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Thermal Stress Evaluation of a Symmetrically Laminated Composite Plate

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ABSTRACT

The purpose of this investigation is to determine the in-plane thermal stresses for a symmetrically laminated, 50"x12"x0.19"composite plate with temperature dependent material properties. For this study only in-plane stresses are investigated. The in-plane equations of motion are solved exactly using a stress function, and the resulting compatibility equation is solved approximately using the Galerkin method. This investigation also serves as proof of concept for the variational method . This method produces accurate results while being less rigorous in a computational sense than the high degree of freedom finite element model required to solve the same problem. Variations with lamina orientation and multiple layered laminates are investigated. Results are given in terms of the in-plane force resultant. The baseline case for this study was aluminum and the in-plane force resultant at the center of the plate was calculated to be 60.251 lb/in. The exact solution for the in-plane force resultant at the center of the plate is 59.667 lb/in, a difference of less than 1 percent. Based on these results additional investigations were accomplished for composite plates. The results from this study will be used as parametric data by NASA-Dryden in verifying finite-element codes and will aid in experimental analysis of thermal loading on composite plates. Furthermore, this study should serve as a reference to an investigation that considers thermally stressed laminates with bending and extensional coupling.

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LIST OF SYMBOLS

a, b	= plate length and width
A	= laminate stiffenesses
C	= lamina stiffness
E_1,E_2	= Young's moduli in longitudinal and transverse material directions
F	= stress function
G_{12}	= shear modulus
h	= plate thickness
n	= order of approximating polynomial
N	= in-plane force resultant
Q	= lamina reduced stiffness
$\bar{Q}$	= transformed lamina reduced stiffness
u,v,w	= displacements
x,y,z	= coordinates
α_1,α_2	= coefficient of thermal expansion in longitudinal and transverse material directions
β	= percent moduli change
γ	= thermal degradation factor
γ_{xy}	= shear strain
ε	= total strain
ζ	= non-dimensional x- direction
δ	= variational operator
ΔT	= change in temperature
η	= non-dimensional y- direction
κ	= curvature
ν_{12}	= Poisson's ratio
θ	= lamination angle
σ	= stress
τ	= shear stress

Superscripts

T	= thermal
o	= midplane

Subscripts

,	= differentiation

1. INTRODUCTION

1.1 MOTIVATION

Hypersonic atmospheric vehicle research is again becoming an area of great interest. The push to develop a vehicle which can take-off, land, and be operated in a fashion similar to current aircraft is one of the interests in this area. Current concepts incorporate composite materials (possibly ceramic and/or metal matrix) as one of the principal structural components. Studies at NASA-Dryden are being conducted to verify the capabilities of composite materials in such an application. The results of this study and others in this area will be used by NASA-Dryden as an analytical tool to verify finite element methods for heated structures and to aid in experimental investigations.

In addition to the mechanical stresses present in composite materials, the thermal stresses must also be investigated, especially for hypersonic flight applications. For composite plates, coupling of the extensional and bending deformations is usually present. However, for this study only extensional deformations are investigated. The purpose of this study is to develop a mathematical model for the in-plane thermal stresses of a symmetrically laminated, rectangular, composite plate with stress free boundary conditions.

For the present study, the classical Kirchoff thin-plate theory is used and the in-plane equation of motion is expressed in terms of a stress function. The resulting partial differential equation is solved using Galerkin's method. The temperature distribution for this study is symmetrical and the relation of the material properties to temperature is assumed to be linear.

1.2 THEORETICAL BACKGROUND

1.2.1 Stress-Strain Relations for Composite Plates

Laminated, fibrous composite plates are typically made of stacked layers of fibers, each layer having all its fibers aligned in the same direction. However, the alignment most often varies with each layer. In order to characterize the stiffness of the resulting plate, it is important to be able to write the stress-strain relation for each layer. In particular the stress-strain relationship must be known with respect to an axis system which is not necessarily aligned in the same direction as any of the fibers in any given layer.

The generalized Hooke's Law relating stresses to strains can be written as

$$\sigma_i = C_{ij}\varepsilon_j \qquad i,j = 1\ldots6 \tag{EQ 1}$$

where C_{ij} is the stiffness matrix and the σ_i are the stress components and, ε_j are the strain components. The stiffness matrix has 36 constants, however, due to symmetry the stiffness matrix is populated by 21 independent constants (REF 2). Thus, in simplest terms the stress-strain relations for any linear elastic material is given by the general expression

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{31}\\ \tau_{12}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46}\\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56}\\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{bmatrix} \quad \text{(EQ 2)}$$

The relations in (EQ 2) are referred to as characterizing anisotropic materials since there are no planes of symmetry for the material properties. An example of this is a thick laminated plate (REF 1).

If there are three orthogonal (mutually perpendicular) planes of symmetry, such as for a single layer of fibers, the material is said to be orthotropic. The stress-strain relation for an orthotropic material is similar to (EQ 2), however, now there are only 9 independent constants instead of 21. The remaining constants are zero. The nonzero constants are C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}, C_{44}, C_{55}, C_{66}. Furthermore, in work done by Lempriere it was shown that the stiffness matrix must be positive-definite ($C_{ij}>0$ for $C_{ij} \neq 0$). This condition is a consequence of the requirement that the sum of the work done by all stress components must be positive in order to avoid the creation of energy (REF 2).

1.2.2 Two Dimensional State of Stress

Two dimensional stress or plane stress is an area of great interest in many different fields. One concept of plane stress is a flat plate. Flat plates are used as a model in a variety of analytical approximations for otherwise untractable problems. In many problems, choosing the z-direction to be normal to the plate surface, the stresses σ_z, τ_{yz}, and τ_{zx} are small with respect to the other stresses. Thus, the stress-strain relationship for an orthotropic material in a state of plane stress is:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} \quad \text{(EQ 3)}$$

This is the relationship relevant to a layer of fibers in a composite laminate, the fibers being aligned in the x-direction (REF 1).

Instead of considering the fibers to be oriented along the (x,y) axis system, let us consider the principal axes of a given lamina, with the 1-direction parallel to the fibers and the 2-direction perpendicular to the fibers. In doing this the C's in (EQ 3) are replaced with Q's and are now referred to as reduced stiffnesses. Furthermore, x and y are replaced with one and two respectively. In terms of engineering constants the reduced stiffnesses are:

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} \qquad Q_{12} = \nu_{12}\frac{E_{21}}{1-\nu_{12}\nu_{21}} = \nu_{21}\frac{E_{12}}{1-\nu_{21}\nu_{12}}$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} \qquad Q_{66} = G_{12} \tag{EQ 4}$$

Where E_1 is the modulus in the 1-direction, ν_{12} is the Poisson's ratio for contraction in the 2-direction due to extension in the 1-direction, and G_{12} is the shear modulus relating shear stress in the 2-direction to shear strain in the 1-direction.

The preceding stress-strain relations are the basis for the stiffness and stress analysis of an individual lamina subjected to forces in its own plane. These relations are essential in the analysis of laminates (REF 2).

1.2.3 Stress-Strain Relations for a Lamina of Arbitrary Orientation

In most problems it is necessary to solve for the stress-strain relationship in a direction other than the principal material coordinates. An example is a laminated plate consisting of lamina oriented at different angles relative to the plate axis system. The stress-strain relationship for a lamina in the principal, (1,2), system transformed to a global (x,y) system is given by:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \tag{EQ 5}$$

where θ is the angle from the x-axis to the 1-axis (REF 2). In some treatments of the subject the quantity $\varepsilon_{xy}=1/2\gamma_{xy}$ is introduced and is termed the tensor expression for shear strain (REF 1). In compact form the stress-strain relationship for a lamina oriented at any angle is:

$$\{\sigma\}_{xy_k} = \left[\overline{Q}\right]_k \{\varepsilon\}_{xy} \tag{EQ 6}$$

where $\overline{Q}_{ij}$ is the transformed reduced stiffnesses instead of the reduced stiffnesses Q_{ij}, and k is the k^{th} layer of a multilayered laminate. The equations for $\overline{Q}_{ij}$ are given in References 1, and 2. Also note that the transformed reduced stiffness matrix has terms in all nine positions instead of the zeros present in the stiffness matrix

1.2.4 Mechanical Properties of Laminated Composite Plates

In the preceding section the stress-strain relations for a lamina of an orthotropic material under plane stress oriented at any angle were developed. These relations are useful when dealing with laminated plates because of the arbitrary orientation of the lamina. Equation 6 can be thought of as the stress-strain relations of the k^{th} layer of a multilayered laminate. The next step is to develop the stress and strain variations through the thickness of a laminate.

1.2.5 Strains for a Laminated Plate

The laminate is presumed to consist of perfectly bonded lamina which are not allowed to slip relative to another. Furthermore, the bonds are presumed to be very thin as well as non-shear deformable. Additional assumptions of the behavior of the laminate are given by the Kirchhoff theory of plates (REF 2).

The development of the strains for a laminated plate are given in Reference 2. Ultimately, the laminate strains are reduced to ε_x, ε_y, and γ_{xy} (i.e.γ_{xz},=γ_{yz},=ε_z=0) by virtue of the Kirchhoff-Love hypothesis. Using the derived displacements u and v for the x and y directions respectively, the strains in matrix form are

$$[\varepsilon] = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{bmatrix} + z \begin{bmatrix} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix}}_{[\varepsilon^0]} + z \underbrace{\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}}_{[\kappa]} \qquad \text{(EQ 7)}$$

Here ε^0_x, ε^0_y, γ^0_{xy}, are the "midplane strains", κ_x and κ_y are the "curvatures" and κ_{xy} is the "twist" (REF 1). Now applying (EQ 6) and (EQ 7) one can determine the stresses in the k^{th} layer in terms of the midplane strains, curvatures, and twist using the relation:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \gamma_{xy} \end{bmatrix}_k = [\bar{Q}]_k \left\{ \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \right\} \qquad \text{(EQ 8)}$$

In this problem only in-plane stresses are investigated so the curvatures and twists are neglected.

1.2.6 Resultant Laminate Forces and Moments

Typically a designer deals with forces and moments instead of stresses. The resultant in-plane forces are calculated by integrating the stresses over the thickness of the plate. The common load parameters for a plate are the edge force intensities, N_x, N_y, and N_{xy} in units (lb/in), which are obtained using the following equation:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k dz \tag{EQ 9}$$

Here N is the total number of layers, h is the total plate thickness and k is the k^{th} layer. Since the transformed reduced stiffness matrix in (EQ 8) is constant for each lamina, upon substitution into (EQ 9), it can be pulled outside the integral, however, it must remain within the summation of force resultants for each layer. In addition, the middle surface strains are not dependent on z so they can be removed from under the integral and summation signs (REF 2). If one defines a matrix

$$[A] = \sum_{k=1}^{N} [\bar{Q}]_k t_k \tag{EQ 10}$$

then the in-plane loads for a laminated plate can be expressed as

$$[N] = [A][\varepsilon^0] \tag{EQ 11}$$

where [A] is referred to as the "extensional stiffness" matrix. Since only in-plane force resultants are evaluated: $[\varepsilon^0] = [\varepsilon]$ (REF 2).

1.2.7 Stress-Strain Temperature Relations

As stated in the introduction, the purpose of this project was to determine the thermal stresses in a laminated composite plate. The equations developed to this point include only the mechanical stresses and do not account for the thermal stresses. Thermal stresses arise from the expansion and contraction of fibers in a solid material. "For the solid to remain continuous, a system of thermal strains and corresponding thermal stresses may be induced, depending on the characteristics of the solid and its temperature distribution" (REF 4).

For plane stress of an orthotropic material in principal material coordinates the stress-strain-temperature relation in terms of the stiffness matrix is given by: (REF 2)

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = [Q] \begin{bmatrix} \varepsilon_1 - \alpha_1 \Delta T \\ e_2 - \alpha_2 \Delta T \\ \gamma_{12} - 0 \end{bmatrix} \tag{EQ 12}$$

Here α is the coefficient of thermal expansion in the principal material directions and ΔT is the relative change in temperature from a datum temperature. For an arbitrary orientation the coordinate transformation accomplished in (EQ 5) must be performed. After the transformation to laminate coordinates a thermal shear term, α_{xy}, is present where only the extensional strains were present before:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = [\bar{Q}]_k \begin{bmatrix} \varepsilon_x - \alpha_x \Delta T \\ \varepsilon_y - \alpha_y \Delta T \\ \gamma_{xy} - \alpha_{xy} \Delta T \end{bmatrix}_k \qquad \text{(EQ 13)}$$

Substituting (EQ 13) into (EQ 9) and defining the following thermal load vector

$$\begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\bar{Q}]_k \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k \Delta T dz \qquad \text{(EQ 14)}$$

one can derive the following in-plane resultant forces:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = [A] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} - \begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix} \qquad \text{(EQ 15)}$$

The thermal forces, N^T, are true thermal forces only when the total strains and curvatures are zero (REF 2).

Rearranging (EQ 15) to solve for the strains results in the following contracted matrix equation

$$[\varepsilon]_{xy} = [a][N]_{xy} + [a][N^T]_{xy} \qquad \text{(EQ 16)}$$

Here [a] is the inverse of the extensional stiffness matrix [A].

This is a relatively short description of the classical laminated plate theory required for the present study.

2. SOLUTION

The purpose of this section to take a step by step approach to the solution, for in-plane loads, of a thermally stressed composite plate.

2.1 FORCE EQUILIBRIUM EQUATIONS

"To obtain the exact stress field in any given solid under the action of external loads, the equations of equilibrium, compatibility equations, and the boundary conditions must all be satisfied" (REF 4). Therefore, in developing the equation for thermal stresses it is wise to start with the equilibrium equations.

