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EXACT CALCULATIONS FOR SEQUENTIAL TESTS BASED ON BERNOULLI TRIALS

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We consider methods of computing exactly the probability of "acceptance" and the "average sample size needed" for the sequential probability ratio test (SPRT) and likewise the newer "2-SPRT," concerning the value of a Bernoulli parameter. The methods permit one to approximate, iteratively, the desired operating characteristics for the test.

Key words: Sequential probability ratio test (SPRT); 2-SPRT; average sample number

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1. INTRODUCTION

Consider a Bernoulli population, with p denoting the proportion of units possessing attribute A. Based on one-at-a-time sampling from this population, we want to make a decision as to whether (the unknown) p is small or large, according to the following specifications. Suppose $0<p_1<p_2<1$. We let α^* denote the desired probability of erroneously not deciding that p is small, when in fact $p = p_1$. Likewise we let β^* denote the desired probability of erroneously not deciding that p is large, when in fact $p = p_2$. Two tests (i.e., decision rules) designed to meet these specifications approximately are: (1) the sequential probability ratio test

(SPRT) (Wald 1947), which approximatley minimizes "average sample number" (ASN) if in fact $p = p_1$ or $p = p_2$; and (2) the 2-SPRT (Lorden 1976), which helps to reduce ASN for values of p intermediate between p_1 and p_2.

In contrast to α^* and β^*, we let α and β denote the actual probabilities attained. Also, let a(p) denote the actual probability of deciding small, and E(p) denote ASN, as functions of p. Our goals are: (1) to attain, with these sorts of tests, α and β as close as possible to α^* and β^*; (2) as part of attaining the first goal, to compute α and β exactly; (3) also, to compute a(p) and E(p) exactly for various values of p.

We will consider two numerical examples:

(1) $p_1 = .01$ and $p_2 = .07$, with $\alpha^* = \beta^* = .05$.

(2) $p_1 = .4$ and $p_2 = .6$, with $\alpha^* = \beta^* = .001$.

2. THE SPRT

As we draw our sample one at a time, let n denote accumulated sample size, and k denote accumulated number of A-units. The SPRT decides low for $k \leq -c_1 + bn$ and decides high for $k \geq c_2 + bn$, whichever happens first. Here we have b, c_1 and c_2 0, with these defined by the calculations

$$B_1 = \log((1 - \alpha^*)/\beta^*) \text{ and } B_2 = \log((1 - \beta^*)/\alpha^*)$$
$$C_1 = \log(p_2/p_1) \text{ and } C_2 = \log((1 - p_1)/(1 - p_2))$$
$$c_1 = B_1/(C_1 + C_2),\ c_2 = B_2/(C_1 + C_2),\ b = C_2/(C_1 + C_2).$$

To compute a(p) and E(p) exactly, the following computations may be implemented. Let r_{nk} denote the probability that in the first n sample units: (1) k A's are obtained, and (2) no decision has been reached. Let u_n and v_n denote probabilities of deciding small and large, respectively, within the first n trials. Initially set r_{00}

$= 1$ and $u_0 = v_0 = 0$. Starting with $n = 0$ and letting n increase, repeatedly (for various k) let

$$R = pr_{nk} + (1 - p)r_{n,k+1} ,$$

with obvious omission of calculations for which r_{nk} or $r_{n,k+1}$ is 0. Then set $r_{n+1,k+1} = R$, except that if decision occurs for $n + 1$ and $k + 1$, add R to the value of (either) u_n or v_n, and set $r_{n+1,j+1} = 0$.

Let $Q_n = 1 - u_n - v_n$. Then, Q_n is the probability that no decision will have been reached in the first n trials. We stop and decide small when Q_n becomes less than a prespecified bound ε (which we have taken to be .00001). Let n_1 denote the correspon-ding value of n for $p = p_1$, n_2 the value for $p = p_2$. Let $n^* = \max(n_1,n_2)$. We base our test on the original SPRT, plus "trunca-tion" and decide small if n reaches n^*. The values of α and β for our test will differ from those for the unaltered SPRT by at most ε.

Having determined our test in this manner, we can compute, for any value of p, the value of a(p) (that is, u_{n^*} in the above notation) and E(p) (based on contributions to u_n and v_n, plus the contribution corresponding to Q_{n^*}). We would use double precision in accumulating u_n, v_n and E(p), and also in the calculation of the quantities r_{nk}. It is important that n^* be of manageable size, and we find that it is; for Example 1 we obtained $n^* = 369$ for the "final iteration." Such iterations are now to be discussed.

