NOAA Technical Report NOS 84 NGS 15

Application of Special Variance Estimators to Geodesy

Rockville, Md.
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NOAA geodetic publications

Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys. Federal Geodetic Control Committee, John O. Phillips (Chairman), Department of Commerce, NOAA, NOS, 1974 reprinted annually, 12 pp (PB265442). National specifications and tables show the closures required and tolerances permitted for first-, second-, and third-order geodetic control surveys. (A single free copy can be obtained, upon request, from the National Geodetic Survey, C18x2, NOS/NOAA, Rockville MD 20852.)

Specifications To Support Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys. Federal Geodetic Control Committee, John O. Phillips (Chairman), Department of Commerce, NOAA, NOS, 1975, reprinted annually, 30 pp (PB261037). This publication provides the rationale behind the original publication, "Classification, Standards of Accuracy, ..." cited above. (A single free copy can be otained, upon request, from the National Geodetic Survey, C18x2, NOS/NOAA, Rockville MD 20852.)

Proceedings of the Second International Symposium on Problems Related to the Redefinition of North American Geodetic Networks. Sponsored by U.S. Department of Commerce; Department of Energy, Mines and Resources (Canada); and Danish Geodetic Institute; Arlington, Va., 1978, 658 pp. (GPO #003-017-0426-1). Fifty-four papers present the progress of the new adjustment of the North American Datum at midpoint, including reports by participating nations, software descriptions, and theoretical considerations.

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NOS NGS-1 Use of climatological and meteorological data in the planning and execution of National Geodetic Survey field operations. Robert J. Leffler, December 1975, 30 pp (PB249677). Availability, pertinence, uses, and procedures for using climatological and meteorological data are discussed as applicable to NGS field operations.

NOS NGS-2 Final report on responses to geodetic data questionnaire. John F. Spencer, Jr., March 1976, 39 pp (PB254641). Responses (20%) to a geodetic data questionnaire, mailed to 36,000 U.S. land surveyors, are analyzed for projecting future geodetic data needs.

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Application of Special Variance Estimators to Geodesy

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Abstract. Special variance estimators are computed and analyzed for a standard geodetic network adjustment. One important estimator requires the computation of noninteger degrees of freedom. An analysis is performed on the results obtained from constraining the coordinates of peripheral network stations by a priori variances.

1. Introduction

This report has two purposes. The first is to discuss and analyze the application of special variance estimators to a standard geodetic triangulation network adjustment. The second is to analyze the results obtained when the coordinates of peripheral network stations are constrained by a wide range of a priori variances. These variances obviously determine the distortions of interior observations caused by constraining the previously adjusted border station coordinates.

Using equations formulated by Theil (1963), a variance estimator is used that requires the computation of "noninteger degrees of freedom." Theil's derivation was underlain by a Bayesian point of view, involving the introduction of prior probability densities on parameters (Bossler 1972). The question of whether these prior densities should be used to incorporate subjective opinion, or should be restricted to more concrete types of prior knowledge, belongs to the study of the possible interpretations of probability, an always controversial subject, and not our concern here. In this paper, we assume prior variances (actually the entire variance-covariance matrix) to be known, although in fact we do vary their common scale throughout its range for the sake of the resulting sensitivity analysis. One might also consider varying the prior values of (or a scale in) the variances of the observations; but in the case at hand, and as assumed by Theil, a good value of this scale factor is already available from a free adjustment.

We follow Theil in distinguishing between observations on the one hand, and parameters having associated prior variances on the other. However, it is worth noting that the latter can be regarded as constituting only another group of observations of a very particular type.

The occurrence of nonintegral degrees of freedom, apart from its intuitive appeal, is to be expected at some point in any generalized variance estimation scheme that involves distinguishing the unknown true values of prior variances from their preliminary values by unknown multiplicative constants (more than one) or variances factors, which are to be estimated from the data at hand. In contrast to the usual derivation (leading to integral degrees of freedom) in which there is only one such factor common to all prior variances, here the unknown factor occurs only in the variances on the observations of distances, directions, and azimuths. Theil's formulas allow us to incorporate the additional information contained in prior variances on parameters into the estimation of this variance factor in a consistent way.

