also, the ARIMA models used for forecast extension in the X-11-ARIMA and X-12-ARIMA programs are of the general form of (3). Regression terms are often added to these models to account for trading-day and other effects. This leads to the RegARIMA models used by X-12-ARIMA (Findley et al. 1998) and the RegComponent models discussed in Bell (2004). As regression effects do not usually affect the unit root properties of the seasonal adjustment methods examined here, we shall not, with one exception, bother including them in the models we present here. The one exception involves trend constants—an overall nonzero mean for the differenced series, w_t . Trend constants do affect the degree of polynomials annihilated and reproduced by some of the various filters, so these are explicitly considered.

We now establish two results on forecasting with models of the general form given by (3). Let $\delta(B) = 1 - \delta_1 B - \dots - \delta_{d+11} B^{d+11} = (1 - B)^{d-1}(1 - B^{12})$. Bell (2004, Section 12.3.3) notes that minimum mean squared error (MMSE) forecasts $\hat{y}_{t|n}$ of y_t for $t > n$ from finite data $\mathbf{y} = (y_1, \dots, y_n)'$ satisfy

$$\hat{y}_{t|n} = \delta_1 \hat{y}_{t-1|n} + \dots + \delta_{d+11} \hat{y}_{t-d-11|n} + \hat{w}_{t|n} \quad (4)$$

where $\hat{y}_{j|n} = y_j$ for $j = 1, \dots, n$, and $\hat{w}_{t|n} = E(w_t | \mathbf{w})$ is the MMSE forecast of w_t under the model (3) given the differenced observed series $\mathbf{w} = (w_{d+12}, \dots, w_n)'$. The result (4) follows under Assumption A of Bell (1984) about starting values for the series y_t , and is consistent with the standard approach to forecasting nonstationary time series used, for example, in Box and Jenkins (1970). (Though Bell (2004) presents the result (4) for forecasting from RegComponent models with finite data, this result, and thus Theorems 1 and 2 below, apply to more general models of the form of (3). They also apply to forecasting from semi-infinite data $\{y_j \text{ for } j \leq n\}$, subject to additional assumptions given in Bell (2004) about how the y_j extending into the infinite past were generated.) Theorem 1 establishes what happens if the forecast procedure defined by (4) is applied to any deterministic function ξ_t that is annihilated by $\delta(B)$. Note that in doing this the model (3) is taken as given, so the forecast procedure does not involve fitting (3) to ξ_t taken as data.

Theorem 1: Forecasting via (4) with the model given by (3) reproduces any deterministic function ξ_t that is annihilated by $\delta(B)$. For $\delta(B) = (1 - B)^d U(B)$, these ξ_t include (a) polynomials in t of degree less than d , (b) fixed seasonal effects, and (c) linear combinations of these two. Furthermore, higher order deterministic functions, such as polynomials of degree d or more, are not reproduced.

Proof: To check if forecasting reproduces a deterministic function ξ_t , we set $y_t^* = \xi_t$,

apply (4) to $\mathbf{y}^* = (y_1^*, \dots, y_n^*)'$, and see if $\hat{y}_{t|n}^* = \xi_t$ for $t > n$. Notice that (4) is the difference equation $\delta(B)\hat{y}_{t|n}^* = \hat{w}_{t|n}^*$ with starting values $\hat{y}_{j|n}^* = y_j^* \equiv \xi_j$ for $j = n - d - 10, \dots, n$. We also set $w_t^* = \delta(B)y_t^* = \delta(B)\xi_t$ for all t , which can be thought of as a difference equation that determines ξ_t for $t > n$ given w_t^* for $t > n$ and starting values ξ_j for $j = n - d - 10, \dots, n$. Since these two difference equations have the same form, with the same operator $\delta(B)$, and the same starting values, their solutions for $t > n$ will be the same, and hence forecasting via (4) will reproduce ξ_t if and only if $\hat{w}_{t|n}^* \equiv \delta(B)\xi_t$ for $t > n$, i.e., if and only if forecasting w_t^* reproduces $\delta(B)\xi_t$. From standard results on conditional expectation in a multivariate normal distribution, $E(w_t|\mathbf{w}) = [\gamma_w(t-1), \dots, \gamma_w(t-n)]\Sigma_w^{-1}\mathbf{w}$ where $\Sigma_w = \text{Var}(\mathbf{w})$. So we need to check if

$$[\gamma_w(t-1), \dots, \gamma_w(t-n)]\Sigma_w^{-1}\mathbf{w}^* = w_t^* \equiv \delta(B)\xi_t \quad \text{for } t > n? \quad (5)$$

where $\mathbf{w}^* = (w_{d+12}^*, \dots, w_n^*)'$. Notice that $\Sigma_w^{-1}\mathbf{w}^*$ is a vector with finite entries that do not depend on t . By the assumed summability of $\gamma_w(k)$, $\gamma_w(k) \rightarrow 0$ as $k \rightarrow \infty$, so that the left-hand-side of (5) goes to 0 as $t \rightarrow \infty$. Hence, for (5) to hold, it must be the case that $w_t^* \equiv \delta(B)\xi_t \rightarrow 0$ as $t \rightarrow \infty$.

Suppose first that ξ_t is such that $w_t^* \equiv \delta(B)\xi_t = 0$ for all t . Then, (5) is satisfied, and such ξ_t are reproduced in forecasting. This establishes results (a)–(c).

Now suppose that $\xi_t = t^h$ for $h \geq d$. From results given in Conte (1965, pp. 84-85), $(1-B)^{d+h}$ is a polynomial of degree $h-d$ with leading term $\frac{h!}{(h-d)!}t^{h-d}$. If $h = d$, $\delta(B)\xi_t = U(B)(1-B)^{d+d} = 12h!$ for all t , which does not go to 0, so (5) will not hold. If $h > d$, then $\delta(B)\xi_t = U(B)(1-B)^{d+h}$ is a polynomial in t of degree $h-d$ with leading term $\frac{12h!}{(h-d)!}t^{h-d}$, so $w_t^* \equiv \delta(B)\xi_t \rightarrow \infty$ as $t \rightarrow \infty$, and again (5) will not hold. Thus, polynomials of degree d or more are not reproduced in forecasting from (4). An analogous argument applies for deterministic functions ξ_t such that $U(B)^2\xi_t = 0$ but $U(B)\xi_t \neq 0$ (or similarly for individual factors of $U(B)$). Thus, the theorem is established.

Since the backward model for y_t has the same form as the usual (forward) model, just with F replacing B in the AR, differencing, and MA operators (Box and Jenkins 1970, pp. 197-198), the results of the theorem also hold for MMSE backcasting.

Models of the general form (3) can be extended by addition of a trend constant, which is a nonzero mean μ_w for the differenced series w_t :

$$(1-B)^d U(B)y_t \equiv w_t = \mu_w + \tilde{w}_t \quad (6)$$

where $\tilde{w}_t = w_t - \mu_w$ has $E(\tilde{w}_t) = 0$, and now \tilde{w}_t follows the model given for w_t by (3). To obtain forecasts from this model, we modify (4) to

$$\hat{y}_{t|n} = \mu_w + \delta_1 \hat{y}_{t-1|n} + \dots + \delta_{d+11} \hat{y}_{t-d-11|n} + \tilde{w}_{t|n} \quad (7)$$

where $\tilde{w}_{t|n} = E(\tilde{w}_t|\mathbf{w})$. We also need to substitute an estimate of μ_w into (7). We could use the sample mean, \bar{w} , of w_t , or a generalized least squares (GLS) estimate

$$\hat{\mu}_w = (\mathbf{1}'\Sigma^{-1}\mathbf{w})/(\mathbf{1}'\Sigma^{-1}\mathbf{1}) \quad (8)$$

where $\mathbf{1}' = (1, \dots, 1)$ and Σ is any positive definite covariance matrix. Setting $\Sigma \propto I$ gives the least squares estimate, \bar{w} . Setting $\Sigma = \Sigma_w$ gives the optimal GLS estimate under model (6).

For the model (6), we have the following analog to Theorem 1. Note that the model defined by (6) and (3) is taken as given except for the estimation of μ_w by (8), which is regarded as part of the forecast procedure. (The use of (8) is for finite data; with data extending into the infinite past, μ_w would be taken as known.)

Theorem 2: Forecasting via (7) with the model given by (6) reproduces (a) polynomials in t up to degree d , (b) fixed seasonal effects, and (c) linear combinations of these two.

Proof: Let $\xi_t = \alpha_0 + \alpha_1 t + \dots + \alpha_d t^d$ and set $y_t^* = \xi_t$. From results of Conte (1965) cited earlier, $w_t^* \equiv \delta(B)\xi_t = U(B)(1-B)^d \xi_t = 12\alpha_d(d!)$ for all t . Thus, $\mu_{w^*} = 12\alpha_d(d!)$, $w^* = 12\alpha_d(d!)\mathbf{1}$, and, for any positive definite Σ , (8) gives $\hat{\mu}_{w^*} = 12\alpha_d(d!) = \mu_{w^*}$. Since $\tilde{w}_t^* \equiv w_t^* - \mu_{w^*} = 0$ for all t , $\tilde{w}_{t|n}^* = 0$ for all t . Analogous to the proof of Theorem 1, we now have $\delta(B)\hat{y}_{t|n}^* = \mu_{w^*}$ and $\delta(B)\xi_t \equiv w_t^* = \mu_{w^*}$ for $t > n$, two difference equations of the same form and with the same starting values ($\hat{y}_{j|n}^* = y_j^* = \xi_j$ for $j = n - d - 10, \dots, n$). So $\hat{y}_{t|n}^* = \xi_t$ for $t > n$ and result (a) holds. If, instead, ξ_t is a sequence of fixed seasonal effects ($U(B)\xi_t = 0$), then $w_t^* = 0$ for all t and the above argument holds with $\hat{\mu}_{w^*} = 0$. Thus, result (b) holds. If $\xi_t = \xi_{1t} + \xi_{2t}$, where $\xi_{1t} = \alpha_0 + \alpha_1 t + \dots + \alpha_d t^d$ and $U(B)\xi_{2t} = 0$, then $w_t^* = 12\alpha_d(d!)$ for all t , and the above argument implies that result (c) holds.

Theorem 2 can be extended to a model where $E(w_t) = \alpha_0 + \alpha_1 t + \dots + \alpha_h t^h$ to show that forecasting with this model reproduces polynomials up to degree $h + d$. The simplest version of this would have $h = d = 1$, which implies a quadratic time trend in the data. As use of quadratic or higher order polynomial trends should be very unusual in practice, we shall not pursue this additional generality here.

3 Unit Root Properties of Model-Based Filters

These results follow directly from expressions for the signal extraction estimates that form the basis of model-based seasonal adjustment. We consider the results for three cases defined by the amount of data used in the signal extraction: the doubly infinite realization $\{y_t \text{ for } t = -\infty, \dots, \infty\}$; the semi-infinite realization $\{y_t \text{ for } t \leq n\}$ for some finite n ; and the finite vector of observations $\mathbf{y} = (y_1, \dots, y_n)'$. The first of these leads to symmetric infinite filters, the second to asymmetric infinite filters, and the third to finite filters (of which at most one will be symmetric).

Bell (1984) presents results on signal extraction with a doubly infinite realization of y_t for models of the form of (1)–(3). The MMSE linear signal extraction estimate of S_t given $\{y_t \text{ for } t = -\infty, \dots, \infty\}$ is

$$\hat{S}_t = \omega_S(B)y_t \quad \text{where} \quad \omega_S(B) = \frac{\gamma_u(B)}{\gamma_w(B)}(1-B)^d(1-F)^d. \quad (9)$$

Analogous to (9), the linear filters for the MMSE estimates of N_t , T_t , and I_t are

$$\omega_N(B) = \frac{\gamma_z(B)}{\gamma_w(B)}U(B)U(F) \quad (10)$$

$$\omega_T(B) = \frac{\gamma_v(B)}{\gamma_w(B)}U(B)U(F) \quad (11)$$

$$\omega_I(B) = \frac{\sigma_I^2}{\gamma_w(B)}U(B)U(F)(1-B)^d(1-F)^d. \quad (12)$$

It can be shown that $\omega_N(B) = 1 - \omega_S(B) = \omega_T(B) + \omega_I(B)$.