We have obtained actual α^* and β, in contrast to desired α^* and β^*. To obtain α and β closer to α^* and β^*, we compensate as follows. Let α_0^* and β_0^* denote the desired α^* and β^*, and α_0 and β_0 denote the realized α and β. With $j = 0$, we: (1) set

$$\alpha_{j+1}^* = \alpha_j^* \, \alpha_0^*/\alpha_j \text{ and } \beta_{j+1}^* = \beta_j^* \, \beta_0^*/\beta_j \; ;$$

(2) use α_1^* and β_1^* in computing c_1, c_2 and b; and (3) with α_1 and β_1 denoting new realized values apply the same idea with j = 1. These iterations can continue until α_j and β_j (i.e., α and β) are close to the originally desired α_0^* and β_0^*.

Using this approach for Example 1, with $\alpha^* = \beta^* = .05$, we obtained $\alpha_0 = .0279$ and $\beta_0 = .0486$. Eventually we obtained $\alpha_7 = .0502$ and $\beta_7 = .0501$, with $\alpha_7^* = .1047$ and $\beta_7^* = .0480$. As a variant of Example 1, we tried $\alpha^* = .10$ and $\beta^* = .02$. We obtained a sort of cycling in our iterations, but were able to obtain (as closest to α^* and β^*) $\alpha_8 = .09958$ and $\beta_8 = .01997$.

3. THE 2-SPRT

For the 2-SPRT one uses halves of 2 different SPRT's. Let p* denote a value of p intermediate between p_1 and p_2. Here we will restrict ourselves to the choice p* = b as defined above. This choice of p* makes sense especially for $\alpha^* = \beta^*$, based on consideration of "information numbers" (Lorden 1976). Along with p* = b, we approximate that a(p*) equals $B_2/(B_1 + B_2)$ (and thus .5 for $\alpha^* = \beta^*$). Accordingly, we have formulated a 2-SPRT which decides small for $k \leq -c_3 + b_3 n$ and decides large for $k \geq c_4 + b_4 n$, whichever happens first. Here we have b_3, c_3, b_4 and $c_4 > 0$, with these defined by the calculations

$$p^* = 1/(1 + \log(p_2/p_1)/\log((1 - p_1)/(1 - p_2)))$$
$$a^* = 1/(1 + \log((1- \alpha^*)/\beta^*)/\log((1 - \beta^*)/\alpha^*)))$$
$$C_{31} = \log(p_2/p^*) \text{ and } C_{32} = \log((1 - p^*)/(1 - p_2))$$
$$C_{41} = \log(p^*/p_1) \text{ and } C_{42} = \log((1 - p_1)/(1 - p^*))$$
$$B_{31} = \log(a^*/\beta^*) \text{ and } B_{42} = \log((1- a^*)/\alpha^*)$$
$$b_3 = C_{32}/(C_{31} + C_{32}) \text{ and } b_4 = C_{42}/(C_{41} + C_{42})$$
$$c_3 = B_{31}/(C_{31} + C_{32}) \text{ and } c_4 = B_{42}/(C_{41} + C_{42}).$$

We are comparing k against a pair of converging straight lines. Accordingly, we easily may find an upper bound on the maximum possible value of n. We may readily compute a(p) and E(p) exactly, using the computational approach for the SPRT. We may also use the above iterative approach for j = 0, 1, ...

Using this method for Example 1, with $\alpha^* = \beta^* = .05$, we obtained $\alpha_4 = .0498$ and $\beta_4 = .0500$, with $\alpha_4^* = .0780$ and $\beta_4^* = .0473$. For our variant of example 1, with $\alpha^* = .10$ and $\beta^* = .02$, we were able to obtain $\alpha_2 = .1005$ and $\beta_2 = .0200$, with $\alpha_2^* = .1385$ and $\beta_2^* = .0198$. For example 2, with $\alpha^* = \beta^* = .001$ (and with $p^* = a^* = .5$), we obtained $\alpha_1 = \beta_1 = .0010$ with $\alpha_1^* = \beta_1^* = .0011$.

4. COMPARISON

We briefly compare the statistical properties of the SPRT and 2-SPRT, although our primary purpose has been to provide computational procedures which permit this comparison and to bring α and β closer to α^* and β^* for both procedures. Almost invariably E(P) is smaller for the SPRT for p close to p_1 or p_2; but for intermediate values such as p^*, in which area E(n) is maximal for both procedures, E(n) is smaller for the 2-SPRT. In Example 1 we obtained final values

p	.01	.02	.03	.04	.07
E(p) SPRT	62.48	73.00	72.17	62.97	35.17
E(p) 2-SPRT	66.79	70.55	67.08	59.85	37.94

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