BUREAU OF THE CENSUS
STATISTICAL RESEARCH DIVISION REPORT SERIES
SRD Research Report Number: Census/SRD/RR-86/03

THE HOMEOMORPHISM EXTENSION PROBLEM FOR TRIANGULATIONS IN CONFLATION

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Report completed: January 17, 1986

Report issued: January 17, 1986

The Homeomorphism Extension Problem for Triangulations in Conflation

Abstract

We discuss the problem of extending a one-to-one correspondence between two equinumerous finite sets of points in the plane to a homeomorphism between two topological spaces containing the sets. This problem arose during the development of a computer system to merge pairs of digitized map files at the Census Bureau. This system is called conflation. Conflation requires three fundamental steps: control point selection, triangulation, and rubber-sheeting. Pairs of points, each pair consisting of a point from each map, are selected. The selected points of one map are then used as the vertices of a specific, well-defined triangulation on that map, and for each of the triangles of this triangulation we create a triangle on the corresponding set of vertices on the other map. The set of triangles on the second map need not form a triangulation there. If they do, we show that a specific extension of the correspondence between the vertices is a homeomorphism. Moreover, the converse is also true. Also, we prove a second characterization for triangulations from which an easy detection algorithm is derived. A description of related problems follows at the end.

INTRODUCTION

We discuss the problem of extending a one-to-one correspondence between two equinumerous finite sets of points in the plane to a specific type of homeomorphism between two topological spaces containing the sets. To understand the importance of the problem, some background information is necessary.

The Census Bureau is developing a computer system for merging two digitized map files. The process of merging these files is known as conflation. Basically, conflation requires three fundamental steps: control point selection, triangulation, and rubber-sheeting. Pairs of points, each pair consisting of a point from each map, are selected. The selected points of one map are then used as the vertices of a specific, well-defined triangulation on that map, and for each of the triangles of this triangulation we create a triangle on the corresponding set of vertices on the other map. The set of triangles on the second map need not form a triangulation there, but there is a one-to-one correspondence between the sets of triangles (see Figure 1). Moreover, for each pair of corresponding triangles there is a unique affine transformation that takes vertices of one triangle to vertices of the other. So we obtain a piecewise-linear function from the first map to the second that depends on the control points selected and on the triangulation on the convex hull of those points. This function is the extension we are interested in. If the triangles on the second map form a triangulation, then the function is one-to-one, onto, and bi-continuous. Thus, we have a piecewise-linear homeomorphism (PLH) between the maps. The PLH is used as the rubber-sheet function that moves one map onto the other. However, the rubber-sheeting process breaks down when the extension results in something other than a homeomorphism between the maps.

In this paper, we define the piecewise-linear extension mentioned above and show that the extension is a homeomorphism precisely when we have a triangulation on the second map. Then we prove a second necessary and sufficient condition for triangulations from which we derive an easy detection algorithm. A description of related problems follows at the end.

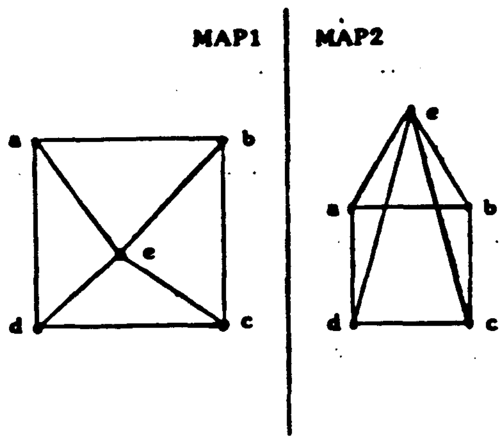

Figure 1: Each map has these triangles: Δ (ade), Δ (aeb), Δ (dce), Δ (bec). For Δ (aeb), a,e,b is a counterclockwise ordering of the vertices in map 1, but clockwise for map 2.

TRIANGULATIONS

We begin with a discussion of triangulations in general. Intuitively, a triangulation of a region is like a jigsaw puzzle where each piece is a triangle. If we keep the set of vertices constant, no new triangles may be created by adding edges. So, formally we can define a triangulation as follows:

Definition 1: Given $n \geq 3$ points in the plane which are not all collinear, let R be the convex hull of the points. Then, a **triangulation** on R is a maximal subdivision of R into triangles where the n points are the vertices and every point in R is in one and only one triangle unless it lies on a triangle edge.