2. Description of the Test Network

A 67-station horizontal triangulation network located in northwest Arkansas (fig. 1) was selected for our purposes. This network was originally designed to provide a greater density of control in a region with sparse horizontal control. The geodetic network consists of 47 new stations encircled by 20 stations which were previously adjusted in several other triangulation arcs. Such a fill-in network is generally referred to as "area work" and is typical of most networks of this size in the United States. Observations were made between May and October, 1971, using first-order procedures. Table 1 lists the types and quantities of the observations used in the adjustment, together with the equations for computing their a priori variances. Prior to these adjustments, all spurious observations were eliminated by extensive preprocessing and procedural checks to assure excellent data quality.

TABLE 1. — Network observations

Type of observations	Quantity	1/assigned weight
Old stations (part of existing arcs)	20	See section 4.2
New stations	47	0
Directions	538*	$\sigma^2 = (0.6)^2 + 2(0.001/D \sin 1'')^2$
Distances	47	$\sigma^2 = (15.0)^2 + (D \times 10^{-6})^2 + (0.00005\Delta H/3)^2$
Azimuths	4	$\sigma^2 = 0.45^2 + 0.80^2 + (\tan \phi/0.80)^2 + (0.40 \sin \phi)^2$

* Adjustment included 136 lists of directions.
ϕ = latitude.
D = distance in meters.
ΔH = height difference in meters.

3. Description of the Problem

Geodesists have performed geodetic adjustments in the past using direct observations of unknown parameters (Schmid and Schmid 1965). In our study we assumed a priori knowledge associated with the 20 points surrounding the border of the triangulation network shown in figure 1. Latitude and longitude coordinates of each point were assumed, along with both *known* and *estimated* variances of these parameters. The data adjustments were based on the standard errors shown in table 2 (sec. 4.2). Results obtained from the combined adjustment (old arc and new area work) are given in section 4. Of singular interest is the computation and analysis of the a posteriori estimator of the variance of unit weight of the direction and distance observations, $\hat{\sigma}^2_{0_L}$. The notation ^ differentiates between the *estimated* and the true (or known) value.

We use here an estimator derived by Theil (1963) that was brought to the attention of the geodetic community by Bossler (1972). It accounts for the a priori knowledge associated with the *known* variances of the known latitudes and longitudes in a certain fashion. The derivation leads to a noninteger value for the degrees of freedom. This notation and the associated concept of a continuous range for the degrees of freedom appeal to initial intuition when pondering the question of whether the assigned latitude and longitude values are observations or unknowns. We believe our study is the first documented large-scale geodetic application of Theil's concept. It is especially reassuring to note that the results achieved with these techniques do not depart significantly from those computed in the traditional manner.

4. Combined Geodetic Adjustment

4.1 Results of the Free Adjustment

A "free" adjustment of the 67 stations indicated that the quality of the data was excellent. Here, free adjustment means that the coordinates of one point were held fixed by heavily weighting that particular latitude and longitude. All azimuths and distances were entered into the adjustment and assigned the weights (the reciprocal of the variances) shown in table 1. After the initial solution one iteration was required, using the NGS TRAV10 program (Schwarz 1978), for this adjustment and all others described in this report.

The estimated variance of unit weight from the free adjustment was

$$\hat{\sigma}^2_{0_L} = \frac{V'PV}{n-u} = 0.968; \quad n-u = 321$$

where

$\hat{\sigma}^2_{0_L}$ is the estimated variance of unit weight,

V is a column vector of residuals, and

P is the observational weight matrix.

Note that $\sigma^2_{0_L} P^{-1} = \Sigma_L$, where Σ_L is defined as the matrix of the *true* variances of the observations. The quantity $n-u$ is normally referred to as "degrees of freedom" and is the number of observations, n, minus the number of unknown parameters, u, to be

