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CONTROLLED ROUNDING OF THREE DIMENSIONAL TABLES

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ABSTRACT

The objective of this report is to present a heuristic procedure to find controlled roundings of three dimensional tables. The problem of three dimension controlled rounding is far more difficult than its two dimension counterpart primarily because the underlying network structure of the two dimensional problem does not exist in three dimensions. In fact, given an arbitrary three dimensional table, a controlled rounding may not exist. We present in this paper a heuristic procedure for finding a controlled rounding for three dimensional tables which has been extremely successful under extensive testing.

I. INTRODUCTION

The objective of this report is to present a heuristic procedure to find controlled roundings of three dimensional tables. The problem of three dimension controlled rounding is far more difficult than its two dimension counterpart primarily because the underlying network structure of the two dimensional problem does not exist in three dimensions. In fact, given an arbitrary three dimensional table, a controlled rounding may not exist. We present in this paper a heuristic procedure for finding a controlled rounding for three dimensional tables which has been extremely successful under extensive testing.

After introductory remarks in Section I, Section II begins with a very explicit description of controlled rounding for a two dimensional table. The fundamental steps in forming a controlled rounding of a two dimensional table are to set up a system of linear equations, formulate a network flow problem, model that system of equations, find a saturated flow through the network and interpret the flow as a controlled rounding of the original table. In Section III we begin a similar process for three dimensional tables. Up to the stage of solving a system of equations, both problems are structurally identical. However, we exhibit three dimension tables which fail to have zero-restricted controlled roundings, weakly zero-restricted controlled roundings, and in fact any controlled roundings at all. The underlying problem is that in three dimensions there is no single network as was employed in the two dimensional case. In Section IV we present a heuristic procedure for finding controlled roundings of three dimensional tables. The model we introduce is based on a sequence of network flow problems; the solution of each reduces the size of the table to be rounded. We then extract a controlled rounding of the original table from this sequence of solutions to the individual network flow problems. In Section V we discuss software developed at the Census Bureau to implement the heuristic procedure introduced in the preceeding section and report on program performance. In Section VI we describe an alternative definition of controlled rounding and discuss programs to implement the alternative procedure. Lastly, Section VII is a brief summary of findings.

We continue this introduction by establishing terminology, notation, and definitions to lay the groundwork for describing procedures for finding a controlled rounding for two dimensional tables. Most simply stated, the problem is as follows.

Given a positive integer $\underline{b}$ and an additive table of non-negative integers

$$A = \begin{array}{c|cccc} a_{00} & a_{01} & a_{02} & \cdots & a_{0C} \\ \hline a_{10} & a_{11} & a_{12} & \cdots & a_{1C} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2C} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{R0} & a_{R1} & a_{R2} & \cdots & a_{RC} \end{array} ,$$

that is,

$$\sum_{j=1}^{C} a_{ij} = a_{i0} \qquad i=1,\ldots,R$$

$$\sum_{i=1}^{R} a_{ij} = a_{0j} \qquad j=1,\ldots, C$$

$$a_{ij} \geq 0 \qquad i=1,\ldots,R \text{ and } j=1,\ldots C$$

find an additive table

$$B = \begin{array}{c|cccc} b_{00} & b_{01} & b_{02} & \cdots & b_{0C} \\ \hline b_{10} & b_{11} & b_{12} & \cdots & b_{1C} \\ b_{20} & b_{21} & b_{22} & \cdots & b_{2C} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ b_{R0} & b_{R1} & b_{R2} & \cdots & b_{RC} \end{array}$$

where

$$\sum_{j=1}^{C} b_{ij} = b_{i0} \qquad i=1,\ldots,R$$

$$\sum_{i=1}^{R} b_{ij} = b_{0j} \qquad j=1,\ldots,C \text{ and}$$

(1) $$b_{ij} = [a_{ij}/b]b \text{ or } [a_{ij}/b]b+b \qquad i=0,\ldots,R \quad j=0,\ldots,C,$$

where [x] stands for the greatest integer less than or equal to x. Such a table B is called a controlled rounding of A. If we replace (1) by

(2) $$|b_{ij} - a_{ij}| < b$$

we say B is a zero-restricted controlled rounding of A. Under a zero-restricted controlled rounding, no multiple of the base changes. If we replace (1) by

(3) $$b_{ij} = \begin{cases} [a_{ij}/b]b \text{ or } [a_{ij}/b]b+b & \text{if } a_{ij} > 0 \\ 0 & \text{if } a_{ij} = 0 \end{cases}$$

we say B is a weakly zero-restricted controlled rounding of A.

All two dimensional tables have zero-restricted controlled roundings, and under many circumstances one prefers a zero-restricted controlled rounding rather than one that is not zero-restricted, for example, in order to obtain an unbiased controlled rounding, see (Cox, 1987). When seeking a controlled rounding that is minimal with respect to some measure of closeness, zero-restricted controlled roundings do not suffice as discussed in (Greenberg, 1988). There exist three dimensional tables which have controlled roundings yet fail to have zero-restricted controlled roundings, so for three dimensional tables controlled rounding procedures must be developed which are not necessarily zero-restricted. Of course, if zero is to change, it can only go to b since a controlled rounding is always positive. The added definition, weakly zero-restricted, recognizes the special role of zero. That is, under a weakly zero-restricted contolled rounding zero is always to remain zero, but a non-zero multiple of the base can increase by the value of the base.

II. STEP-BY-STEP PROCEDURE FOR SOLVING THE TWO DIMENSIONAL CONTROLLED ROUNDING PROBLEM

In this Section, we go though a step by step procedure for finding two dimensional (not necessarily zero-restricted) controlled roundings -- including a step in which the problem is reduced to the solution of a zero-one network flow problem. This being done, we set the stage for the formulation of a heuristic solution to the three dimensional controlled rounding problem. The presentation here follows closely along the lines of (Greenberg, 1988). For a further discussion of the two dimensional problem, we refer the reader to (Cox and Ernst, 1982) and (Cox, 1987).

One first reduces the problem modulo the base. That is, write

(4) $$A = bD + R$$

where D and R are RxC matrices and

$$a_{ij} = bd_{ij} + r_{ij}$$

where $0 \leq r_{ij} < b$ for $i=1,\ldots,R$ and $j=1,\ldots,C$.

Define

$$d_{i0} = \sum_{j=1}^{C} d_{ij} \qquad i=1,\ldots,R$$

$$d_{0j} = \sum_{i=1}^{R} d_{ij} \qquad j=1,\ldots,C$$

$$d_{00} = \sum_{i=1}^{R} \sum_{j=1}^{C} d_{ij} = \sum_{i=1}^{R} d_{i0} = \sum_{j=1}^{C} d_{0j}$$

and

$$r_{i0} = \sum_{j=1}^{C} r_{ij} \qquad i=1,\ldots,R$$

$$r_{0j} = \sum_{i=1}^{R} r_{ij} \qquad j=1,\ldots,C$$

$$r_{00} = \sum_{i=1}^{R} \sum_{j=1}^{C} r_{ij} = \sum_{i=1}^{R} r_{i0} = \sum_{j=1}^{C} r_{0j} .$$

With these added definitions, we have additivity of the following system of tables--including marginal positions:

$$\begin{array}{c|cccc} a_{00} & a_{01} & a_{02} & \cdots & a_{0C} \\ \hline a_{10} & a_{11} & a_{12} & \cdots & a_{1C} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2C} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{R0} & a_{R1} & a_{R2} & \cdots & a_{RC} \end{array} =_{(b)} \begin{array}{c|cccc} d_{00} & d_{01} & d_{02} & \cdots & d_{0C} \\ \hline d_{10} & d_{11} & d_{12} & \cdots & d_{1C} \\ d_{20} & d_{21} & d_{22} & \cdots & d_{2C} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ d_{R0} & d_{R1} & d_{R2} & \cdots & d_{RC} \end{array} + \begin{array}{c|cccc} r_{00} & r_{01} & r_{02} & \cdots & r_{0C} \\ \hline r_{10} & r_{11} & r_{12} & \cdots & r_{1C} \\ r_{20} & r_{21} & r_{22} & \cdots & r_{2C} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ r_{R0} & r_{R1} & r_{R2} & \cdots & r_{RC} \end{array} ,$$

so equation (4) applies to the entire tables.

Forming a (zero-restricted, weakly zero-restricted) controlled rounding, S of R, the sum

$$B = bD+S$$

will be a (zero-restricted, weakly zero-restricted) controlled rounding of A. Thus, our objective is to form a controlled rounding of R noting that $r_{ij} < b$ for i=1,...,R and j=1,...,C.

Next, "fold-in" the RxC table R to form an R+1 by C+1 table in which all marginals are multiples of the base by adding a slack row and slack column. That is, we form the table

$$C = \begin{array}{c|ccccc}
c_{00} & c_{01} & c_{02} & \cdots & c_{0C} & c_{0,C+1} \\
\hline
c_{10} & c_{11} & c_{12} & \cdots & c_{1C} & c_{1,C+1} \\
c_{20} & c_{21} & c_{22} & \cdots & c_{2C} & c_{2,C+1} \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
c_{R0} & c_{R1} & c_{R2} & \cdots & c_{RC} & c_{R,C+1} \\
c_{R+1,0} & c_{R+1,1} & c_{R+1,2} & \cdots & c_{R+1,C} & c_{R+1,C+1}
\end{array}$$

where

$$c_{ij} = r_{ij} \qquad i=1,\ldots,R \text{ and } j=1,\ldots,C$$

$$c_{i,C+1} = [(\sum_{j=1}^{C} c_{ij})/b]b+b - \sum_{j=1}^{C} c_{ij} \qquad i=1,\ldots,R$$

$$c_{R+1,j} = [(\sum_{i=1}^{R} c_{ij})/b]b+b - \sum_{i=1}^{R} c_{ij} \qquad j=1,\ldots,C$$

$$c_{R+1,C+1} = \sum_{i=1}^{R} \sum_{j=1}^{C} c_{ij} - [\sum_{i=1}^{R} \sum_{j=1}^{C} c_{ij}]$$

$$c_{i0} = [(\sum_{j=1}^{C} c_{ij})/b]b+b \qquad i=1,\dots,R$$

$$c_{0j} = [(\sum_{i=1}^{R} c_{ij})b/]b+b \qquad j=1,\dots,C$$

$$c_{R+1,0} = \sum_{j=1}^{C} c_{R+1,j} \qquad j=1,\dots,C+1$$

$$c_{0,C+1} = \sum_{i=1}^{R} c_{i,C+1} \qquad i=1,\dots,R+1$$

$$c_{00} = \sum_{i=1}^{R+1} \sum_{j=1}^{C+1} c_{ij} = \sum_{i=1}^{R+1} c_{i0} = \sum_{j=1}^{C+1} c_{0j} \ .$$

Note that all marginal values of C are multiples of b.

Let us take a 4x4 table, A, and follow the steps through to this point with rounding base b=3:

A=

119	24	40	18	37
15	4	8	3	0
41	7	13	1	20
19	1	5	9	4
44	12	14	5	13

3D=

102	21	33	15	33
12	3	6	3	0
36	6	12	0	18
15	0	3	9	3
39	12	12	3	12

R=

17	3	7	3	4
3	1	2	0	0
5	1	1	1	2
4	1	2	0	1
5	0	2	2	1

It is easy to see that

$$A = 3D + R$$

and

$$C = \begin{array}{r|rrrrr} 36 & 6 & 9 & 6 & 6 & 9 \\ \hline 6 & 1 & 2 & 0 & 0 & 3 \\ 6 & 1 & 1 & 1 & 2 & 1 \\ 6 & 1 & 2 & 0 & 1 & 2 \\ 6 & 0 & 2 & 2 & 1 & 1 \\ 12 & 3 & 2 & 3 & 2 & 2 \end{array} .$$

Returning to the main development, our objective is to reassign values for c_{ij} for $i=1,\ldots,R+1$ and $j=1,\ldots,C+1$ as either 0 or b while maintaining additively to the marginals. Having done this, and calling the new values f_{ij} and the new table F (with the same marginals as C), observe

$$F = \begin{array}{c|cccccc} c_{00} & c_{01} & c_{02} & \cdots & c_{0C} & c_{0,C+1} \\ \hline c_{10} & f_{11} & f_{12} & \cdots & f_{1C} & f_{1,C+1} \\ c_{20} & f_{21} & f_{22} & \cdots & f_{2C} & f_{2,C+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ c_{R0} & f_{R1} & f_{R2} & \cdots & f_{RC} & f_{R,C+1} \\ c_{R+1,0} & f_{R+1,1} & f_{R+1,2} & \cdots & f_{R+1,C} & f_{R+1,C+1} \end{array}$$

is a controlled rounding of C.

To be somewhat more formal, we solved the following system of equations for f_{ij} for $i=1,\ldots,R+1$ and $j=1,\ldots,C+1$:

$$\sum_{j=1}^{C+1} f_{ij} = c_{i0} \qquad i=1,\ldots,R+1$$

$$\sum_{i=1}^{R+1} f_{ij} = c_{0j} \qquad j=1,\ldots,C+1$$

$$f_{ij} \in \{0,b\} \qquad i=1,\ldots,R+1 \text{ and } j=1,\ldots,C+1.$$

This system does have solutions, and one of the ways to find a solution is to obtain a saturated flow through the complete directed bipartite capacitiated network shown in Figure 1, see (Cox, Fagan, Hemmig, Greenberg, 1986 and Gondran and Minoux, 1984). Nodes correspond to marginal constraints, all arcs flow from left to right, and the directed arc between node n_{i0} and n_{0j} corresponds to cell (i,j) in table C, c_{ij}, where i=1,...,R and j=1,...,C.