The unit root factors appearing in the model-based filters given in (9)–(12), and their implications, are summarized in Table 1 below.

Table 1. Unit Root Factors in Symmetric Infinite Model-Based Filters

Filter	Unit root factors	Annihilates	Reproduces
$\omega_S(B)$	$(1-B)^d(1-F)^d$	polynomials of degree $<2d$	fixed seasonal effects
$\omega_N(B)$	$U(B)U(F)$	fixed seasonal effects	polynomials of degree $<2d$
$\omega_T(B)$	$U(B)U(F)$	fixed seasonal effects	polynomials of degree $<2d$
$\omega_I(B)$	$U(B)U(F)(1-B)^d(1-F)^d$	fixed seasonal effects and polynomials of degree $<2d$	—

As noted in the Introduction, we could replace “fixed seasonal effects” in Table 1 by the more general “deterministic functions ξ_t such that $U(B)U(F)\xi_t = 0$,” but this additional generality is not of much interest.

The most common case in model-based seasonal adjustment has $d = 2$. For this case we see from Table 1 that the symmetric nonseasonal and trend filters will reproduce cubic polynomials of t . For the less common case of $d = 1$, the symmetric nonseasonal and trend filters will reproduce only linear functions of time. Other values of d are extremely uncommon.

Notice from Table 1 that the various filters all include double the unit root factors needed to remove nonstationarities present in the other components according to the model (2). Thus, $\omega_S(B)$ contains not just the $(1-B)^d$ needed to remove the nonstationarities in the trend component, it contains $(1-B)^d(1-F)^d$. Similarly, $\omega_N(B)$ and $\omega_T(B)$ contain not just the $U(B)$ needed to remove the seasonal component nonstationarities, they contain $U(B)U(F)$. Finally, while $U(B)(1-B)^d$ alone would remove both the seasonal and trend nonstationarities, $\omega_I(B)$ contains $U(B)U(F)(1-B)^d(1-F)^d$.

In the context of time series modeling, application of more differences than needed to render a series stationary is termed “overdifferencing” (Harvey 1981). The overdifferenced series follows a model that includes $(1-B)^k$ as an MA polynomial, where k is the excessive number of differences applied. Application of an extra $U(B)$ factor produces $U(B)$ as an MA polynomial in the model for the filtered series. This overdifferencing by symmetric filters is worth noting if one considers time series modeling of estimated components. (Such modeling faces other issues, however, including

nonstationarities induced by end effects from the different asymmetric filters applied at different time points. See Bell (1995) for related discussion.)

In the context of seasonal adjustment, the overdifferencing implicit in the results of Table 1 implies that the spectral densities of \hat{S}_t and \hat{I}_t will have zeros at frequency zero, while the spectra of \hat{N}_t , \hat{T}_t , and \hat{I}_t will have zeros at the seasonal frequencies ($2\pi j/12$ for $j = 1, \dots, 6$). The latter can be called “overadjustment,” a term that refers more generally to dips at the seasonal frequencies (not necessarily to zero) in the spectra of \hat{N}_t , \hat{T}_t , or \hat{I}_t . Evidence of overadjustment (from examination of estimated spectra of estimated components) has long been considered as potentially indicative of problems with the seasonal adjustment. For example, Granger (1978a) proposed absence of dips or peaks in the spectrum of \hat{N}_t as one criterion for a good seasonal adjustment. However, Sims (1978) and Tukey (1978), in discussing Granger’s paper, both pointed out that this was an unrealistic criterion because such “overadjustment” simply follows (for model-based adjustment) as a consequence of MMSE prediction (as reflected by the results in Table 1). This led Granger (1978b), in response, to reverse himself and agree with their conclusion that overadjustment does not necessarily indicate a problem. In any case, any assessment of the spectral properties of the estimated components from a model-based adjustment should take into account the results of Table 1. In doing this, note that estimated components from long but finite series will (approximately) show properties of symmetric infinite filtering in the center of the series, but will show properties of asymmetric infinite filtering nearer the ends (see Table 2 below), so that the estimated spectra of \hat{N}_t , \hat{T}_t , or \hat{I}_t will show a mixture of these properties. This may produce dips, though not actual zeros, at the seasonal frequencies.

Something not obvious from the filter expressions in (9)–(12) is whether the filters contain additional unit root factors beyond those listed in Table 1. From (12), the only additional factors in the numerator of $\omega_I(B)$ would come from factors in the denominator of $\gamma_w(B)$. These would be AR operators in the model for w_t , for which unit root factors are ruled out by the stationarity assumption. Thus, the only unit root factors in $\omega_I(B)$ are $U(B)U(F)(1 - B)^d(1 - F)^d$. For $\omega_S(B)$, $\omega_N(B)$, and $\omega_T(B)$, additional unit root factors are possible if they appear in the MA polynomials of the ARIMA models for S_t , N_t , or T_t . For example, Hillmer and Tiao (1982, p. 67) examine a model for which the canonical trend component indeed has a factor of $(1 + B)$ in its MA polynomial. On the other hand, the spectral density of $z_t = (1 - B)^d N_t$, which is given by $(2\pi)^{-1} \gamma_z(e^{i\lambda}) = (2\pi)^{-1} \{ \gamma_v(e^{i\lambda}) + |1 - e^{i\lambda}|^{2d} \sigma_I^2 \}$ for $\lambda \in [-\pi, \pi]$, will be positive for all λ as long as $\gamma_v(1) > 0$ and $\sigma_I^2 > 0$. This would imply no additional unit roots in $\omega_N(B)$. Apart from these results, potential additional unit root factors in the filters considered can obviously be examined for any particular model, but as general results are difficult to give, we shall not seek further results of this sort here or in the subsequent sections.

Bell and Martin (2004) obtained expressions for asymmetric MMSE signal extraction filters with a semi-infinite sample for ARIMA component models. For estimating S_{n-m} and N_{n-m} using data $\{y_t \text{ for } t \leq n\}$, the optimal filters can be written (Bell

and Martin 2004, eq. (10))

$$\begin{aligned}\omega_S^{(m)}(B) &= (1-B)^d F^m \frac{\sigma_1^2 r_S^{(m)}(B)}{\sigma_a^2 \theta(B)} \\ \omega_N^{(m)}(B) &= U(B) F^m \frac{\sigma_2^2 r_N^{(m)}(B)}{\sigma_a^2 \theta(B)}\end{aligned}\tag{13}$$

where σ_1^2 and σ_2^2 are the innovation variances in the ARIMA models for S_t and N_t , and $\theta(B)$ and σ_a^2 are the MA polynomial and innovation variance in the ARIMA model for y_t . Also, $r_S^{(m)}(B)$ and $r_N^{(m)}(B)$ are finite polynomials in B that depend on m , the distance from the time point of interest to the end of the series. They can be determined as discussed by Bell and Martin. We see that $\omega_S^{(m)}(B)$ includes the factor $(1-B)^d$, while $\omega_N^{(m)}(B)$ includes the factor $U(B)$. The analogous results for the asymmetric trend and irregular filters show that these include as factors $(1-B)^d$ and $(1-B)^d U(B)$, respectively. Note that these results all apply for any value of m .

Kohn and Ansley (1987) and Bell and Hillmer (1988) (see also Bell (2004)) provided signal extraction results with a finite sample for models of the form of (1)–(3). Kohn and Ansley developed results for use with a modified Kalman filter and a smoother, while Bell and Hillmer provided results in the form of matrix expressions. McElroy (2005) simplified Bell and Hillmer’s results to provide explicit expressions for the “filter matrix,” each of whose rows provides the weights for the filter producing the signal extraction estimate at a particular time point. Let the signal extraction estimate of the vector $\mathbf{S} = (S_1, \dots, S_n)'$ using $\mathbf{y} = (y_1, \dots, y_n)'$ be $\hat{\mathbf{S}} = \Omega_S \mathbf{y}$. McElroy’s results show that

$$\Omega_S = [\Delta'_S \Sigma_u^{-1} \Delta_S + \Delta'_N \Sigma_z^{-1} \Delta_N]^{-1} \Delta'_N \Sigma_z^{-1} \Delta_N\tag{14}$$

where $\Sigma_u = \text{Var}(\mathbf{u})$, $\Sigma_z = \text{Var}(\mathbf{z})$, Δ_S is the $(n-11) \times n$ matrix such that $\Delta_S \mathbf{S} = \mathbf{u} = (u_{12}, \dots, u_n)'$ (i.e., the rows of Δ_S contain the coefficients of $U(B)$, which are 12 ones, in positions that shift left to right as one goes down the rows), and Δ_N is the $(n-d) \times n$ differencing matrix such that $\Delta_N \mathbf{N} = \mathbf{z} = (z_{d+1}, \dots, z_n)'$ (i.e., the rows of Δ_N contain the coefficients of the expansion of $(1-B)^d$).

Since the last matrix in (14) is Δ_N , the finite filter for signal extraction estimation of S_t for any $t = 1, \dots, n$ contains $(1-B)^d$. Analogous results for the other components show that the finite filters for estimating T_t and N_t both contain $U(B)$, while those for estimating I_t contain $(1-B)^d U(B)$. These results on unit root factors in the finite signal extraction filters agree with those obtained by Bell and Martin (2004) for semi-infinite signal extraction filters. These results and their implications are summarized in Table 2.

Table 2. Unit Root Factors in Asymmetric Model-Based Filters

Filter	Unit root factors	Annihilates	Reproduces
$\omega_S^{(m)}(B)$	$(1 - B)^d$	polynomials of degree $< d$	fixed seasonal effects
$\omega_N^{(m)}(B)$	$U(B)$	fixed seasonal effects	polynomials of degree $< d$
$\omega_T^{(m)}(B)$	$U(B)$	fixed seasonal effects	polynomials of degree $< d$
$\omega_I^{(m)}(B)$	$U(B)(1 - B)^d$	fixed seasonal effects and polynomials of degree $< d$	—

Contrasting these results with those of Table 1, we see that the asymmetric and finite filters contain only half the unit root factors of the symmetric filters. Thus, $\omega_N^{(m)}(B)$, $\omega_T^{(m)}(B)$, and $\omega_I^{(m)}(B)$ annihilate, while $\omega_S^{(m)}(B)$ reproduces, fixed seasonal effects, but not the more general deterministic function ξ_t such that $U(B)^2\xi_t = 0$ that was noted for the corresponding symmetric filters. Also, $\omega_S^{(m)}(B)$ and $\omega_I^{(m)}(B)$ annihilate, while $\omega_N^{(m)}(B)$ and $\omega_T^{(m)}(B)$ reproduce, polynomials only up to degree $d - 1$, rather than polynomials of degree up to $2d - 1$ as for the corresponding symmetric filters. Thus, for $d = 2$ these are linear functions of time (rather than cubic polynomials in t), and for $d = 1$ these are just the constant function (rather than linear functions of t).

The unit root factors shown in Table 2 are just those needed to remove nonstationarities present in the other components according to the model (2). Thus, in contrast to the results for symmetric infinite filters, no “overdifferencing” or “overadjustment” occurs with the asymmetric model-based filters.