The force equilibrium equations for a 2-dimensional state of plane stress with zero body forces are:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \qquad \text{(EQ 17)}$$

where N_x, N_y, and N_{xy}, are in-plane loads with units of force/length.

The "Airy" stress function F defined such that:

$$\left[N_x, N_y, N_{xy}\right] = \left[\frac{\partial^2 F}{\partial y^2}, \frac{\partial^2 F}{\partial x^2}, -\frac{\partial^2 F}{\partial x \partial y}\right] \qquad \text{(EQ 18)}$$

identically satisfies the equilibrium equations.

2.2 COMPATIBILITY EQUATION

The compatibility equation is an equation that gives a relationship between the derivatives of the strain components (REF 5). The compatibility equation for a two dimensional strain field can be derived from the strain displacement relations given in section 1.2.5. Differentiate the third equation in (EQ 7) with respect to x and y to obtain

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial x^2 y} \qquad \text{(EQ 19)}$$

Substituting the first two equations from (EQ 7) into (EQ 19) one can obtain the compatibility equation for a two dimensional strain field:

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} \qquad \text{(EQ 20)}$$

Substitution of (EQ 16 &18) into (EQ 20) results in the partial differential equation form of the compatibility equation:

$$\frac{\partial^2}{\partial y^2}\left(a_{11}\frac{\partial^2 F}{\partial y^2}+a_{12}\frac{\partial^2 F}{\partial x^2}-a_{16}\frac{\partial^2 F}{\partial x\partial y}\right)+\frac{\partial^2}{\partial x^2}\left(a_{12}\frac{\partial^2 F}{\partial y^2}+a_{22}\frac{\partial^2 F}{\partial x^2}-a_{26}\frac{\partial^2 F}{\partial x\partial y}\right)$$

$$-\frac{\partial^2}{\partial x\partial y}\left(a_{16}\frac{\partial^2 F}{\partial y^2}+a_{26}\frac{\partial^2 F}{\partial x^2}-a_{66}\frac{\partial^2 F}{\partial x\partial y}\right) = -\frac{\partial^2}{\partial y^2}(a_{11}N_x^T+a_{12}N_y^T+a_{16}N_{xy}^T) \tag{EQ 21}$$

$$-\frac{\partial^2}{\partial x^2}(a_{12}N_x^T+a_{22}N_y^T+a_{26}N_{xy}^T)+\frac{\partial^2}{\partial x\partial y}(a_{16}N_x^T+a_{26}N_y^T+a_{66}N_{xy}^T)$$

2.3 VARIATIONAL METHOD

The phrase 'variational methods' refers to methods that make use of variational principles to determine approximate solutions to partial differential equations. In this investigation, a variational formulation, which will be recognized as the principle of minimum complementary potential energy, is used to approximate a solution to the compatibility equation described in section 2.2.

2.3.1 Variational Formulation

The first step in the variatonal formulation is to multiply (EQ 20) with all the terms on one side of the equality with the variation δF and integrate over the domain:

$$\int_A \delta F\,(\varepsilon_{x,yy}+\varepsilon_{y,xx}-\gamma_{xy,xy})\,dA = 0 \tag{EQ 22}$$

where it is understood that all the strains are expressed in terms of the stress function F.

Equation 22 can be written in an alternative form by applying the Green-Gauss theorem twice to trade the differentiation between F and δF so they both have the same order derivatives (REF 5). In applying the Green-Gauss theorem the first step is to rewrite (EQ 22) in the form:

$$\int_A \hat{\nabla}\bullet\hat{A}\,dA = \oint_S \hat{n}\bullet\hat{A}\,dS \tag{EQ 23}$$

where

$$\hat{A} = \delta F\,(\varepsilon_{y,x}-\frac{1}{2}\gamma_{xy,y})\,\hat{\imath}+\delta F\,(\varepsilon_{x,y}-\frac{1}{2}\gamma_{xy,x})\,\hat{\jmath}$$

$$\hat{n} = n_x\hat{\imath}+n_y\hat{\jmath} \tag{EQ 24}$$

Applying the Green-Gauss theorem twice results in

$$\int_A (\delta F_{,xx}\varepsilon_y + \delta F_{,yy}\varepsilon_x - \delta F_{,xy}\gamma_{xy})\, dA = 0 \qquad \text{(EQ 25)}$$

Expanding (EQ 25) for the strains results in the equation:

$$\int_A [\delta F_{,yy}(a_{11}F_{,yy} + a_{12}F_{,xx} - a_{16}F_{,xy} + a_{11}N^T{}_x + a_{12}N^T_y + a_{16}N^T_{xy})$$

$$+ \quad \delta F_{,xx}(a_{12}F_{,yy} + a_{22}F_{,xx} - a_{26}F_{,xy} + a_{12}N^T_x + a_{22}N^T_y + a_{26}N^T_{xy})$$

$$+ \quad \delta F_{,xy}(a_{16}F_{,yy} + a_{26}F_{,xx} - a_{66}F_{,xy} + a_{16}N^T_x + a_{26}N^T_y + a_{66}N^T_{xy})\ ]\, dA = 0 \qquad \text{(EQ 26)}$$

2.3.2 Galerkin's Method

The integral equation, (EQ 25), is the variational equivalent of the in-plane compatibility equation, (EQ 20), and essentially represents a Galerkin solution for the partial differential equation shown explicitly in (EQ 21).

The first step in the Galerkin method is to approximate the unknown function with a suitable set of admissible functions. For the present formulation the stress function F is approximated as:

$$F \approx \sum_{k,l=1}^{n} F_{kl}\zeta^k\eta^l \qquad \text{(EQ 27)}$$

where ζ^k and η^l are required to satisfy the following three conditions:

a) be continuous, as required by the variational principle being used

b) satisfy the specified essential boundary conditions

c) be linearly independent and complete (REF 5)

For this problem the essential boundary conditions are zero and a series of polynomials were chosen as the approximating function because polynomials can satisfy arbitrary boundary conditions, and F_{kl} are the coordinates or components of the approximation.

The assumed solution in (EQ 27) is in the form of a finite linear combination of undetermined parameters. Approximating the continuous function in (EQ 25) by a finite linear combination of functions introduces some error. Therefore, the solution obtained is an approximation of the true solution for the equation of motion described in (EQ 25 & 26).

The variation of (EQ 27) is given by

$$\delta F = \sum_{i,j=1}^{n} \delta F_{ij}\zeta^i\eta^j \qquad \text{(EQ 28)}$$

2.3.3 Matrix Form of the Compatibility Equation

Taking the variation of (EQ 18) it is clear that $\{\delta N_x, \delta N_y, \delta N_{xy}\} = \{\delta F_{,yy}, \delta F_{,xx}, -\delta F_{,xy}\}$. Thus, the matrix form of (EQ 25) is

$$\int_A \{\delta N\}^T \{\varepsilon\}\, dA = 0 \qquad \text{(EQ 29)}$$

where $\{\delta N^T\}$ is the transpose of the matrix in (EQ 18). Substituting (EQ 16) into (EQ 29) to obtain the matrix equation

$$\int_A [\{\delta N\}^T [a]\, \{N\} + \{\delta N\}^T [a]\, \{N^T\}]\, dA = 0 \qquad \text{(EQ 30)}$$

Since (EQ 30) is in matrix form it is necessary to put (EQ 28) into matrix form also. This is accomplished by using a row vector of length N, where N=(n+1)*(n+1) and n is the order of the polynomial in (EQ 27) and (EQ 28), and a column vector of length N in the following fashion

$$F = \sum_{i,j=1}^{n} F_{ij}\zeta^i\eta^j = [1, \eta, \eta^2, \ldots \zeta^0\eta^n, \zeta, \zeta\eta, \zeta\eta^2, \ldots \zeta^1\eta^n, \zeta^2, \zeta^2\eta, \ldots \zeta^n\eta^n]$$
$$x\,[F_{00}, F_{01}, F_{02}, \ldots, F_{0n}, F_{10}, F_{11}, F_{12}, \ldots, F_{1n}, F_{20}, \ldots F_{nn}]^T \qquad \text{(EQ 31)}$$

Here the row vector can be denoted by [H] and the column vector by {F}. However, since $\{\delta N\}$ is composed of second order derivatives, and it relates directly to $\{\delta F\}$, it is necessary to differentiate F in (EQ 31) according to the relation in (EQ 25). Only [H] needs to be differentiated since {F} does not depend on x and y, therefore

$$F_{,yy} = [H]_{,yy}\{F\}$$
$$F_{,xx} = [H]_{,xx}\{F\} \qquad \text{(EQ 32)}$$
$$F_{,xy} = [H]_{,xy}\{F\}$$

If a matrix, [B], is defined to contain the three row vectors, $[H]_{,yy}$, $[H]_{,xx}$, and $[H]_{,xy}$, then [B] would be a 3xN matrix, and {N}=[B]{F}. Similarly $\{\delta N\}=[B]\{\delta F\}$. From this relation it is clear that $\{\delta N\}^T=\{\delta F\}^T[B]^T$. If these results for {N}and $\{\delta N\}$ are substituted in (EQ 29) the matrix equation is given by

$$\int_A (\{\delta F\}^T [B]^T [a]\, [B]\, \{F\} + \{\delta F\}^T [B]^T [a]\, \{N^T\})\, dA = 0 \qquad \text{(EQ 33)}$$

Since $\{\delta F\}^T$ is common in both factors of the integrand it can be factored to one side. In addition, for arbitrary and linearly independent δF_{ij} (i,j=1,n) it follows that the remaining integrand must be equal to zero. If the thermal terms are brought to the right hand side of the equation and the stress function component matrix is brought outside the integral because it is constant then

$$\int_A ([B]^T [a]\ [B])\, dA \{F\} = -\int_A ([B]^T [a]\ \{N^T\})\, dA \qquad \text{(EQ 34)}$$

Thus we obtain N linearly independent simultaneous equations for the N unknowns in {F}. Once the coefficients {F} are determined the in-plane force resultants can be determined at any point within the plate by using the relation {N}=[B]{F}.

2.4 NUMERICAL INTEGRATION

In order to evaluate the [A]{x}={b} problem shown in (EQ 34) it is necessary to integrate the [A] and {b} matrices. The integration could be done explicitly, however, an alternate method that yields exact results for polynomial functions is Gaussian quadrature.

Most numerical integration schemes are based on predetermined x-values. However, Gauss observed that if we remove the requirement that the function be evaluated at predetermined x-values, then a three term formula will contain six parameters: the three unknown x-values and the three weights (REF 7). This should correspond to an interpolating polynomial of degree 5. In addition, the tabulated values for the Gaussian quadrature procedure are derived for a symmetric interval of integration from -1 to 1. Clearly, Gaussian quadrature formulas can only be applied when the function is explicitly known. Since the integrand in (EQ 34) is approximated by a polynomial it can be calculated at any point and is readily suited to this method of numerical integration.

For this problem the limits of integration were not from -1 to 1. The [B] matrices are in terms of ζ and η, with the origin of this axis system placed at the center of the plate. The limits of integration for this axis system were from -a/2 to a/2, and -b/2 to b/2, where a and b are the length and width of the plate respectively. Therefore ζ and η were set equal to 2x/a and 2y/a. Now the limits of integration are from -1 to 1 and a factor placed in front of the integral accounts for the change of variable effects for dA=dxdy=$\frac{ab}{4}d\zeta d\eta$.

3. TESTING & ANALYSIS

The purpose of this section is to explain how the program to calculate the coefficients in (EQ 34) was tested and verified. In addition, the dependence of the material properties on temperature will be discussed.

3.1 PROGRAM VERIFICATION

This phase in any program intensive application is critical because only through careful checking and the use of multiple test procedures can one be sure that the program is doing what it was designed to do. Typically, a known problem is solved by some method other than the method used in the program and compared to the results of the program. In this section, verifications with hand calculations and verification with an exact integration program will be presented.

3.1.1. Verification With Hand Calculations

The program "Polystress" was checked by hand for a second order polynomial approximation of the integrand in (EQ 34). For a second order polynomial the [B] matrix is:

$$[B] = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 2\zeta & 0 & 0 & 2\zeta^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2\eta & 2\eta^2 \\ 0 & 0 & 0 & 0 & 1 & 2\eta & 0 & 2\zeta & 4\zeta\eta \end{bmatrix} \qquad \text{(EQ 35)}$$

Note that [B] is a 3xN where ($N=(n+1)^2$). The transpose of [B] was multiplied by an assumed inverted extensional stiffness matrix [a]. The resulting matrix was then multiplied with [B] and exactly integrated over the interval -1 to 1. This matrix was then compared to the results of numerically integrating the same equation and they were identical to the sixth significant figure. The same procedure was followed for the thermal load vector on the right hand side of (EQ 34) and again the results were identical to the sixth significant figure.

3.1.2 Comparison with Exact Integration Program

This program was written as a supplement for an exact integration code that would calculate not only the in-plane resultant forces but also the out-of-plane resultant moments for a thermally loaded fibrous composite plate. At the present time the code was exactly integrating the extensional stiffness matrix with no temperature dependent relations included. Thus, I could compare my results with the exact integration program. Clearly, the hand calculations and the results from the numerical integration could be used to verify the exact integration program as well. As it turned out all three methods produced the same results and thus the numerical and exact integration phases of the respective programs had been verified.

3.2 TEMPERATURE DEPENDENT MATERIAL PROPERTIES

The modulus of elasticity, E, and the coefficient of thermal expansion, α, are not temperature independent. In data given by (REF 8) the modulus of elasticity decreases almost linearly with increasing temperature and the coefficient of thermal expansion increases almost linearly with increasing temperature. References 9 and 10 also use this type of linear relationship for their temperature dependent properties.