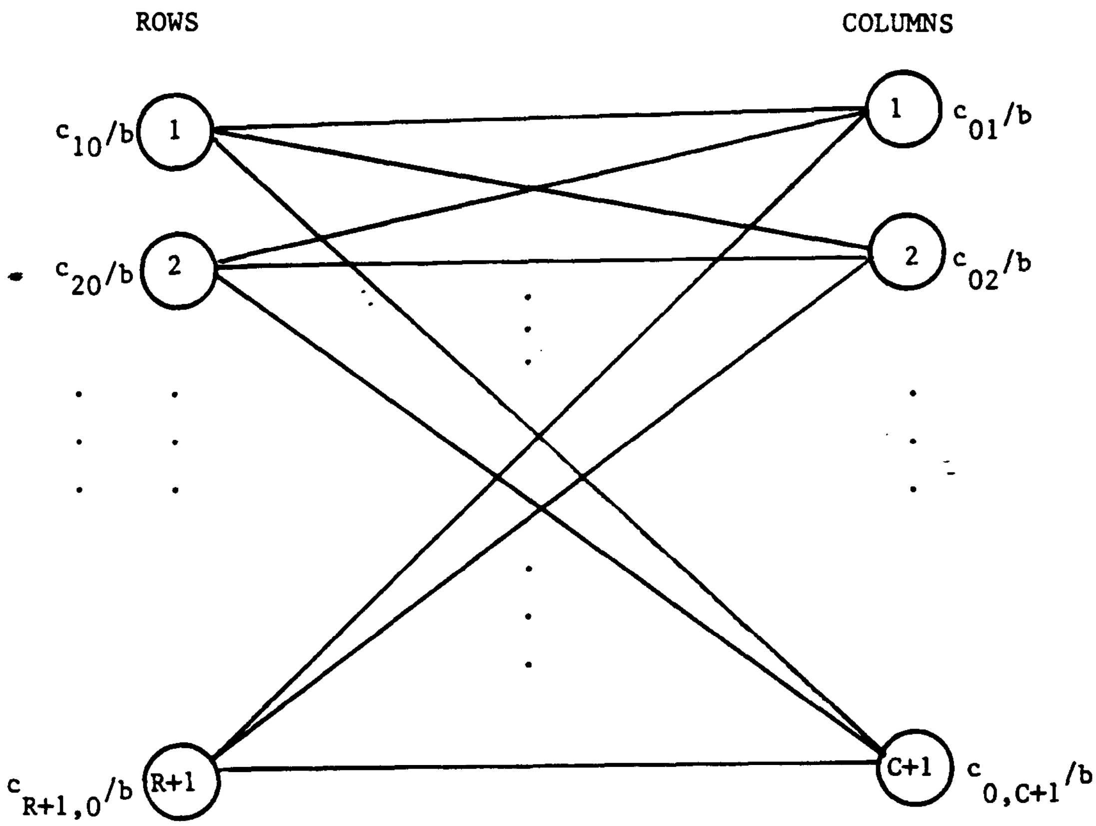

FIGURE 1

The nodes on the left correspond to sources, nodes on the right correspond to sinks, supplies (row marginal values) and demands (column marginal values) are shown alongside each source and sink respectively, and each arc has upper capacity equal to one. A saturated flow does exist, and we set f_{ij} equal to the flow over arc (n_{i0}, n_{0j}) times b.

The table

$$
S = \begin{array}{c|cccc}
f_{00} & f_{01} & f_{02} & \cdots & f_{0C} \\
\hline
f_{10} & f_{11} & f_{12} & \cdots & f_{1C} \\
f_{20} & f_{21} & f_{22} & \cdots & f_{2C} \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
f_{R0} & f_{R1} & f_{R2} & \cdots & f_{RC}
\end{array}
$$

where

$$f_{i0} = \sum_{j=1}^{C} f_{ij} \qquad i=1,\ldots,R$$

$$f_{0j} = \sum_{i=1}^{R} f_{ij} \qquad j=1,\ldots,R$$

$$f_{00} = \sum_{i=1}^{R} \sum_{j=1}^{C} f_{ij} = \sum_{i=1}^{R} f_{i0} = \sum_{j=1}^{C} f_{0j}$$

is a controlled rounding of R and

$$B = bD + S$$

is a controlled rounding of A.

Continue with the base 3 example started earlier:

$$
A = \begin{array}{c|cccc}
119 & 24 & 40 & 18 & 37 \\
\hline
15 & 4 & 8 & 3 & 0 \\
41 & 7 & 13 & 1 & 20 \\
19 & 1 & 5 & 9 & 4 \\
44 & 12 & 14 & 5 & 13
\end{array}
\qquad
3D = \begin{array}{c|cccc}
102 & 21 & 33 & 15 & 33 \\
\hline
12 & 3 & 6 & 3 & 0 \\
36 & 6 & 12 & 0 & 18 \\
15 & 0 & 3 & 9 & 3 \\
39 & 12 & 12 & 3 & 12
\end{array}
$$

R =

17	3	7	3	4
3	1	2	0	0
5	1	1	1	2
4	1	2	0	1
5	0	2	2	1

C =

36	6	9	6	6	9
6	1	2	0	0	3
6	1	1	1	2	1
6	1	2	0	1	2
6	0	2	2	1	1
12	3	2	3	2	2 .

Obtain a base 3 controlled rounding, F of C, extract the upper left RxC subtable with derived marginals, S, (which will be a base 3 controlled rounding of R), and form

$$B = 3D + S$$

which will be a base 3 controlled rounding of A. Below we display five base 3 controlled roundings, F of C, the corresponding controlled rounding, S of R, and finally the controlled rounding, B of A (which was our objective all along).

F_i — S_i — B_i

1.

F_1:

36	6	9	6	6	9
6	0	3	0	0	3
6	3	0	0	3	0
6	0	3	0	0	3
6	0	0	3	3	0
12	3	3	3	0	3

S_1:

18	3	6	3	6
3	0	3	0	0
6	3	0	0	3
3	0	3	0	0
6	0	0	3	3

B_1:

120	24	39	18	39
15	3	9	3	0
42	9	12	0	21
18	0	6	9	3
45	12	12	6	15

2.

F_2:

36	6	9	6	6	9
6	3	3	0	0	0
6	0	3	0	0	3
6	3	0	0	3	0
6	0	0	3	0	3
12	0	3	3	3	3

S_2:

18	6	6	3	3
6	3	3	0	0
3	0	3	0	0
6	3	0	0	3
3	0	0	3	0

B_2:

120	27	39	18	36
18	6	9	3	0
39	6	15	0	18
21	3	3	9	6
42	12	12	6	12

3.

36	6	9	6	6	9
6	0	0	0	3	3
6	0	3	0	0	3
6	0	3	3	0	0
6	3	3	0	0	0
12	3	0	3	3	3

18	3	9	3	3
3	0	0	0	3
3	0	3	0	0
6	0	3	3	0
6	3	3	0	0

120	24	42	18	36
15	3	6	3	3
39	6	15	0	18
21	0	6	12	3
45	15	15	3	12

One can see that each of the tables in the "B" column is a base 3 controlled rounding of A. Note that B_1 is a zero-restricted controlled rounding, B_2 is weakly zero-restricted, and B_3 is the only table in which a true zero is given the value 3. In all, a wide range of possible roundings exist.

III. CONTROLLED ROUNDING IN THREE DIMENSIONS

In Section II, we defined two dimensional controlled rounding, showed how it can be couched as a solution to a set of linear equalities, and finally how to find the solution by solving a zero-one capacitated network flow problem. In this Section we define the three dimensional controlled rounding problem and show how the exact solution found in two dimensions fails to extend to three dimensions. In the following chapters we carefully develop a heuristic procedure to solve the three dimension controlled rounding problem. We begin by establishing notation, terminology, and appropriate definitions. As will be observed, the structure and setting up of the three dimensional problem is identical to the two dimensional problem. The major departure arises when we must solve the derived system of equations to find the actual controlled rounding.

A simple way of representing an RxCxL three dimensional table is shown below. Above the dashed line we have the two dimensional levels k=1,...L adding to the totals face below the line.

REPRESENTING A THREE DIMENSION TABLE

R rows C columns L levels

a_{001}	a_{011}	$a_{021} \cdots a_{0C1}$
a_{101}	a_{111}	$a_{121} \cdots a_{1C1}$
a_{201}	a_{211}	$a_{221} \cdots a_{2C1}$
.	.	.
.	.	.
.	.	.
a_{R01}	a_{R11}	$a_{R21} \cdots a_{RC1}$

a_{002}	a_{012}	$a_{022} \cdots a_{0C2}$
a_{102}	a_{112}	$a_{122} \cdots a_{1C2}$
a_{202}	a_{212}	$a_{222} \cdots a_{2C2}$
.	.	. ···
.	.	.
.	.	.
a_{R02}	a_{R12}	$a_{R22} \cdots a_{RC2}$

···

a_{00L}	a_{01L}	$a_{02L} \cdots a_{0CL}$
a_{10L}	a_{11L}	$a_{12L} \cdots a_{1CL}$
a_{20L}	a_{21L}	$a_{22L} \cdots a_{2CL}$
.	.	. .
.	.	. .
.	.	. .
a_{R0L}	a_{R1L}	$a_{R2L} \cdots a_{RCL}$

---

a_{000}	a_{010}	$a_{020} \cdots a_{0C0}$
a_{100}	a_{110}	$a_{120} \cdots a_{1C0}$
a_{200}	a_{210}	$a_{220} \cdots a_{2C0}$
.	.	.
.	.	.
.	.	.
a_{R00}	a_{R10}	$a_{R20} \cdots a_{RC0}$

WHERE

(5) $$\sum_{k=1}^{L} a_{ijk} = a_{ij0} \qquad i=1,\dots,R \text{ and } j=1,\dots,C$$

(6) $$\sum_{j=1}^{C} a_{ijk} = a_{i0k} \qquad i=1,\dots,R \text{ and } k=1,\dots,L$$

(7) $$\sum_{i=1}^{R} a_{ijk} = a_{0jk} \qquad j=1,\dots,C \text{ and } k=1,\dots,L$$

(8) $$0 \leq a_{ijk} \qquad i=1,\dots,R \;\; j=1,\dots,C \;\; k=1,\dots,L$$

(9) $$\sum_{j=1}^{C} a_{0jk} = \sum_{i=1}^{R} a_{i0k} = a_{00k} \qquad k=1,\dots,L$$

(10) $$\sum_{k=1}^{L} a_{0jk} = \sum_{i=1}^{R} a_{ij0} = a_{0j0} \qquad j=1,\dots,C$$

(11) $$\sum_{k=1}^{L} a_{i0k} = \sum_{j=1}^{C} a_{ij0} = a_{i00} \qquad i=1,\dots,R$$

(12) $$\sum_{i=1}^{R} a_{i00} = \sum_{j=1}^{C} a_{0j0} = \sum_{k=1}^{L} a_{00k} = a_{000}$$

Given a positive integer, b, the objective is to find a three dimensional additive table, B, (adapting the representation and notation used for A above) which satisfies (5)-(12) and

(13) $b_{ijk}=[(a_{ijk}/b]b$ or $[(a_{ijk}/b)]b+b$ $i=0,\ldots,R$ $j=0,\ldots,C$, $k=0,\ldots L$,

where [ ] is the greatest integer function.

In this paper we present a heuristic iterative procedure which starts with a three dimensional table A and rounding base b and attempts to find a controlled rounding B. If no solutions are found after a specified number of iterations the program indicates its inability to find a solution. As is shown by several examples in Section IV, there exist tables A and rounding base b for which solutions to the system of equations above do not exist. These are carefully contrived examples. All three dimensional tables we have encountered through a random generation process or based on actual 1980 Decennial Census tabulations do have controlled roundings. Except for specially constructed examples, under extensive testing this heuristic has always yielded controlled roundings in an efficient manner.

We begin exactly as in Section II. If A is an RxCxL table, write

(14) $$A = bD + R$$

where D and R are also RxCxL tables and

$$a_{ijk} = bd_{ijk} + r_{ijk}$$

where $0 \leq r_{ijk} < b$ for $i=1,\ldots,R$ $j=1,\ldots,C$ and $k=1,\ldots,L$.

Defining

$$d_{ij0},\ d_{i0k},\ d_{0jk},\ d_{i00},\ d_{0j0},\ d_{00k},\ d_{000}$$

$$r_{ij0},\ r_{i0k},\ r_{0jk},\ r_{i00},\ r_{0j0},\ r_{00k},\ r_{000}$$

as one would expect, we can extend the additivity in (14) above to include marginal positions. If we form a (zero-restricted, weakly zero-restricted) controlled rounding, S and R, the sum

$$B = bD + S$$

is a (zero-restricted, weakly zero-restricted) controlled rounding of A. Thus, our objective reduces to finding a controlled rounding of R observing that $r_{ijk} < b$ for $i,j,k > 0$. From this point onward, without loss of generality we can assume that b=1 and $c_{ijk} < 1$ for $i,j,k > 0$ by dividing all entries in the table R by b. We continue under this assumption.