Any convex hull of a finite set of points in the plane can be triangulated. In any triangulation every edge on the boundary of the hull is an edge for one triangle, and every edge which is not on the boundary is an edge for two triangles. Therefore, the entire region R is covered by triangles which abutt edge to edge and vertex to vertex but do not overlap. Among other things, this implies that none of the edges in a triangulation cross each other except possibly at their endpoints (see Figure 2). So, we say that two edges in a triangulation have a **non-trivial intersection** if the point of intersection is a non-endpoint for at least one of the edges.

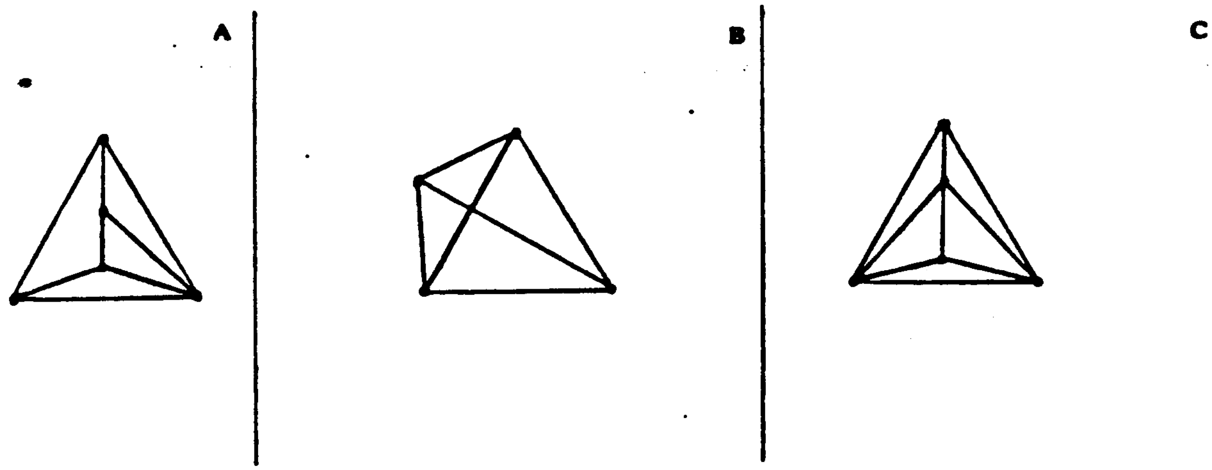

Figure 2: A,B are not triangulations, C is. **A)** Number of triangles is not maximal, **B)** Non-trivial intersection of edges, **C)** A triangulation.

The number of triangles and edges in a triangulation of the covex hull of n points with m points in the boundary of the hull is fixed, where $3 \leq m \leq n$. The formulas are as follows:

(1) $$NT = 2(n-1) - m$$

(2) $$NE = 3(n-1) - m$$

where NT is number of triangles and NE is number of edges. The proofs of these are omitted, but the results follow from the Euler characteristic for planar graphs (see Lefschetz).

PIECEWISE-LINEAR HOMEOMORPHISM

Now we lay the foundation for discussing the extension problem. First, we define the specific extension that we are interested in studying. Then, we prove a theorem showing that the extension is a homeomorphism precisely when there is a triangulation on the image space.

Let S_1 and S_2 be two finite sets of $n \geq 3$ points in the plane which are not all collinear and α be a one-to-one correspondence between the elements of S_1 and S_2. Let R_1 and R_2 be the convex hulls of S_1 and S_2, respectively, and let $H_i = \{h_1^i, \ldots, h_{mi}^i\}$ be the boundary of R_i. Note that H_i is the boundary of a simple convex m_i-gon.

We triangulate R_1 then use α to create edges and triangle-boundaries on R_2 using the points in S_2 as vertices. For the edges, this is done as follows: For every $\alpha(s_1)$, $\alpha(s_2) \in S_2$, $\alpha(s_1)$ and $\alpha(s_2)$ are connected by an edge if and only if s_1 and s_2 are connected by an edge. The triangles are created analogously. It is important to note that the region in R_2 determined by the induced triangles need not be all of R_2. It should be added, also, that these triangles do not necessarily form a triangulation of R_2. We will denote the set of edges and triangle-boundaries in R_1 as T and the set of edges and triangle-boundaries in R_2 as $\alpha(T)$. Also, for any edge, e, or triangle-boundary, t, in T, denote $\alpha(e)$ and $\alpha(t)$ as the corresponding edge and triangle-boundary in $\alpha(T)$, respectively.