In (REF 10) the temperature dependence of the modulus of elasticity is expressed in the form

$$E_1(T) = E_1^o(1-\gamma T) = E_1(1-\beta(\frac{T}{T_1}))$$

$$E_2(T) = E_2^o(1-\gamma T) = E_2(1-\beta(\frac{T}{T_1})) \qquad \text{(EQ 36)}$$

where $\beta=\gamma T_1$ and $E^o{}_1$ and $E^o{}_2$ are the values of the moduli of elasticity along the 1 and 2 directions at the reference temperature T_0. Here T denotes the temperature excess above the reference temperature at any point and β denotes the percent change of the modulus of elasticity over the specified temperature interval ($T_0<T<T_1$) Values of γ were calculated for a 25%, and 50% reduction in the modulus of elasticity. These same values were used for the corresponding increase of the thermal expansion coefficient. For changes of 25% and 50% the values for γ are: 0.0005, and 0.001 respectively, for $T_0=75$ and $T_1=550$ degrees Fahrenheit.

Since the variational method allows for a continuous solution, a quadratic curve fit which very closely matches an experimental temperature distribution obtained from NASA-Dryden was employed (REF 11). Figure 1 displays the temperature distribution used for this study which is given by the relation T(x)= dT(x) + 75.0. A one dimensional curve fit was accomplished due to time constraints and because this is a parametric study to give baseline theoretical results on very specific problem to be used for a more general problem at NASA. The curve fit used in the program is from Reference 12.

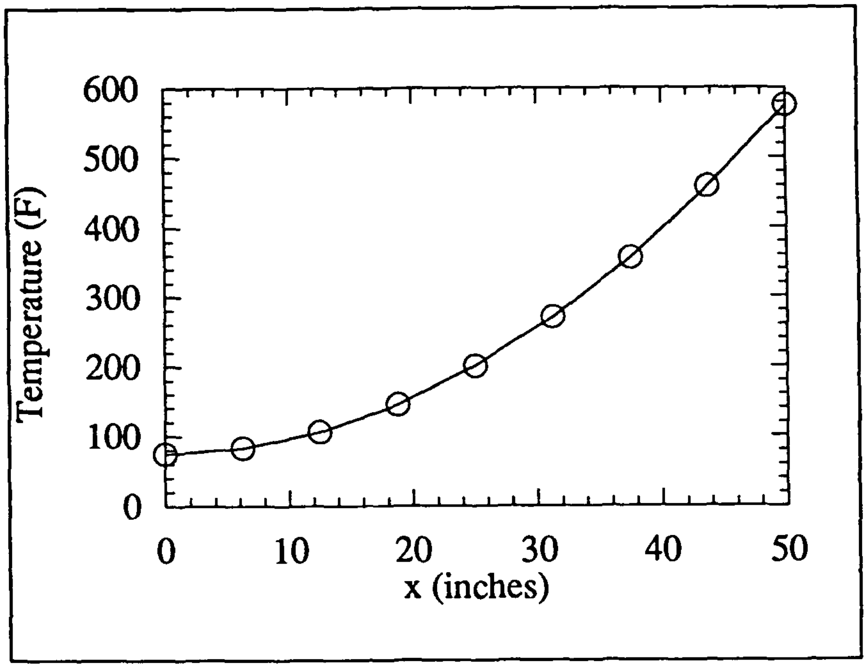

FIGURE 1. Computational Temperature Distribution

The materials scheduled for testing were Graphite/Epoxy, Boron/Epoxy, E-Glass/ Epoxy, and Aluminum. Properties for these materials were taken from References 2 and 8. These materials have certain temperature limitations. However, since this was a preliminary parametric study of the trends for a thermally loaded composite plate this limitation was overlooked. The materials selected could have been changed, or the temperature distribution reduced, but the materials are typical for test cases studied in most applications.

3.3 CONVERGENCE

In most approximation routines it is necessary to show that the computer code is converging to a single solution, whether or not this is a true solution depends on the method and comparison with experimental data. For the present study, the solution should converge as the order of the polynomial approximation increases.

For the present study, convergence was verified by taking the square root of the sum of squares for the in-plane loads at a specified number of points within the plate and summing them for the entire plate. This process was repeated for different order polynomials. Since this method is dependent on the number of points, the same number of points was used for all the test cases: 77 grid points, with 11 in the x-direction and 7 y locations for each x location . Contour plots for eighth and tenth order approximating polynomials are shown in Attachment 1. Clearly, the results for the two different order of polynomials are very close and would appear to support the premise that the solution is converging. However, additional orders of polynomial approximations were done to verify that convergence had been reached. Using the sum of the square root of the sum of squares, a percent difference of 2.5 was calculated for the entire plate for polynomial approximations of 8 and 10 respectively. The difference between an eighth and tenth order approximation at the center of the plate, x=25 inches and y=6 inches, is approximately 2%.

Additional convergence test cases were accomplished for higher order polynomials. However, the coefficient matrix [A] which is equivalent to $[B]^T[a][B]$ in (EQ 35) becomes ill-conditioned upon inversion, thus the accuracy is reduced. The difference between a tenth and twelfth order approximation was approximately 1.8%, and the difference between higher order polynomials was slightly worse because the [A] matrix became too ill-conditioned. A tenth order polynomial approximation was chosen for the test cases because it was the best trade-off between time and percent convergence.

The results from the contour plots closely matched experimental data from Reference 11 on similar test cases. This analysis has shown the verification of the program "Polystress". Finally, the program has solved a problem with a known solution quite accurately, and it can now be applied to problems whose solutions are unknown with some validity.

3.4 TEST MATRIX

Typically a test matrix is set up prior to the test period for any project. This is to ensure that the actual testing time is used efficiently. If something peculiar is found while processing the data additional testing can be accomplished, but usually only after the initial matrix is completed.

The test matrix for this problem is shown in Table 1. The tests in Table 1 will be performed for a fibrous, laminated, composite plate with dimensions: length= 50 in, width= 12 in, thickness=.19 in. A tenth order polynomial will be used to approximate the solution, and an eighth order polynomial will be used to verify convergence and input as required.

TABLE I. Test Matrix

Material	Layup	# Layers	Gamma	Theta Range
Graphite/Epoxy	[0]	1	0.00	----
Boron/Epoxy	[0]	1	0.00	----
E-Glass/Epoxy	[0]	1	0.00	----
Aluminum	[0]	1	0.00	----
Graphite/Epoxy	[θ]	1	0.0005	0 to 90
Boron/Epoxy	[θ]	1	0.0005	0 to 90
E-Glass/Epoxy	[θ]	1	0.0005	0 to 90
Aluminum	[0]	1	0.0005	----
Graphite/Epoxy	[+θ/−θ/+θ]	3	0.0005	0 to 90
Boron/Epoxy	[+θ/−θ/+θ]	3	0.0005	0 to 90
E-Glass/Epoxy	[+θ/−θ/+θ]	3	0.0005	0 to 90
E-Glass/Epoxy	[+θ/−θ/+θ/−θ/+θ]	5	0.0005	0 to 90
Graphite/Epoxy	[0]	1	0.001	----
Boron/Epoxy	[0]	1	0.001	----
E-Glass/Epoxy	[0]	1	0.001	----
Aluminum	[0]	1	0.001	----

Here γ is the factor applied to the modulus of elasticity and the coefficient of thermal expansion. The cases investigated are all for symmetric laminates since there is no coupling between extensional and bending stiffnesses for this layup.

Two types of symmetric laminates are studied: 1) laminates with (single/multiple) specially orthotropic layers, and 2) laminates with multiple generally orthotropic layers. Specially orthotropic implies that the stiffness matrix [Q] and the reduced stiffness matrix $[\bar{Q}]$ are identical, (i.e. the principal material coordinates of the lamina is aligned with global laminate axis for the plate) (REF 2). Item 2 contains the special classification: regular symmetric angle-ply laminate. This specification refers to orthotropic lamina, of equal thicknesses, with opposite signs of the angle of orientation of the principal material axis relative to the laminate axis system for the problem.

4. RESULTS & DISCUSSION

The primary purpose of this study is to perform a parametric investigation of the principal in-plane force resultants for a symmetrically laminated, 50"x12"x0.19"composite, plate for a thermal loading condition. The boundary conditions for the middle surface stresses for this study are free-free. This condition implies that both the midsurface stress components normal to the boundary and tangential to the boundary are zero. This is the conventional stress configuration (REF 6). Four different materials were investigated.

TABLE 2. Material Properties

Property	Graphite/Epoxy	Boron/Epoxy	E-Glass/Epoxy	Aluminum
E_1	30e6 psi	30e6 psi	7.8e6 psi	10.3e6 psi
E_2	0.75e6 psi	3.0e6 psi	2.6e6 psi	10.3e6 psi
ν_{12}	0.25	0.30	0.25	0.33
G_{12}	1.3e6 psi	1.0e6 psi	.375e6 psi	Isotropic
α_1	-0.21e-6 /F°	3.5e-6 /F°	3.5e-6 /F°	12.8e-6 /F°
α_2	16.0e-6 /F°	11.4e-6 /F°	11.4e-6 /F°	12.8e-6 /F°

Properties were taken from References 1, 2, and 8.

4.1 COMPARISON OF PRINCIPAL IN-PLANE FORCE RESULTANTS

Variations in lamina orientation angles and number of layers are considered for the free-free, laminated, composite, plate. The principal, in-plane force resultants for a single-lamina and a three-layered, regularly symmetric angle ply laminate as a function of orientation angle are shown in Figures 2, and 3 respectively. Principal in-plane force resultant, is essentially the maximum in-plane force resultant for the plate regardless of whether it is N_x, N_y, N_{xy}. In addition, the positive and negative principal in-plane force resultants were graphed separately to aid in analysis.

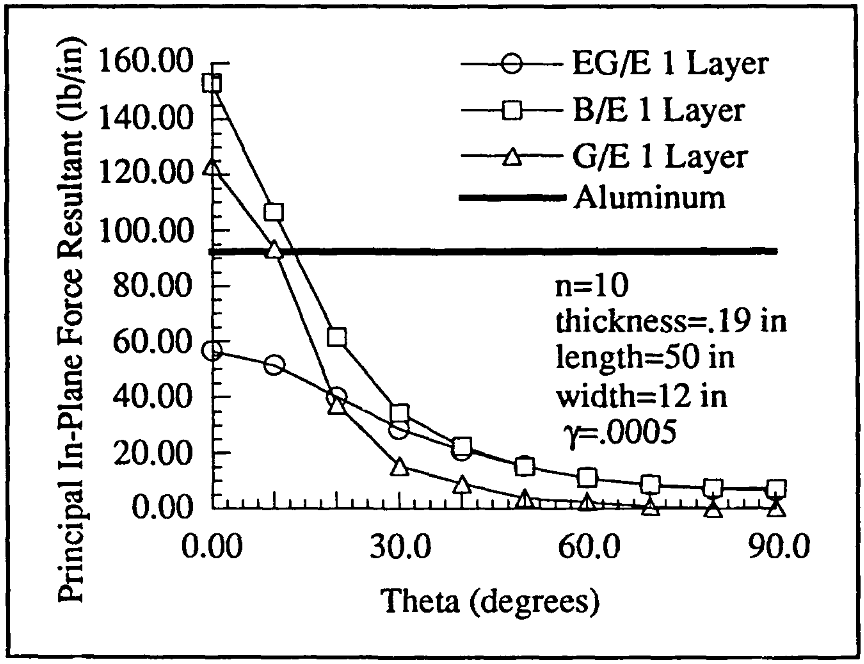

FIGURE 2. Principal (+) in-plane force resultant for a single lamina.

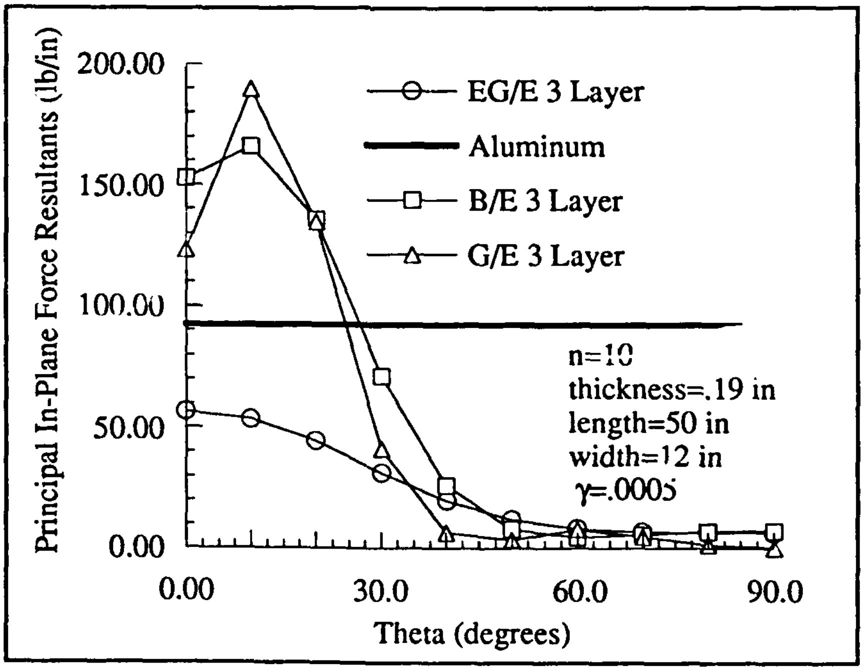

FIGURE 3. Principal (+) in-plane force resultant for a 3-layered laminate.