Next, "fold in" the RxCxL table R to obtain the (R+1)x(C+1)x(L+1) table C where:

$$c_{ijk} = r_{ijk} \qquad i=1,\ldots,R \qquad j=1,\ldots,C \quad k=1,\ldots,L$$

$$c_{i,j,L+1} = [\sum_{k=1}^{L} c_{ijk}] + 1 - \sum_{k=1}^{L} c_{ijk} \qquad i=1,\ldots,R \quad j=1,\ldots,C$$

$$c_{i,C+1,k} = [\sum_{j=1}^{C} c_{ijk}] + 1 - \sum_{j=1}^{C} c_{ijk} \qquad i=1,\ldots,R \quad k=1,\ldots,L$$

$$c_{R+1,j,k} = [\sum_{i=1}^{R} c_{ijk}] + 1 - \sum_{i=1}^{R} c_{ijk} \qquad j=1,\ldots,C \quad k=1,\ldots,L$$

$$c_{i,C+1,L+1} = \sum_{j=1}^{C} \sum_{k=1}^{L} c_{ijk} - [\sum_{j=1}^{C} \sum_{k=1}^{L} c_{ijk}] \qquad i=1,\ldots,R$$

$$c_{R+1,j,L+1} = \sum_{i=1}^{R} \sum_{k=1}^{L} c_{ijk} - [\sum_{i=1}^{R} \sum_{k=1}^{L} c_{ijk}] \qquad j=1,\ldots,C$$

$$c_{R+1,C+1,k} = \sum_{i=1}^{R} \sum_{j=1}^{C} c_{ijk} - [\sum_{i=1}^{R} \sum_{j=1}^{C} c_{ijk}] \qquad k=1,\ldots,L$$

$$c_{R+1,C+1,L+1} = [\sum_{i=1}^{R} \sum_{j=1}^{C} \sum_{k=1}^{L} c_{ijk}] + 1 - \sum_{i=1}^{R} \sum_{j=1}^{C} \sum_{k=1}^{L} c_{ijk}$$

$$c_{ij0} = [\sum_{k=1}^{L} c_{ijk}] + 1 \qquad i=1,\ldots,R \quad j=1,\ldots,C$$

$$c_{i0k} = [\sum_{j=1}^{C} c_{ijk}] + 1 \qquad i=1,\ldots,R \quad k=1,\ldots,L$$

$$c_{0jk} = [\sum_{i=1}^{R} c_{ijk}] + 1 \qquad j=1,\ldots,C \quad k=1,\ldots,L$$

$$c_{i00} = \sum_{k=1}^{L} c_{i0k} = \sum_{j=1}^{C} c_{ij0} \qquad i=1,\ldots,R$$

$$c_{0j0} = \sum_{k=1}^{L} c_{0jk} = \sum_{i=1}^{R} c_{ij0} \qquad j=1,\ldots,C$$

$$c_{00k} = \sum_{j=1}^{C} c_{0jk} = \sum_{k=1}^{L} c_{i0k} \qquad k=1,\ldots,L$$

$$c_{000} = \sum_{i=1}^{R+1} \sum_{j=1}^{C+1} \sum_{k=1}^{L+1} c_{ijk}$$

$$= \sum_{j=1}^{R+1} \sum_{k=1}^{L+1} c_{0jk} = \sum_{i=1}^{R+1} \sum_{k=1}^{L+1} c_{i0k} = \sum_{i=1}^{R+1} \sum_{j=1}^{C+1} c_{ij0}$$

$$= \sum_{i=1}^{R+1} c_{i00} = \sum_{j=1}^{C+1} c_{0j0} = \sum_{k=1}^{L+1} c_{00k} .$$

As can be seen, this is simply the more complicated counterpart of the C table defined in Section II used with two dimensional controlled rounding. Let us take a 3x3x3 table and follow the steps through to this point. Suppose the base b=3, and the table A is as follows:

	60	17	24	19	52	19	14	19	40	16	9	15	|	152	52	47	53
	28	8	7	13	13	0	3	10	16	6	3	7	|	57	14	13	30
A =	12	9	2	1	24	11	8	5	14	8	0	6	|	50	28	10	12
	20	0	15	5	15	8	3	4	10	2	6	2	|	45	10	24	11

3D=

51	15	21	15
24	6	6	12
9	9	0	0
18	0	15	3

42	15	12	15
12	0	3	9
18	9	6	3
12	6	3	3

33	12	9	12
15	6	3	6
12	6	0	6
6	0	6	0

126	42	42	42
51	12	12	27
39	24	6	9
36	6	24	6

R =

9	2	3	4
4	2	1	1
3	0	2	1
2	0	0	2

10	4	2	4
1	0	0	1
6	2	2	2
3	2	0	1

7	4	0	3
1	0	0	1
2	2	0	0
4	2	0	2

26	10	5	11
6	2	1	3
11	4	4	3
9	4	0	5 .

As discussed above, we seek a base 3 controlled rounding, S of R. Before creating the C table, we divide all entries in R by b=3 to reduce the problem to one in which all marginals are integers and all internal entries of C are less than or equal to one. Having done this, we get C:

7	1	2	2	2
2	2/3	1/3	1/3	2/3
2	0	2/3	1/3	1
1	0	0	2/3	1/3
2	1/3	1	2/3	0

8	2	1	2	3
1	0	0	1/3	2/3
3	2/3	2/3	2/3	1
2	2/3	0	1/3	1
2	2/3	1/3	2/3	1/3

7	2	1	2	2
1	0	0	1/3	2/3
1	2/3	0	0	1/3
2	2/3	0	2/3	2/3
3	2/3	1	1	1/3

9	2	3	3	1
2	1/3	2/3	1	0
3	2/3	2/3	1	2/3
2	2/3	1	1/3	0
2	1/3	2/3	2/3	1/3

31	7	7	9	8
6	1	1	2	2
9	2	2	2	3
7	2	1	2	2
9	2	3	3	1 .

The objective is to replace all internal entries of C with the values zero or one while maintaining table additivity to the marginals. So doing provides a controlled rounding, F of C, as in Section II, which leads to a controlled rounding, B of A. In the next section a heuristic procedure is provided for finding F. Below one such zero-one assignment is displayed, and this example is continued to obtain a controlled rounding of A. One controlled rounding, F of C is:

7	1	2	2	2
2	1	0	1	0
2	0	1	0	1
1	0	0	0	1
2	0	1	1	0

8	2	1	2	3
1	0	0	0	1
3	0	1	1	1
2	1	0	0	1
2	1	0	1	0

7	2	1	2	2
1	0	0	0	1
1	1	0	0	0
2	1	0	1	0
3	0	1	1	1

9	2	3	3	1
2	0	1	1	0
3	1	0	1	1
2	0	1	1	0
2	1	1	0	0

31	7	7	9	8
6	1	1	2	2
9	2	2	2	3
7	2	1	2	2
9	2	3	3	1

Extracting the upper left corner of F (that is, f_{ijk} i=1,...,R j=1,...,C k=1,...,L), deriving the marginals by appropriate summations, and finally multiplying back by b=3, we obtain the table S which is a controlled rounding of R:

S =

9	3	3	3
6	3	0	3
3	0	3	0
0	0	0	0

9	3	3	3
0	0	0	0
6	0	3	3
3	3	0	0

9	6	0	3
0	0	0	0
3	3	0	0
6	3	0	3

27	12	6	9
6	3	0	3
12	3	6	3
9	6	0	3 .

As the final step, we form

$$B = 3D+S$$

to obtain a controlled rounding of A:

B=

60	18	24	18
30	9	6	15
12	9	3	0
18	0	15	3

51	18	15	18
12	0	3	9
24	9	9	6
15	9	3	3

42	18	9	15
15	6	3	6
15	9	0	6
12	3	6	3

153	54	48	51
57	15	12	30
51	27	12	12
45	12	24	9 .

Observe that B is a zero-restricted controlled rounding of A. In general, a three dimensional table does <u>not</u> have a zero-restricted controlled rounding. Consider the 2x2x2 table, A, discussed in (Cox and Ernst, 1982):

A =

2	1	1
1	1	0
1	0	1

2	1	1
1	0	1
1	1	0

4	2	2
2	1	1
2	1	1 .

This table does not have a zero-restricted controlled rounding base 2, however a <u>weakly zero-restricted</u> controlled rounding does exist:

B =

4	2	2	2	0	2	6	2	4
2	2	0	2	0	2	4	2	2
2	0	2	0	0	0	2	0	2 .

The following 6x4x3 table fails to have a zero-restricted <u>or</u> a weakly zero-restricted base 1 controlled rounding:

G=

4	1	1	1	1	4	1	1	1	1	4	1	1	1	1
1	0	1/2	0	1/2	0	0	0	0	0	1	0	1/2	0	1/2
1	1/2	0	1/2	0	0	0	0	0	0	1	1/2	0	1/2	0
0	0	0	0	0	1	1/2	0	0	1/2	1	1/2	0	0	1/2
0	0	0	0	0	1	0	1/2	1/2	0	1	0	1/2	1/2	0
1	1/2	1/2	0	0	1	1/2	1/2	0	0	0	0	0	0	0
1	0	0	1/2	1/2	1	0	0	1/2	1/2	0	0	0	0	0

12	3	3	3	3
2	0	1	0	1
2	1	0	1	0
2	1	0	0	1
2	0	1	1	0
2	1	1	0	0
2	0	0	1	1 .

The most direct way to see that G does not have a zero-restricted controlled rounding is to consider the system of equations

$$\sum_{\substack{i=1 \\ (g_{ijk} \neq 0)}}^{6} x_{ijk} = g_{0jk} \qquad j=1,\ldots,4 \qquad k=1,\ldots,3$$

$$\sum_{\substack{j=1 \\ (g_{ijk} \neq 0)}}^{4} x_{ijk} = g_{i0k} \qquad j=1,\ldots,6 \qquad k=1,\ldots,3$$

$$\sum_{\substack{k=1 \\ (g_{ijk}\neq 0)}}^{3} x_{ijk} = g_{ij0} \qquad i=1,\ldots,6 \qquad j=1,\ldots,4.$$

A zero-one solution to this system would be a zero-restricted controlled rounding of G. However, by writing this system in matrix form,

$$DX = E,$$

one can verify that the matrix D is non-singular. The original table entries g_{ijk} $(i,j,k > 0)$ are the only values which satisfy this system (i.e., allow for an additive table). Thus, A cannot have a zero-restricted controlled rounding. This example was suggested to us by Gale (1987) and is based on a multicommodity flow problem failing to have integer solutions discussed in (Gale, 1960, pp. 173-174).

To show that G cannot have a weakly zero-restricted controlled rounding we proceed as follows. First observe that if B is any controlled rounding of G, then b_{000} must equal 13. For otherwise all marginal values of B would be the same as marginals of G and hence all zeros of G would also be zeros of B; and we saw above that the table G is the only table satisfing those marginals with zeros as they are placed in G. Thus $b_{000} = 13$.

If $b_{000} = 13$, then one of b_{001}, b_{002}, and b_{003} must equal 5 and the other two must equal 4. Assume $b_{001} = 5$, $b_{002} = b_{003} = 4$ and that the zero cells in G must remain zero in B. To find B one must complete the partially completed table below:

5				
	0	b_{121}	0	b_{141}
	b_{211}	0	b_{231}	0
	0	0	0	0
	0	0	0	0
	b_{511}	b_{521}	0	0
	0	0	b_{631}	b_{641}

Level 1

4	1	1	1	1
0	0	0	0	0
0	0	0	0	0
1	b_{312}	0	0	b_{342}
1	0	b_{422}	b_{432}	0
1	b_{512}	b_{522}	0	0
1	0	0	b_{632}	b_{642}

Level 2

4	1	1	1	1
1	0	b_{123}	0	b_{143}
1	b_{213}	0	b_{233}	0
1	b_{313}	0	0	b_{343}
1	0	b_{422}	b_{433}	0
0	0	0	0	0
0	0	0	0	0

Level 3

where b_{ijk}'s denote values to be assigned. Note that $b_{312} + b_{313} \geq 1$ since $b_{310} \geq 1$ and $b_{311} = 0$, so b_{312} or b_{313} must equal 1. By setting $b_{312} = 1$, there is only one way to complete Level 2:

4	1	1	1	1
0	0	0	0	0
0	0	0	0	0
1	1	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	0	0	1

Level 2

4	1	1	1	1
1	0	0	0	1
1	0	0	1	0
1	1	0	0	0
1	0	1	0	0
0	0	0	0	0
0	0	0	0	0

Level 3 .

Since $b_{420} \geq 1$ and $b_{421} = b_{422} = 0$, then b_{423} must be set to 1 leading to a unique Level 3 shown above. But $b_{341} + b_{342} + b_{343} = 0 < b_{340} = 1$. Thus, <u>one cannot assign</u> $b_{312} = 1$. A similar analysis shows that b_{313} cannot be set to 1 so that b_{001} cannot be set to 5. A comparable analysis shows that neither b_{002} nor b_{003} can be set to 5 while leaving zeros in G remain as zeros in B. Accordingly, there is <u>no weakly zero-restricted rounding of G</u>.

However, a controlled rounding of G does exist, namely:

5	2	1	1	1
1	0	1	0	0
1	1	0	0	0
0	0	0	0	0
0	0	0	0	0
1	1	0	0	0
2	0	0	1	1

4	1	1	1	1
0	0	0	0	0
0	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	0
1	1	0	0	0

4	1	1	1	1
1	0	0	0	1
1	0	0	1	0
1	1	0	0	0
1	0	1	0	0
0	0	0	0	0
0	0	0	0	0

13	4	3	3	3
2	0	1	0	1
2	1	0	1	0
2	1	0	0	1
2	0	1	1	0
2	1	1	0	0
3	1	0	1	1 .

Note <u>zeros</u> changed to <u>ones</u> are in positions (6,1,2) and (6,1,0). Six marginal integer positions are increased by one, namely:

(0,0,1), (6,0,1), (0,1,1), (0,0,0), (6,0,0), and (0,1,0).

The controlled rounding exhibited here was obtained using software developed by the authors at the Census Bureau based on the three dimensional heuristic which we discuss in the next section.

Not all three dimensional tables do have a controlled rounding, and over the next several pages we present a procedure for constructing tables failing to have a controlled rounding due to Ernst (1987). Start with the following three dimensional table, E, found by Ernst working independently:

3	0	1	1	1
0	0	0	0	0
1	0	1/2	0	1/2
1	0	0	1/2	1/2
1	0	1/2	1/2	0

3	1	0	1	1
1	0	0	1/2	1/2
0	0	0	0	0
1	1/2	0	0	1/2
1	1/2	0	1/2	0

3	1	1	0	1
1	0	1/2	0	1/2
1	1/2	0	0	1/2
0	0	0	0	0
1	1/2	1/2	0	0

3	1	1	1	0
1	0	1/2	1/2	0
1	1/2	1/2	0	0
1	1/2	0	1/2	0
0	0	0	0	0

12	3	3	3	3
3	0	1	1	1
3	1	1	0	1
3	1	0	1	1
3	1	1	1	0 .