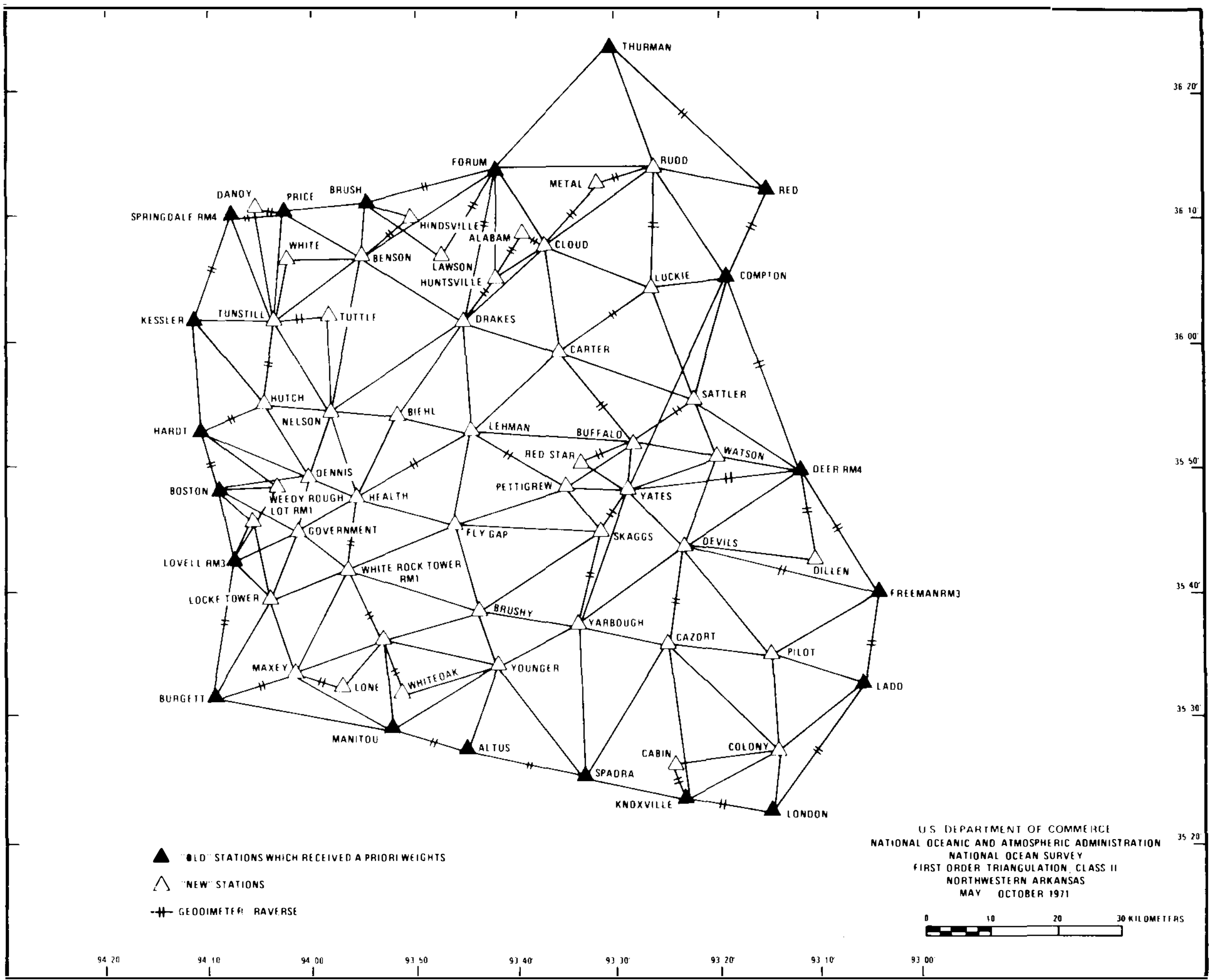

FIGURE 1.—First-order, class II triangulation in northwestern Arkansas.

solved. Incidentally, this value (0.968) is well within the χ^2 allowable limits at the 99-percent level.

4.2 Results of Combined Adjustment

After the free adjustment, a series of weighted adjustments was performed. For each, a common standard error (in meters) was assigned to the latitudes and longitudes of the 20 stations around the border of the triangulation scheme shown in figure 1. The values ranged from

$$\Sigma^{1/2}=(\Sigma_{\phi,\lambda})^{1/2}=\begin{bmatrix}0.0001 & 0\\ 0 & 0.0001\end{bmatrix}$$

$$\text{to}\ \ (\Sigma_{\phi,\lambda})^{1/2}=\begin{bmatrix}100.0 & 0\\ 0 & 100.0\end{bmatrix}.$$

The reason for performing these adjustments was to "fit" the 47-station net into the larger net defined by the 20 border points shown in figure 1. Thirty-five adjustments were performed in which the elements of the diagonal matrix $(\Sigma_{\phi,\lambda})^{1/2}$ generally were varied by increments of 0.01 meter (table 2).