In both Figures 2 and 3 the magnitude of the principal resultant force is greatest at 0, and conversely at 180 degrees. This is a result of the one dimensional thermal loading applied to the plate. When the lamina is aligned with the load the in-plane stresses are the greatest because the modulus of elasticity for the lamina is aligned with the load instead of being at an angle relative to the load.

Furthermore, the relation between the material properties, Table 1, and the relations expressed in Figure 2 is pronounced. Initially, Boron/Epoxy has a higher force resultant than any of the other materials because the moduli of elasticity for Boron/Epoxy are as high or higher than any of the other materials. Also as the orientation angle exceeds approximately 45 degrees the force resultants for Boron/Epoxy and E-Glass/Epoxy are nearly identical. This result is due to the same coefficients of thermal expansion being used for both materials, and the relative decline in importance of E_1 and the increasing importance of E_2 as the orientation angle approaches 90 and E_2 is aligned in the x-direction of the laminate. Note the similarity in the values for E_2 for both materials. The isotropic case for aluminum was shown on both graphs as a reference for the other values. Tabular values for this case and the other test cases are in Attachment 2. Similar results to Figures 2 and 3 are given in Reference 2.

Figure 3 illustrates the same trends as Figure 2, however, the magnitude of the principal resultant forces is greater at ten degrees than at zero for all the composite materials except E-Glass/Epoxy. Another interesting observation involves the comparison of the unidirectional lamina at different orientations in Figure 2 with the angle-ply data in Figure 3. For angles larger than 0^o but less than 45^o, the angle-ply has about a 50 to 60 percent higher resultant force than does the unidirectional lamina. However, at approximately 40 to 45 degrees and higher, the single lamina has a greater resultant force. Again, E-Glass/Epoxy exhibits the same trends but on an order of magnitude lower. The trends for higher loads at 10^o than at 0^o, and the differences between the data for the single ply and the data for the multi-layer angle ply, are the results of mechanical and thermal interactions between the layers (REF 2).

The results for Graphite/Epoxy are more dramatic than any of the other cases. This is because the coefficient of thermal expansion in the 1 principal direction is less than zero (REF 1). This implies that the expansion in the 2 principal direction is large enough to cause a contraction in the 1-direction. This reaction produces thermal reactions between the layers which causes the resultant forces to be larger than normal. As the angle is increased, the contribution of the α_2 term is reduced, and the relatively low value of elastic moduli in the 2-direction causes the rapid decline of the resultant forces.

The relative invariant behavior of E-Glass/Epoxy is a result of the relatively small change between the elastic moduli in the 1 and 2 principal material directions. Compared to the other materials, except of course Aluminum, the difference between E_1, and E_2, is order of magnitudes lower. This difference explains the results in Figures 2 and 3.

Figures 4 and 5 illustrate the principal negative force resultants for the same layup as Figures 2 and 3.

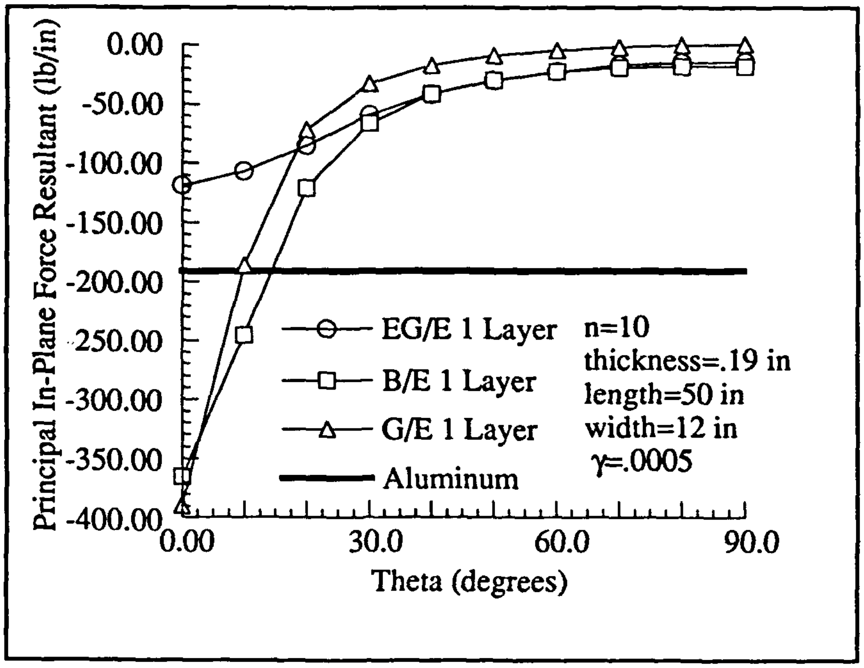

FIGURE 4. Principal (-) in-plane force resultant for a single lamina.

The trends for the data in Figure 4 are the same as those presented for Figure1.

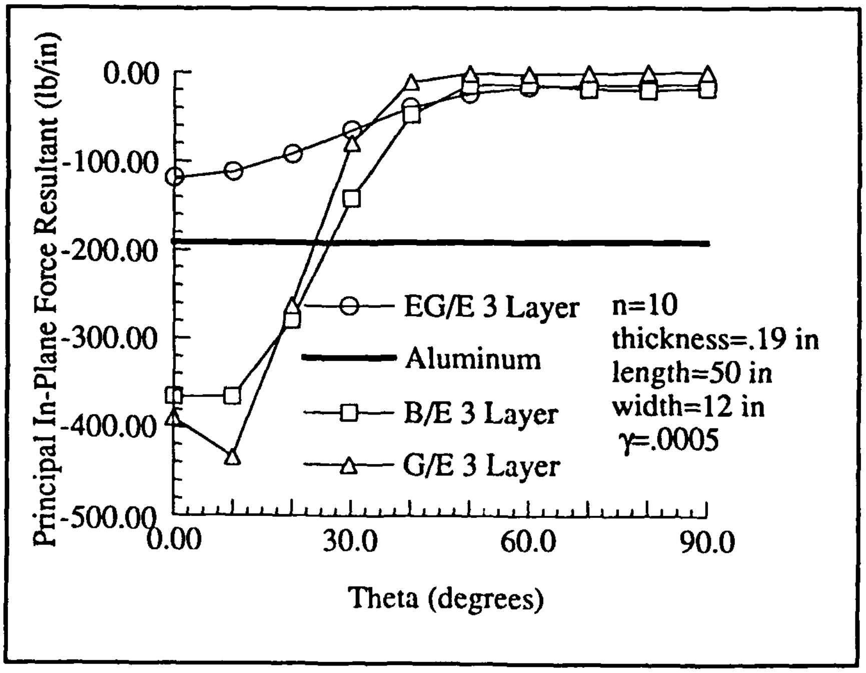

FIGURE 5. Principal (-) in-plane force resultant for a 3-layered laminate.

The data in Figures 4 and 5 follows the same trends as described earlier for the principal, positive, force resultant.

4.2 E-GLASS/EPOXY MULTI-LAYER COMPARISON

A comparison of the effect of the number of layers on the principal in-plane force resultants was accomplished for the free-free, E-Glass/Epoxy, laminated plate. Figures 6, and 7 show the results for 1, 3, and 5 layer angle ply laminates as a function of lamina orientation angle.

As discussed earlier for the comparison between the single lamina in Figure 2 and the 3-layered laminate in Figure 3, the interlaminar mechanical and thermal interactions account for the multi-layered laminate having larger in-plane resultant forces than the unidirectional lamina for orientation angles greater than 0^o and less than approximately 45^o degrees. For angles greater than 45^o the single-layered lamina has higher force resultants. The interlaminar effects explanation is also valid for Figures 6 and 7. From 0^o to approximately 40^o the resultant forces for the 3 and 5 layer laminates are higher than those for the single layer case. For angles greater than 40^o, the single-layer forces are greater. Also the difference between the force resultants for the 3 and 5 layer laminates is much less than that for the 1 and 3 layer laminates. One would expect the difference for each successive layer to get smaller and smaller based on these assumptions.

To verify this assumption a test case was ran for a [20/-20/20/-20/20/-20/20] E-Glass/ Epoxy laminate. The principal in-plane force resultants are: 45.129 lb/in and -92.251 lb/in respectively. The previous results for the E-Glass/Epoxy 1-layer, 3-layer, and 5-layer laminate are: 39.998 & -85.159, 44.433 & -91.055, and 45.030 & -92.128 lb/in respectively. These results are shown graphically in Figure 8.

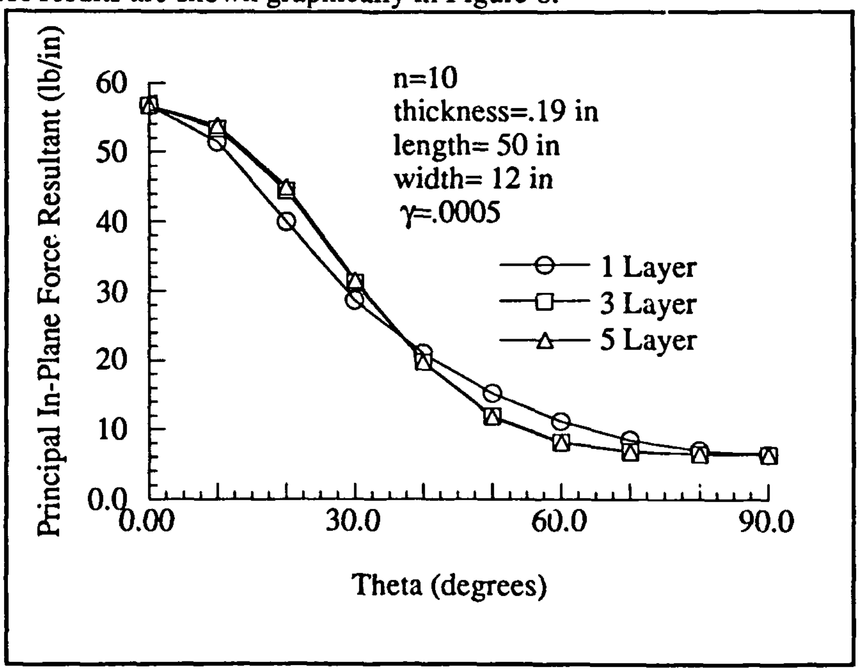

FIGURE 6. E-Glass/Epoxy multiple layer comparison (+).

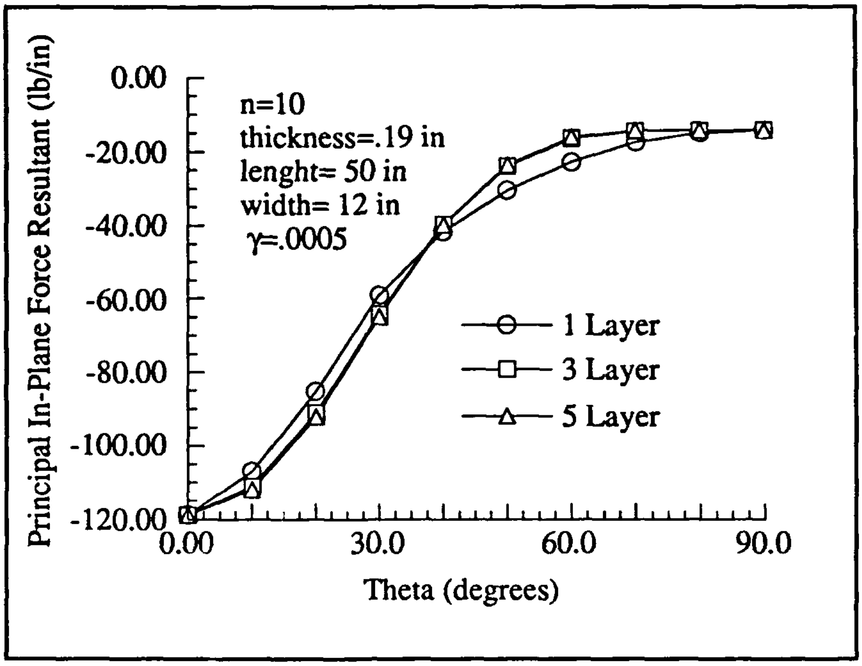

FIGURE 7. E-Glass/Epoxy multiple layer comparison (-).

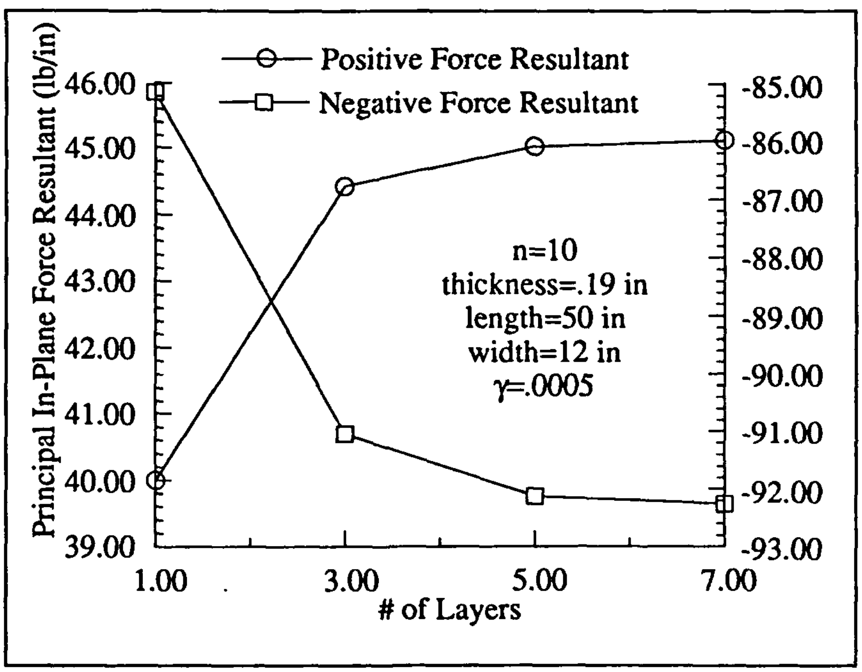

FIGURE 8. Effect of multiple layers on the force resultants for E-Glass/Epoxy.