By employing an analysis as for the example above, one can see that there does not exist a controlled rounding which leaves all zero values fixed. However, there does exist a controlled rounding, H^1 of E, shown below, and using arguments similiar to those above for the Gale example, G, we observe that for every controlled rounding of E, the grand total, E_{000} rounds to <u>13</u>:

$H_1 =$

3	0	1	1	1
0	0	0	0	0
1	0	1	0	0
1	0	0	1	0
1	0	0	0	1

3	1	0	1	1
1	0	0	1	0
0	0	0	0	0
1	0	0	0	1
1	1	0	0	0

3	1	1	0	1
1	0	0	0	1
1	1	0	0	0
0	0	0	0	0
1	0	1	0	0

4	1	1	1	1
1	0	1	0	0
1	0	0	0	1
1	1	0	0	0
1	0	0	1	0

13	3	3	3	4
3	0	1	1	1
3	1	1	0	1
3	1	0	1	1
4	1	1	1	1 .

To exhibit a three dimensional table which fails to have a controlled rounding Ernst proceeded as follows. Form the 12x12x4 table:

$E^1 =$

E	0	0
0	E	0
0	0	E

where in this representation "E" consists of the internal entries of the table E above and 0 represents the 4x4x4 zero matrix. The marginals of E^1 are not shown, but they are the same as those of E respecting the block structure. Suppose that table B is a controlled rounding of E^1 and represent the (internal entries) of B as

$B =$

B^1	B^2	B^3
B^4	B^5	B^6
B^7	B^8	B^9

.

Since integers cannot decrease under controlled rounding, the sum of internal entries in each of B^1, B^5 and B^9 must be at least 12. If

$$B^2 = B^3 = B^4 = B^6 = B^7 = B^8 = 0,$$

then because of block structure of E^1, each of B^1, B^5 and B^9 must be a controlled rounding of E so the sum of internal entries of B^1, B^5 and B^9 is 13 by the argument above. Thus the total of all internal entries of B must be 39. But this is impossible because the grand total of E^1 is 36 and under any controlled rounding cell values can increase by at most one unit.

We continue this line of argument by observing that since integers cannot decrease under controlled rounding, the sum of internal entries in B^1, B^5 and B^9 must be at least 12. Thus no <u>two</u> elements from the set

$$\{B^2\ B^3\ B^4\ B^6\ B^7\ B^8\}$$

can contain a one. Hence exactly one element from that set must contain a one based on the discussion above. Without loss of generality we can assume that B^2 contains a single one, and so because of the block structure of B, B^9 must be a controlled rounding of E. Thus the sum of internal entries of B^9 must equal 13 and (as by the arguments above) the sum of all elements of B must equal 38. This also contradicts the fact that B is a controlled rounding of E^1. As all possibilities have been examined, we can conclude that E^1 has no controlled rounding.

To create a smaller three dimensional table which fails to have a controlled rounding we form the 8x8x4 table, E^2,

$$E^2 = \begin{array}{|c|c|} \hline E & 0 \\ \hline 0 & E \\ \hline \end{array}$$

where "E" in this representation consists of the internal entries of table E and 0 is the 4x4x4 zero matrix. The marginals of E^2 (not shown) are the same as those of E respecting the block structure, and the totals face of E^2 (Level 0) is:

24	3	3	3	3	3	3	3	3
3	0	1	1	1	0	0	0	0
3	1	1	0	1	0	0	0	0
3	1	0	1	1	0	0	0	0
3	1	1	1	0	0	0	0	0
3	0	0	0	0	0	1	1	1
3	0	0	0	0	1	1	0	1
3	0	0	0	0	1	0	1	1
3	0	0	0	0	1	1	1	0 .

Let B be an arbitrary controlled rounding of E^2 and write the internal entries of B (in block form) as:

$$B^0 = \begin{array}{|c|c|} \hline B^1 & B^2 \\ \hline B^3 & B^4 \\ \hline \end{array},$$

and let

$$b^s_{000} = \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{4} b^s_{ijk} \qquad s=1,\ldots,4 ,$$

with other marginals defined similarly. Note that

$$b_{000} \le 25, \quad b^1_{000} \ge 12, \quad b^4_{000} \ge 12, \quad \text{and } b_{000} = \sum_{s=1}^{4} b^s_{000} .$$

If $b^2_{000} = b^3_{000} = 0$, then B^1 and B^4 are both controlled roundings of E^1 so

$$b^1_{000} = b^4_{000} = 13$$

which contradicts the fact that

$$b^1_{000} + b^4_{000} = b_{000} \le 25.$$

The totals b^2_{000} and b^3_{000} cannot both be greater than zero since, once again the relation

$$b_{000} = \sum_{s=1}^{4} b^s_{000} \le 25$$

will be violated. Without loss of generality we can assume that

$$b^2_{000} = 1 \text{ and } b^3_{000} = 0,$$

and let

$$b^2_{i_1 j_1 k_1} = 1$$

for a unique $1 \le i_1 \le 4$, $1 \le j_1 \le 4$, and $1 \le k_1 \le 4$ and

$$b^2_{ijk} = 0$$

otherwise. It follows that

$$b^1_{000} = b^4_{000} = 12,$$

and

$$b_{i0k} = b^1_{i0k} = e_{i0k} \quad (i,k)\neq(i_1,k_1),\ i=1,\dots,4 \quad k=1,\dots,4.$$

If

$$b_{i_1 0 k_1} = 2$$

then

$$e_{i_1 0 k_1} = 1 \text{ and } b^1_{i_1 0 k_1} = 1$$

and as

$$b^1_{i0k} = b_{i0k}\varepsilon\{0,1\} \quad \text{for } (i,k)\neq(i_1,k_1) \quad i=1,\dots,4 \quad k=1,\dots,4,$$

B^1 would be a controlled rounding of E with $b^1_{000} = 12$, which as shown above is impossible. Thus, we can assume

$$b_{i_1 0 k_1} = 1,$$

and so

$$b^1_{i_1 0 k_1} = 0.$$

If

$$e_{i_1 0 k_1} = 0$$

then B^1 would be a controlled rounding of E with $b^1_{000} = 12$, which, once again is impossible. Thus, assume

$$e_{i_1 0 k_1} = 1.$$

Since no marginal values of E^2 can decrease under a controlled rounding,

$$b^1_{ij0} = e_{ij0} \text{ and } b^1_{0jk} = e_{0jk} \quad i=1,\dots,4 \quad j=1,\dots,4 \quad k=1,\dots,4.$$

Thus

$$b^1_{i_1 00} = \sum_{k=1}^{4} b^1_{i_1 0k} = \sum_{j=1}^{4} b^1_{i_1 j0} = \sum_{j=1}^{4} e_{i_1 j0} = e_{i_1 00} \ (=3)$$

and so there exists $1 \le k_0 \le 4$ such that $k_0 \neq k_1$ and

$$b^1_{i_1 0k_0} = e_{i_1 0k_0} + 1.$$

But

$$b^1_{00k_0} = \sum_{i=1}^{4} b^1_{i0k_0} = \sum_{j=1}^{4} b^1_{0jk_0} = \sum_{j=1}^{4} e_{0jk_0} = e_{00k_0} \ (=3),$$

so there exists, $0 \le i_0 \le 4$ such that $i_0 \neq i_1$ and

$$b^1_{i_0 0k_0} = e_{i_0 0k_0} - 1.$$

However, by construction, since

$$i_0 \neq i_1 \text{ and } k_0 \neq k_1,$$

it follows that

$$b_{i_0 jk_0} = 0 \text{ for all } j=5,\dots,8$$

and so

$$b_{i_0 0k_0} = b^1_{i_0 0k_0} < e_{i_0 0k_0}$$

This would be a contradiction of the fact that integer values cannot decrease under a controlled rounding. Thus the table E^2 fails to have a controlled rounding.

The fact that example E^2 failed to have a controlled rounding was more difficult to see than for E^1, however, we did want to find as small a table as possible having no controlled rounding. In addition, the following conjectures seem plausible.

Conjectures: Every three dimensional table with integer marginals having at least one dimension less than 4 does have a controlled rounding. Every table having fewer than 256 interior cells does have a controlled rounding.

A rather subtle extension of the examples above, E^1 and E^2, was also observed by Ernst (1987). We couch the discussion in terms of E^2. Note there are 256 internal cells in E^2, and let $\varepsilon = 1/300$. Form the table E^3 having internal entries

$$e^3_{ijk} = e^2_{ijk} + \varepsilon$$

and derive marginals by addition. All table values of E^3 (including marginals) fail to be integer, in fact,

$$e^2_{ijk} < e^3_{ijk} < e^2_{ijk} + 1.$$

One can see that E^3 fails to have a controlled rounding by noting that any controlled rounding of E^3 is also a controlled rounding of E^2. The table E^3 differs from E^2 to the extent that no entries of E^3 are integers. That is, the failure of E^2 to have a controlled rounding is not a function of the behavior of integer table values under controlled rounding.

We could have used the Gale example, G, rather than the Ernst example, E in constructing counter examples by forming

$G^1 =$

G	0	0
0	G	0
0	0	G

and

$G^2 =$

G	0
0	G

and using the identical analysis show that G^1 and G^2 fail to have a controlled rounding. The crucial factor is that for every controlled rounding of both E and G, the grand totals, e_{000} and g_{000}, must increase by one unit. By letting ε = 1/400 we could have defined G^3 where

$$g^3_{ijk} = g^2_{ijk} + \varepsilon$$

and obtained another non-integer table failing to have a controlled rounding.

Even though there do exist three dimensional tables which have no controlled roundings, they must constitute rather rare events. After randomly generating many three dimensional tables and also examining tables that arise in real applications as will be discussed below, all were observed to have controlled roundings.

IV. FINDING CONTROLLED ROUNDINGS IN THREE DIMENSIONS

In Section III, we started with an RxCxL table, A, for which we wish to find a controlled rounding and derived an (R+1)x(C+1)x(L+1) table, C, having integer marginals and internal entries less than or equal to one. If we assign zeros and ones to the internal entries of C and maintain additivity to the marginals, the revised table will lead to a controlled rounding of A as in Section II. The objective of this section is to develop procedures for such an assignment. For ease of exposition, notation, and diagram drawing, we let C be of size 4x3x3. It will be clear how this procedure plays out for arbitrary R, C and L greater than or equal to 2. Let C be written:

c_{001}	$^*c_{011}$	$^*c_{021}$	$^*c_{031}$
$^*c_{101}$	c_{111}	c_{121}	c_{131}
$^*c_{201}$	c_{211}	c_{221}	c_{231}
$^*c_{301}$	c_{311}	c_{321}	c_{331}
c_{401}	c_{411}	c_{421}	c_{431}

c_{002}	$^*c_{012}$	$^*c_{022}$	$^*c_{032}$
$^*c_{102}$	c_{112}	c_{122}	c_{132}
$^*c_{202}$	c_{212}	c_{222}	c_{232}
$^*c_{302}$	c_{312}	c_{322}	c_{332}
c_{402}	c_{412}	c_{422}	c_{432}

c_{003}	$^*c_{013}$	$^*c_{023}$	$^*c_{033}$
$^*c_{103}$	c_{113}	c_{123}	c_{133}
$^*c_{203}$	c_{213}	c_{223}	c_{233}
$^*c_{303}$	c_{313}	c_{323}	c_{333}
c_{403}	c_{413}	c_{423}	c_{433}

---

c_{000}	c_{010}	c_{020}	c_{030}
c_{100}	c_{110}	c_{120}	c_{130}
c_{200}	c_{210}	c_{220}	c_{230}
c_{300}	c_{310}	c_{320}	c_{330}
c_{400}	$^*c_{410}$	$^*c_{420}$	$^*c_{430}$

Set up the <u>directed</u> network in Figure 2, in which nodes correspond to marginal positions (marginal constraints) and a <u>directed</u> arc between two nodes corresponds to the cell determined by the corresponding two marginals. Sources are on the left, sinks on the right, all arcs are directed and flow from <u>left</u> to <u>right</u> and the corresponding supplies and demands are shown next to the appropriate source or sink. Each arc has capacity equal to <u>one unit</u>.