We plotted Theil's estimator, $\hat{\sigma}^2_{0_T}$, which may be a more appropriate estimate for $\hat{\sigma}^2_{0_L}$, as a function of the a priori standard errors. This is shown in figure 2. The method of computing $\hat{\sigma}^2_{0_T}$ for these cases is discussed in the following section. Note that the crossover point (the point at which the curve crosses $\hat{\sigma}^2_{0_T}=1.00$) occurs at $(\Sigma_{\phi,\lambda})^{1/2}\approx 0.25$ m, which is a reasonable value for the a priori standard errors in the latitude and longitude of the border points. This value can be considered as the point at which the "new" work was not unduly distorted before it was combined with the old work. This is also evidenced by examination of the residuals and coordinate shifts. For the average distance between the border points (≈ 17 km) the error (0.25 m) amounts to $\approx 1/50{,}000$ ($\sqrt{2}\times 0.25/17{,}000$), which is compatible

TABLE 2.—Assigned a priori standard errors and their associated integer and noninteger degrees of freedom

$\sigma_\phi = \sigma_\lambda$ (meters)	Integer degrees of freedom	Noninteger degrees of freedom
0.0001	359	359.00
.0010	359	358.86
.0100	359	350.05
.0300	359	335.32
.0500	359	329.75
.0600	359	328.19
.0700	359	327.04
.0800	359	326.18
.0900	359	325.51
.1000	359	324.98
.1100	359	324.56
.1200	359	324.22
.1300	359	323.93
.1400	359	323.69
.1500	359	323.49
.1600	359	323.32
.1800	359	323.05
.2000	359	322.83
.2500	359	322.48
.2700	359	322.37
.2800	359	322.33
.2900	359	322.29
.3000	359	322.25
.3100	359	322.21
.3200	359	322.18
.3300	359	322.14
.3500	359	322.08
.4000	359	321.96
.4500	359	321.86
.5000	359	321.77
.5500	359	321.70
.7500	359	321.48
1.0000	359	321.32
10.0000	359	321.00
100.0000	359	321.00

with the quality of the surrounding network (caused primarily by past distortions).

4.3 Discussion of the Combined Adjustment

Weighting the "old" border points is an attractive alternative to fixing the border points and distorting the new observations. The main disadvantage of this approach is that the coordinates of the border points would change. Imagine coordinates established in 1983 through a new adjustment (as proposed by the countries of North America) being changed again in 1984 as a result of incorporating new observations. Such a solution would clearly be unappealing. However, if new observations were badly distorted, this too would be objectionable.

An attractive alternative is given by Blaha (1974). For the conterminous United States, it is expected that we will be able to fit future observations into the 1984 framework without distorting them and, at the same time, hold the 1983 coordinates rigorously fixed. For areas other than the United States, where the triangulation is less dense, the method of weighting the parameters, as described in our combined adjustment, provides several advantages.

5. Theil's Estimators

In all geodetic adjustments that use the standard least-squares model shown below, the a priori or given estimates for the variance-covariance matrix, Σ_L, must be accepted or an improved estimator determined for $\sigma^2_{0_L}$ in the adjustment process itself. $\sigma^2_{0_L}$ is the scalar portion of Σ_L.

$$\underset{(n\times 1)}{L} = \underset{(n\times u)}{A}\ \underset{(u\times 1)}{X} + \underset{(n\times 1)}{\epsilon}$$

where

$$E(L) = AX$$

$$E(\epsilon\epsilon') = \sigma^2_{0_L} P^{-1} = \Sigma_L .$$

E, a linear operator, is the expectation operator,
L is an observation vector,
A is a matrix of constants,
X is a true vector of parameters to be estimated, and
ϵ is a stochastic vector of errors.

The notation $\underset{(n\times 1)}{L}$ defines a vector of n rows and one column; hence, $\underset{(n\times u)}{A}$ describes a matrix of n rows and u columns.

There are many possibilities for estimating $\sigma^2_{0_L}$. For example, Rao (1965) has shown that the estimator

$$\hat{\sigma}^2_{0_L} = \frac{V'PV}{n-u} \text{ is unbiased, i.e., } E(\hat{\sigma}^2_{0_L}) = \sigma^2_{0_L}.$$

This is the best estimator in the Gauss-Markov sense, although other estimators may be more efficient or possess other desirable attributes.