A couple additional points are worth noting. One is that the results in Table 2 cover the case of symmetric finite filters, i.e., the filters used to produce signal extraction estimates at $t = (n + 1)/2$ for n odd, though with one qualification. The one qualification comes from an observation by Findley and Martin (2006, p. 29) that symmetric finite filters that include a $1 + B$ factor must also include a $1 + F$ factor ($1 + B$ twice). Thus, for this particular case, the $U(B)$ factors in Table 2 become $U(B)(1 + F)$. Another possible exception to the results of Table 2 could occur if the “infinite” filters given by (9)–(12) actually turned out to be finite, but this would require unusual and unrealistic restrictions on the models (including no MA operator in the model (3), a condition violated by all the common model-based approaches to seasonal adjustment). Thus, the reduction in unit root factors from Table 1 to Table 2 is essentially due to the fact that the symmetric infinite filters given by (9)–(12) extend beyond the reach of the data (which is also what leads to asymmetry of filters.) This point leads us to consider an alternative approach to doing asymmetric signal extraction with semi-infinite or finite samples.

Unit root factors in finite model-based filters follow from matrix expressions given by McElroy (2009). These results turn out to be the same as in Table 2 (with one qualification noted shortly), so all the remarks of the preceding paragraph apply. Thus, the reduction in unit root factors from Table 1 to Table 2 is essentially due to the fact that the symmetric infinite filters extend beyond the reach of the data (which

is also what leads to asymmetric filters.) The one qualification involves the case of symmetric finite filters, i.e., the filters used to produce signal extraction estimates at $t = (n + 1)/2$ for n odd. For this particular case, Findley and Martin (2006, p. 29) observe that such filters that include a $1 + B$ factor must also include a $1 + F$ factor ($1 + B$ twice), so that the $U(B)$ factors in Table 2 then become $U(B)(1 + F)$.

The alternative approach to asymmetric signal extraction is to forecast (and back-cast) extend the series as far as necessary to apply the symmetric filters given by (9)–(12). This approach was proposed by Cleveland (1972), who noted that it produces $E(S_t|\mathbf{y})$, etc., the MMSE estimators based on the finite data \mathbf{y} . As the symmetric model-based filters can generally be assumed to be infinite, this approach would seem to require an infinite number of forecasts and backcasts for exact calculations. However, an algorithm suggested by G. Tunncliffe-Wilson (reported in Burman (1980)) provides a way to get the exact results with a finite set of calculations. We now review how consideration of this alternative approach can be used to infer the results in Table 2. We do so because this is fairly instructive, and because this approach is also used with X-11 filters in Section 5.

Consider asymmetric signal extraction to estimate S_t . From Theorem 1 given in Section 2, forecast extension via (4) reproduces fixed seasonal effects. Subsequent application of the symmetric seasonal filter $\omega_S(B)$ also reproduces fixed seasonal effects, hence, the combination of the two procedures, which performs asymmetric signal extraction to estimate S_t , reproduces fixed seasonal effects. Similarly, while application of the symmetric seasonal adjustment filter $\omega_N(B)$ reproduces polynomials up to degree $2d - 1$, forecast extension via (4) reproduces polynomials only up to degree $d - 1 < 2d - 1$. Hence, the combination of these two procedures will reproduce polynomials only up to degree $d - 1$. The other results in Table 2 can be similarly inferred. Notice that the limiting factor here is what is reproduced by the forecast extension, since the symmetric signal extraction filters reproduce polynomials of higher degree (and the more general version of fixed seasonal effects discussed earlier) than does the forecast extension. This fact will be relevant to the derivation in Section 5 of unit root properties of asymmetric filters obtained from X-11 symmetric filters with full forecast and backcast extension.

If the model with a trend constant, (6), is used, we would revise (2) to $(1 - B)^d T_t \equiv v_t = \mu_v + \tilde{v}_t$, and note that $\mu_w = E(w_t) = E[U(B)v_t + (1 - B)^d U(B)I_t] = U(B)E(v_t) = 12\mu_v$. We can then think of $E(y_t) = [\mu_w / (12(d!))]t^d$, since this gives $E[\delta(B)y_t] = U(B)(1 - B)^d E(y_t) = \mu_w$ as desired. We could add lower degree polynomial terms to $E(y_t)$, but these are not identifiable in the model (6) since they get differenced out. We can write the semi-infinite sample signal extraction estimate of S_t from (13) as

$$\begin{aligned} \hat{S}_t &= (1 - B)^d F^m \frac{\sigma_{1r_S}^{2(m)}(B)}{\sigma_a^2 \theta(B)} [y_t - E(y_t)] \\ &= F^m \frac{\sigma_{1r_S}^{2(m)}(B)}{\sigma_a^2 \theta(B)} [(1 - B)^d y_t - \mu_w / 12] \end{aligned} \quad (15)$$

since $(1 - B)^d E(y_t) = \mu_w / 12$. If we apply (15) setting $y_t \equiv \xi_t = \alpha_0 + \alpha_1 t + \dots + \alpha_d t^d$, then we have $(1 - B)^d y_t = \alpha_d (d!)$ and $\mu_w = 12\alpha_d (d!)$ (note proof of Theorem 2 in

Section 2), so $\hat{S}_t = 0$. Similar reasoning applies to finite sample signal extraction. So asymmetric model-based signal extraction estimation of S_t (and similarly I_t) from the trend constant model (6) annihilates polynomials up to degree d , not just degree $d - 1$, while the corresponding signal extraction estimation of N_t and T_t reproduces polynomials of degree d . Hence, for the trend constant model (6), in Table 2 we can replace d by $d + 1$ everywhere. Symmetric signal extraction estimation of S_t and I_t (from a doubly infinite realization of y_t) with model (6) still annihilates polynomials up to degree $2d - 1 \geq d$, since the symmetric filter $\omega_S(B)$ contains $(1 - B)^d(1 - F)^d$ (and hence $E(y_t) = [\mu_w/(12(d!))]t^d$ is simply annihilated by $\omega_S(B)$ in computing \hat{S}_t .)

4 Unit Root Properties of X-11 Symmetric Filters

Wallis (1974) lists the filtering steps used in X-11 seasonal adjustment with the additive decomposition (1) (and the log-additive decomposition when y_t is a logged series). See also Ladiray and Quenneville (2001, Section 2.4). For monthly series these are:

1. Detrend y_t by subtracting a 2×12 MA (a 2-term MA of a 12-term MA).
2. Take a first seasonal MA (e.g., a 3×3) of the result as a preliminary estimate of S_t .
3. Adjust the preliminary seasonal to sum more nearly to 0 over 12 months by subtracting a 2×12 MA.
4. Subtract the result of step 3 from y_t to get a preliminary seasonally adjusted series.
5. Subtract a Henderson trend MA of this from y_t for a more refined detrending than at step 1.
6. Apply a second seasonal MA (e.g., a 3×5) to the result of step 5.
7. Adjust the result of step 6 as in step 3 by subtracting a 2×12 MA – the result, \hat{S}_t^{X11} , estimates S_t .
8. The seasonally adjusted series is $\hat{N}_t^{X11} = y_t - \hat{S}_t^{X11}$; the trend estimate, \hat{T}_t^{X11} , is obtained by applying a Henderson trend MA to \hat{N}_t^{X11} ; and the irregular estimate is $\hat{I}_t^{X11} = \hat{N}_t^{X11} - \hat{T}_t^{X11}$.

The above sequence of steps is potentially applied three times, but the first two cycles of this are used only as part of adjusting the series for extreme values and for trading-day variation. Thus, the above gives the basic X-11 filtering steps as implemented in the X-11-ARIMA program (Dagum 1980) and in the X-12-ARIMA program (Findley et al. 1998, U.S. Census Bureau 2009). The example seasonal MAs mentioned at steps 3 and 6 are the default choices of the original X-11 program (Shiskin, Young,

and Musgrave 1967), though the default option in X-11-ARIMA and X-12-ARIMA is to determine the seasonal MAs empirically. As the particular choices of MAs will not affect the unit root results of interest here, we shall not go into details of this (for which, see Ladiray and Quenneville 2001, Chapter 3).

Bell and Monsell (1992) note that the above steps can be expressed symbolically to produce the following expression for the X-11 symmetric seasonal filter, $\omega_S^{X11}(B)$:

$$\omega_S^{X11}(B) = [1 - \mu(B)]\lambda_2(B) [1 - H(B)\{1 - [1 - \mu(B)]\lambda_1(B)[1 - \mu(B)]\}] \quad (16)$$

where

$$\begin{aligned} \mu(B) &= 2 \times 12 \text{ trend MA} = \frac{1}{24}F^6(1+B)U(B) = \frac{1}{24}(F^6 + 2F^5 + \dots + 2B^5 + B^6) \\ \lambda_1(B) &= \text{first seasonal MA, e.g., } \frac{1}{9}(F^{12} + 1 + B^{12})(F^{12} + 1 + B^{12}) \\ \lambda_2(B) &= \text{second seasonal MA, e.g., } \frac{1}{15}(F^{12} + 1 + B^{12})(F^{24} + F^{12} + 1 + B^{12} + B^{24}) \\ H(B) &= \text{Henderson trend MA.} \end{aligned}$$

For quarterly series we change 12 to 4 and 24 to 8 in the above expressions, and the $F^6(1+B)$ to $F^2(1+B)$ in the definition of $\mu(B)$. The Henderson trend MAs are discussed by Macaulay (1931, p. 54), Kenny and Durbin (1982), Dagum (1985), and Ladiray and Quenneville (2001, Chapter 3). Given the symmetric seasonal filter $\omega_S^{X11}(B)$, the X-11 symmetric filters for estimating the remaining components are as follows:

$$\omega_N^{X11}(B) = 1 - \omega_S^{X11}(B) \quad (17)$$

$$\omega_T^{X11}(B) = H(B)\omega_N^{X11}(B) \quad (18)$$

$$\omega_I^{X11}(B) = [1 - H(B)]\omega_N^{X11}(B). \quad (19)$$

Bell and Monsell (1992) take the expressions (16)–(19) and use polynomial multiplication routines to determine the symmetric filter weights for most of the available choices of the seasonal and Henderson trend MAs. They then plot the filter weights along with the squared gains of the corresponding transfer functions. To determine unit root properties of the filters $\omega_S^{X11}(B)$, $\omega_N^{X11}(B)$, $\omega_T^{X11}(B)$, and $\omega_I^{X11}(B)$, one might take their filter weights and attempt to find the zeros of the corresponding polynomials numerically. Because of the high degree of the polynomials, however (168 for the seasonal adjustment filter with the X-11 default seasonal MAs as above and a 13-term Henderson trend MA), such an approach may not work or would at least leave some doubts about numerical precision. Instead, unit root properties of the X-11 filters can be inferred from the expressions (16)–(19) using knowledge of the unit root properties of $\mu(B)$, the seasonal MAs, and the Henderson trend MAs. The latter properties can be determined via numerical zero finding, or by polynomial division (by $(1-B)$ d times to check for a $(1-B)^d$ factor, or by $(1-B^{12})$ or $U(B)$ to check for these factors). This was done and the results are stated as Lemma 1.

Lemma 1: The moving averages used in the X-11 symmetric filters have the following unit root properties (for monthly data):

- (a) $\mu(B)$, the 2×12 MA, contains $(1 + B)U(B)$
- (b) $1 - \mu(B)$ contains $(1 - B)(1 - F)$
- (c) $1 - \lambda(B)$ contains $(1 - B^{12})(1 - F^{12}) = (1 - B)(1 - F)U(B)U(F)$ for any of the X-11 seasonal MAs $\lambda(B)$
- (d) $1 - H(B)$ contains $(1 - B)^2(1 - F)^2$ for any of the Henderson trend MAs, $H(B)$.

No other factors of $U(B)$, nor additional $(1 - B)$ factors, are contained by these MAs or their complements. For quarterly series change 12 to 4 in (a) and (c).

Note that result (a) follows directly from the definition of $\mu(B)$. Also, since the Henderson trend MAs are explicitly designed to reproduce cubic polynomials (Kenny and Durbin 1982), result (d) must hold.

By manipulating the expressions (16)–(19) and using the results of Lemma 1, we can establish the unit root properties of the X-11 symmetric filters. These are listed in Table 3. Derivation of these results is deferred to Appendix A.