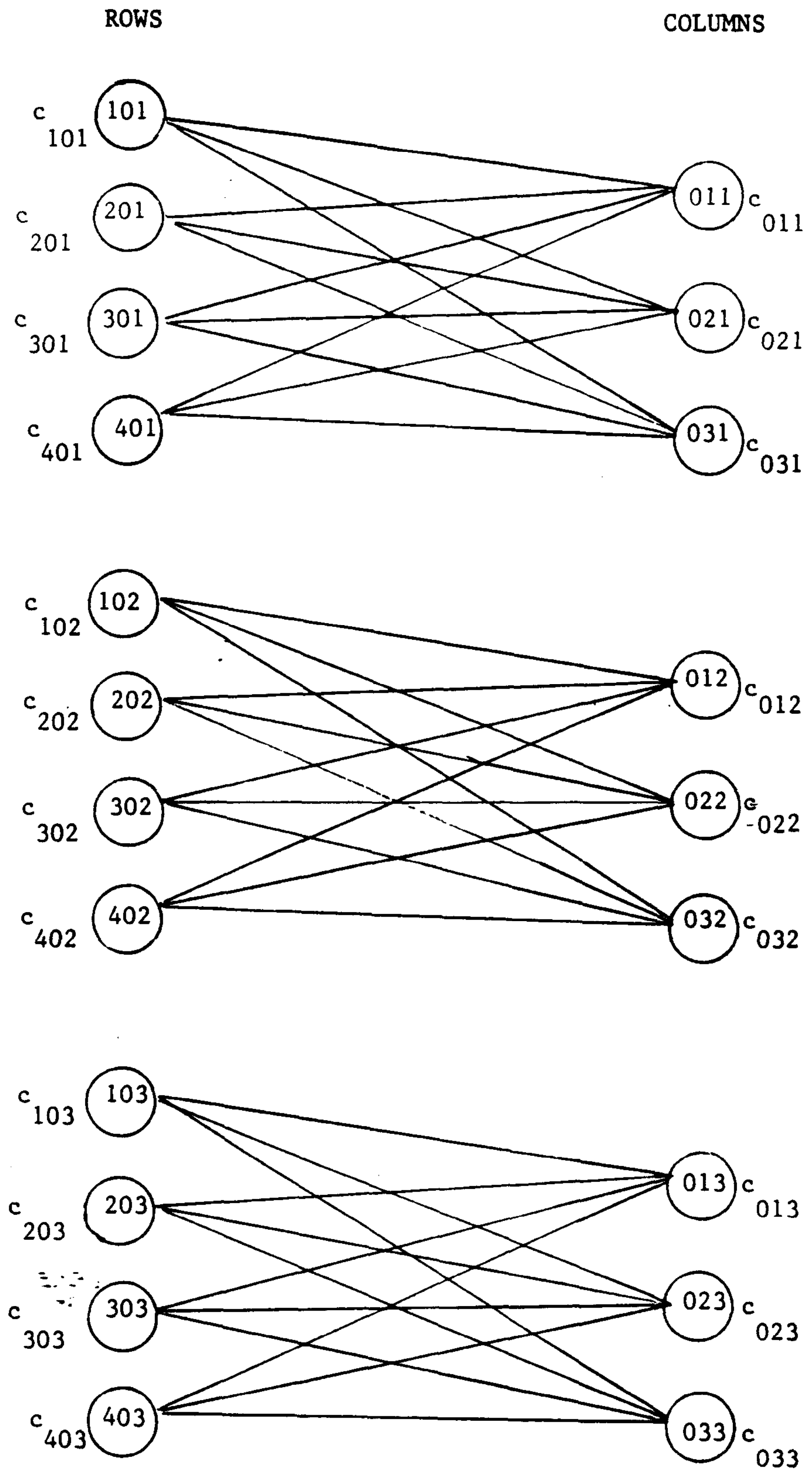
ROWS
COLUMNS
c 101
101
c 201
201
c 301
301
c 401
401
011
c 011
021
c 021
031
c 031
c 102
102
c 202
202
c 302
302
c 402
402
012
c 012
022
c 022
032
c 032
c 103
103
c 203
203
c 303
303
c 403
403
013
c 013
023
c 023
033
c 033

FIGURE 2

Note that all cells in table C are represented, however not all marginal constraints. In particular, the constraints that force additivity of shafts to totals on Level 0 are not represented. The total supply equals the total demand for this network, and the same is true for each of the connected components--one connected component for each face. There does exist a saturated flow on this network--namely the values c_{ijk}--so there must also exist a saturated zero-one flow whicn satisfies each two dimensional level (as in Section II), however the marginal constraints of the form c_{ij0} are not, in general, satisfied.

We will form a new network shown in Figure 3 which does satisfy some of the constraints of the form c_{ij0}.

VERTICAL SHAFTS

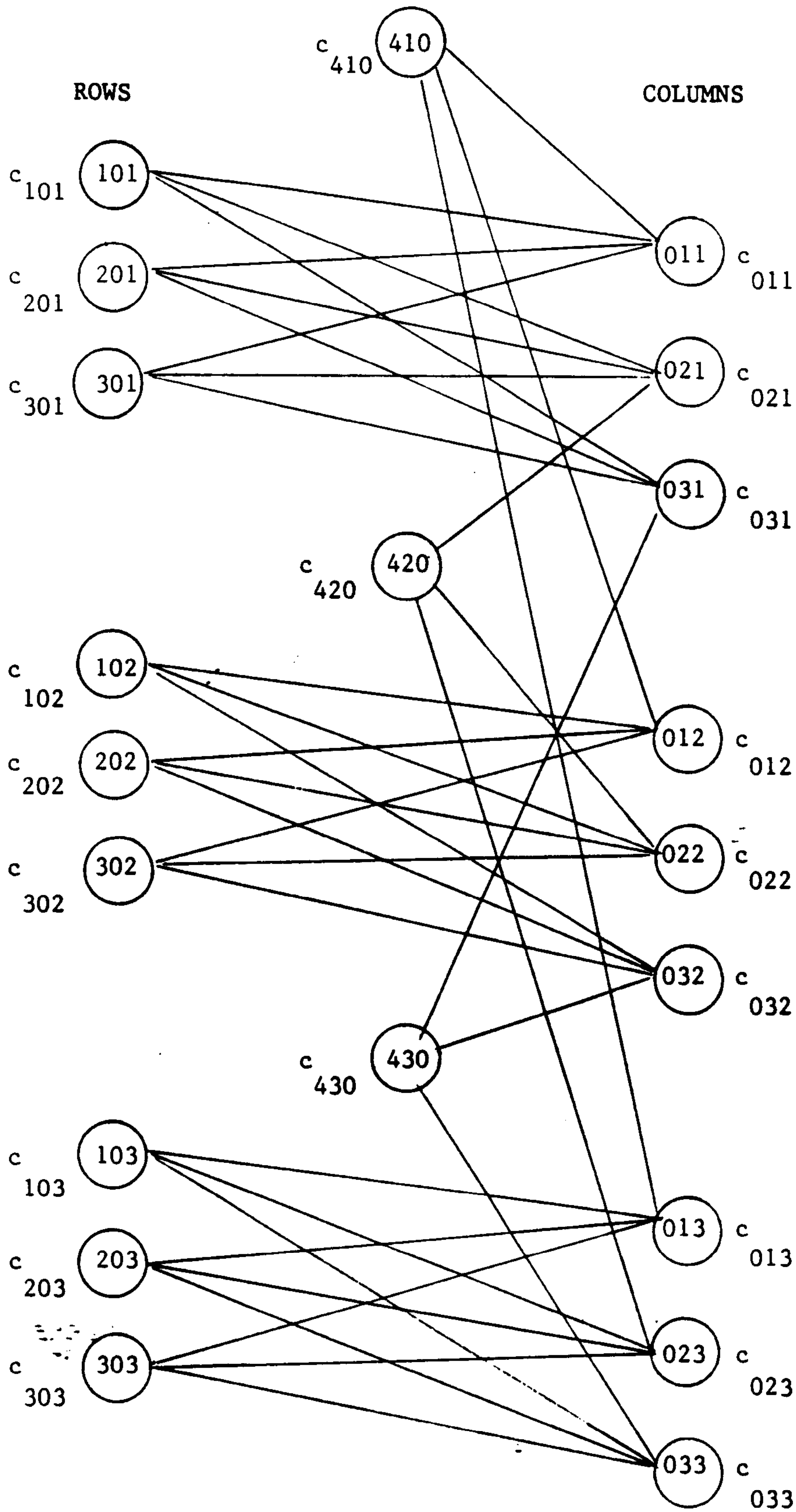

FIGURE 3

Note that for this network all cells of C are still represented by a directed arc with flow from left to right and the marginal constraints represented correspond to the marginals starred in the table above. In this figure we refer to nodes representing constraints forcing additivity to level totals as vertical shafts. Since

$$c_{401} + c_{402} + c_{403} = c_{410} + c_{420} + c_{430} = c_{400}$$

the total supply equals the total demand. All arcs have upper capacity of one and note that the values c_{ijk} provide a saturated flow across this network. Thus there is a zero-one flow and a zero-one assignment of c_{ijk} satisfying all starred marginal positions in the representation of table C on page 30. Since each cell is represented and the sum of row totals equals the sum of column totals at each level, the constraints c_{301}, c_{302} and c_{303} are redundant and satisfied as well. This new network does everything the former did and in addition satisfies some level zero contraints.

Solving this network, calling the zero-one variables g_{ijk}, and assigning the corresponding zero-one values in C, our objective is to reduce the size of the problem by "pulling off" the back face of C, namely, the face above the starred Level 0 constraints. We form the new table constraints consisting of marginals:

d_{001}	$^*d_{011}$	$^*d_{021}$	$^*d_{031}$
$^*c_{101}$			
$^*c_{201}$			
c_{301}			

d_{002}	$^*d_{012}$	$^*d_{022}$	$^*d_{032}$
$^*c_{102}$			
$^*c_{202}$			
c_{302}			

d_{003}	$^*d_{013}$	$^*d_{023}$	$^*d_{033}$
$^*c_{103}$			
$^*c_{203}$			
c_{303}			

---

d_{000}	d_{010}	d_{020}	d_{030}
c_{100}	c_{110}	c_{120}	c_{130}
c_{200}	c_{210}	c_{220}	c_{230}
c_{300}	$^*c_{310}$	$^*c_{320}$	$^*c_{330}$

where

$$d_{0jk} = c_{0jk} - g_{4jk} \quad \text{for } j=1,\ldots,3 \quad k=0,\ldots,3.$$

Also observe that

$$\sum_{i=1}^{3} c_{i0k} = \sum_{j=1}^{3} d_{0jk} = d_{00k} \qquad k=0,\ldots,3.$$

Set up the network in Figure 4 representing the family of starred constraints from the "partial" table above and observe the total supply equals the total demand. Even though, we do not know that this network does have a saturated flow, we attempt to find one by finding a saturated flow across the network in Figure 4 below.

VERTICAL SHAFTS

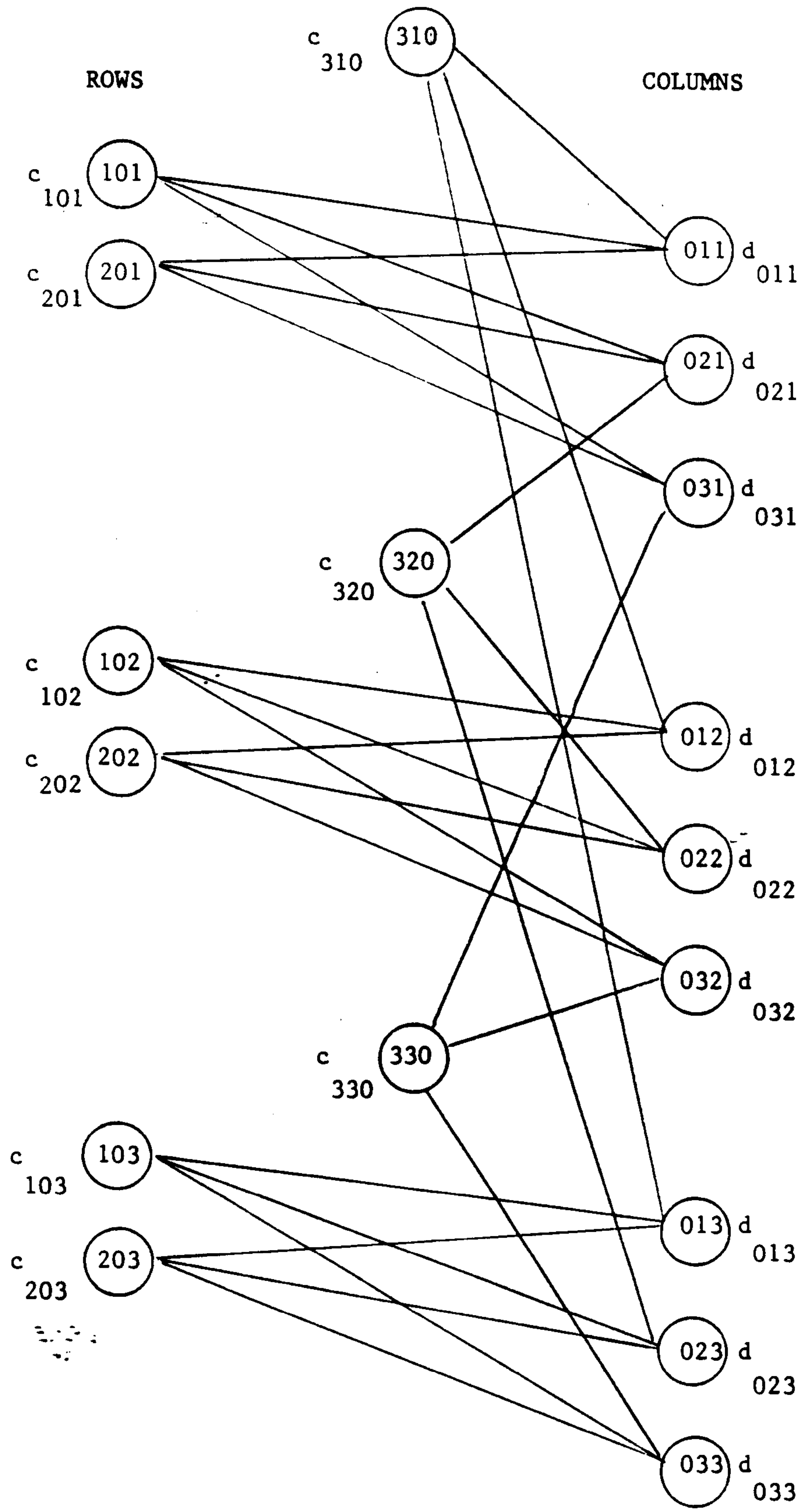

FIGURE 4

If a solution does exist, call the solution g_{ijk} once again, and use it to reduce the size of the problem as above and continue. If at any stage a saturated flow along the network does not exist, terminate processing. Otherwise continue until the last network is solved.

The solutions of each network flow problem are used for a zero-one assignment in the table C to obtain additivity in all directions. The controlled rounding, F of C, is obtained from the multi-stage process as follows. The values

$$f_{4jk} \quad j=1,\ldots,3 \text{ and } k=1,\ldots,3$$

in F are set equal to the values g_{4jk} obtained in solving the <u>first</u> network above. The values

$$f_{3jk} \quad j=1,\ldots,3 \text{ and } k=1,\ldots,3$$

are set equal to g_{3jk} obtained in the solution to <u>second</u> network above. Continuing in this fashion the values

$$f_{2jk} \quad j=1,\ldots,3 \text{ and } k=1,\ldots,3$$

are set equal to g_{2jk} obtained from the <u>third</u> network above and

$$f_{1jk} \quad j=1,\ldots,3 \text{ and } k=1,\ldots,3$$

are obtained from the last network. The derived table F will be additive in all directions, and will be a controlled rounding of C. Just as in Section II, we derive a controlled rounding, B of A, from the controlled rounding, F of C.