4.3 EFFECT OF GAMMA ON IN-PLANE FORCE RESULTANTS

Recall that γ is the factor applied to the moduli of elasticity and the coefficient of thermal expansion to account for temperature dependency. Figure 9 displays the results for the investigation of the four different materials as specially orthotropic single lamina.

The magnitudes of the principal in-plane force resultants increased as γ increased. However, according to Hooke's Law, (EQ 1), the stress is directly related to the moduli of elasticity. Thus it would seem that the stress should decrease with an increase in γ since γ is the temperature degradation factor applied to the moduli of elasticity. But, γ is also applied to the coefficient of thermal expansion, α, which increases with temperature. Therefore, the effects of α on the thermal load vector must be great enough to overcome the effects of γ on the moduli of elasticity.

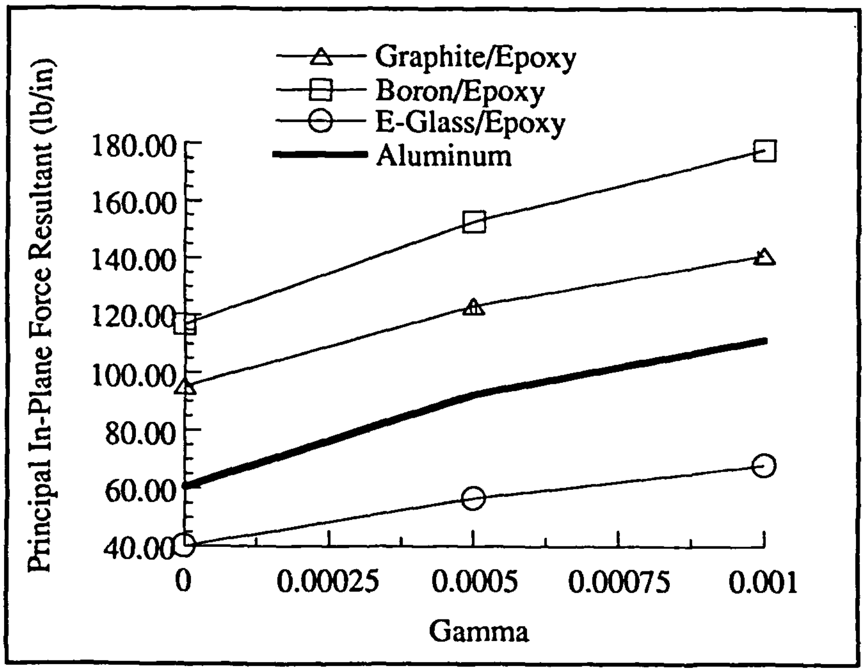

FIGURE 9. Effects of gamma on principal in-plane force resultant.

4.4 CONTOUR PLOTS FOR THE FOUR DIFFERENT MATERIALS

Contour plots for the four different materials investigated during this study are presented in Attachment 3. The plots shown are the force resultant in the x-direction, N_x, for zero γ, and are presented to give the basic characteristics of each material in a graphical instead of tabular method. In addition, the contour plot for each material is symmetrical because the gradient of the temperature distribution used for this study was symmetrical.

The contour plot for Graphite/Epoxy exhibits the largest range of force resultants for any of the materials, and E-Glass/Epoxy exhibits the smallest range. These results are functions of the material properties given in Table 2.

5. CONCLUSION & RECOMMENDATIONS

An analytical approach for determining the in-plane force resultants for symmetrically laminated, composite plates, subjected to a one dimensional thermal loading has been presented. The Galerkin method was used to approximately solve the compatibility equation. A linear response of the material properties with temperature was employed. The results for the baseline case of aluminum for this study were within 1 percent of an exact solution for the same case. In addition, the results demonstrate the dependency of the in-plane force resultants with orientation angle, and the relation between single-layer laminas and multi-layer laminates. The inter-lamina effects for the multi-layered laminate resulted in higher force resultants up to an orientation angle of 45 degrees as compared to the force resultants for the unidirectional laminate. The temperature dependency of the engineering constants resulted in higher force resultants for an increase in γ, the temperature dependency factor. These investigations should provide an excellent database for experimental studies of composite plates at NASA-Dryden. Further studies will investigate out-of-plane motion, for a two dimensional temperature distribution using exact integration techniques.

6. REFERENCES

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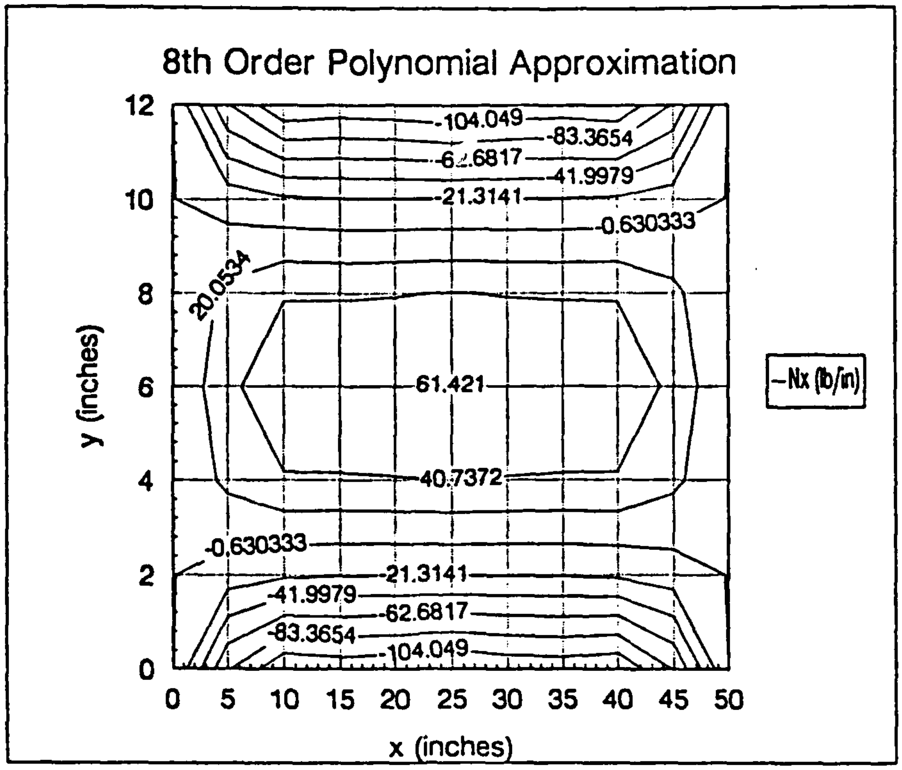
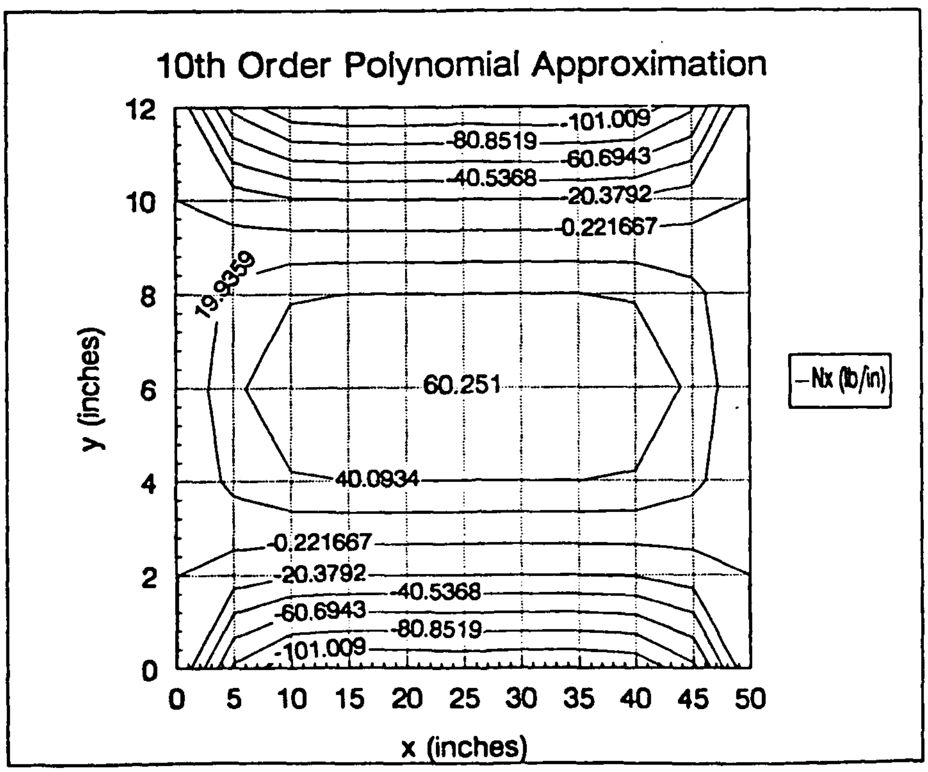
Attachment 1. Convergence Contour Plots



Contour Plots for the Principal In-Plane Force Resultants of Aluminum for different order approximations.



Attachment 2. Table of Results

Principal In-Plane Force Resultant

Length=50 in, Width=12 in, Thickness= .19 in, n=10

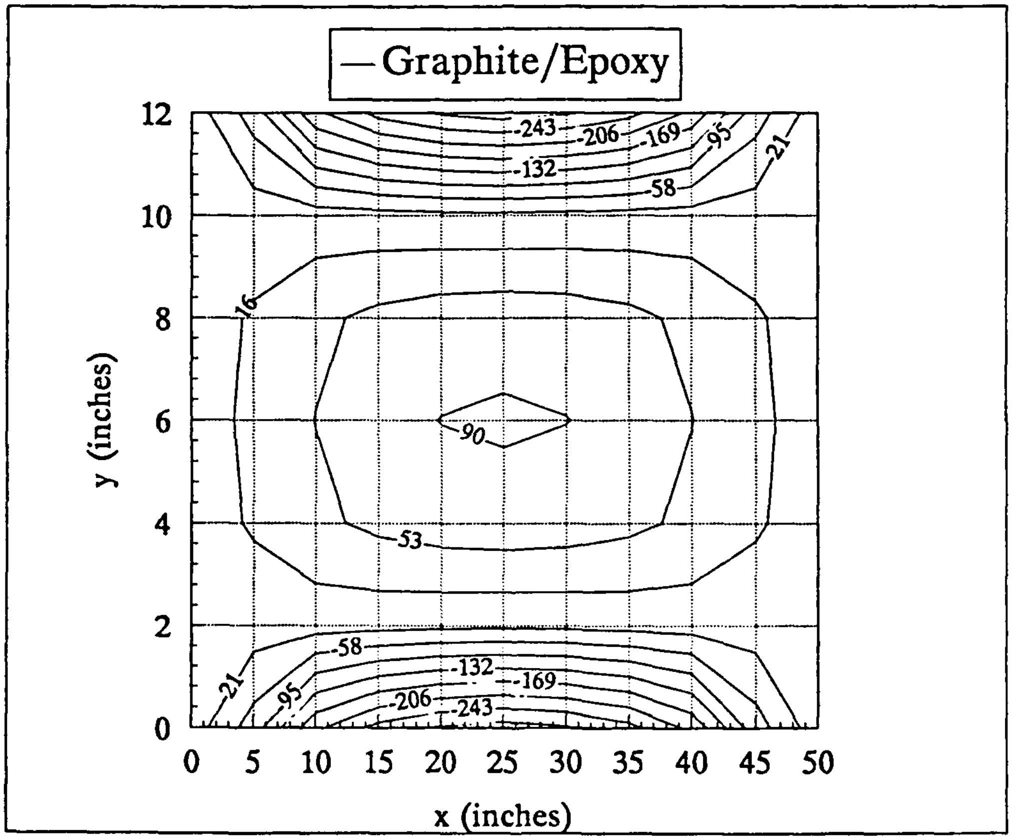
Case #	Material	Layup	Gamma	Pos	Neg
1	Graphite/Epoxy	[0]	0.00	95.078	-297.794
2	Graphite/Epoxy	[0]	0.0005	123.255	-389.614
3	Graphite/Epoxy	[0]	0.001	141.107	-458.437
4	Boron/Epoxy	[0]	0.00	116.701	-264.507
5	Boron/Epoxy	[0]	0.0005	152.819	-365.965
6	Boron/Epoxy	[0]	0.001	177.907	-434.965
7	E-Glass/Epoxy	[0]	0.00	40.120	-80.860
8	E-Glass/Epoxy	[0]	0.0005	56.615	-118.856
9	E-Glass/Epoxy	[0]	0.001	67.984	-143.959
10	Aluminum	[0]	0.00	60.251	-121.167
11	Aluminum	[0]	0.0005	92.166	-191.157
12	Aluminum	[0]	0.001	111.402	-229.105
13	Graphite/Epoxy	[10]	0.0005	93.460	-185.353
14	Graphite/Epoxy	[20]	0.0005	37.331	-71.444
15	Graphite/Epoxy	[30]	0.0005	15.573	-32.953
16	Graphite/Epoxy	[40]	0.0005	9.033	-17.200
17	Graphite/Epoxy	[50]	0.0005	3.989	-9.216
18	Graphite/Epoxy	[60]	0.0005	2.366	-4.844
19	Graphite/Epoxy	[70]	0.0005	0.936	-2.327
20	Graphite/Epoxy	[80]	0.0005	0.147	-0.432
21	Graphite/Epoxy	[90]	0.0005	0.340	-0.109
22	Boron/Epoxy	[10]	0.0005	106.450	-245.463
23	Boron/Epoxy	[20]	0.0005	61.323	-120.464
24	Boron/Epoxy	[30]	0.0005	34.410	-66.828