The correspondence α maps S_1 to S_2. We now define an extension of α which maps R_1 to R_2. Call this extension $\bar{\alpha}$. The question we will ask is whether $\bar{\alpha}$ is a homeomorphism. We introduce a new notation as follows:

$$\bar{T} = \{ \bar{t} \mid t \in T \} \qquad \text{and}$$

$$\bar{\alpha}(\bar{T}) = \{ \bar{\alpha}(\bar{t}) \mid \alpha(t) \in \alpha(T) \}$$

where $\bar{t}$ and $\overline{\alpha(t)}$ are the triangles bounded by t and $\alpha(t)$, respectively. From now on we will refer to t and $\alpha(t)$ as triangles, too, for simplicity. No confusion should result from this.

On each pair of corresponding triangles, we define the affine transformation that takes each vertex to its corresponding one as follows: Let $t \in T$ and $\alpha(t) \in \alpha(T)$ be a corresponding pair of triangles. We denote the affine transformation from $\bar{t}$ to $\overline{\alpha(t)}$ as

$\bar{\alpha}_t$. Let v_1, v_2, v_3, $\alpha(v_1)$, $\alpha(v_2)$, $\alpha(v_3)$ be the vertices of t and $\alpha(t)$, respectively. For any $x \in \bar{t}$, there exists non-negative real numbers β_1, β_2, β_3 such that

$$x = \sum_{i=1}^{3} \beta_i \ v_i \quad \text{and} \quad 1 = \sum_{i=1}^{3} \beta_i .$$

The β_i are called the simplicial coordinates for x. Then

$$\bar{\alpha}_t(x) = \sum_{i=1}^{3} \beta_i \ \alpha(v_i) .$$

The map $\bar{\alpha}_t$ is a homeomorphism between the triangles taken as subsets of the plane with the standard $\mathbf{R}^2$ topology. Note that $\bar{\alpha}_t$ is unique and it takes edges to corresponding edges. The map $\bar{\alpha}$ is defined as follows: For any $x \in R_1$, there exists $\bar{t}$ such that $x \in \bar{t}$, then

$$\bar{\alpha}(x) = \bar{\alpha}_t(x) .$$

We should emphasize at this point that $\bar{\alpha}$ depends not only on S_1, S_2, and α, but also on the triangulation of R_1. For any two distinct triangulations of R_1, applying the procedure just described for generating $\bar{\alpha}$ gives rise to two different extensions.

It is easy to show that when a pair of triangles of $\bar{T}$ share an edge, the corresponding pair in $\bar{\alpha}(\bar{T})$ share the corresponding edge and that the affine transformations for adjacent pairs must agree on the edge common to the pair. Therefore, $\bar{\alpha}$ is well-defined and continuous. In addition, each affine transformation is invertible; if there is an inverse of $\bar{\alpha}$, it is continuous. The only questions that need to be answered are whether $\bar{\alpha}$ is onto and one-to-one.

To conclude these remarks, we prove a theorem relating homeomorphisms, triangulations, and our function $\bar{\alpha}$.

Theorem 1: $\bar{\alpha}$ is a homeomorphism between R_1 and R_2 if and only if $\bar{\alpha}(\bar{T})$ is a triangulation of R_2.

Proof: Assume $\bar{\alpha}$ is a homeomorphism. Therefore, the boundary of R_1 is mapped onto the boundary of R_2, i.e. H_2 is the image of H_1. In particular, $m_1 = m_2$. By (1) and

(2), $\bar{\alpha}(\bar{T})$ has the correct (maximum) number of triangles and edges to be a triangulation on R_2. Because $\bar{\alpha}$ is one to one, no triangles fold over each other, and there are no non-trivial intersections of edges. Because $\bar{\alpha}$ is onto, all of R_2 is covered by triangles of $\bar{\alpha}(\bar{T})$. Therefore, each point in R_2 lies in one and only one triangle unless it lies on an edge. Because $\bar{\alpha}$ is bicontinuous, the set of triangles meet edge to edge and all the adjacency relations are preserved. Therefore, $\bar{\alpha}(\bar{T})$ is a triangulation on R_2.