We illustrate this process by continuing with a base 3 rounding of

	60	17	24	19	52	19	14	19	40	16	9	15	152	52	47	53
A =	28	9	7	13	13	0	3	10	16	6	3	7	57	14	13	30
	12	9	2	1	24	11	8	5	14	8	0	6	50	28	10	12
	20	0	15	5	15	8	3	4	10	2	6	2	45	10	24	11

Writing

$$A = 3D+R$$

dividing all entries of R by 3, and forming C, we seek a zero-one assignment for the table, C:

7	*1	*2	*2	*2
*2	2/3	1/3	1/3	2/3
*2	0	2/3	1/3	1
*1	0	0	2/3	1/3
2	1/3	1	2/3	0

8	*2	*1	*2	*3
*1	0	0	1/3	2/3
*3	2/3	2/3	2/3	1
*2	2/3	0	1/3	1
2	2/3	1/3	2/3	1/3

7	*2	*1	*2	*2
*1	0	0	1/3	2/3
*1	2/3	0	0	1/3
*2	2/3	0	2/3	2/3
3	2/3	1	1	1/3

9	*2	*3	*3	*1
*2	1/3	2/3	1	0
*3	2/3	2/3	1	2/3
*2	2/3	1	1/3	0
2	1/3	2/3	2/3	1/3

31	7	7	9	8
6	1	1	2	2
9	2	2	2	3
7	2	1	2	2
9	*2	*3	*3	*1 .

Setting up the network for this table similar to Figure 3 we find a saturated flow shown in Figure 5 (displaying only those arcs with a positive flow).

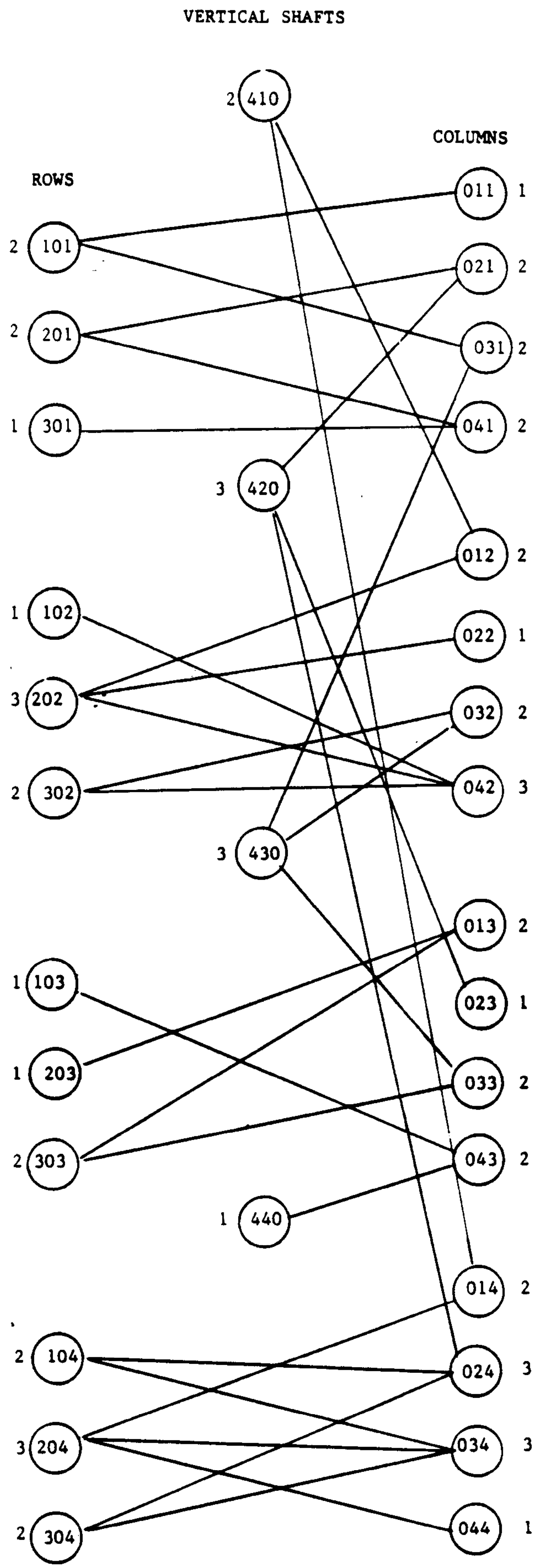
VERTICAL SHAFTS
2 410
COLUMNS
ROWS
011 1
2 101
021 2
2 201
031 2
1 301
041 2
3 420
012 2
1 102
022 1
3 202
032 2
2 302
042 3
3 430
013 2
1 103
023 1
1 203
033 2
2 303
043 2
1 440
014 2
2 104
024 3
3 204
034 3
2 304
044 1

FIGURE 5

Interpreting this flow as table entries for yields:

G_1 =

7	1	2	2	2
2	1	0	1	0
2	0	1	0	1
1	0	0	0	1
2	**0**	**1**	**1**	**0**

8	2	1	2	3
1	0	0	0	1
3	1	1	0	1
2	0	0	1	1
2	**1**	**0**	**1**	**0**

7	2	1	2	2
1	0	0	0	1
1	1	0	0	0
2	1	0	1	0
3	**0**	**1**	**1**	**1**

9	2	3	3	1
2	0	1	1	0
3	1	0	1	1
2	0	1	1	0
2	**1**	**1**	**0**	**0**

31	7	7	9	8
6	1	1	2	2
9	2	2	2	3
7	2	1	2	2
9	**2**	**3**	**3**	**1** .

Removing the "back-face" (consisting of all bottom rows in **bold**) we next have the following partial table (only marginal positions are displayed) in which to insert zeros and ones:

5	*1	*1	*1	2
*2				
*2				
1				

6	*1	*1	*1	*3
*1				
*3				
2				

4	*2	*0	*1	*1
*1				
*1				
2				

7	*1	*2	*3	*1
*2				
*3				
2				

22	5	4	6	7
6	1	1	2	2
9	2	2	2	3
7	*2	*1	*2	*2 .

Setting up the next network including only the starred marginals we obtain the saturated flow shown in Figure 6.

VERTICAL SHAFTS

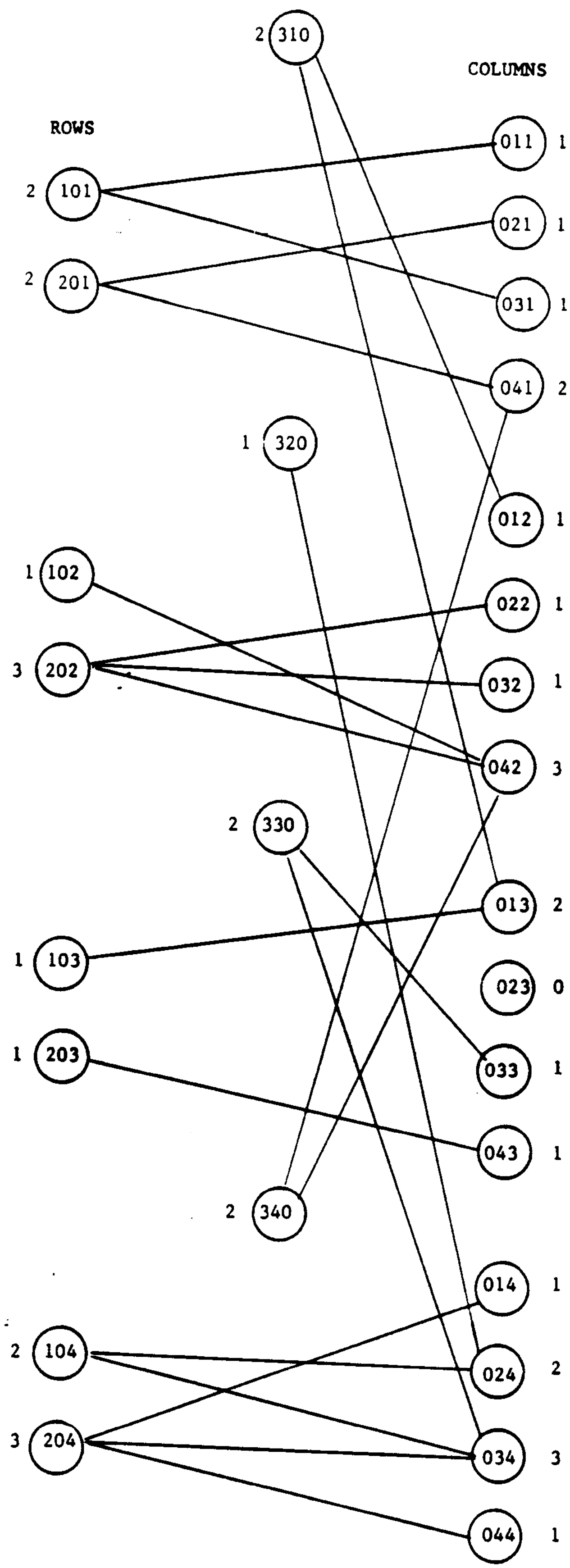

FIGURE 6

Interpreting this flow as entries for G_2 as indicated above yields:

G_2 =

5	1	1	1	2
2	1	0	1	0
2	0	1	0	1
1	**0**	**0**	**0**	**1**

6	1	1	1	3
1	0	0	0	1
3	0	1	1	1
2	**1**	**0**	**0**	**1**

4	2	0	1	1
1	1	0	0	0
1	0	0	0	1
2	**1**	**0**	**1**	**0**

7	1	2	3	1
2	0	1	1	0
3	1	0	1	1
2	**0**	**1**	**1**	**0**

22	5	3	6	7
6	1	1	2	2
9	2	2	2	3
7	**2**	**1**	**2**	**2** .

Removing the "back-face" again (represented above in **bold**) and subtracting the back face from the appropriate marginals, we next have the following table to solve:

4	*1	*1	*1	*1
*2				
2				

4	*0	*1	*1	*2
*1				
3				

2	*1	*0	*0	*1
*1				
1				

5	*1	*1	*2	*1
*2				
3				

15	3	3	4	5
6	1	1	2	2
9	*2	*2	*2	*3 .

Proceeding as before we set up the appropriate network, obtain a saturated flow and interpret the solution as table G_3:

G_3 =

4	1	1	1	1
2	1	0	1	0
2	**0**	**1**	**0**	**1**

4	0	1	1	2
1	0	0	0	1
3	**0**	**1**	**1**	**1**

2	1	0	0	1
1	0	0	0	1
1	**1**	**0**	**0**	**0**

5	1	1	2	1
2	0	1	1	0
3	**1**	**0**	**1**	**1**

15	3	3	4	5
6	1	1	2	2
9	**2**	**2**	**2**	**3**

followed by:

$$G_4 = \begin{array}{c|cccc} 2 & *1 & *0 & *1 & *0 \\ \hline 2 & 1 & 0 & 1 & 0 \end{array} \quad \begin{array}{c|cccc} 1 & *0 & *0 & *0 & *1 \\ \hline 1 & 0 & 0 & 0 & 1 \end{array} \quad \begin{array}{c|cccc} 1 & *0 & *0 & *0 & *1 \\ \hline 1 & 0 & 0 & 0 & 1 \end{array} \quad \begin{array}{c|cccc} 2 & *0 & 1 & *1 & *0 \\ \hline 2 & 0 & 1 & 1 & 0 \end{array}$$

$$\begin{array}{c|cccc} 6 & 1 & 1 & 2 & 2 \\ \hline 6 & *1 & *1 & *2 & *2 \end{array} .$$

Note that solving G_3 suffices since the only solution to G_4 -- if one exists -- is the first rows at each level of the G_3 solution. The network in Figure 7 exhibits the solution giving rise to G_3, from which one can also read off the solution yielding G_4.

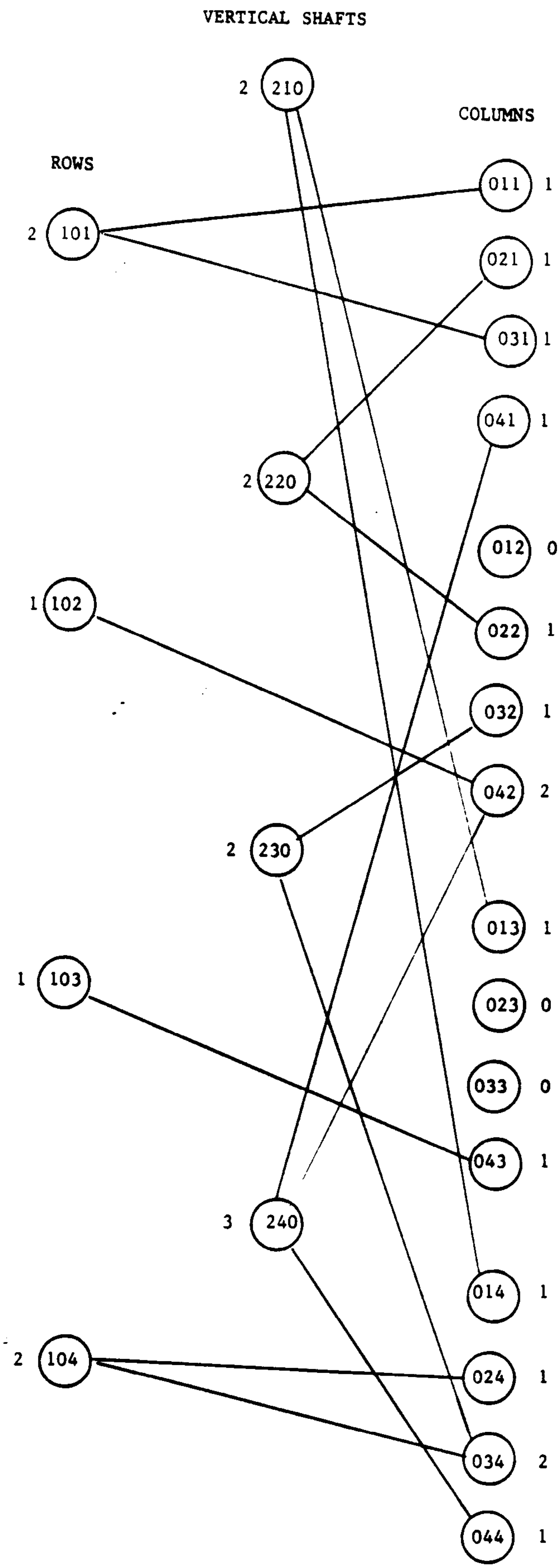
VERTICAL SHAFTS
2 210
COLUMNS
ROWS
011 1
2 101
021 1
031 1
041 1
2 220
012 0
1 102
022 1
032 1
042 2
2 230
013 1
1 103
023 0
033 0
043 1
3 240
014 1
2 104
024 1
034 2
044 1