Theil (1963) developed an estimator which is intuitively appealing. A summary of the computation procedures for Theil's derivation follows: The model L = AX can be partitioned further, thereby obtaining two sets of matrix equations:

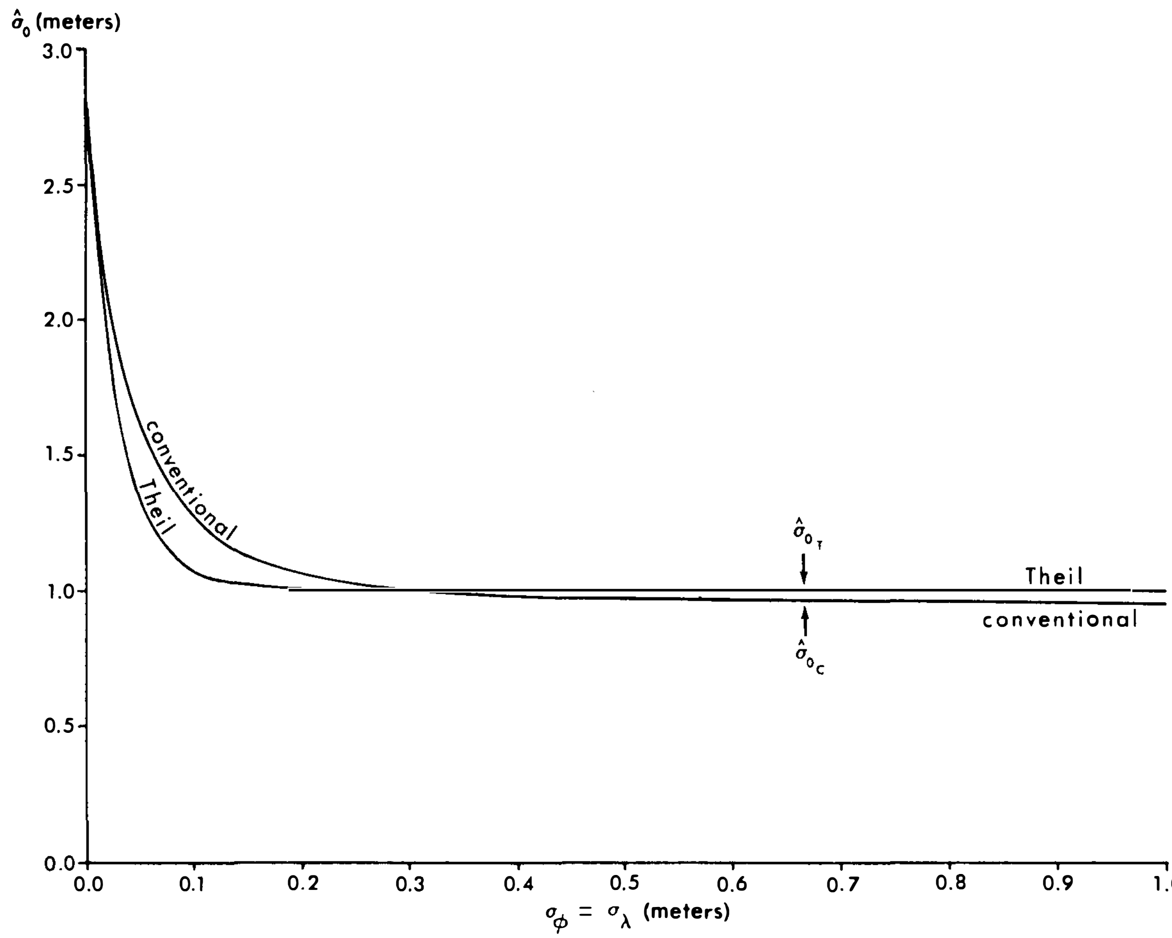

FIGURE 2.—Estimators $\hat{\sigma}_{0_T}$ and $\hat{\sigma}_{0_c}$ as functions of the a priori standard errors σ_ϕ and σ_λ.

$$\underset{(n\times 1)}{L} = \underset{(n\times u)}{A}\ \underset{(u\times 1)}{X} + \underset{(n\times 1)}{\epsilon} \qquad (1)$$

where

$$\mathrm{E}(\epsilon\epsilon') = \sigma^2_{0_L}\, P^{-1} = \Sigma_L.$$

$$E(\epsilon) = 0$$
$$E(\epsilon\eta') = 0$$

and

$$\underset{(u_1\times 1)}{L_X} = \underset{(u_1\times 1)}{X} + \underset{(u_1\times 1)}{\eta} \qquad (2)$$

where

$$\mathrm{E}(\eta\eta') = \sigma^2_{0_X}\, P_X^{-1} = \Sigma_X.$$

$$E(\eta) = 0.$$

Equations (2) represent the direct observations of the *unknown* parameters **X** that will be incorporated into a combined adjustment of equations (1) and (2). Σ_X is the true variance-covariance matrix of the latitudes and longitudes of the 20 border stations.