Assume $\bar{\alpha}(\bar{T})$ is a triangulation on R_2. For any $x \varepsilon R_2$, x lies in one and only one triangle unless it lies on an edge. So, the triangles in $\alpha(T)$ cover all of R_2, therefore $\bar{\alpha}$ is onto. Also, from this and the definition of $\bar{\alpha}$, we have the pre-image of any $x \varepsilon R_2$ contains just one point in R_1. Therefore, $\bar{\alpha}$ is one-to-one. From the definition of $\bar{\alpha}$ we know that $\bar{\alpha}$ is continuous. Since $\bar{\alpha}$ is one-to-one and onto, it has an inverse. Since each piece of $\bar{\alpha}$ is a homeomorphism and since $\bar{\alpha}(\bar{T})$ is a triangulation on R_2, we can apply the same argument to the inverse of $\bar{\alpha}$ that we applied to $\bar{\alpha}$ to conclude that $\bar{\alpha}$ is bicontinuous. Therefore, $\bar{\alpha}$ is a homeomorphism. □

In the proof of the theorem we showed that if $\bar{\alpha}$ is a homeomorphism then $\bar{\alpha}(H_1) = H_2$. Since α agrees with $\bar{\alpha}$ on vertices and edges of T, then $\alpha(H_1) = H_2$. Therefore, we have a

Corollary: If $\bar{\alpha}$ is a homeomorphism between R_1 and R_2, then $\alpha(H_1) = H_2$.

We emphasize this fact because we will need it as an assumption in the next section. Note that it is a necessary condition, but <u>not</u> a sufficient condition, for $\bar{\alpha}(\bar{T})$ to be a triangulation on R_2 (see Figures 3 and 4).

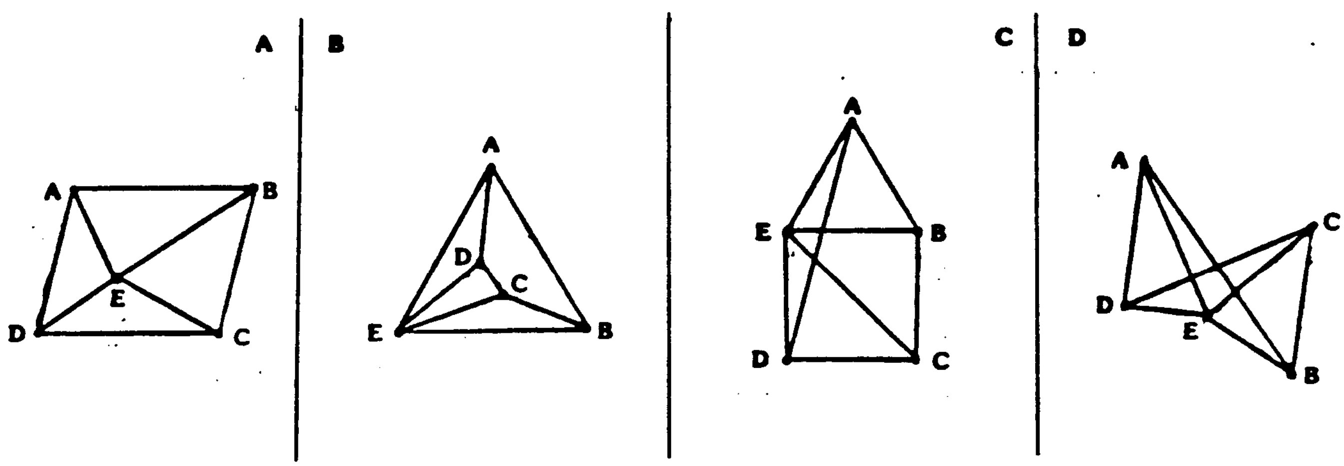

Figure 3: What can go wrong if $\alpha(H_1) \neq H_2$. **A)** Triangles on R_1, **B)** R_2 where $m_1 > m_2$, **C)** R_2 where $m_1 < m_2$, **D)** R_2 where $m_1 = m_2$ and α maps each vertex in H_1 to one in H_2.