Table 3. Unit Root Factors in Symmetric X-11 Linear Filters

Filter	Unit root factors	Annihilates	Reproduces
$\omega_S^{X11}(B)$	$(1 - B)^3(1 - F)^3$	polynomials of degree ≤ 5	fixed seasonal effects
$\omega_N^{X11}(B)$	$U(B)(1 + F)$	fixed seasonal effects	polynomials of degree ≤ 5
$\omega_T^{X11}(B)$	$U(B)(1 + F)$	fixed seasonal effects	polynomials of degree ≤ 3
$\omega_I^{X11}(B)$	$U(B)(1 + F)(1 - B)^2(1 - F)^2$	fixed seasonal effects and polynomials of degree ≤ 3	—

Comparing Tables 1 and 3, several interesting differences emerge.

First, since it would be very unusual to have a value of d greater than 2 in model (2), we see that $\omega_S^{X11}(B)$ contains more $1 - B$ factors (effectively 6) than would a symmetric model-based seasonal filter, $\omega_S(B)$. Consequently, the X-11 symmetric seasonal adjustment filters will reproduce polynomials up to degree 5, while symmetric model-based adjustment filters will only reproduce polynomials up to degree 3 (if $d = 2$) or 1 (if $d = 1$).

Second, unlike the model-based symmetric filters, the X-11 symmetric seasonal adjustment, trend, and irregular filters do not include $U(B)U(F)$. Instead, they include only $U(B)(1 + F)$ ($\propto \mu(B)$). In fact, no symmetric infinite model-based MMSE filter can produce this result, which is one reason that no models can be found that will exactly reproduce the X-11 symmetric filters. However, if one computes transfer functions of the filters $\omega_N^{X11}(B)/U(B)$ (i.e., the adjustment filter with its $U(B)$ factor

removed), one finds that these transfer functions very nearly reach 0 at all the seasonal frequencies. (They are exactly 0 at the frequency π due to $\omega_N^{X11}(B)$ containing the additional $1 + F$ factor, whose zero is at $F = -1 = e^{i\pi}$.) This shows that the X-11 symmetric seasonal adjustment filters, and thus the X-11 symmetric trend and irregular filters as well, very nearly include $U(B)U(F)$. This may be partly why Cleveland and Tiao (1976), Burrige and Wallis (1984), and Planas and Depoutot (2002) were successful at finding models that could well-approximate X-11 symmetric filters.

Third, note that $\omega_I^{X11}(B)$ includes $(1 - B)^2(1 - F)^2$, whereas $\omega_S^{X11}(B)$ includes $(1 - B)^3(1 - F)^3$. Though not shown in Table 3, $1 - \omega_T^{X11}(B)$ also includes just $(1 - B)^2(1 - F)^2$. This contrasts with the results in Table 1, where $\omega_S(B)$, $\omega_I(B)$, and $1 - \omega_T(B)$ all include the same $(1 - B)^d(1 - F)^d$ factors. Hence, for X-11, the symmetric trend filter reproduces, and the symmetric irregular filter annihilates, polynomials of lower degree than are annihilated by the symmetric seasonal filter, whereas for model-based filters the degrees of the polynomials annihilated or reproduced by these filters are all the same.

Fourth, because $\omega_S^{X11}(B)$ and $\omega_I^{X11}(B)$ contain as many or more $(1 - B)$ factors as do the corresponding symmetric model-based filters, the remarks of Section 3 about “overdifferencing” by the symmetric model-based filters apply also to $\omega_S^{X11}(B)$ and $\omega_I^{X11}(B)$. Also, since $\omega_N^{X11}(B)$, $\omega_T^{X11}(B)$, and $\omega_I^{X11}(B)$ very nearly include $U(B)U(F)$, the remarks of Section 3 about “overadjustment” apply to these X-11 filters. Though, as noted in Section 3, such overadjustment does not necessarily pose a problem, these results should nevertheless be kept in mind when analyzing X-11 component estimates such as X-11 seasonally adjusted series. (For short-to-medium length series, depending on the effective length of the X-11 filters used, the properties of the X-11 component estimates will also be strongly affected by the different properties of the asymmetric X-11 linear filters given later in Tables 4–6.)

Young (1968) provided alternative approximations to X-11 symmetric filters. For the seasonal filter, Young (1968, eq. (4)) omitted steps 3 and 7 from the steps outlined by Wallis (1974), yielding the following approximation obtained by omitting the first two $1 - \mu(B)$ terms from (16). We denote this filter $\omega_S^Y(B)$:

$$\omega_S^Y(B) = \lambda_2(B) [1 - H(B)\{1 - \lambda_1(B)[1 - \mu(B)]\}]. \quad (20)$$

Corresponding approximations to the X-11 symmetric seasonal adjustment, trend, and irregular filters start with $\omega_S^Y(B)$ and follow as in (17)–(19). Young argued for considering these filters when applied to logged data as an approximation to X-11’s multiplicative decomposition. Though Wallis’s (1974) representation of X-11 linear filters is exact, it is so only for additive and log-additive decompositions, and it appears that whether Young’s approximation (20), or Wallis’s exact version with a log-additive decomposition, provides a better approximation to multiplicative X-11 has not been studied.

Unit root results for Young’s approximate filters differ some from the results of Table 3. First, $\omega_S^Y(B)$ contains only $(1 - B)(1 - F)$, so $\omega_S^Y(B)$ annihilates, and $\omega_N^Y(B)$ reproduces, only linear polynomials in t (not the polynomials up to degree 5 of the

Wallis representation). Second, $1 - \omega_T^Y(B)$ also contains $(1 - B)(1 - F)$, and while this is less than the $(1 - B)^2(1 - F)^2$ contained by $1 - \omega_T^{X11}(B)$, it is consistent with the result for $\omega_S^Y(B)$, as is the case for model-based symmetric filters. (Note also that the model-based symmetric seasonal and irregular filters for $d = 1$ also contain just the factors $(1 - B)(1 - F)$.) Third, due to the presence of $1 - H(B)$ (note (19)), $\omega_I^Y(B)$ includes $(1 - B)^2(1 - F)^2$, and so annihilates cubic polynomials in t , matching the result in Table 3, but differing from the result for $\omega_S^Y(B)$. Fourth, $\omega_N^Y(B)$, $\omega_T^Y(B)$, and $\omega_I^Y(B)$ all include $U(B)(1 + B)$, matching the corresponding results of Table 3.

5 Unit Root Properties of X-11 Asymmetric Filters with Full Forecast Extension

To deal with the issue of symmetric X-11 filters not being applicable near the ends of time series, Dagum (1975) proposed extending series with forecasts and backcasts from ARIMA models, leading to the X-11-ARIMA method (Dagum 1980). Pierce (1980) and Geweke (1978) pointed out that extending series with optimal (MMSE) forecasts and backcasts, i.e., appending sufficient forecasts and backcasts to the series so the symmetric filters could be applied at $t = 1, \dots, n$, would minimize mean squared revisions of the seasonally adjusted data. In practice, the true model for a series is unknown, so optimal forecasts cannot be achieved, but the motivating idea behind these papers was that one could find a model good enough that using its forecasts and backcasts could at least reduce the size of the revisions. (As noted in Section 2, the programs X-11-ARIMA and X-12-ARIMA use models of the form of (3) or (6).) As implemented in X-11-ARIMA and X-12-ARIMA, the default option is not full forecast extension, but rather extension with one year of forecasts. (Backcast extension is usually of less concern. It can be requested in X-12, but the default is no backcast extension.) With this approach, the original X-11 asymmetric filters still play a role. In this section, we consider unit root properties of X-11 filters obtained using full forecast and backcast extension. Unit root properties of X-11 filters obtained with partial forecast and backcast extension will be discussed in Section 6.

Unit root properties of X-11 filters obtained using full forecast extension follow from the results of Table 3, Theorem 1 of Section 2, and the approach described at the end of Section 3. Thus, $(1 - B)$ and $U(B)$ factors occur in the various filters according to the lesser of (i) the degree to which they occur in the corresponding symmetric X-11 filter, and (ii) the degree to which polynomials or fixed seasonal effects are reproduced in forecasting by the model used. The results are given in Table 4 below. We use a tilde to distinguish these asymmetric X-11 filters from the X-11 symmetric filters considered in the previous section, e.g., we write $\tilde{\omega}_S^{X11}(B)$ instead of $\omega_S^{X11}(B)$.

Notice that the results in Table 4 are the same as those in Table 2 for asymmetric model-based filters, with just the noted exceptions that would occur for large values of d that should never occur in practice anyway. This is because the X-11 symmetric filters $\omega_S^{X11}(B)$ and $\omega_I^{X11}(B)$ contain as many or more $1 - B$ factors (see Table 3) than are ordinarily contained by the symmetric model-based signal extraction filters (Table 1), and so the forecasting results are what limit the number of $(1 - B)$

factors in both the asymmetric model-based and asymmetric X-11 filters with full forecast extension. Also, notice that the X-11 symmetric filters $\omega_N^{X11}(B)$, $\omega_T^{X11}(B)$, and $\omega_I^{X11}(B)$ contain $U(B)$.

Table 4. Unit Root Factors in Asymmetric X-11 Linear Filters with Full Forecast Extension

Filter	Unit root factors*	Annihilates*	Reproduces*
$\tilde{\omega}_S^{X11}(B)$	$(1 - B)^d$	polynomials of degree $< d$	fixed seasonal effects
$\tilde{\omega}_N^{X11}(B)$	$U(B)$	fixed seasonal effects	polynomials of degree $< d$
$\tilde{\omega}_T^{X11}(B)$	$U(B)$	fixed seasonal effects	polynomials of degree $< d$
$\tilde{\omega}_I^{X11}(B)$	$U(B)(1 - B)^d$	fixed seasonal effects and polynomials of degree $< d$	—

* For $\tilde{\omega}_S^{X11}(B)$ and $\tilde{\omega}_N^{X11}(B)$, the results for $1 - B$ factors and polynomials assume that $d \leq 6$; for $d > 6$, change d to 6 in these rows of the table. For $\tilde{\omega}_T^{X11}(B)$ and $\tilde{\omega}_I^{X11}(B)$, the results for $1 - B$ factors and polynomials assume that $d \leq 4$; for $d > 4$, change d to 4 in these rows of the table.

As was the case for asymmetric model-based filters, the unit root factors shown in Table 4 are just those needed to remove nonstationarities present in the other components according to the model (2). Thus, in contrast to the results for symmetric X-11 filters, no “overdifferencing” or “overadjustment” occurs with the asymmetric X-11 filters with full forecast extension.

The same reasoning applies to models (6) with trend constants. From Theorem 2 of Section 2, adding the trend constant to the model increases the degree of polynomials reproduced by forecasting by 1. Hence, when full forecast extension uses model (6), we can increase d to $d + 1$ in Table 4. Per the note to Table 4, this assumes that $d \leq 5$ for the first two rows of the table, and $d \leq 3$ for the last two rows.

Note that the results in Table 4 apply only at those time points for which the symmetric filters cannot be applied and forecast extension is needed. Let the symmetric seasonal filter be written as $\omega_S^{X11}(B) = \sum_{j=-r}^r \omega_{S,j}^{X11} B^j$ with the $2r + 1$ symmetric weights $\omega_{S,j}^{X11} = \omega_{S,-j}^{X11}$. We call r the “half-length” of the symmetric filter. Then, the results in Table 4 for $\tilde{\omega}_S^{X11}(B)$ and $\tilde{\omega}_N^{X11}(B)$ apply for $t = 1, \dots, r$ and $t = n + 1 - r, \dots, n$, while for $t = r + 1, \dots, n - r$, the symmetric seasonal and adjustment filters are used (so the results given in Table 3 apply). For $\tilde{\omega}_T^{X11}(B)$ and $\tilde{\omega}_I^{X11}(B)$, the results in Table 4 apply for $t = 1, \dots, r + p$ and $t = n + 1 - r - p, \dots, n$, where p is the half-length of $H(B) = \sum_{j=-p}^p H_j B^j$, the symmetric Henderson trend MA. The value of r varies with alternative choices of the seasonal and Henderson trend MAs, a point we discuss further in the next section.