Case#	Material	Layup	Gamma	Pos	Neg
25	Boron/Epoxy	[40]	0.0005	27.406	-41.508
26	Boron/Epoxy	[50]	0.0005	15.297	-30.075
27	Boron/Epoxy	[60]	0.0005	11.202	-23.153
28	Boron/Epoxy	[70]	0.0005	8.803	-19.325
29	Boron/Epoxy	[80]	0.0005	7.520	-18.562
30	Boron/Epoxy	[90]	0.0005	7.197	-18.305
31	E-Glass/Epoxy	[10]	0.0005	51.472	-106.982
32	E-Glass/Epoxy	[20]	0.0005	39.998	-85.159
33	E-Glass/Epoxy	[30]	0.0005	28.629	-59.117
34	E-Glass/Epoxy	[40]	0.0005	21.013	-41.738
35	E-Glass/Epoxy	[50]	0.0005	15.249	-30.594
36	E-Glass/Epoxy	[60]	0.0005	11.183	-22.900
37	E-Glass/Epoxy	[70]	0.0005	8.500	-17.514
38	E-Glass/Epoxy	[80]	0.0005	6.959	-15.043
39	E-Glass/Epoxy	[90]	0.0005	6.455	-14.364
40	Graphite/Epoxy	[10/-10/10]	0.0005	189.613	-433.888
41	Graphite/Epoxy	[20/-20/20]	0.0005	134.863	-263.107
42	Graphite/Epoxy	[30/-30/30]	0.0005	41.275	-78.245
43	Graphite/Epoxy	[40/-40/40]	0.0005	6.474	-11.168
44	Graphite/Epoxy	[50/-50/50]	0.0005	3.513	-1.765
45	Graphite/Epoxy	[60/-60/60]	0.0005	7.723	-2.227
46	Graphite/Epoxy	[70/-70/70]	0.0005	4.835	-1.266
47	Graphite/Epoxy	[80/-80/80]	0.0005	1.386	-0.378
48	Graphite/Epoxy	[90/-90/90]	0.0005	0.340	-0.109
49	Boron/Epoxy	[10/-10/10]	0.0005	165.984	-365.550
50	Boron/Epoxy	[20/-20/20]	0.0005	135.700	-279.620
51	Boron/Epoxy	[30/-30/30]	0.0005	71.374	-141.720

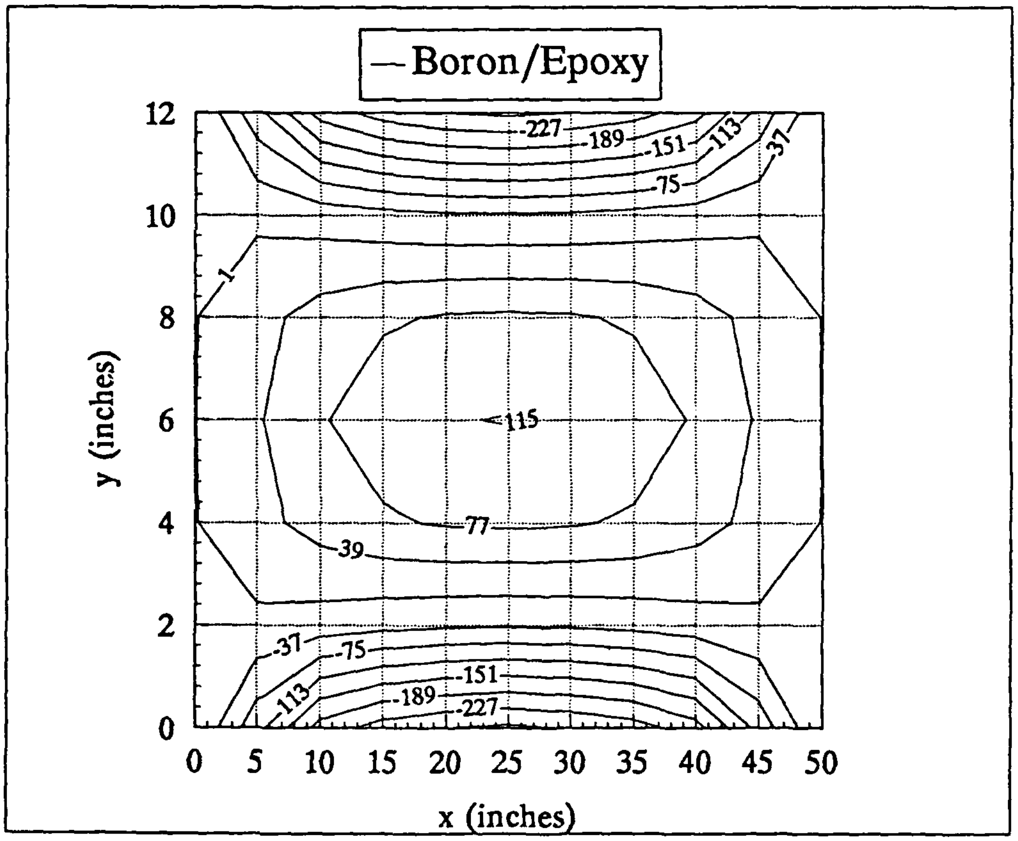
Case #	Material	Layup	Gamma	Pos	Neg
52	Boron/Epoxy	[40/-40/40]	0.0005	26.095	-47.760
53	Boron/Epoxy	[50/-50/50]	0.0005	7.792	-14.778
54	Boron/Epoxy	[60/-60/60]	0.0005	4.753	-13.693
55	Boron/Epoxy	[70/-70/70]	0.0005	5.826	-18.869
56	Boron/Epoxy	[80/-80/80]	0.0005	6.962	-20.227
57	Boron/Epoxy	[90/-90/90]	0.0005	7.197	-18.304
58	E-Glass/Epoxy	[10/-10/10]	0.0005	53.408	-111.120
59	E-Glass/Epoxy	[20/-20/20]	0.0005	44.433	-91.055
60	E-Glass/Epoxy	[30/-30/30]	0.0005	31.244	64.339
61	E-Glass/Epoxy	[40/-40/40]	0.0005	19.760	39.684
62	E-Glass/Epoxy	[50/-50/50]	0.0005	11.991	-24.000
63	E-Glass/Epoxy	[60/-60/60]	0.0005	8.193	-16.454
64	E-Glass/Epoxy	[70/-70/70]	0.0005	6.853	-14.559
65	E-Glass/Epoxy	[80/-80/80]	0.0005	6.494	-14.346
66	E-Glass/Epoxy	[90/-90/90]	0.0005	6.421	-14.289
67	E-Glass/Epoxy	[10/-10/10/-10/10]	0.0005	53.866	-112.070
68	E-Glass/Epoxy	[20/-20/20/-20/20]	0.0005	45.030	-92.128
69	E-Glass/Epoxy	[30/-30/30/-30/30]	0.0005	31.627	-64.973
70	E-Glass/Epoxy	[40/-40/40/-40/40]	0.0005	19.790	-39.812
71	E-Glass/Epoxy	[50/-50/50/-50/50]	0.0005	11.849	-23.661
72	E-Glass/Epoxy	[60/-60/60/-60/60]	0.0005	8.047	-16.102
73	E-Glass/Epoxy	[70/-70/70/-70/70]	0.0005	6.770	-14.471
74	E-Glass/Epoxy	[80/-80/80/-80/80]	0.0005	6.495	-14.375
75	E-Glass/Epoxy	[90/-90/90/-90/90]	0.0005	6.455	-14.364
76	E-Glass/Epoxy	[25/-25/25]	0.0005	38.807	-78.003
77	E-Glass/Epoxy	[32/-32/32]	0.0005	28.533	-58.944
78	E-Glass/Epoxy	[33/-33/33]	0.0005	27.333	-56.304

Case #	Material	Layup	Gamma	Pos	Neg
79	E-Glass/Epoxy	[45/-45/45]	0.0005	15.340	-30.713
80	E-Glass/Epoxy	[65/-65/65]	0.0005	7.325	-14.921
81	E-Glass/Epoxy	[75/-75/75]	0.0005	6.612	-14.413
82	E-Glass/Epoxy	[95/-95/95]	0.0005	6.438	-14.305
83	E-Glass/Epoxy	[110/-110/110]	0.0005	6.853	-14.559
84	E-Glass/Epoxy	[125/-125/125]	0.0005	9.671	-19.382
85	E-Glass/Epoxy	[135/-135/135]	0.0005	15.340	30.713
86	E-Glass/Epoxy	[145/-145/145]	0.0005	25.059	-51.184
87	E-Glass/Epoxy	[150/-150/150]	0.0005	31.244	-64.339
88	E-Glass/Epoxy	[165/-165/165]	0.0005	49.659	-102.502
89	E-Glass/Epoxy	[175/-175/175]	0.0005	55.603	-116.328
90	E-Glass/Epoxy	[180/-180/180]	0.0005	56.615	-118.856

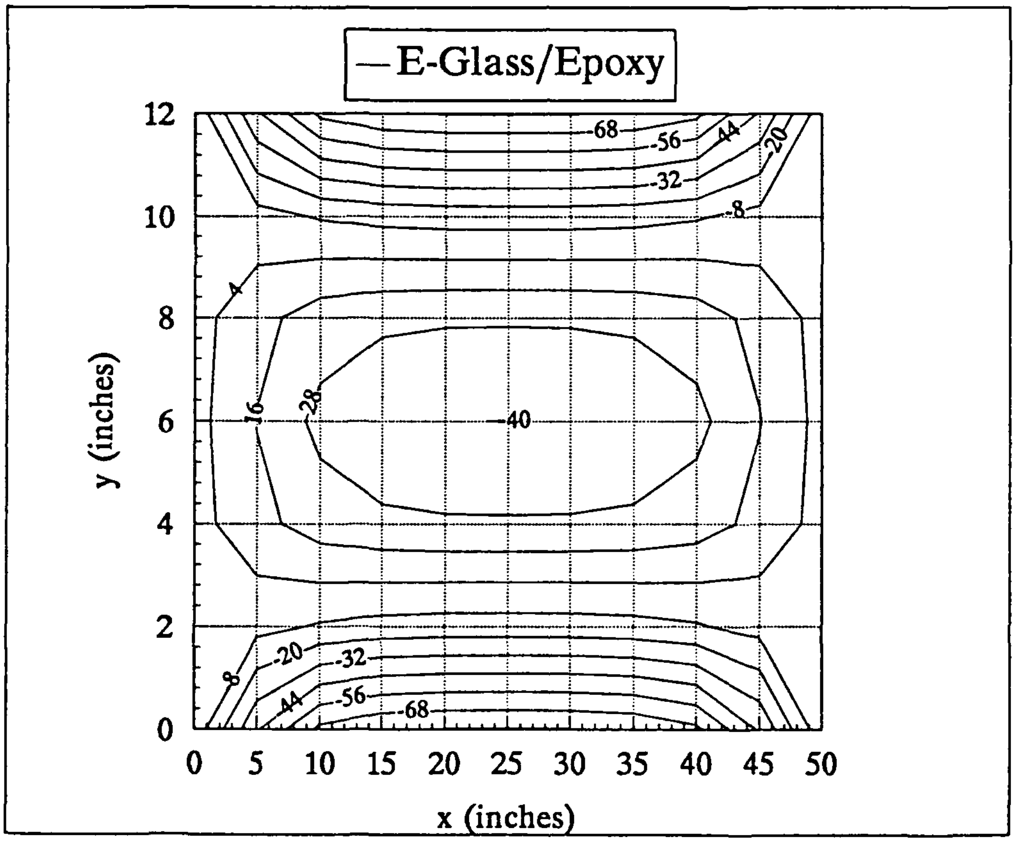
Attachment 3. Contour Plots For Different Materials



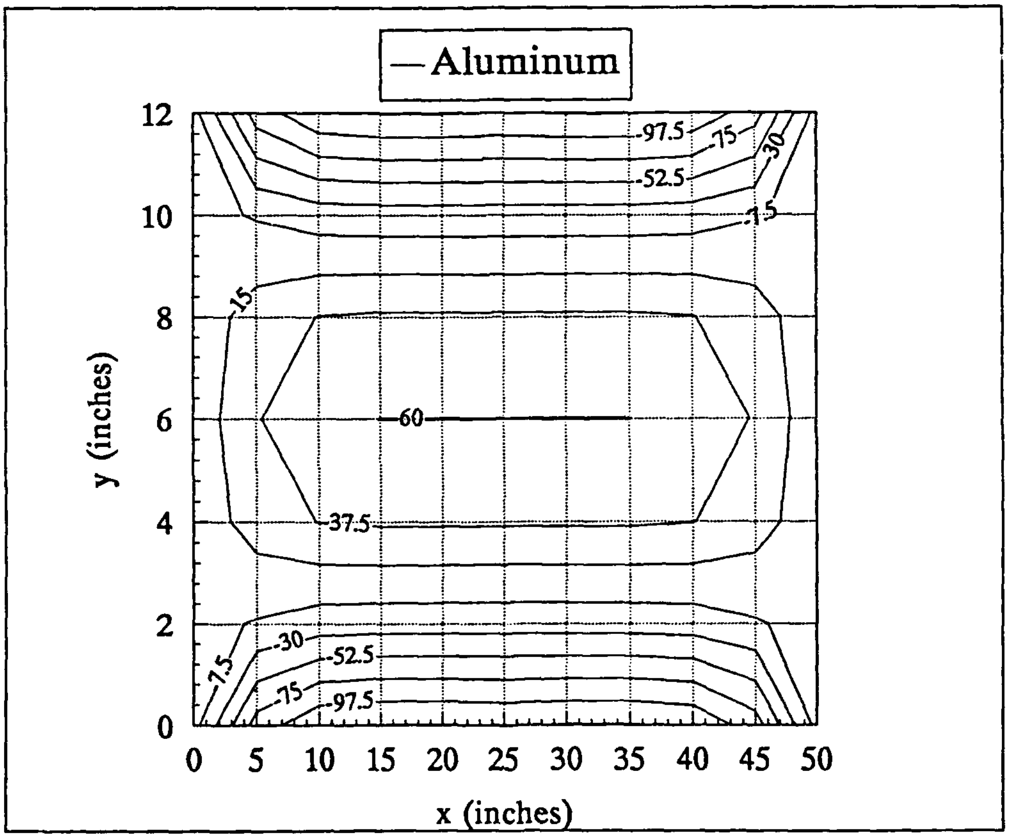
Graphite/Epoxy
12
10
8
6
4
2
0
y (inches)
0 5 10 15 20 25 30 35 40 45 50
x (inches)
-243
-206
-169
-132
-95
-58
-21
16
53
90