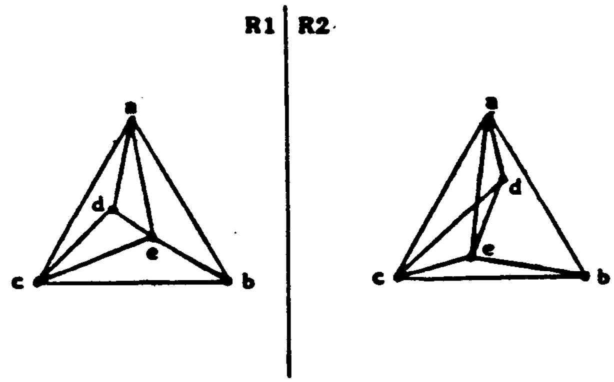

Figure 4: Even though $\alpha(H_1) = H_2$, $\bar{\alpha}(\bar{T})$ is not a triangulation of R_2.

PRELIMINARIES

From now on, we will assume that $\alpha(H_1) = H_2$. To be in a position to prove the main result, we need a definition and some facts first.

For any triangle, $\Delta(v_1\ v_2\ v_3)$, where $v_i = (x_i, y_i)$ for $i = 1,2,3$, let

$$P(v_1, v_2, v_3) = (x_1 - x_2)(y_1 + y_2) + (x_2 - x_3)(y_2 + y_3) + (x_3 - x_1)(y_3 + y_1).$$

Definition 2: If $\Delta(v_1\ v_2\ v_3)$ is a non-degenerate triangle, then the **orientation** of $\Delta(v_1\ v_2\ v_3)$ is **positive** if $P(v_1, v_2, v_3) > 0$ and **negative** otherwise.

For non-degenerate triangles, $P(v_1, v_2, v_3)$ is positive (negative) if and only if v_1, v_2, v_3 is a counterclockwise (clockwise) ordering of the vertices about the interior of the triangle. Notice that any even permutation of the vertices of a triangle leaves the value of P unchanged and any odd permutation of the vertices reverses sign.

Whenever the ordering of the vertices is understood or isn't necessary, we will leave off mention of them and write P. If $t \epsilon T$ and $t = \Delta(s_1\ s_2\ s_3)$, then we write $P_\alpha(s_1, s_2, s_3)$ instead of $P(\alpha(s_1), \alpha(s_2), \alpha(s_3))$ for $\alpha(t)\ \epsilon\ \alpha(T)$. With this notation, we can specify an order for the vertices of a triangle in $\alpha(T)$ by specifying an order for its corresponding triangle in T.

A useful fact is the following formula. The area of a simple polygon in terms of the coordinates of the vertices is

3) $$A = \frac{1}{2}\left| \sum_{i=1}^{n} (x_i - x_{i+1})(y_i + y_{i+1}) \right|,$$

where $n \geq 3$ is the number of vertices in the polygon, (x_i, y_i) is the i-th vertex, and $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. We can remove the absolute values signs above if and only if the vertices as indexed by i are in counterclockwise order around the polygon.

Because any simple polygon can be subdivided into triangles, we have

(4) $$A = \frac{1}{2}\sum_{i=1}^{N} \left| \sum_{j=1}^{3} (w_{i,j} - w_{i,j+1})(z_{i,j} + z_{i,j+1}) \right|,$$

where N is the number of triangles in the subdivision and $(w_{i,j}, z_{i,j})$ is the j-th vertex in the i-th triangle. Again the absolute value signs can be removed if and only if each triangle has its vertices listed in counterclockwise order.

If all sets of vertices are listed counterclockwise in (3) and (4), then (4) collapses into (3) because each interior edge is counted twice, once in each direction. The two terms are negatives of each other, so they cancel. Thus, we are left with the terms in (3).

We can apply (3) and (4) to any triangulation on R_1. If we orient H_1 and the vertices of the triangles in counterclockwise order, then we can remove the absolute value signs and still have equality. For the set of triangles on R_2, we get similar formulas by applying α to each vertex and substituting the new coordinates of the corresponding vertex in each term. Call these new formulas $\alpha(3)$ and $\alpha(4)$. Then, considering them without the absolute value signs, $\alpha(3) = \alpha(4)$ because each interior edge (i.e. an edge not in H_2) is counted twice, once in each direction. Note that this is true regardless of whether the triangles on R_2 form a triangulation and regardless of the order of the vertices of the triangles. It is not true in general, however, that $\alpha(3) = \alpha(4)$ with the absolute value signs replaced.