6 Unit Root Properties of Original X-11 Asymmetric Filters

To deal with the inapplicability of the symmetric filters except in the middle of sufficiently long time series, the original X-11 program (Shiskin, Young, and Musgrave 1967) provided families of asymmetric seasonal and trend MAs used in place of the symmetric versions of $\lambda_1(B)$, $\lambda_2(B)$, and $H(B)$ in equation (16). Ladiray and Quenneville (2001, Chapter 3) discuss these asymmetric MAs and give their filter weights. There is also need for an “asymmetric version” of $\mu(B)$; we discuss this in Appendix B. The asymmetric seasonal and trend MAs are also needed by the X-11 procedures of X-11-ARIMA and X-12-ARIMA for use when there is no or only partial forecast extension.

The asymmetric X-11 filters carry out the same sequence of operations listed at the start of Section 4 (Wallis 1982), and so can still be loosely represented by the expressions (16)–(19). However, the MAs in the asymmetric filters are generally time-varying, in that when insufficient observations are available to apply a symmetric MA at a given time point, the appropriate asymmetric MA is used. Because of this, we cannot simply expand (16)–(19) as polynomials in B with fixed weights. Therefore, while the unit root properties of the asymmetric filters do indeed depend on the unit root properties of the asymmetric MAs, we cannot simply examine (16)–(19) for the presence of $(1 - B)$ factors, etc., in the asymmetric MAs to directly determine their degree in the asymmetric X-11 filters. We instead examine (in Appendix B) what results from applying X-11 asymmetric filters to sequences ξ_t representing either fixed seasonal effects or polynomial functions of time ($1, t, t^2$, etc.).

The unit root factors in the asymmetric versions of X-11’s 3-term, 3×3 , 3×5 , 3×9 , and 3×15 seasonal MAs were found numerically. Filter weights were taken from the X-11 code in the X-12-ARIMA program. Ladiray and Quenneville (2001, p.45) give weights for the asymmetric 3×3 , 3×5 , and 3×9 MAs, though, for the 3×9 , the weights given are approximations that do not quite preserve the unit root factors of the MAs actually used in the program. Ladiray and Quenneville (2001, pp. 40–44) also provide weights for the asymmetric Henderson trend MAs, and note that these MAs reproduce only constants, not linear functions. Collecting these results gives the following lemma:

Lemma 2: The asymmetric seasonal ($\lambda_t(B)$) and Henderson trend ($H_t(B)$) moving averages used in X-11 have the following unit root properties (for monthly series):

- (a) $1 - \lambda_t(B)$ contains $(1 - B^{12}) = (1 - B)U(B)$ for any of the X-11 asymmetric seasonal MAs, $\lambda_t(B)$.
- (b) $1 - H_t(B)$ contains $(1 - B)$ for any of the asymmetric Henderson trend MAs, $H_t(B)$.

No other factors of $U(B)$, nor additional $(1 - B)$ factors, are contained by the $1 - \lambda_t(B)$ and $1 - H_t(B)$. For quarterly series change 12 to 4 in (a).

Using Lemma 2, Appendix B derives the unit root factors in the original X-11 asymmetric filters (denoted as $\omega_{S_t}^{X11}(B)$, etc.) Table 5 below gives the results for the concurrent filters ($t = n$).

We see that the $U(B)$ operator is included in the X-11 concurrent seasonal adjustment, trend, and irregular filters, as was the case in Table 2 (model-based asymmetric filters) and Table 4 (asymmetric X-11 filters with full forecast extension). The additional $1+B$ factor of the X-11 symmetric filters (Table 3) is not included, nor are any other additional factors of $U(B)$. But the main difference from all the previous results is that the original X-11 concurrent seasonal and irregular filters include only $1 - B$, and no higher power of this. Thus, these X-11 filters annihilate, and the corresponding seasonal adjustment and trend filters reproduce, only constants, not polynomials in

Table 5. Unit Root Factors in Original X-11 Concurrent Linear Filters

Filter	Unit root factors	Annihilates	Reproduces
$\omega_{S_n}^{X11}(B)$	$(1 - B)$	constants	fixed seasonal effects
$\omega_{N_n}^{X11}(B)$	$U(B)$	fixed seasonal effects	constants
$\omega_{T_n}^{X11}(B)$	$U(B)$	fixed seasonal effects	constants
$\omega_{I_n}^{X11}(B)$	$U(B)(1 - B)$	fixed seasonal effects and constants	—

time of degree 1 or higher. From Table 2, this agrees with the results for asymmetric (and finite) model-based filters when $d = 1$.

The presence of $U(B)$ in $\omega_{N_n}^{X11}(B)$, $\omega_{T_n}^{X11}(B)$, and $\omega_{I_n}^{X11}(B)$ without an additional $1 + F$ factor means that the original X-11 concurrent filters do not “overadjust.” The presence of just the single $1 - B$ factor in $\omega_{S_n}^{X11}(B)$ and $\omega_{I_n}^{X11}(B)$, however, means that these filters will under-difference the series unless the appropriate model for the data has just $d = 1$. For the more common case where the model assumes $d = 2$, this fact precludes some calculations one might wish to do with original X-11 concurrent filters. Consider, for example, the concurrent seasonal adjustment error, which is $N_t - \omega_{N_n}^{X11}(B)y_t = \omega_{S_n}^{X11}(B)N_t - \omega_{N_n}^{X11}(B)S_t$. This error is nonstationary if $d = 2$, since $\omega_{S_n}^{X11}(B)$ contains only one difference. Hence, given a model for y_t with $d = 2$, the MSE of a seasonal adjustment using an original X-11 concurrent filter cannot be calculated. For this reason, Chu, Tiao, and Bell (2007) made seasonal adjustment MSE calculations, and Bell and Kramer (1999) developed an approximate approach to computing X-11 seasonal adjustment variances, only for X-11 filters with full forecast extension.

The results in Tables 4 and 5 leave open the question, “What unit root factors are contained by the original asymmetric X-11 filters other than the concurrent filters (i.e., for $1 < t < n$)?” Appendix B also obtains these results. We show there that all the original X-11 asymmetric seasonal adjustment, trend, and irregular filters contain $U(B)$ for every t . For $1 - B$ factors, the results depend on t and on the half-lengths

of the seasonal and trend MAs used. Let the half-lengths of the symmetric seasonal MAs $\lambda_1(B)$ and $\lambda_2(B)$ be m_1 and m_2 , respectively. Note that these MAs involve $2m_1 + 1$ and $2m_2 + 1$ weights, including many weights that are zero because the only nonzero weights in the seasonal MAs are at the seasonal lags and leads. Again, let p denote the half-length of the symmetric $H(B)$. The half-length, r , of the full symmetric seasonal filter can then be seen from (16) to be $r = 18 + m_1 + m_2 + p$. As noted in Section 5, the symmetric filter $\omega_S^{X11}(B)$ applies for $t = r + 1, \dots, n - r$, and this contains $(1 - B)^6$. Table 6 shows how the $(1 - B)$ factors in $\omega_{S_t}^{X11}(B)$ vary across the other values of t .

Table 6. $(1 - B)$ Factors in Original X-11 Asymmetric Seasonal Filters

Time points	$\omega_{S_t}^{X11}(B)$ contains	$\omega_{S_t}^{X11}(B)$ annihilates, $\omega_{N_t}^{X11}(B)$ reproduces
$t = 1, \dots, 6 + m_2 + p$ $t = n + 1 - (6 + m_2 + p), \dots, n$	$(1 - B)$	constants
$t = 7 + m_2 + p, \dots, r$ $t = n + 1 - r, \dots, n - (6 + m_2 + p)$	$(1 - B)^3$	polynomials up to degree 2
$t = r + 1, \dots, n - r$ (symmetric filter applies)	$(1 - B)^6$	polynomials up to degree 5

We see that, near the ends of the series ($t = 1, \dots, 6 + m_2 + p$ and $t = n + 1 - (6 + m_2 + p), \dots, n$), $\omega_{S_t}^{X11}(B)$ contains only $1 - B$, while for $t = 7 + m_2 + p, \dots, r$ and $t = n + 1 - r, \dots, n - (6 + m_2 + p)$, $\omega_{S_t}^{X11}(B)$ contains $(1 - B)^3$. The latter provides a transition to the $(1 - B)^6$ contained by the symmetric seasonal filter, which applies for $t = r + 1, \dots, n - r$. Such a transition does not occur for model-based filters nor for X-11 filters with full forecast extension. In both of these cases, all the asymmetric filters contain the same unit root factors.

For the original asymmetric X-11 trend and irregular filters, we need to extend the first two ranges of time points in Table 6 by p months. Thus, for $t = 1, \dots, 6 + m_2 + 2p$ and for $t = n + 1 - (6 + m_2 + 2p), \dots, n$, $\omega_{T_t}^{X11}(B)$ reproduces, and $\omega_{I_t}^{X11}(B)$ annihilates, only constant polynomials. For $t = 7 + m_2 + 2p, \dots, r + p$ and for $t = n + 1 - (r + p), \dots, n - (6 + m_2 + 2p)$, $\omega_{T_t}^{X11}(B)$ reproduces, and $\omega_{I_t}^{X11}(B)$ annihilates, polynomials up to degree 2. For $t = r + p + 1, \dots, n - (r + p)$, the symmetric filters apply, and $\omega_T^{X11}(B)$ reproduces, and $\omega_I^{X11}(B)$ annihilates, polynomials up to degree 3 (Table 3).

For quarterly series we change 6 to 2 in the ranges of time points in Table 6 (and $7 = 6 + 1$ to $3 = 2 + 1$). At the beginning of the series, the first two time point ranges for the quarterly trend and irregular asymmetric filters are then $t = 1, \dots, 2 + m_2 + 2p$ and $t = 3 + m_2 + 2p, \dots, r + p$. At the end of the series the corresponding time point ranges are $t = n + 1 - (2 + m_2 + 2p), \dots, n$ and $t = n + 1 - (r + p), \dots, n - (2 + m_2 + 2p)$. The range where the symmetric quarterly trend and irregular filters apply is still $t = r + p + 1, \dots, n - (r + p)$.

When X-11 is applied to a time series partially extended with forecasts and backcasts, that is, with fewer than r forecasts and backcasts (the number required for application of the symmetric seasonal and nonseasonal filters to the extended series), we can infer the unit root properties of the resulting implied asymmetric filters from the results given above, including Table 6 and the theorems of Section 2. Assume that the forecasting model is of the form of (3), that is, that the differencing in the model is $\delta(B) = (1 - B)^{d-1}(1 - B^{12}) = (1 - B)^d U(B)$. We will consider $d = 1$ or 2. Since (i) from Theorem 1 the forecast extension will reproduce constants and fixed seasonal effects, and (ii) from the above discussion and Table 6, for every t , the filter $\omega_{N_t}^{X11}(B)$ will reproduce constants and annihilate fixed seasonal effects, it follows that X-11 seasonal adjustment with the extended series will also reproduce constants and annihilate fixed seasonal effects (with either $d = 1$ or $d = 2$). In fact, for $d = 1$ the forecast extension will reproduce only constants, not polynomials of higher degree, and so in this case X-11 seasonal adjustment with the extended series will also not reproduce polynomials of degree one or higher.