FIGURE 7

The rounding, F of C, is the composite of the bold rows at each level above (from G_1, G_2, G_3 and G_4). These rows formed the "back-faces" at each iteration. That is, we form:

F =

7	1	2	2	2	8	2	1	2	3	7	2	1	2	2	9	2	3	3	1
2	1	0	1	0	1	0	0	0	1	1	0	0	0	1	2	0	1	1	0
2	0	1	0	1	3	0	1	1	1	1	1	0	0	0	3	1	0	1	1
1	0	0	0	1	2	1	0	0	1	2	1	0	1	0	2	0	1	1	0
2	0	1	1	0	2	1	0	1	0	3	0	1	1	1	2	1	1	0	0

31	7	7	9	8
6	1	1	2	2
9	2	2	2	3
7	2	1	2	2
9	2	3	3	1

and observe that F is a controlled rounding of C. Note that in this example, neither G_1 nor G_2 adds properly to the totals level, whereas the final F does. Multiply all elements of F by the base b=3 and form the controlled rounding S of R and finally form

$$B = 3D+S$$

to obtain a controlled rounding of A. For this base 3 example,

S =

9	3	3	3	9	3	3	3	9	6	0	3	27	12	6	9
6	3	0	3	0	0	0	0	0	0	0	0	6	3	0	3
3	0	3	0	6	0	3	3	3	3	0	0	12	3	6	3
0	0	0	0	3	3	0	0	6	3	0	3	9	6	0	3,

and

B =

60	18	24	18	51	18	15	18	42	18	9	15	153	54	48	51
30	9	6	15	12	0	3	9	15	6	3	6	57	15	12	30
12	9	3	0	24	9	9	6	15	9	0	6	51	27	12	12
18	0	15	3	15	9	3	3	12	3	6	3	45	12	24	9

as already exhibited on page 20, Section III.

One major problem in this process is that the procedure might not terminate successfully. For any of the networks other than the first, there is no guaranttee that we can create a saturated flow. In fact, we may fail to create a network with a saturated flow after any stage because of a poor choice of values asigned to the "back-face" -- even if a more suitable set of values for the "back-face" does exist. Consider the table C (after folding-in)

5	2	2	1	5	2	1	2	4	1	2	1
*2	13/16	15/16	4/16	*2	14/16	3/16	15/16	*2	5/16	14/16	13/16
*2	15/16	6/16	11/16	*1	9/16	4/16	3/16	*1	8/16	6/16	2/16
1	4/16	11/16	1/16	2	9/16	9/16	14/16	1	3/16	12/16	1/16

14	5	5	4
6	2	2	2
4	2	1	1
4	*1	*2	*1 .

We can derive a G_1 for this table as below noting that only the starred marginal constraints are satisfied:

G_1 =

5	2	2	1	5	2	1	2	4	1	2	1	14	5	5	4
2	1	1	0	2	1	0	1	2	1	1	0	6	2	2	2
2	1	0	1	1	0	0	1	1	0	1	0	4	2	1	1
1	0	1	0	2	1	1	0	1	0	0	1	4	1	2	1 .

Subtracting off the last row of each level from the appropriate column total, we have the following table to solve:

4	2	1	1
2			
2			

Level 1

3	1	0	2
2			
1			

Level 2

3	1	2	0
2			
1			

Level 3

10	4	3	3
6	2	2	2
4	2	1	1

Level 0

Levels two and three have deterministic solutions:

3	1	0	2
2	1	0	1
1	0	0	1

Level 2

3	1	2	0
2	1	1	0
1	0	1	0

Level 3

so there is no way to assign values of zero or one to Level 1 to achieve additivity to the 2 in marginal postion c_{210}. Thus the procedure outlined above breaks down. However, there does exist a controlled rounding of C, namely:

F =

5	2	2	1
2	1	1	0
2	1	0	1
1	0	1	0

5	2	1	2
2	1	0	1
1	0	1	0
2	1	0	1

4	1	2	1
2	0	1	1
1	1	0	0
1	0	1	0

14	5	5	4
6	2	2	2
4	2	1	1
4	1	2	1

If the last rows of G_1 had been

1	0	1	0	2	1	0	1	1	0	1	0

rather than

1	0	1	0	2	1	1	0	1	0	0	1

the iterative procedure would have run to a successful conclusion. The question is, how do we assign values to the last row at each level to encourage the process to terminate? Observe that in the networks in Figures

3-7, we made no use of costs on arcs. In fact, no objective function was involved, we only sought a single feasible saturated flow.

According to a result of Gale (1957), given marginals of a two-way table for which there does exist a zero-one assignment for the interior cells, one can fill up the table, one row at a time, by placing ones in the columns with the largest residual totals. We cannot apply this rule directly because additivity to the marginals of the form c_{ij0} must also be maintained when attempting to resolve each two dimensional level. Instead, we assigned costs to arcs corresponding to the back-face in a manner to encourage assigning ones to those columns (at each level) having the greatest residual marginal value. In particular, for those arcs in Figure 4 corresponding to cells on the back face -- arcs having initial node of the form (i,j,0) and terminal node of the form (0,j,k) -- the cost was set to $-d_{0jk}$. All other arcs had the cost of zero. This cost assignment did encourage convergence.

Software was developed to implement this procedure and most tables ran to a successful termination. For those tables that did not terminate, we observed that a number of arcs had the same costs (i.e., there were ties on the residual column marginals). To address this, a random component was added to all costs on the back face. In particular, for arc ((i,j,0), (0,j,k)) corresponding to a vertical shaft in the figures, the cost was set to

$$-100d_{0jk} - r$$

where r is a random integer between one and fifty. Using this new cost assignment, if a table did not reach a successful termination, the table was reset to run from the very beginning at which time alternative random values are assigned. Under this new regimen, tables did run to a successful termination in one or more repetitions. In the next section we discuss extensions to this basic procedure and software developed at the Census Bureau for three dimensional controlled rounding following this strategy, and we report on performance.

V. SOFTWARE FOR THREE DIMENSIONAL CONTROLLED ROUNDING

In the previous section we presented the basic methodology underlying a heuristic for three dimensional controlled rounding. In this section we go into further details and report on testing results and program performance.

As discussed in Section II and III, beyond seeking an arbitrary controlled rounding, we can search for controlled roundings which are weakly zero-restricted or better yet zero-restricted. To that end, we set up three programs for controlled rounding: program ZR to find zero-restricted controlled roundings, program WZR to find weakly zero-restricted controlled roundings (some of which may be zero-restricted), and program NZR which will find a controlled rounding that is not necessarily weakly zero-restricted.

Under the program ZR, we set up the network as in Figure 4, however, all arcs corresponding to cells that are multiples of the base in A are treated in a special manner. These cells will appear in C as zero corresponding to an interior cell of A or second order marginal (a_{i00}, a_{0j0}, a_{00k}) and they will appear as one corresponding to an integer first order marginal (a_{0jk}, a_{i0k}, or a_{ij0}) or third order marginal (a_{000}). Arcs corresponding to a zero cell in C are forced to have a zero flow (in fact, they are removed from the network) while arcs corresponding to a one in C are forced to have a flow of one. The cost along each remaining back face arc (as discussed earlier) is:

$$-100d_{0jk} - r$$

where r is a random integer between 1 and 50. Arcs not on the back face have cost zero. This cost assignment is to assist program termination by encouraging a one to be placed in a column with the greatest residual "demand", as in the sense of Gale (1957).

The next program, WZR, has its network as in Figure 4, however all arcs corresponding to true zeros on the even order marginals are forced to have a zero flow (regarding an interior cell as a 0-order marginal) and all arcs corresponding to true zeros on the odd order marginals are forced to have a flow of one unit. Arcs corresponding to non-zero multiples of the base in A have no corresponding restriction (in contrast to program ZR). Although we do allow these arcs to have a flow of one for even order marginals and zero for the odd order marginals so that corresponding non-zero multiple of the base b can be increased in A, one generally would like to discourage such behavior. Accordingly, the cost for an arc on the back-face in WZR is:

$$\alpha_{ijk}(100)^2(-2)^n - 100d_{0jk} - r,$$

where

$$\alpha_{ijk} = \begin{cases} 0 & \text{if } a_{ijk} \text{ is not a multiple of the base} \\ 1 & \text{if } a_{ijk} \text{ is a non-zero multiple of the base} \end{cases}$$

n is the order of a cell,

and

r is a random integer between 1 and 50.

That is, n=0 for an interior cell and n = 1, 2, or 3 for a first, second, or third order marginal respectively. Since a first or third order marginal which is a multiple of the base does not change when the corresponding arc is set to one, we encourage these arcs be set to one by giving them a smaller cost. An interior value or second order marginal which is a multiple of the base does not change when the corresponding arc is set to zero so we encourage these arcs to be set to zero by giving them a relatively higher cost. Combining these observations, the factor $(-1)^n$ is a part of the cost expression. Since we would prefer to increase an integer which is an interior cell rather than a first order marginal, a first order marginal rather than a second order marginal, and so on, the factor $(2)^n$ is included in the cost function.

Although the assignment of zero-one on the non-back-face positions is wiped clean after each iteration, one would prefer not to set an arc corresponding to an even order non-back-face non-zero multiple of a base position to one or set an arc corresponding to an odd order non-back-face non-zero multiple of the base to zero. The reason for this is one may be led down the path that will force such an interior position to be set to one (or zero) when it reaches the back-face. To discourage this behavior, the non-back-face arcs corresponding to a non-zero multiple of the base in A are given the cost:

$$a_{ijk}(100)^2(-2)^n.$$

Even though costs on the back-face dominate, setting costs on the non-back-face help direct the program to solutions which tend not to change non-zero multiples of the base.

The final program is call NZR, and in this program the network includes all arcs shown in Figure 4. Zero cells in A can be increased to the value of the base and non-zero multiples of the base can be increased by the value of the base. In order to avoid such changes to the extent possible, costs on arcs are set as follows. The cost for the arc corresponding to cell c_{ijk} is:

on back-face

$$\alpha_{ijk}(100)^2(-2)^n - 100d_{0jk} - r$$

on non-back-face

$$\alpha_{ijk}(100)^2(-2)^n$$

where

$$\alpha_{ijk} = \begin{cases} 0 & \text{if } a_{ijk} \text{ is not a multiple of the base} \\ 1 & \text{if } a_{ijk} \text{ is a non-zero multiple of the base} \\ 10,000 & \text{if } a_{ijk} = 0 \end{cases}$$

and

n is the order of a cell, r is a random integer between 1 and 50.

The rationale for this costing is an extension of the arguments provided above for WZR. Note that the cost function above can be viewed as a single cost function which accommodates all programs: ZR, WZR, and NZR. We are experimenting with variations on this cost function for added program efficiency.

A single main program has been set up which takes a three dimensional table as input, attempts to round employing up to a fixed number of repetitions of ZR, followed by up to a fixed number of repetitions of WZR, followed by up to a fixed number of iterations of NZR. If a controlled rounding was not found after repetitions of NZR the table is printed out for examination and analysis. As will be discussed more below, virtually all tables were resolved within ZR. In the body of the main program, after a table is read in, the derived table C is constructed following along the lines in Section III and a sequence of network is defined following the lines in Section IV. Software for solving a network flow problem is called as a subroutine to solve the sequence of networks using the unified cost function, and the results of the sequence of saturated flows is interpreted as the solution of the controlled rounding as in Section V.

The network flow software, called Minimum Cost Flow (MCF), was developed at the University of Texas, see (Glover and Klingman 1982). After reading in an RxCxL table, the dimensions are reordered if necessary so that R is less than C and L. Since R equals the number of back-faces that must be "pulled-off" in reducing the table size (the next iteration will be on an (R-1)xCxL table) it was reasonable to expect that this ordering of dimensions would reduce the running time of the program because fewer iterations would be required. After running a variety of tables, this was very clearly seen to be the case.

We have tested this program and report here on a family of simulations to impart the flavor of program performance. These tests were executed on a SPERRY 8600, 1180 series at the Census Bureau. All CPU time in the tables below are given in terms of 1108 CPU equivelants. (Actual 1180 run times are about one-third the CPU times displayed).

For the tests we report on below, five sets of 2x2x5 tables were randomly generated, each set having 1,000 tables. The sets were generated to contain:

0%   25%   50%   75%   and   90%

zero cells. The three subroutines discussed above were linked; however, all 5,000 tables successful zero-restricted controlled roundings were found in ZR (the first of the linked programs). Since that was the case, we severed the linked programs and tested the same randomly generated tables under ZR, WZR, and NZR independently. In the next three tables we display the results of this testing.