The last facts we will need are computational and are easily derived. If we apply (3) to a triangle on R_1, then the area of the triangle as given by (3) is

(5) $$A = 1/2 \left| P \right|$$

It follows from (4) and (5) that the area of R_1 (or H_1) is given by

(6) $$A = \frac{1}{2} \sum_{t \varepsilon T} | P |_t .$$

Of course, if the vertices of every triangle are oriented counterclockwise, then $P > 0$ for every triangle and the absolute value signs can be removed in (6). We have similar formulas for the triangles on R_2.

MAIN RESULT

Now, we prove the main result. It gives us a practical way of determining whether $\bar{\alpha}$ is a homeomorphism. The algorithm follows from this theorem.

Theorem 2: $\bar{\alpha}(\bar{T})$ is a triangulation if and only if either α preserves orientation on every triangle in T or α reverses orientation on every triangle in T.

Proof: Assume that $\bar{\alpha}(\bar{T})$ is a triangulation. Let $s,t \varepsilon T$ be adjacent triangles and $\alpha(s)$, $\alpha(t)$ ε $\alpha(T)$ be the corresponding adjacent triangles. Let $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$, and $v_3 = (x_3, y_3)$ be the vertices of t such that $P(v_1, v_2, v_3) > 0$. Let $v_4 = (x_4, y_4)$ be the other vertex for s and v_2, v_3 be the common vertices between s and t (see Figure 5). Suppose α reverses the orientation of t. Then, $P_\alpha(v_1, v_2, v_3) < 0$ for $\alpha(t)$. $\bar{\alpha}(\bar{T})$ is a triangulation of R_2, so $\alpha(v_1)$ and $\alpha(v_4)$ lie on opposite sides of the common edge between $\alpha(s)$ and $\alpha(t)$. For s, $P(v_4, v_3, v_2) > 0$. Thus, $P_\alpha(v_4, v_3, v_2) < 0$ for $\alpha(s)$ (see Figure 6). Therefore, α reverses the orientation for s. Therefore, we conclude that α reverses orientation for all triangles in T.

figure 5

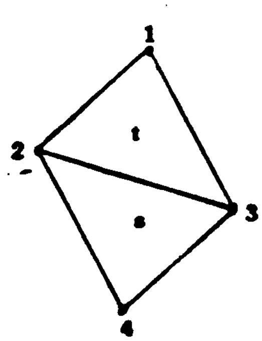

figure 6

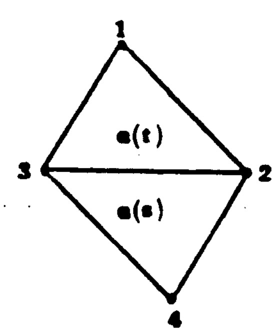

Figure 5: A pair of adjacent triangles in T: s and t.

Figure 6: The corresponding pair of adjacent triangles in $\alpha(T)$: $\alpha(s)$ and $\alpha(t)$.

Assume α preserves orientation on every triangle in T. Fix a listing of the vertices of every triangle in T such that $P > 0$ for each triangle. Therefore, $P_\alpha > 0$ for every triangle in $\alpha(T)$. If we list the vertices of H_1 in counterclockwise order, then since $\alpha(H_1) = H_2$ we can list the vertices of H_2 in counterclockwise order. Applying (3) to the vertices of H_2, we get

(7)
$$\text{Area of } R_2 = \frac{1}{2} \left| \sum_{i=1}^{m} (x_i - x_{i+1})(y_i + y_{i+1}) \right|$$

$$= \frac{1}{2} \sum_{i=1}^{m} (x_i - x_{i+1})(y_i + y_{i+1}) ,$$

where $(x_i, y_i) = h_i^2 = \alpha(h_i^1)$.

From the discussion just before the statement of the theorem above, (4) applied to the vertices of the triangles on R_2, and (7), we get

(8)
$$\frac{1}{2} \sum_{i=1}^{m} (x_i - x_{i+1})(y_i + y_{i+1})$$

$$= \frac{1}{2} \sum_{i=1}^{NT} \sum_{j=1}^{3} (w_{i,j} - w_{i,j+1})(z_{i,j} + z_{i,j+1}) ,$$

where $(w_{i,j}, z_{i,j})$ is the vertex in R_2 corresponding to the j-th vertex of the i-th triangle on R_1.