Consider now the case of $d = 2$ and assume one year of both forecast and backcast extension. (One year forecast extension, but no backcast extension, is the default choice in X-11-ARIMA and X-12-ARIMA.) For simplicity, we'll consider what happens at the beginning of the time series. Parallel results hold at the end. Theorem 1 of Section 2 says that forecast and backcast extension reproduce linear functions of time, but not higher order polynomials. With the one-year backcast extension, the adjustment filters that apply at $t = 1, 2, \dots$ are $\omega_{N_{t+12}}^{X11}(B)$. (The filters $\omega_{S_j}^{X11}(B)$, and conversely, $\omega_{N_j}^{X11}(B)$, for $j = 1, \dots, 12$ would apply only for producing (back) projected seasonal factors, which is not of concern here.) The results of Table 6 then provide the unit root properties of the seasonal and seasonal adjustment filters, but with the time point ranges (at the beginning of the series) shifted by subtracting 12. We thus see the following. (i) For $t = 1, \dots, m_2 + p - 6$, $\omega_{S_{t+12}}^{X11}(B)$ contains $1 - B$, so that seasonal adjustment for these time points reproduces only constants. (ii) For $t = m_2 + p - 5, \dots, r - 12$, $\omega_{S_{t+12}}^{X11}(B)$ contains $(1 - B)^3$, which annihilates quadratics, but since forecasting only reproduces linear functions, seasonal adjustment for these time points reproduces only linear functions. (iii) For $t = r - 11, \dots, r$, the symmetric filter $\omega_S^{X11}(B)$ is applied to the extended series. While it contains $(1 - B)^6$, again the limiting factor is the forecast and backcast extension, so seasonal adjustment for these time points still reproduces only linear functions. Note the corresponding results in the first two rows of Table 4 for the case of $d = 2$. (iv) For $t = r + 1, \dots, n - r$, the symmetric seasonal and seasonal adjustment filters apply using only observed data (no forecast or backcast extension needed), and seasonal adjustment reproduces polynomials up to degree 5.

Similar reasoning can be used to infer properties of the X-11 trend and irregular filters when applied with limited forecast and backcast extension, as well as to infer unit root properties of X-11 filters when applied to series extended with more or fewer forecasts (and backcasts). We could also obtain results for values of $d > 2$, but as noted earlier such results would be of little practical relevance.

Finally, it should be noted that the effective half-length of an X-11 symmetric

seasonal filter is, in practical terms, much less than the $r = 18 + m_1 + m_2 + p$ on which the results given here are based. This is because the X-11 filter weights are quite small beyond a certain point much less than r . From the plots given in Bell and Monsell (1992) of X-11 symmetric filter weights (covering filters generated from the 3×1 , 3×3 , default 3×5 , and 3×9 seasonal MAs, and the 9-, 13-, and 23-term Henderson trend MAs), one might judge that the effective half-length of an X-11 symmetric seasonal filter is about m_2 , or perhaps, to be safe, $m_2 + 12$ ($m_2 + 4$ for quarterly series). Thus, while the “asymmetric” X-11 seasonal filters $\omega_{S_t}^{X11}(B)$ include $(1 - B)^6$ only for $t = r + 1, \dots, n - r$, they may come close to doing so for $t = m_2 + 1, \dots, r$ and $t = n - r + 1, \dots, n - m_2$.

7 Illustration

We illustrate the results of Section 6 on reproduction of polynomials for X-11 seasonal adjustment and trend filters with no forecast extension. For simplicity, quarterly rather than monthly seasonal adjustment was done. The input series to X-11 were polynomials of degrees 1 through 5 covering 15 years plus one quarter, or 61 observations. These were of the form $y_t = 30 \times [(t - 31)/30]^k$ for $t = 1, \dots, 61$ and $k = 1, \dots, 5$. The series values thus ranged from $-30, \dots, 30$ for odd powers, and from 30 down to 0 and then back up to 30 for even powers. This makes the average absolute quarter-to-quarter change equal to 1 in all cases, so that errors in the seasonal adjustments and trend estimates – the differences between these values and the input polynomial trends – generally reflect errors relative to average absolute quarter-to-quarter changes.

To keep the X-11 filters relatively short, we specified a 3×3 seasonal MA and a 5-term Henderson trend MA. For these MA choices, $m_1 = m_2 = 2 \times 4 = 8$ and $p = 2$, so $r = 6 + m_1 + m_2 + p = 24$ is the half-length of the symmetric X-11 seasonal adjustment filter, which thus (Table 6) reproduces all the polynomials up to degree 5 for time points 25, \dots , 37 ($= 61 - 24$). From Table 6 (with the modifications noted for quarterly series), the X-11 asymmetric seasonal adjustment filters will reproduce polynomials up to degree 2 at time points $3 + m_2 + p = 13, \dots, 24 = r$, and similarly at time points 38, \dots , 49. At time points 1, \dots , 12 and 50, \dots , 61, only constants are exactly reproduced by the X-11 seasonal adjustment filters.

The half-length of our X-11 trend filter is $r + p = 26$, so it reproduces polynomials up to degree 3 (note Table 3) for time points 27, \dots , 35. It reproduces linear and quadratic polynomials at time points 15, \dots , 26 and 36, \dots , 47. At time points 1, \dots , 14 and 48, \dots , 61, the trend filters reproduce only constants.

Figure 1 displays the results. The seasonal adjustment errors are displayed in the left column of plots for the input polynomials of degrees 1 to 5. The corresponding trend estimation errors are displayed in the right column of plots. The dotted vertical lines in the plots are the limits of the intervals over which the filters reproduce the respective polynomials. No dotted vertical lines appear in the last two plots in the right column, since for these cases none of the input polynomial values are reproduced.

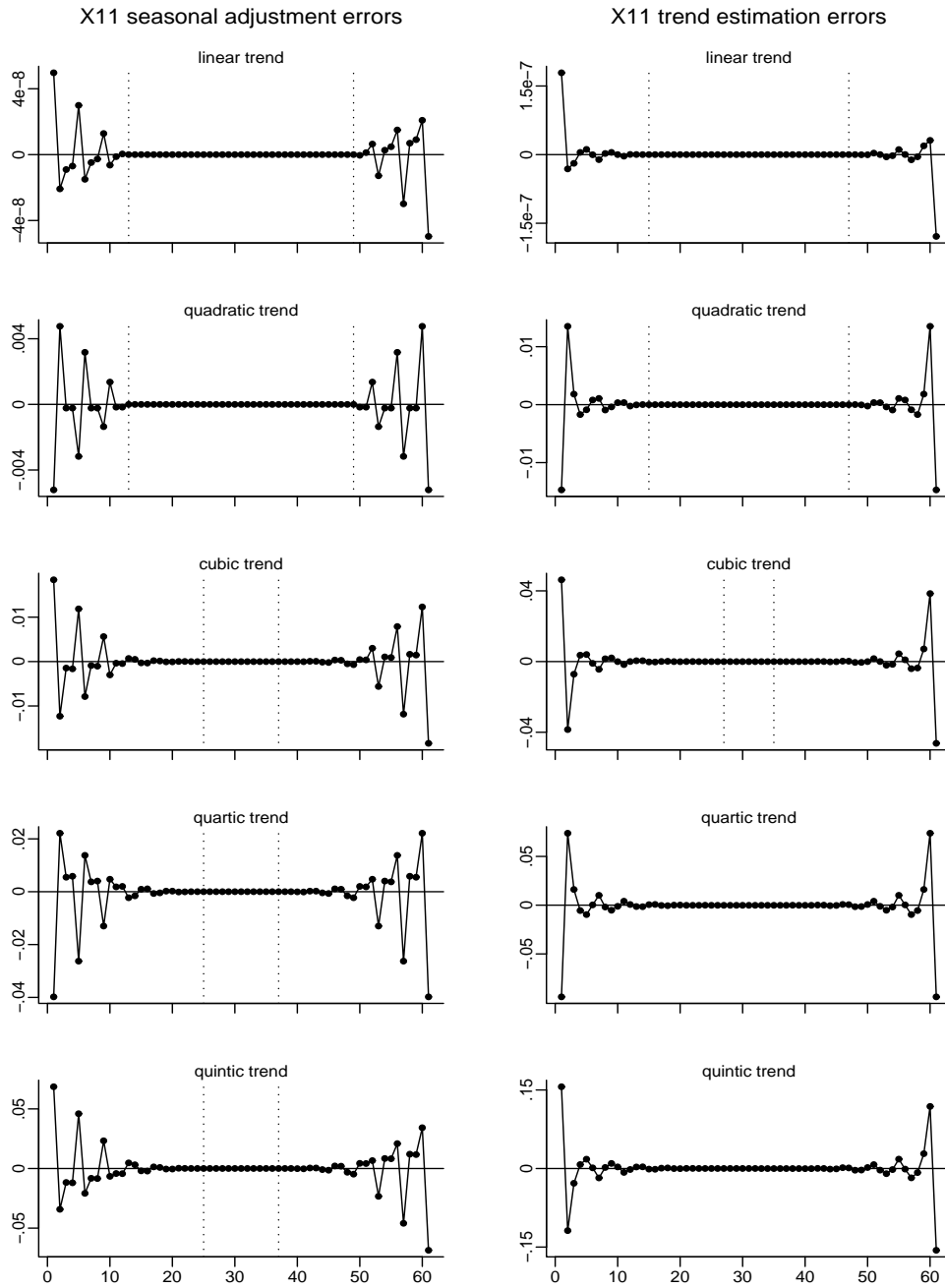


Figure 1: Seasonal adjustment and trend estimation errors for X-11 quarterly filters with no forecast extension applied to polynomials of degrees one through five. The X-11 filters use 3×3 seasonal MAs and a 5-term Henderson trend MA. The polynomials are of the form $y_t = 30 \times [(t - 31)/30]^k$ for $t = 1, \dots, 61$ and $k = 1, \dots, 5$. The errors are zero within the spans denoted by the dotted vertical lines.

Though many of the plotted points outside the region denoted by the dotted vertical lines appear to fall on the horizontal axis, these values are not exactly zero, just too small to visually detect their differences from zero on the plots.

For the linear trend plots in the first row, we see the magnitude of the errors in both the seasonally adjusted values and trend estimates is quite small, even at the very ends of the series. This illustrates the point made in Section 6 that while the X-11 asymmetric filters exactly reproduce only constants near the ends of the series, they come very close to reproducing linear polynomials. As we look down the rows of plots we notice that the magnitude of the errors increases with the degree of the input polynomial, and, especially for the higher degrees, the errors are not so trivial as they were for the linear trend. There is generally a seasonal pattern to the errors and, apart from this, the magnitude of the errors tends to be larger nearer to the ends of the series. This latter point is unsurprising given that, for polynomials of degree 2 and higher, the absolute rates of change in the series increase as one approaches either end of the series.

8 Conclusions

We have presented here an essentially complete catalog of results on unit root factors in commonly used seasonal, seasonal adjustment, trend, and irregular linear filters, both model-based and from X-11 with or without forecast extension (full or partial). The unit root factors of interest are differencing operators $((1 - B)^d$ for some $d > 0$) and seasonal summation operators ($U(B) = 1 + B + \dots + B^{11}$ for monthly data), as these determine the extent to which the various filters annihilate or reproduce (i) polynomials in time and (ii) fixed seasonal effects. Differences between the results for various cases were noted. For example, symmetric filters include more (higher order) unit root factors than do the corresponding asymmetric filters. Also, X-11 asymmetric seasonal filters obtained with full forecast extension from a model with both a seasonal and nonseasonal difference contain $(1 - B)^2$, as do model-based asymmetric seasonal filters, but X-11 asymmetric seasonal filters with partial or no forecast extension have only a single $(1 - B)$ factor.

We can summarize the results related to fixed seasonal effects by saying that all the seasonal filters examined here, both symmetric and asymmetric, would reproduce fixed seasonal effects (and so the corresponding seasonal adjustment, trend, and irregular filters would annihilate them). Symmetric model-based seasonal filters go further and reproduce deterministic functions ξ_t such that $U(B)^2\xi_t = 0$, but this extension is of little relevance for practice. For polynomial functions of time, model-based symmetric filters from models with both a seasonal and nonseasonal difference reproduce cubics, as do X-11 symmetric trend filters, while X-11 symmetric seasonal adjustment filters reproduce quintic (fifth degree) polynomials. Corresponding asymmetric model-based and X-11 filters with full forecast extension reproduce just linear time trends, while X-11 filters with partial or no forecast extension reproduce only constants. For models with only a seasonal difference, symmetric model-based adjustment and trend filters reproduce linear time trends, while the corresponding

asymmetric model-based and X-11 filters (with or without forecast extension) reproduce only constants.