Boron/Epoxy
-227
-189
-151
-113
-75
-37
-1
-115
-77
-39
y (inches)
x (inches)
0
2
4
6
8
10
12
0 5 10 15 20 25 30 35 40 45 50



— E-Glass/Epoxy
y (inches)
x (inches)
0
2
4
6
8
10
12
0 5 10 15 20 25 30 35 40 45 50
-68
-56
-44
-32
-20
-8
4
16
28
40



—Aluminum
12
10
8
6
4
2
0
y (inches)
0 5 10 15 20 25 30 35 40 45 50
x (inches)
-97.5
-75
-52.5
-30
-7.5
15
-60
-37.5

Attachment 4. Program Listing

```
      PROGRAM POLYSTRESS
c ************************************************
c ** POLYNOMIAL RITZ SOLUTION FOR IN PLANE STRESS *
c ** EVALUATION WITH A THERMAL LOADING CONDITION  *
c **                        *
c ** THE IN PLANE FORCE RESULTANTS ARE WRITTEN TO *
c ** FILE CALLED SDISTRIBUTION.DAT                *
c **                                              *
c ** Darren Knipp                                 *
c ************************************************
      implicit real*8 (a-h,o-z)
      parameter(in=200)
      dimension am(in,in), bm(in,in), cm(in,in), av(in), bv(in)
      dimension bmat(in,in),bmgrid(in,in)
      dimension amd(in,in),avd(in),tk(25),theta(25)
      dimension x(9),T(9),TE(9),break(9),tcoef(4,9)
      common /worksp/rwksp
      real rwksp(2*in**2+3*in)
      call iwkin(2*in**2+3*in)

c **    open output data files ***********

      open(unit=9,file='extstiff.dat',status='unknown')
      open(unit=10,file='thermload.dat',status='unknown')
      open(unit=11,file='stressf.dat',status='unknown')
      open(unit=12,file='distrib.dat',status='unknown')

c ****** Temperature data ***********
c **
c **    dT(x) = 0.2*x^2, T(x)=dT(x)+75.***********
      data x/0.,6.25,12.5,18.75,25.,31.25,37.5,43.75,50./
      data T/75.,82.81,106.25,145.31,200.,270.31,356.25,457.81,575./
      data TE/75.,82.81,106.25,145.31,200.,270.31,356.25,457.81,575./

c *********** DATA INPUT **************

      call input(a,b,h,ip,nf,nx,ny,nl,tk,theta,tfactor,gfac)

c **    scale the room temperature wrsp to room temp. of 75 deg. **
      do 5 i=1,9
 5      T(i)=tfactor*(T(i)-75.)+75.

c **    curve fit the temperature data **
      call dcsakm(9,x,T,break,tcoef)

c *********** COMPUTE THE STRESS FUNCTION COEFFICIENTS ***

      call sfunction(a,b,h,nf,tk,theta,nl,am,bm,cm,av,bv,bmat,
     +                  amd,avd,break,tcoef,ip,gfac)

c **    calculate the stress distribution and write to output file

      call sdistribution(a,b,nf,nx,ny,av,bmgrid)

      stop
       end

c ************************************************************
      subroutine input(a,b,h,ip,nf,nx,ny,nlayer,tlayer,theta,
     +                        tfactor,gfactor)
c **
c **    This subroutine prompts the user for the problem information.
c **
c **************************************************************
      implicit real*8(a-h,o-z)
      character*30 name1,mat
      dimension tlayer(25),theta(25)
      pi=3.141592653589793

c **    Define the problem **

      print*,' Define material properties:  1, Graphite/Epoxy'
```

```
      print*,'                                    2, Boron/Epoxy'
      print*,'                                    3, E-Glass/Epoxy'
      print*,'                                    4, Aluminum'
      read(5,*)ip
      print*,' '
      print*,' Input the output data filename within single quotes '
      read(5,*) name1
      open(unit=3,file=name1,status='unknown')
      print*,' '
      print*,' Input order of polynomial for stress function'
      read(5,*) nf
      print*,' Input the plate length and width '
      read(5,*) a,b
      print*,'Input the number of layers:  nl (nl max=25) '
      read(5,*) nlayer
      print*,'For each layer input: thickness and theta '
      do 10 i=1,nlayer
       read(5,*) tlayer(i),theta(i)
  10    continue
      print*,'Input factor to apply to temperature distribution'
      print*,'(0.0 = room temp., 1.0 = actual lab temp.)'
      read(5,*) tfactor
      print*,'Input factor to apply to property variance with temp.'
      print*,'(0<= gfactor =<1)'
      read(5,*) gfactor
      print*,'Dimension of your grid for the stress distribution:x,y'
      read(5,*)nx,ny

      if (ip .eq. 1) then
       mat='Graphite/Epoxy'
      endif
      if (ip .eq. 2) then
       mat='Boron/Epoxy'
      endif
      if (ip .eq. 3) then
       mat='E-Glass/Epoxy'
      endif
      if (ip .eq. 4) then
       mat='Aluminum'
      endif
c **    echo input data **

      write(3,100)
      write(3,150) mat
      write(3,*)
       +'__________________________________________________________'
      write(3,*) ' '
      write(3,*) '           ----- Input Data -----'
      write(3,*)
       +'__________________________________________________________'
      write(3,*) ' '
      write(3,200) nlayer
      write(3,300)
      do 20 i=1,nlayer
       write(3,400) i,tlayer(i),theta(i)
 20     continue

c **    compute total thickness and transform ply angles to radians **

      h=0.0
      do 30 i=1,nlayer
       theta(i)=theta(i)*pi/180
       h=h+tlayer(i)
 30     continue
      write(3,450) h

      return

 100    format(/' Composite Laminate Stiffness'/)
 150    format(' Material: ',a20/)
 200    format(' Number of Layers = ',I3/)
 300    format(' Laminate Geometry '/,' Layer        t        theta')
```

```
 400    format(i4,4x,2f9.4)
 450    format(/'Laminate Thickness = ',f9.4)

      end
c *************************************************
        subroutine sfunction(a,b,h,n,tk,theta,nl,rm,rmb,tm,rv,rvb,bmat,
     + rmd,rvd,break,tcoef,ip,gfac)
c **
c **    This subroutine calculates the stress function
c **    coefficients: Fij.
c **
c **    variables:
c **         input:
c **            a     -- plate length x direction
c **            b     -- plate length y direction
c **            n     -- order of stress function polynomial
c **            h     -- plate thickness
c **         output:
c **            rm    -- [B]trans*[a]*[B] order N*N
c **            rv    -- row vector containing the coefficients
c **************************************************
     implicit real*8 (a-h,o-z)
     dimension rm((n+1)*(n+1),(n+1)*(n+1))
     dimension rmd((n+1)*(n+1),(n+1)*(n+1))
     dimension rv((n+1)*(n+1))
     dimension rvd((n+1)*(n+1))
     dimension tm((n+1)*(n+1),(n-3)*(n-3))
     dimension rmb((n-3)*(n-3),(n-3)*(n-3))
     dimension rvb((n-3)*(n-3))
     dimension bmat(3,(n+1)*(n+1)),amat(3,3),amv(3),tk(25)
     dimension point(10),weight(10),break(9),tcoef(4,9),theta(25)
     nn=(n+1)*(n+1)
     m=3
     nnb=(n-3)*(n-3)
     rmfac=a*b/4
     rvfac=-a*b/4
     ng=10

c **    initialize rm matrix to zero ******

        do 10 i=1,nn
      rv(i)=0.0
         do 10 j=1,nn
          rm(i,j)=0.0
   10   continue

c ****  call subroutine for numerical integration points and weights

        call gquad(point,weight)

c **    evaluate matrices at integration point *****

     do 30 i=1,ng
      zeta=point(i)
      wz=weight(i)
      call astiff(a,zeta,tk,theta,nl,break,tcoef,ip,gfac,amat,amv)
      j=0
      do 30 j=1,ng
       eta=point(j)
       we=weight(j)
       call bmatrix(a,b,n,zeta,eta,bmat)
       call btab(amat,bmat,rmd,m,nn)
       call bta(1,nn,m,amv,bmat,rvd)
       ii=0
       do 20 ii=1,nn
        rv(ii)=rv(ii)+rvd(ii)*wz*we
       jj=0
        do 20 jj=1,nn
        rm(ii,jj)=rm(ii,jj)+rmd(ii,jj)*wz*we
 20       continue
 30     continue
```

```
      do 40 i=1,nn
       rv(i)=rvfac*rv(i)
       do 40 j=1,nn
        rm(i,j)=rmfac*rm(i,j)
 40     continue

c **     write rm(stiffness matrix) & rv(load vector) to output data file

      write(10,100) (rv(i),i=1,nn)
      do 50 j=1,nn
       write(9,100) (rm(i,j),i=1,nn)
 50     continue
 100     format(6e12.5)

c **     solve matrix equation including the linear transformation

      call transform(n,tm)
      call btab(rm,tm,rmb,nn,nnb)
      call bta(1,nnb,nn,rv,tm,rvb)

      call dlslsf(nnb,rmb,nnb,rvb,rvb)
      call mply(tm,rvb,rv,nn,1,nnb)

c **     write rv to data file (fcoefficients)

      write(11,100) (rv(i),i=1,nn)

      return
      end

c ***********************************************************
      SUBROUTINE GQUAD(POINT,H)
c **
c **     This subroutine provides the integration points and
c **     weights for the Gaussian Quadrature procedure.
c **
c **     variables:
c **           ouput:
c **            point  -- location for numerical integration
c **            weight -- weight for each point
c **
c *************************************************************
      REAL*8 H(10), POINT(10)
C **      INTEGRATION POINTS AND WEIGHTS ***
      POINT(1)=.973906528517172
      POINT(2)=.865063366688985
      POINT(3)=.679409568299024
      POINT(4)=.433395394129247
      POINT(5)=.148874338981631
      POINT(6)=-.973906528517172
      POINT(7)=-.865063366688985
      POINT(8)=-.679409568299024
      POINT(9)=-.433395394129247
      POINT(10)=-.148874338981631
      H(1)=.066671344308688
      H(2)=.149451349150581
      H(3)=.219086362515982
      H(4)=.269266719309996
      H(5)=.295524224714753
      H(6)=.066671344308688
      H(7)=.149451349150581
      H(8)=.219086362515982
      H(9)=.269266719309996
      H(10)=.295524224714753
      RETURN
      END

c ***********************************************************
      subroutine bmatrix(a,b,n,zeta,eta,bm)
c **
c **     This subroutine forms the bmatrix.  The bmatrix consists
c **     of three row vectors {[H,yy]/[H,xx]/[H,xy]}.
```

```
c **
c **    variables:
c **         input:
c **            a    -- plate length (in)
c **            b    -- plate width (in)
c **            zeta -- nondimensionalized x-location
c **             eta -- nondimensionalized y-location
c **         output:
c **              bm -- bmatrix of order 3xN N=(n+1)*(n+1)
c ***********************************************************
      implicit real*8(a-h,o-z)
      dimension bm(3,(n+1)*(n+1))

c **    initialize constants ****************

      r1=4/b**2
      r2=4/a**2
      r3=4/(a*b)
      icol=0

c **    begin loop to formulate the bmatrix ***

      do 10 i=0,n
       do 10 j=0,n
        icol=icol+1
        iyy=j*(j-1)
        ixx=i*(i-1)
        ixy=i*j
        if (iyy .eq. 0) then
           bm(1,icol)=0.0
        else
           bm(1,icol)=iyy*eta**(j-2)*zeta**i*r1
        endif
        if (ixx .eq. 0) then
           bm(2,icol)=0.0
        else
           bm(2,icol)=ixx*zeta**(i-2)*eta**j*r2
        endif
        if (ixy .eq. 0) then
           bm(3,icol)=0.0
        else
           bm(3,icol)=ixy*zeta**(i-1)*eta**(j-1)*r3
        endif
 10    continue
      return
      end

c **********************************************************
      subroutine astiff(a,zeta,tk,theta,nl,break,tcoef,ip,gfac,
     +                      ainv,aintv)
c **
c **    This subroutine calculates the extensional stiffness
c **    matrix for a specially orthotropic, isotropic plate.
c **
c **    variables:
c **         input:
c **            theta  --  orientation angle
c **               nl  --  number of layers
c **               tk  --  lamina thickness
c **
c **         output:
c **             ainv  --  inverted stiffness matrix
c **            aintv  --  [ai]*[Nthermal]
c ***********************************************************