From (4), (5), and (6) applied to the triangles on R_2 and the fact that $P_\alpha > 0$ for each triangle, we get

(9)
$$\frac{1}{2} \sum_{i=1}^{NT} \sum_{j=1}^{3} (w_{i,j} - w_{i,j+1})(z_{i,j} + z_{i,j+1})$$

$$= \frac{1}{2} \sum_{i=1}^{NT} \left| \sum_{j=1}^{3} (w_{i,j} - w_{i,j+1})(z_{i,j} + z_{i,j+1}) \right| .$$

Combining (7), (8), and (9) we get that the sum of the areas of the triangles on R_2 is the area of the region bounded by H_2, i.e. R_2. Therefore, any point in R_2 is in one and only one triangle unless it lies on an edge. By (1) and (2) we get that the number of triangles is the maximum. Therefore, the set of triangles on R_2, i.e. $\bar{\alpha}(\bar{T})$, is a triangulation. We can prove the result when α reverses orientation on every triangle in T in a similar manner. □

ALGORITHM

We are now in a position to present an easy detection algorithm for determining whether or not $\bar{\alpha}(\bar{T})$ is a triangulation. The theoretical basis for the algorithm is Theorem 2. The algorithm runs in O(NT) time, where NT is the number of triangles.

The only data structures necessary for this procedure are as follows:

a) For each triangle in T, a list of the vertices;

b) Table for α, i.e. a list of corresponding pairs of vertices, one from S_1 and one from S_2 in each pair.

Procedure

a) For each triangle in the list
   1) Calculate $P \cdot P_\alpha$
   2) If $P \cdot P_\alpha < 0$ then **Stop.** $\bar{\alpha}(\bar{T})$ is not a triangulation on R_2.

b) **eof.** $\bar{\alpha}(\bar{T})$ is a triangulation of R_2.

CONCLUSION

We have proved a necessary and sufficient condition for determining whether or not the extension $\bar{\alpha}$ is a homeomorphism. It turns out that this is equivalent to knowing whether or not $\bar{\alpha}(\bar{T})$ is a triangulation on R_2. We then outlined a very simple detection algorithm for determining if indeed $\bar{\alpha}$ is a homeomorphism.

The full story, however, is far from complete. For we can ask the following question. If it turns out that $\bar{\alpha}$ is not a homeomorphism, is it possible to modify S_1 and S_2 in such a way that the modified $\bar{\alpha}$ is a homeomorphism? We say "modify" here because it is not clear whether we should add or remove points from S_1 and S_2 in order to effect this change. The two figures below (Figures 7 and 8) indicate how either method could be used. It is possible, too, that a combination of both methods will be useful. In any case, once it is determined that certain modifications will work, a second question follows.

Given that the answer to the first question is yes, is it possible to develop an effective method, i.e. one from which an algorithm can be developed?

From the discussion above, we can formally state two research problems as follows:

Problem 1: Is it possible to modify S_1 and S_2 such that the modified $\bar{\alpha}$ is a homeomorphism?

Problem 2: Given that the answer to problem 1 is 'yes', can we find an <u>effective</u> modification method?