It is difficult to draw any general conclusions about whether the differences that exist in unit root factors between model-based and X-11 filters favor one or the other, or are generally neutral. Such conclusions, when possible, would presumably depend on the properties of the time series being seasonally adjusted. We can say, however, that while the relation between unit root factors for model-based symmetric and asymmetric filters stems from established statistical principles of MMSE linear projection, the relation between unit root factors for X-11 symmetric and asymmetric filters (with or without forecast extension) is ad-hoc. This could raise concerns for some X-11 filters in certain specific instances. On the other hand, the illustration of Section 7, showing that X-11 asymmetric seasonal adjustment and trend filters without forecast extension come very close to reproducing linear polynomials (though exactly reproducing only constants), means we must be cautious in how we interpret the exact results.

This last remark also reminds us that the exact results presented should nonetheless hold approximately in other settings where one filter approximates another. For example, the results on unit roots of model-based symmetric filters strictly apply only to the case of seasonal adjustment using a doubly infinite realization of a time series ($\{y_t$ for $t = -\infty, \dots, \infty\}$), a situation never exactly realized in practice. However, model-based filter weights from the models considered here die out with increasing lead or lag, so that in the middle of a sufficiently long series the symmetric filters nearly apply, and then we can expect the unit root results for symmetric filters to hold approximately. Similarly, X-11 filters tend to be rather long, so that a rather large number of forecasts and backcasts would be needed to exactly achieve full forecast extension. However, the ends of X-11 symmetric filters contain a large number of very small (in magnitude) weights, so that the effective length of the filters is considerably less than their exact length. Hence, the results on unit roots in X-11 filters with full forecast extension will apply approximately with much less forecast extension than is required for the results to apply exactly. Precisely how long a series needs to be in order to be considered “sufficiently long,” or how many forecasts are really needed to approximate “full forecast extension,” will depend on the particular fitted models and filters being used, so this must be judged on a case-by-case basis.

9 Appendix A: Derivation of Unit Root Properties of X-11 Symmetric Filters

We start with $\omega_S^{X11}(B)$. From (16) we may rewrite it as

$$\omega_S^{X11}(B) = [1 - \mu(B)]\lambda_2(B)[1 - H(B)] + [1 - \mu(B)]^3\lambda_2(B)H(B)\lambda_1(B). \quad (21)$$

Using Lemma 1 (b) and (d) we see that $\omega_S^{X11}(B)$ contains $(1 - B)^3(1 - F)^3$. Hence, it annihilates polynomials up to degree 5, and $\omega_N^{X11}(B) = 1 - \omega_S^{X11}(B)$ reproduces polynomials up to degree 5. To see if $\omega_N^{X11}(B)$ contains $U(B)$, we manipulate it using

(21) as follows:

$$\begin{aligned}
\omega_N^{X11}(B) &= [1 - \lambda_2(B)] + \mu(B)\lambda_2(B) + [1 - \mu(B)]\lambda_2(B)H(B) \\
&\quad - [1 - \mu(B)][1 - 2\mu(B) + \mu(B)^2]\lambda_2(B)H(B)\lambda_1(B) \\
&= [1 - \lambda_2(B)] + \mu(B)\lambda_2(B) + [1 - \mu(B)]\lambda_2(B)H(B)[1 - \lambda_1(B)] \\
&\quad + [1 - \mu(B)][2\mu(B) - \mu(B)^2]\lambda_2(B)H(B)\lambda_1(B) \\
&= [1 - \lambda_2(B)] + [1 - \mu(B)][1 - \lambda_1(B)]\lambda_2(B)H(B) \\
&\quad + \mu(B)\lambda_2(B)[1 + 2H(B)\lambda_1(B)] \\
&\quad - \lambda_2(B)H(B)\lambda_1(B)\{2\mu(B)^2 + [1 - \mu(B)]\mu(B)^2\} \\
&= [1 - \lambda_2(B)] + [1 - \mu(B)][1 - \lambda_1(B)]\lambda_2(B)H(B) \\
&\quad + \mu(B)\lambda_2(B)[1 + 2H(B)\lambda_1(B)] - \mu(B)^2\lambda_2(B)H(B)\lambda_1(B)[3 - \mu(B)].
\end{aligned}$$

From Lemma 1 (a) and (c), all four terms above contain $U(B)$. In fact, all but the third term clearly contain $U(B)U(F)$. Whether or not the third term contains $U(B)U(F)$ (equivalently, contains $U(B)^2$) comes down to whether $1 + 2H(B)\lambda_1(B)$ contains $U(B)$. This can be checked by seeing if $B = e^{2\pi ij/12}$ for $j = 1, \dots, 6$, the zeroes of $U(B)$, are also zeroes of $1 + 2H(B)\lambda_1(B)$. (This is for monthly filters. For quarterly filters consider only $j = 3, 6$.) Since $\lambda_1(e^{2\pi ij/12}) = 1$ for $j = 1, \dots, 6$ for any of the X-11 seasonal MAs $\lambda_1(B)$, the result depends on whether $H(e^{2\pi ij/12}) = -0.5$. In fact, for none of the Henderson MAs does the transfer function go as low as -0.5 , therefore, $1 + 2H(B)\lambda_1(B)$ contains no factors of $U(B)$. This shows that $\omega_N^{X11}(B)$ contains $U(B)$ but not $U(B)U(F)$, though it does contain an additional $(1 + F)$ factor since this is part of $\mu(B)$. Since $\omega_N^{X11}(B)$ contains $U(B)$, it annihilates fixed seasonals, and $\omega_S^{X11}(B) = 1 - \omega_N^{X11}(B)$ reproduces fixed seasonals.

Considering now $\omega_T^{X11}(B) = H(B)\omega_N^{X11}(B)$, from the immediately preceding results $\omega_T^{X11}(B)$ contains $U(B)(1 + F)$, but, from Lemma 1 (which notes that $H(B)$ contains no factors of $U(B)$), it does not contain $U(B)U(F)$. Since

$$1 - \omega_T^{X11}(B) = [1 - H(B)] + H(B)[1 - \omega_N^{X11}(B)] = [1 - H(B)] + H(B)\omega_S^{X11}(B),$$

from Lemma 1 (d) and the above results for $\omega_S^{X11}(B)$, $1 - \omega_T^{X11}(B)$ contains $(1 - B)^2(1 - F)^2$, but no higher powers of $(1 - B)$. Thus, $\omega_T^{X11}(B)$ reproduces polynomials up to degree 3.

Finally, for $\omega_I^{X11}(B) = [1 - H(B)]\omega_N^{X11}(B)$, using Lemma 1 (d) and the above results for $\omega_N^{X11}(B)$, we see that $\omega_I^{X11}(B)$ contains $(1 - B)^2(1 - F)^2$ but no higher powers of $(1 - B)$, and also that $\omega_I^{X11}(B)$ contains $U(B)(1 + F)$ but not $U(B)U(F)$. Thus, $\omega_I^{X11}(B)$ annihilates polynomials up to degree 3 as well as fixed seasonal effects.

10 Appendix B: Derivation of Unit Root Properties of Original X-11 Asymmetric Filters

We write the original X-11 asymmetric seasonal filter that applies at time point $t \in \{1, \dots, n\}$ as

$$\omega_{S_t}^{X11}(B) = [1 - \mu_t^{(3)}(B)]\lambda_{2t}(B) \left[1 - H_t(B)\{1 - [1 - \mu_t^{(2)}(B)]\lambda_{1t}(B)[1 - \mu_t^{(1)}(B)]\} \right] \quad (22)$$

where the t subscript on the MAs ($\lambda_{1t}(B)$, $\lambda_{2t}(B)$, $H_t(B)$, and $\mu_t^{(i)}(B)$ for $i = 1, 2, 3$) indicates that, when the symmetric MAs ($\lambda_1(B)$, $\lambda_2(B)$, $H(B)$, and $\mu(B)$) cannot be applied, asymmetric versions of these MAs appropriate for the time point will be used. We shall examine what happens as the successive parts of $\omega_{S_t}^{X11}(B)$ are applied to a deterministic function ξ_t representing either fixed seasonal effects or polynomials of various degrees. This depends on the unit root properties of the asymmetric MAs $\lambda_{1t}(B)$, $\lambda_{2t}(B)$, and $H_t(B)$ that were given in Lemma 2, as well as on the properties of the ‘‘asymmetric versions’’ of $\mu(B)$, labeled $\mu_t^{(i)}(B)$, $i = 1, 2, 3$, whose actions we describe first. For concreteness we do this for the monthly case; an essentially similar discussion applies to the quarterly case.

Wallis (1982, p. 79), in summarizing the steps of X-11 with asymmetric filters, noted how the $\mu_t^{(i)}(B)$ are handled. This occurs at steps 1, 3, and 7 of the asymmetric analog to the steps listed at the beginning of Section 4 for the symmetric filters. The actions of the $\mu_t^{(i)}(B)$ are detailed below in the context of the full set of filtering steps. Ladiray and Quenneville (2001, Chapter 4) illustrate these actions in the presentation of an example of X-11 seasonal adjustment.

- (i) At step 1, X-11 simply omits computing values of $\mu_t^{(1)}(B)y_t$ for $t = 1, \dots, 6$ and $t = n - 5, \dots, n$, computing only $\mu_t^{(1)}(B)y_t = \mu(B)y_t$ for $t = 7, \dots, n - 6$ (where the symmetric $\mu(B)$ applies). It then computes $S_t^{(1)} \equiv \lambda_{1t}(B)[1 - \mu(B)]y_t$ only for $t = 7, \dots, n - 6$, using the asymmetric versions of $\lambda_{1t}(B)$ as needed, where $S_t^{(1)}$ denotes the first ‘‘uncentered’’ estimate of the seasonal. There are thus no values of $S_t^{(1)}$ for $t = 1, \dots, 6$ and $t = n - 5, \dots, n$.
- (ii) At step 3, X-11 first computes $\mu_t^{(2)}(B)S_t^{(1)} = \mu(B)S_t^{(1)}$ for $t = 13, \dots, n - 12$. Then, to provide a value of $\mu_t^{(2)}(B)S_t^{(1)}$ for $t = 7, \dots, 12$, it substitutes the value of $\mu(B)S_{13}^{(1)}$. It makes the analogous substitution of $\mu(B)S_{n-12}^{(1)}$ for $\mu_t^{(2)}(B)S_t^{(1)}$ at $t = n - 11, \dots, n - 6$. It then computes the ‘‘preliminary seasonal’’ as $S_t^{(p)} = S_t^{(1)} - \mu_t^{(2)}(B)S_t^{(1)}$ for $t = 7, \dots, n - 6$. To provide values of $S_t^{(p)}$ for $t = 1, \dots, 6$, X-11 substitutes $S_{t+12}^{(p)}$, the value for the same month in the adjacent year. It makes the analogous substitution of $S_{t-12}^{(p)}$ for $S_t^{(p)}$ for $t = n - 5, \dots, n$. It now has values of $S_t^{(p)}$ for all t , and it then progresses through computation of $S_t^{(2)} = \lambda_{2t}(B) \left[y_t - H_t(B)(y_t - S_t^{(p)}) \right]$, the second uncentered estimate of the seasonal, using asymmetric $\lambda_{2t}(B)$ and $H_t(B)$ as needed.