      implicit real*8(a-h,o-z)
      dimension ainv(3,3),aintv(3),therm(3),break(9),tcoef(4,9)
      dimension tk(25),theta(25),amat(3,3),q(3,3),f(3)
      pi=3.141592653589793

c **    convert zeta position to x position and evaluate temperature
c **    curve fit at the x position
```

```
      x=.5*a*(zeta+1.)
      temp=dcsval(x,8,break,tcoef)
      dt=temp-75.

c **     call property routine **

      call prop(dt,ip,gfac,e1,e2,a1,a2,rnu12,g12)
      print*,dt,ip,gfac,e1,e2,a1,a2,rnu12,g12
c **     initialize stiffness matrix and thermal load vectors **

      do 10 i=1,3
       f(i)=0.0
       do 10 j=1,3
        amat(i,j)=0.0
 10      continue

c **     form stiffness matrices
      do 50 ic=1,nl
       c=dcos(theta(ic))
       c2=c*c
       c3=c2*c
       c4=c2*c2
       s=dsin(theta(ic))
       s2=s*s
       s3=s2*s
       s4=s2*s2
c **     compute reduced stiffnesses **
       rnu21=(e2/e1)*rnu12
       q11=e1/(1-rnu12*rnu21)
       q12=rnu12*e2/(1-rnu12*rnu21)
       q22=e2/(1-rnu12*rnu21)
       q66=g12
c **     compute transformed reduced stiffnesses **
       q(1,1)=q11*c4+2*(q12+2*q66)*s2*c2+q22*s4
       q(1,2)=(q11+q22-4*q66)*s2*c2+q12*(s4+c4)
       q(2,2)=q11*s4+2*(q12+2*q66)*s2*c2+q22*c4
       q(1,3)=(q11-q12-2*q66)*s*c3+(q12-q22+2*q66)*s3*c
       q(2,3)=(q11-q12-2*q66)*s3*c+(q12-q22+2*q66)*s*c3
       q(3,3)=(q11+q22-2*q12-2*q66)*s2*c2+q66*(s4+c4)
       q(2,1)=q(1,2)
       q(3,1)=q(1,3)
       q(3,2)=q(2,3)
c **     compute transformed thermal expansion coefficients **
       ax=a1*c2+a2*s2
       ay=a1*s2+a2*c2
       axy=2*s*c*(a1-a2)
c **     compute terms for thermal forces **
       therm(1)=(q(1,1)*ax+q(1,2)*ay+q(1,3)*axy)*tk(ic)*dt
       therm(2)=(q(2,1)*ax+q(2,2)*ay+q(2,3)*axy)*tk(ic)*dt
       therm(3)=(q(3,1)*ax+q(3,2)*ay+q(3,3)*axy)*tk(ic)*dt
c **     form stiffness matrix and thermal load vector ntv **
       do 20 i=1,3
        f(i)=f(i)+therm(i)
        do 20 j=1,3
         amat(i,j)=amat(i,j)+q(i,j)*tk(ic)
 20       continue

 50      continue
      do 15 i=1,3
 15      print*,(amat(i,j),j=1,3)
      print*,(f(i),i=1,3)

c **     invert the stiffness matrix **

      call inv(amat,ainv)

c **     calculate ai*f **

      call mply(ainv,f,aintv,3,1,3)

      return
      end
```

```
c ***************************************************************
      subroutine prop(dt,ip,gfac,e1,e2,a1,a2,rnu12,g12)
c **
c **    This subroutine accounts for the material property
c **    variation wrsp to temperature.
c **
c **    variables:
c **        input:
c **           dt       -- change in temperature
c **           ip       -- material property specification
c **          gfac       -- material property variation with
c **                          temperature factor
c **
c **        output:
c **           e1       -- modulus of elasticity (fiber direction)
c **           e2       -- modulus of elasticity (perp. to 1)
c **           a1       -- coefficient of thermal expansion (1)
c **           a2       -- coefficient of thermal expansion (2)
c **        rnu12       -- Poisson's ratio
c **          g12       -- shear modulus
c ****************************************************************
      implicit real*8(a-h,o-z)

c **    specify which material properties to use **
c **    properties taken from Jones 'Mechanics of Composite **
c **    Materials' p.70 & 199.

      if (ip .eq. 1) then

c **    Graphite/Epoxy material properties **

       e1rt=30.0e6
      e2rt=.75e6
      rnu12=.25
      g12rt=.375e6
      a1rt=-0.21e-6
      a2rt=16.0e-6
      fac=1.0-gfac*dt
      fac2=1.0+gfac*dt
      e1=e1rt*fac
      e2=e2rt*fac
      a1=a1rt*fac2
      a2=a2rt*fac2
      g12=g12rt*fac

      endif

      if (ip .eq. 2) then

c **    Boron/Epoxy  material properties **

       e1rt=30.0e6
      e2rt=3.0e6
      rnu12=.30
      g12rt=1.0e6
      a1rt=3.5e-6
      a2rt=11.4e-6
      fac=1.0-gfac*dt
      fac2=1.0+gfac*dt
      e1=e1rt*fac
      e2=e2rt*fac
      a1=a1rt*fac2
      a2=a2rt*fac2
      g12=g12rt*fac

      endif

      if (ip .eq. 3) then

c **    E-Glass/Epoxy material properties **
```

```
       e1rt=7.8e6
      e2rt=2.6e6
      rnu12=.25
      g12rt=1.25e6
      a1rt=3.5e-6
      a2rt=11.4e-6
      fac=1.0-gfac*dt
      fac2=1.0+gfac*dt
      e1=e1rt*fac
      e2=e2rt*fac
      a1=a1rt*fac2
      a2=a2rt*fac2
      g12=g12rt*fac

      endif

      if (ip .eq. 4) then

c **    Aluminum material properties (isotropic case) **

      e1rt=10.3e6
      rnu12=.33
      g12rt=e1rt/(2*(1+rnu12))
      a1rt=12.8e-6
      fac=1.0-gfac*dt
      fac2=1.0+gfac*dt
      e1=e1rt*fac
      e2=e1
      a1=a1rt*fac2
      a2=a1
      g12=g12rt*fac

      endif

      return
      end

c ************************************************************
      SUBROUTINE INV(XM,XINV)
c **
c **    This subroutine computes the inverse of a 3x3 matrix.
c **
c *************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION XM(3,3),XINV(3,3),C(3,3)

c **    XI=ADJ(X)/DET(X) **
c **    ADJ(X)=TRANSPOSE OF COFACTOR MATRIX **

      C(1,1)=XM(2,2)*XM(3,3)-XM(2,3)*XM(3,2)

c **    account for - when taking the det of x by switching order
c **    of multiplication for the odd sum i+j of x(i,j)

      C(2,1)=XM(1,3)*XM(3,2)-XM(1,2)*XM(3,3)
      C(3,1)=XM(1,2)*XM(2,3)-XM(1,3)*XM(2,2)
      C(1,2)=XM(2,3)*XM(3,1)-XM(2,1)*XM(3,3)
      C(2,2)=XM(1,1)*XM(3,3)-XM(1,3)*XM(3,1)
      C(3,2)=XM(1,3)*XM(2,1)-XM(1,1)*XM(2,3)
      C(1,3)=XM(2,1)*XM(3,2)-XM(2,2)*XM(3,1)
      C(2,3)=XM(1,2)*XM(3,1)-XM(1,1)*XM(3,2)
      C(3,3)=XM(1,1)*XM(2,2)-XM(1,2)*XM(2,1)

      DET=XM(1,1)*C(1,1)+XM(2,1)*C(2,1)+XM(3,1)*C(3,1)

      DO 10 I=1,3
       DO 10 J=1,3
        XINV(I,J)=C(J,I)/DET
   10    CONTINUE

      RETURN
      END
```

```
c ***********************************************************************
      subroutine sdistribution(a,b,n,nx,ny,fcoef,bgrid)
c **
c **    This subroutine determines the stress distribution for
c **    the requested grid parameters.
c **
c **    variables:
c **        input:
c **            a     -- plate length
c **            b     -- plate width
c **            n     -- order of polynomial
c **            nx    -- # of grid points in x-direction
c **            ny    -- # of grid points in y-direction
c **          fcoeff -- stress function coefficients
c **
c **        output:
c **
c ***********************************************************************
      implicit real*8(a-h,o-z)
      dimension bgrid(3,(n+1)*(n+1))
      dimension fcoef((n+1)*(n+1)),eload(3)
      nn=(n+1)*(n+1)

c **    put origin of the grid in the center of the plate to utilize
c **    the symmetry of the problem

c      dx=a/(2*(nx-1))
c      dy=b/(2*(ny-1))
c      xloc=a/2-dx
c      yloc=b/2

c **    to verify symmetry put the origin of the grid at the left edge

      dx=a/(nx-1)
      dy=b/(ny-1)
      xloc=0.0-dx
      yloc=0.0

c **    calculate the load intensities {N} for each grid point

      do 10 i=1,nx
       xloc=xloc+dx
       zeta=(2*xloc/a)-1
       yloc=0.0
       do 10 j=1,ny
        eta=(2*yloc/b)-1
        call bmatrix(a,b,n,zeta,eta,bgrid)
        call mply(bgrid,fcoef,eload,3,1,nn)
        write(12,100) xloc,yloc,(eload(ii),ii=1,3)
        yloc=yloc+dy
 10      continue

 100    format(1x,f7.3,f7.3,2x,3f18.6)

      return
      end

c ***********************************************************************
      SUBROUTINE BTAB(A,B,R,M,N)
c **
c **    This subroutine calculates [B]trans*[a]*[B] matrix.
c **
c **    variables:
c **          input:
c **            A -- matrix of order M*M
c **            B -- matrix of order M*N
c **            N -- order (n+1)*(n+1)
c **         output:
```

```
c **            R -- matrix of order N*N
c **
c ****************************************************************
      implicit real*8(a-h,o-z)
      DIMENSION A(M,M), B(M,N),R(N,N)

      DO 40 I=1,N
       DO 30 J=1,N
        DY=0.0
         DO 20 K=1,M
          IF (B(K,I) .EQ. 0.0) GO TO 20
          CY=0.0
          DO 10 L=1,M
           IF (B(L,J) .EQ. 0.0) GO TO 10
           CY=CY+A(K,L)*B(L,J)
 10          CONTINUE
          DY=DY+CY*B(K,I)
 20         CONTINUE
         R(I,J)=DY
 30       CONTINUE
 40     CONTINUE
      RETURN
      END

c ****************************************************************
      SUBROUTINE BTA(L,M,N,A,B,C)
c **
c **    This subroutine calculates [B]transpose*[A].
c **
c **    variables:
c **         input:
c **            A -- matrix of order N*L
c **            B -- matrix of order N*M
c **         output:
c **            C -- matrix of M*L
c ****************************************************************
      INTEGER I,J,K,L,M,N
      REAL*8 A(N,L),B(N,M),C(M,L),DY

      DO 10 I=1,M
       DO 10 J=1,L
        DY=0.0
        DO 20 K=1,N
 20         DY=DY+B(K,I)*A(K,J)
        C(I,J)=DY
 10     CONTINUE
      RETURN
      END

c ***************************************************************
      SUBROUTINE MPLY(A,B,R,M,N,L)
c **
c **    This subroutine calculates [a]*[b] and stores it in [r].
c **
c **    variables:
c **         input:
c **            A -- matrix of order M*L
c **            B -- matrix of order L*N
c **         output:
c **            R -- matrix of order M*N
c **************************************************************
      IMPLICIT REAL*8(A-H,O-Y)
      DIMENSION A(M,L), B(L,N), R(M,N)

      DO 20 I=1,M
       DO 20 J=1,N
        Y=0.0
        DO 10 K=1,L
 10         Y=Y+A(I,K)*B(K,J)
        R(I,J)=Y
 20     CONTINUE
      RETURN
```

```
      END
c ***************************************************************
      SUBROUTINE TRANSFORM(N,TM)
c **
c **    This subroutine calls the transformation matrix subroutine.
c **
c **    variables:
c **        input:
c **            N  -- order of stress function polynomial
c **        output:
c **            TM -- transformation matrix
c ***************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION TM((N+1)*(N+1),(N-3)*(N-3))
      II=0
      DO 10 I=0,N
       DO 10 J=0,N
        II=II+1

        JJ=0
        DO 10 K=4,N
         DO 10 L=4,N
          JJ=JJ+1

          CALL TMAT(I,K,AFAC)
          CALL TMAT(J,L,BFAC)
          TM(II,JJ)=AFAC*BFAC
 10      CONTINUE
      RETURN
      END

c ***************************************************************
      SUBROUTINE TMAT(I1,I2,FAC)
c **
c **    This subroutine forms the linear transformation matrix
c **    to account for the zero edge boundary conditions.
c **
c **    variables:
c **        input:
c **            I1  --
c **            I2  --
c **        output:
c **            FAC --
c ***************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      ICOUNT=I1+1
      GOTO(10,20,30,40)ICOUNT
      IF (I1.EQ.I2) THEN
      FAC=1.0
      ELSE
      FAC=0.0
      END IF
      RETURN
 10    N2=(-1)**I2
      N4=(-1)**(I2-1)
      FAC=(-2*(1+N2)+I2*(1-N4))/4.
      RETURN
 20    N2=(-1)**I2
      N4=(-1)**(I2-1)
      FAC=(3*(N2-1)+I2*(1+N4))/4.
      RETURN
 30    N4=(-1)**(I2-1)
      FAC=I2*(N4-1)/4
      RETURN
 40    N2=(-1)**I2
      N4=(-1)**(I2-1)
      FAC=(1-N2-I2*(1+N4))/4
      RETURN
      END
```