PROCEDURE ZR --- TABLE SIZE 2x2x5

Percent zeros	0%	25%	50%	75%	90%
Number of Tables	1000	1000	1000	1000	1000
Number remaining after:					
one iteration	153	229	196	119	13
five iterations	17	27	31	32	5
ten iterations	3	5	5	4	3
fifteen iterations	0	0	2	1	1
Most repititions needed for a single table	13	15	16	16	18
Time in CPU seconds (MCF Time)/(Total Time) including IO and diagnostic information	96/210	93/215	73/194	48/162	25/122

TABLE 1

PROCEDURE WZR -- TABLE SIZE 2x2x5

Percent zeros	0%	25%	50%	75%	90%
Number of tables	1000	1000	1000	1000	1000
Number remaining after					
one repitition	1	26	44	27	6
five repititions	0	1	3	4	0
ten repititions	0	0	0	0	0
fifteen repitions	0	0	0	0	0
Most repititions needed for a single table	2	7	6	7	5
Time in CPU seconds (MCF Time)/(Total Time) including IO and diagnostic information	104/201	97/195	89/189	74/173	61/157
Number of multiples of base increased	160	268	245	133	20
Number of tables in which a multiple of base was increased	149	206	184	103	15

TABLE 2

PROCEDURE NZR -- TABLE SIZE 2x2x5

Percent zeros	0%	25%	50%	75%	90%
Number of tables	1000	1000	1000	1000	1000
Number remaining after					
one repitition	1	3	0	0	0
five repititions	0	0	0	0	0
ten repititions	0	0	0	0	0
fifteen repititions	0	0	0	0	0
Most repititions needed for a single table	2	4	1	1	1
Time in CPU seconds (MCF Time)/(Total Time) including IO and diagnostic information	104/199	99/194	92/187	86/181	79/174
Number of multiples of base increased	160	295	345	205	38
Number of Tables in which multiples of base was increased	149	233	241	133	22
Number of zeros increased to one	0	51	108	101	25
Number of tables in which true zero increased to one	0	46	83	60	14

TABLE 3

All information in these tables should be regarded as approximate. In several runnings of these same tables with different seeds for the random number generator, the number of repetitions, number of zeros changed, and so on, varied considerably. The objective in presenting these tables is to impart a broad sense of how these programs performed.

In the process of evaluating this heuristic, we observed that the cost function employed was not always giving a result desired. Consider the following 2x2x2 table, A, for which we seek a base 3 controlled rounding:

A =

10	10	0	12	4	8		22	14	8
0	0	0	4	4	0		4	4	0
10	10	0	8	0	8		18	10	8

R =

1	1	0	3	1	2		4	2	2
0	0	0	1	1	0		1	1	0
1	1	0	2	0	2		3	1	2

C =

12	3*	3*	6*	9	3*	3*	3*	15	6*	6*	3*		36	12	12	12
3*	0	0	3	3*	1	0	2	6*	2	3	1		12	3	3	6
3*	1	0	2	3*	0	2	1	3*	2	1	0		9	3	3	3
6	2	3	1	3	2	1	0	6	2	2	2		15	6*	6*	3*

For a base 3 weakly zero-restricted controlled rounding, F of C, we must have at least:

F =

12	3	3	6	9	3	3	3	15	6	6	3		36	12	12	12
3	**0**	**0**		3		**0**		6		**3**			12	3	3	6
3		**0**		3	**0**			3					9	3	3	3
6				3			**0**	6					15	6	6	3 .

Using the cost function as described above and the starred constraints, the last row at each level must look like:

F =

12	3	3	6	9	3	3	3	15	6	6	3		36	12	12	12
3	**0**	**0**		3		**0**		6		**3**			12	3	3	6
3		**0**		3	**0**			3					9	3	3	3
6	0	3	3	3	3	0	0	6	3	3	0		15	6	6	3,

yielding the unique

F =

12	3	3	6
3	**0**	**0**	3
3	3	**0**	0
6	0	3	3

9	3	3	3
3	0	**0**	3
3	**0**	3	0
3	3	0	0

15	6	6	3
6	3	**3**	0
3	0	0	3
6	3	3	0

36	12	12	12
12	3	3	6
9	3	3	3
15	6	6	3

From which we obtain the controlled rounding

B =

12	12	0
0	0	0
12	12	0

12	3	9
3	3	0
9	0	9

24	15	9
3	3	0
21	12	9

This rounding is weakly zero-restricted but not zero-restricted. In fact, no zero-restricted controlled rounding could be found for this table using the program ZR under the cost function described above because the last row was forced. However, a base 3 zero-restricted rounding, B of A, does exist, namely:

B =

9	9	0
0	0	0
9	9	0

12	3	9
3	3	0
9	0	9

21	12	9
3	3	0
18	9	9

The problem is that the cost function above based in part on residual column totals does not allow finding such a solution. To remedy this problem, in addition to the cost strategy described earlier, we also assign costs which do not take the residual column totals into consideration. Recall that for each of the programs ZR, WZR and NZR the cost functions above could have been expressed in a unified manner:

<u>back face</u>

$$a_{ijk}(100)^2(-2)^n-100d_{0jk}-r$$

<u>non-back face</u>

$$a_{ijk}(100)^2(-2)^n$$

where r, a_{ijk}, d_{0jk} and n are defined earlier. The contribution of column total residuals is the term

$$-100d_{0jk},$$

which when removed yields costs:

back face

$$a_{ijk}(100)^2(-2)^n-r$$

non-back face

$$a_{ijk}(100)^2(-2)^n.$$

The cost strategy when running any of the programs, ZR, WZR or NZR is as follows. Allow for at most five repititions of the program for a given table under the first cost function above (which includes the residual column totals term), and if the program fails to obtain a controlled rounding then begin using the second cost function (which does not include the residual column totals term) for subsequent repetitions. Benefit of this modification to the cost function showed up when running larger tables than reported on above.

We now summarize findings on program performance for somewhat larger tables. As one would expect larger tables took longer to run and required more repetitions before yielding a controlled rounding. The number of overall iterations was reduced by using the combination of cost functions as described above rather than using only the cost functions with column residual totals. Using the linked program as discussed earlier, we obtained controlled roundings for all but two 2x8x10 tables in ZR and all but eleven 4x6x8 tables in ZR; they all yielded controlled roundings in WZR. In the tables below we display results from running 500 2x8x10 and 500 4x6x8 randomly generated tables.

PROGRAM ZR-WZR-NZR LINKED -- TABLE SIZE 2x8x10

Percent zeros	0%	25%	50%	75%	90%
Number of tables	100	100	100	100	100
Number remaining after					
one repitition	14	59	74	73	43
twenty-five repititions	0	7	16	5	0
fifty repititions	0	3	11	2	0
one hundred repititions	0	1	1	0	0
Most repititions needed for a single table	4	100 ZR 1 WZR	100 ZR 27 WZR	56	14
Time in CPU seconds (MCF Time)/(Total Time) including IO and diagnostic information	64/110	235/387	408/707	144/309	26/94
Number of multiples of base increased	0	26	6	0	0
Number of Tables in which multiples of base was increased	0	1	1	0	0
Number of zeros increased to one	0	0	0	0	0
Number of tables in which true zero increased to one	0	0	0	0	0

TABLE 4

PROGRAM ZR-WZR-NZR LINKED -- TABLE SIZE 4x6x8

Percent zeros	0%	25%	50%	75%	90%
Number of tables	100	100	100	100	100
Number remaining after					
one repitition	36	71	92	91	47
twenty-five repititions	0	25	27	18	0
fifty repititions	0	3	14	5	0
one hundred repitions	0	1	5	0	0
Most repititions needed for a single table	12	100 ZR 3 WZR	100 ZR 43 WZR	70	25
Time in CPU seconds (MCF Time)/(Total Time) including IO and diagnostic information	146/221	620/899	997/1525	400/720	56/157
Number of multiples of base increased	0	17	56	0	0
Number of Tables in which multiples of base was increased	0	1	5	0	0
Number of zeros increased to one	0	0	0	0	0
Number of tables in which true zero increased to one	0	0	0	0	0

TABLE 5

As noted earlier, all information in these tables should be regarded as approximate. In several runnings of these same tables with different seeds for the random number generator, the number of repetitions, number of zeros changed, and so on, varied considerably. As earlier, the objective in presenting these tables is to impart a broad sense of how these procedures perform.

VI. ALTERNATIVE PROCEDURE FOR CONTROLLED ROUNDING

In an earlier report, (Greenberg, 1988), we describe an alternative procedure for controlled rounding of two dimensional tables in which a non-zero multiple of the base can either increase or decrease. We outline this procedure here and show how it is applied to three dimensional tables.

If A is an RxCxL table, we say the RxCxL table B is a <u>controlled rounding base b</u> if

$$|b_{ijk}-a_{ijk}| \leq b \quad \text{for } i=1,\ldots,R \quad j=1,\ldots,C \quad k=1,\ldots,L.$$

The rounding is said to be <u>zero-restricted</u> if

$$|b_{ijk}-a_{ijk}|<b$$

and is <u>weakly zero-restricted</u> if

$$|b_{ijk}-a_{ijk}|\leq b \qquad \text{if } a_{ijk}>0$$

$$b_{ijk}=0 \qquad \text{if } a_{ijk}=0 \text{ .}$$

Every controlled rounding under the old definition is a controlled rounding (resp., weakly zero-restricted, zero-restricted controlled rounding) under the new definition, however the converse is not true. Returning to the first example by Ernst (page 23), E,

3	0	1	1	1	3	1	0	1	1	3	1	1	0	1	3	1	1	1	0
0	0	0	0	0	1	0	0	1/2	1/2	1	0	1/2	0	1/2	1	0	1/2	1/2	0
1	0	1/2	0	1/2	0	0	0	0	0	1	1/2	0	0	1/2	1	1/2	1/2	0	0
1	0	0	1/2	1/2	1	1/2	0	0	1/2	0	0	0	0	0	1	1/2	0	1/2	0
1	0	1/2	1/2	0	1	1/2	0	1/2	0	1	1/2	1/2	0	0	0	0	0	0	0

11	3	3	2	3
3	0	1	1	1
3	1	1	0	1
2	1	0	1	0
3	1	1	0	1 .

The following table, H^2, is a controlled rounding of E under the new definition but not the old definition:

3	0	1	1	1
0	0	0	0	0
1	0	1	0	0
1	0	0	1	0
1	0	0	0	1

2	1	0	1	0
1	0	0	1	0
0	0	0	0	0
0	0	0	0	0
1	1	0	0	0

3	1	1	0	1
1	0	0	0	1
1	1	0	0	0
0	0	0	0	0
1	0	1	0	0

3	1	1	0	1
1	0	1	0	0
1	0	0	0	1
1	1	0	0	0
0	0	0	0	0

11	3	3	2	3
3	0	1	1	1
3	1	1	0	1
2	1	0	1	0
3	1	1	0	1 .

The composite table

$$E^2 = \begin{array}{|c|c|} \hline E & 0 \\ \hline 0 & E \\ \hline \end{array}$$

as discussed in Section V had no controlled rounding under the standard definition, however it does have a controlled rounding under the new definition, namely,

$$B = \begin{array}{|c|c|} \hline H^1 & 0 \\ \hline 0 & H^2 \\ \hline \end{array}$$

where H^1 is the interior of the controlled rounding of E displayed in Section V (page 23) and H^2 is the interior of the controlled rounding of E displayed above.

The example, E^3 (page 29), formed by adding 1/300 to all cells of E^2 (page 25), which failed to have a controlled rounding under the old definition also fails to have a controlled rounding under the new definition. In fact, if a table has no cells which are a multiple of the base, the two procedures for controlled rounding (new and old) are identical.

In (Greenberg, 1988) we show how to set up a network flow problem for controlled rounding of two dimensional tables under the new definition. Basically, corresponding to each cell which is a non-zero multiple of the base there are two arcs in the network formulation rather than one. Costs are assigned arcs to encourage a non-zero multiple of the base not change in obtaining a controlled rounding; and it is equally "costly" to decrease a non-zero multiple of the base (by the base value) as it is to increase a non-zero multiple of the base. For zero-restricted controlled roundings to two definitions are the same. For two dimensional controlled rounding algorithms and software to extend the usual definition of controlled rounding to obtain roundings corresponding to the new definition are explicitly described.

The same modifications applied to procedures presented here for three dimension controlled rounding allow the development of algorithms and software to obtain three dimensional controlled roundings corresponding to the new definition. We omit the details here, however, they can easily be carried though by referring to the earlier report. We developed computer programs for three dimensional controlled rounding under the new definition and report below on performance.

Since every controlled rounding under the new definition is also a controlled rounding under the old definition, we expected the new technique to be more efficient in finding non-zero-restricted controlled roundings. In particular, for higher dimensional tables we hoped that under the new procedures fewer repetitions would be needed to resolve a problem table. We also hoped that tables failing to have a controlled rounding under the old definition would have controlled roundings under the new definition. As it turns out, the average number of repetitions needed for finding non-zero-restricted controlled roundings when running a family of randomly generated higher dimensional tables (2x8x10 and 4x6x8) under the new definition was indeed somewhat lower than under the old definition. However, since more arcs are required under the new definition (two per cell for non-zero multiples of the base) the overall time for program performance increased. The increase offset any savings due to fewer repetitions, and program performance for the new procedure was less efficient than for the old.

As noted above, there exist examples of tables having controlled roundings under the new definition but not under the old. But the examples were rather contrived, and we saw no instances of this phenomenon when examining randomly generated tables or tables arising from actual 1980 Decennial Census tabulations. For all the reasons above, we did not pursue extensive program development of the new method of controlled rounding for three dimensional tables, but rather kept the original ZR-WZR-NZR linked programs as our basic three dimension controlled rounding software.

VII. SUMMARY

In this report we exhibit an effective and efficient heuristic procedure for three dimensional controlled rounding. We discuss factors motivating this heuristic and provide examples of program performance and program limitations. Under extensive, testing software developed for this methodology has performed extremely well based on randomly generated tables and tabulation arrays taken from the 1980 Decennial Censuses.

These procedures were developed at the Bureau of the Census within the framework of Disclosure Avoidance research for the 1990 Decennial Censuses. One option for the public release of standard tabulation files which will maintain the confidentiality of respondents is to round all table values. The application of controlled rounding within the setting of disclosure avoidance is discussed in considerable detail in (Greenberg, 1986). Software based on the methods presented in this report was extensively tested and evaluated on standard tabulation files taken from the 1980 Decennial Census. Based on these tests, it was determined that if controlled rounding would be the disclosure avoidance methodology used on the 1990 Decennial Census standard tabulation files, this software would be an effective and efficient vehicle for three dimensional controlled rounding implementation.

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