(iii) At step 7, X-11 can now compute the final seasonal estimate for $t = 7, \dots, n-6$ as $\hat{S}_t^{X11} \equiv [1 - \mu_t^{(3)}(B)]S_t^{(2)} = S_t^{(2)} - \mu(B)S_t^{(2)}$. To extend results to $t = 1, \dots, 6$, X-11 substitutes $\mu(B)S_7^{(2)}$ for $\mu_t^{(3)}(B)S_t^{(2)}$, and thus computes $\hat{S}_t^{X11} = S_t^{(2)} - \mu(B)S_7^{(2)}$ for $t = 1, \dots, 6$. The analogous substitution is made in computing \hat{S}_t^{X11} for $t = n-5, \dots, n$. The remaining operations for computing \hat{N}_t^{X11} , \hat{T}_t^{X11} , and \hat{I}_t^{X11} are then carried out.

We now let ξ_t be a sequence of fixed seasonal effects and consider what happens with application of $\omega_{S_t}^{X11}(B)$ from (22). First,

$$[1 - \mu_t^{(1)}(B)]\xi_t = \begin{cases} - & t = 1, \dots, 6 \text{ and } t = n-5, \dots, n \\ \xi_t & t = 7, \dots, n-6 \end{cases}$$

The result for $t = 7, \dots, n-6$ follows because there we apply $1 - \mu(B)$, and, from Lemma 1, $\mu(B)$ contains $U(B)$, and $U(B)\xi_t = 0$. Then, $S_t^{(1)} \equiv \lambda_{1t}(B)[1 - \mu_t^{(1)}(B)]\xi_t = \xi_t$ for $t = 7, \dots, n-6$ because each $\lambda_{1t}(B)$ reproduces fixed seasonal effects. Now, for $t = 13, \dots, n-12$, $\mu_t^{(2)}(B)S_t^{(1)} = \mu(B)\xi_t = 0$, and the zero value for $t = 13$ is extended to $t = 7, \dots, 12$, and similarly for $t = n-11, \dots, n-6$. Thus, $S_t^{(p)} \equiv [1 - \mu_t^{(2)}(B)]S_t^{(1)} = S_t^{(1)} = \xi_t$ for $t = 7, \dots, n-6$, and then the substitutions for $t = 1, \dots, 6$ and $t = n-5, \dots, n$ extend this to all t . Now, $\xi_t - S_t^{(p)} = 0$ for all t , and these zeros are reproduced by $H_t(B)$, so that $\xi_t - H_t(B)(\xi_t - S_t^{(p)}) = \xi_t$ for all t . Applying $\lambda_{2t}(B)$ then still yields ξ_t , as does applying $[1 - \mu(B)]$ for $t = 7, \dots, n-6$. For $t = 1, \dots, 6$, $\mu_t^{(3)}(B)\xi_t = \mu(B)\xi_7 = 0$, and similarly for $t = n-5, \dots, n$, so $[1 - \mu_t^{(3)}(B)]\xi_t = \xi_t$ for all t . We thus see that $\omega_{S_t}^{X11}(B)$ reproduces fixed seasonals. Consequently, $\omega_{N_t}^{X11}(B)$ annihilates fixed seasonals, and so do $\omega_{T_t}^{X11}(B)$ and $\omega_{I_t}^{X11}(B)$.

We now turn to application of $\omega_{S_t}^{X11}(B)$ to polynomials of various orders. As noted earlier, we let m_1 and m_2 denote the half-lengths of the symmetric seasonal MAs $\lambda_1(B)$ and $\lambda_2(B)$, and let p denote the half-length of the symmetric $H(B)$, so the half-length of the full symmetric seasonal filter $\omega_S^{X11}(B)$ is $r = 18 + m_1 + m_2 + p$, and $\omega_S^{X11}(B)$ has $2r + 1$ weights. Thus, $\omega_S^{X11}(B)$ applies for $t = r + 1, \dots, n - r$, and, from Table 3, for these time points $\omega_S^{X11}(B)$ annihilates polynomials up to degree 5, but not those of degree 6 or higher. We need to consider what happens when applying $\omega_{S_t}^{X11}(B)$ for $t = 1, \dots, r$ and $t = n - r + 1, \dots, n$.

To proceed further, we split $\omega_{S_t}^{X11}(B)$ into two parts:

$$\begin{aligned} \omega_{S_t}^{X11}(B) &= [1 - \mu_t^{(3)}(B)]\lambda_{2t}(B)[1 - H_t(B)] \\ &+ [1 - \mu_t^{(3)}(B)]\lambda_{2t}(B)H_t(B)[1 - \mu_t^{(2)}(B)]\lambda_{1t}(B)[1 - \mu_t^{(1)}(B)] \end{aligned} \quad (23)$$

and consider application of the two parts separately. First, note that both parts annihilate constants for all t since both $1 - H_t(B)$ and $[1 - \mu_t^{(2)}(B)]\lambda_{1t}(B)[1 - \mu_t^{(1)}(B)]$ (with the substitutions needed at $t = 1, \dots, 12$ and $t = n-11, \dots, n$) do so. Hence, $\omega_{S_t}^{X11}(B)$ annihilates constants for all t . We now consider what both parts of (23) do when applied to polynomials of degree $1, \dots, 5$.

Consider the first part of (23). For $t = 6 + m_2 + p + 1, \dots, n - (m_2 + p + 6)$, the symmetric version, $[1 - \mu(B)]\lambda_2(B)[1 - H(B)]$ applies, which, from Lemma 1, includes $(1 - B)^3(1 - F)^3$, and so it annihilates polynomials up to degree 5. Now consider applying the first part of (23) to $\xi_t = t$ for $t \leq 6 + m_2 + p$. First, for $t > p$ the symmetric $H(B)$ is used, so that, for these t , $1 - H_t(B)$ annihilates $\xi_t = t$, but this is not the case for $t \leq p$. Then, $\lambda_{2t}(B)[1 - H_t(B)]t = 0$ for t where $\lambda_{2t}(B)$ does not reach to $t = 1, \dots, p$, which is for $t > m_2 + p$; for $t \leq m_2 + p$, $\lambda_{2t}(B)[1 - H_t(B)]t \neq 0$. Applying $1 - \mu_t^{(3)}(B)$ to this result yields zero when it applies only to the zero values, which is for $t = 6 + m_2 + p + 1, \dots, n - (m_2 + p + 6)$. Similarly, $[1 - \mu_t^{(3)}(B)]\lambda_{2t}(B)[1 - H_t(B)]t \neq 0$ for $t > n - (m_2 + p + 6)$. So the first part of $\omega_{S_t}^{X_{11}}(B)$ annihilates t only for $t = 6 + m_2 + p + 1, \dots, n - (m_2 + p + 6)$, where it actually annihilates polynomials up to degree 5. The same reasoning applies to t^h for $2 \leq h \leq 5$.

Consider now the second part of (23). Its half-length is r , and for $t = r + 1, \dots, n - r$ its symmetric version is applied. From Lemma 1, this contains $(1 - B)^3(1 - F)^3$ and so it annihilates polynomials up to degree 5. For $t < r$, consider first the application of $[1 - \mu_t^{(2)}(B)]\lambda_{1t}(B)[1 - \mu_t^{(1)}(B)]$ to a polynomial ξ_t . Since $1 - \mu(B)$ contains $(1 - B)(1 - F)$, where $[1 - \mu_t^{(1)}(B)]\xi_t$ is computed we get 0 for linear ξ_t , and a constant for quadratic ξ_t . These are reproduced by asymmetric $\lambda_{1t}(B)$ (which is an MA), yielding a constant for $t = 7, \dots, n - 6$. This constant is then annihilated by $1 - \mu_t^{(2)}(B)$ to give zero for all t (with the substitutions made at the ends of the series). So $[1 - \mu_t^{(2)}(B)]\lambda_{1t}(B)[1 - \mu_t^{(1)}(B)]\xi_t = 0$ for up to quadratic ξ_t . Subsequent application of $[1 - \mu_t^{(3)}(B)]\lambda_{2t}(B)H_t(B)$ still yields zero. Similar reasoning applies for $t > n - r$. So $[1 - \mu_t^{(3)}(B)]\lambda_{2t}(B)H_t(B)[1 - \mu_t^{(2)}(B)]\lambda_{1t}(B)[1 - \mu_t^{(1)}(B)]$ contains $(1 - B)^3$ for all t .

Now $[1 - \mu_t^{(1)}(B)]t^3 = [1 - \mu(B)]t^3$ is a linear function where it is computed, and this is reproduced by symmetric $\lambda_1(B)$ but not (from Lemma 2(a)) by asymmetric $\lambda_{1t}(B)$. Hence, we get zero from subsequent application of $1 - \mu_t^{(2)}(B)$ only for $t = 13 + m_1, \dots, n - (12 + m_1)$. When $[1 - \mu_t^{(3)}(B)]\lambda_{2t}(B)H_t(B)$ is applied to the result, we won't get zero when its weights extend below $13 + m_1$ (or above $n - (12 + m_1)$). The half-length of $[1 - \mu(B)]\lambda_2(B)H(B)$ is $6 + m_2 + p$, and so we get $[1 - \mu_t^{(3)}(B)]\lambda_{2t}(B)H_t(B)[1 - \mu_t^{(2)}(B)]\lambda_{1t}(B)[1 - \mu_t^{(1)}(B)]t^3 = 0$ only for $t \geq 13 + m_1 + 6 + m_2 + p = r$ and $t < n - r$, which is the range where the symmetric $\omega_S^{X_{11}}(B)$ applies. Thus, for $t = 1, \dots, r$ and $t = n - r + 1, \dots, n$, $[1 - \mu_t^{(3)}(B)]\lambda_{2t}(B)H_t(B)[1 - \mu_t^{(2)}(B)]\lambda_{1t}(B)[1 - \mu_t^{(1)}(B)]$ contains only $(1 - B)^3$.

Putting the results for the two parts of (23) together, we see that (i) for $t = 1, \dots, 6 + m_2 + p$ and $t = n - (5 + m_2 + p), \dots, n$, $\omega_{S_t}^{X_{11}}(B)$ contains $(1 - B)$, (ii) for $t = 7 + m_2 + p, \dots, r$ and $t = n - r + 1, \dots, n - (6 + m_2 + p)$, $\omega_{S_t}^{X_{11}}(B)$ contains $(1 - B)^3$, and (iii) for $t = r + 1, \dots, n - r + 1$, $\omega_{S_t}^{X_{11}}(B) = \omega_S^{X_{11}}(B)$, which contains $(1 - B)^5$. The properties of $\omega_{N_t}^{X_{11}}(B) = 1 - \omega_{S_t}^{X_{11}}(B)$ for reproducing polynomials follow immediately from these results.

For $\omega_{T_t}^{X_{11}}(B) = H_t(B)\omega_{N_t}^{X_{11}}(B)$ and $\omega_{I_t}^{X_{11}}(B) = [1 - H_t(B)]\omega_{N_t}^{X_{11}}(B)$, note the fol-

lowing. (i) Asymmetric $H_t(B)$ will reproduce constants, as will $\omega_{N_t}^{X11}(B)$, so $\omega_{T_t}^{X11}(B)$ will reproduce, and $\omega_{I_t}^{X11}(B)$ will annihilate, constants for all t . (ii) Symmetric $H(B)$ reproduces cubics, so for t where the $H_t(B)$ applied last is the symmetric $H(B)$, and where $\omega_{N_t}^{X11}(B)$ reproduces quadratics, $\omega_{T_t}^{X11}(B)$ will reproduce quadratics. This occurs for $t = 7 + m_2 + 2p, \dots, r + p$ and for $t = n + 1 - (r + p), \dots, n - (6 + m_2 + 2p)$. For these t , $\omega_{I_t}^{X11}(B)$ contains $(1 - B)^3$ and annihilates quadratics. (iii) For $t = r + p + 1, \dots, n - (r + p)$, the symmetric X-11 trend and irregular filters apply, and from Table 3 $\omega_T^{X11}(B)$ reproduces cubics while $\omega_I^{X11}(B)$ contains $(1 - B)^4$ and annihilates cubics.

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