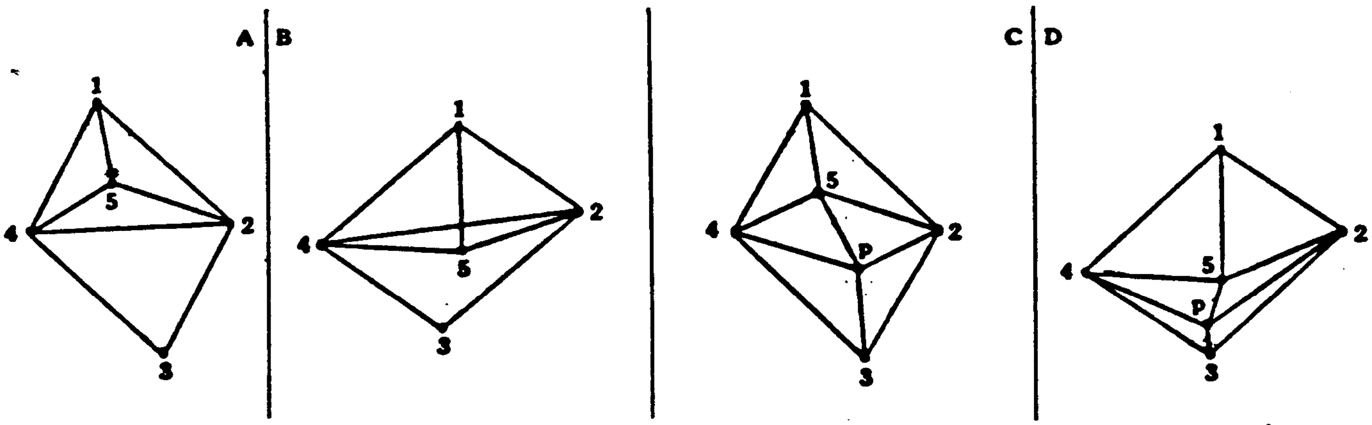

Figure 7: How adding a vertex can create a triangulation on R_2.
A) R_1, B) R_2, C) R_1' with point P added, D) R_2'.

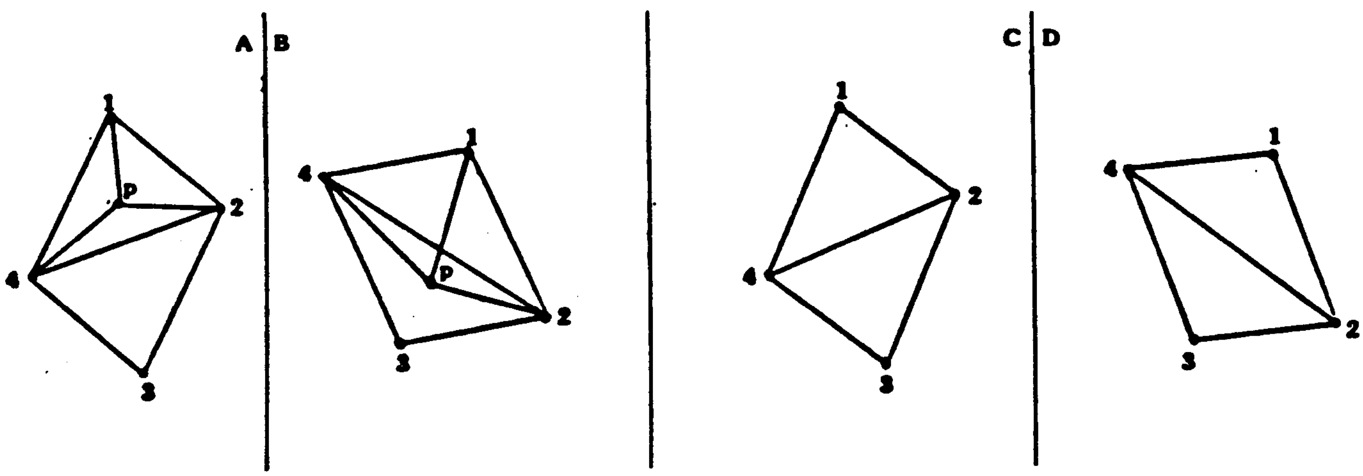

Figure 8: How removing a vertex can create a triangulation on R_2.
A) R_1, B) R_2, C) R_1' with point P in R_1 removed, D) R_2' .

These two problems assume that we will use the same extension method as presented in the paper. There is nothing sacred about extending to the piecewise-linear function. Instead of trying to modify S_1 and S_2, it might be possible to modify the extension procedure in such a way that a homeomorphism is obtained. Finding another way to create an extension could be a rather difficult problem though, owing to the fact that triangulations are rather easy to use and they lead naturally into our choice of an extension.

Another possiblity is to change the triangles rather than modify the vertex sets. In our application we use a particular triangulation procedure, the Delaunay Triangulation. This is a well-defined procedure, meaning that for a given set of points the same triangles are produced regardless of the order of processing the points. By relaxing the strict mathematical definition for the triangulation procedure we will be able to have different sets of triangles on a given set of points. By judiciously choosing which triangles to change, we could remove all triangle pairs where the orientation is reversed. This would mean our extension is a homeomorphism.

These ideas and others will be examined in the near future to find an algorithm for creating an extension which is a homeomorphism.

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