To compute Theil's estimator, the standard least-squares solution is first determined using only the data from equation (1), which yields

$$\hat{X} = (A'PA)^{-1}\, A'PL \quad \text{and} \quad \hat{\sigma}^2_{0_L} = \frac{V'PV}{n-u}\ ;$$

where

$$V = \mathbf{A}\hat{\mathbf{X}} - \mathrm{L}.$$

Then,

$$\tilde{X} = \left[\frac{1}{\hat{\sigma}^2_{0_L}} A'PA + \Sigma_X^{-1}\right]^{-1} \left[\frac{1}{\hat{\sigma}^2_{0_L}} A'PL + \Sigma_x^{-1}\mathrm{L}_x\right].$$

Finally,

$$\hat{\sigma}^2_{0_T} = \frac{(A\tilde{X}-L)'P\,(A\tilde{X}-L)}{n-\tilde{u}}$$

where

$$\tilde{u} = \mathrm{tr}\left[\hat{\sigma}^{-2}_{0_L}\, A'PA\,\left(\hat{\sigma}^{-2}_{0_L}\, A'PA + \Sigma_X^{-1}\right)^{-1}\right].$$

It is possible that $\hat{\sigma}^2_{0_T}$ may be a more appropriate estimate of $\sigma^2_{0_L}$. It must be emphasized that Σ_X was considered to be *known*. The final estimated covariance matrix for the parameters, **X**, is given by

$$\Sigma_{\tilde{X}_T} = \left[\frac{1}{\hat{\sigma}^2_{0_T}} A'PA + \Sigma_X^{-1}\right]^{-1}.$$

Note that ũ, the number of unknowns, is *not* an integer value due to the influence of the a priori information (L_X). If

$$\Sigma_X^{-1} = 0 \text{ (no a priori information)},$$
$$\tilde{u} = \mathrm{tr}\left[\hat{\sigma}^{-2}_{0_L}\, A'PA\, [\hat{\sigma}^{-2}_{0_L}\, A'PA]^{-1}\right] = u.$$

Theil (1963: p. 413) shows that this (biased) estimator ($\hat{\sigma}^2_{0_T}$) contains a bias that has a higher order of smallness than 1/n and, therefore, is quite adequate for large samples. For a complete derivation of $\hat{\sigma}^2_{0_T}$ and other interesting attributes of this estimator, the reader is referred to Theil (1963). Table 2 lists the values for $n-u+u_1$ and $n-\tilde{u}$, the integer and non-integer degrees of freedom, respectively.

Another commonly computed estimator was obtained to allow a comparison of the two final variance-covariance matrices—Theil's and the one most commonly computed. This estimator is denoted by $\hat{\sigma}^2_{0_C}$. Actually, the estimators $\hat{\sigma}^2_{0_T}$ and $\hat{\sigma}^2_{0_C}$ are not directly comparable because $\hat{\sigma}^2_{0_T}$ is an alternate method for $\hat{\sigma}^2_{0_L}$, whereas $\hat{\sigma}^2_{0_C}$ is normally used to scale the final entire variance-covariance matrix. This will be shown later. Nevertheless, because of their similar properties, both standard errors of unit weight ($\hat{\sigma}_{0_C}$ and $\hat{\sigma}_{0_T}$) corresponding to these estimates are plotted in figure 2.

To summarize, the following values were computed:

$\hat{\sigma}^2_{0_L}$ = a constant (0.968) representing the most frequently computed estimate of the variance for the triangulation observations without constraints.

$\hat{\sigma}^2_{0_T}$ = Theil's estimator for the variance of the distance and direction (triangulation) observations. This estimator varied as a function of the a priori variances assigned to the 20 border stations.

$\hat{\sigma}^2_{0_C}$ = the normally computed "geodetic" estimator used in this type of problem, including triangulation observations and constraints (see below for definition).

$$\hat{\sigma}^2_{0_C} = \frac{(\bar{A}\hat{\bar{X}}-\bar{L})'\,\bar{P}\,(\bar{A}\hat{\bar{X}}-\bar{L})}{n-u+u_1}$$

where

$$\bar{L} = \begin{bmatrix} L \\ L_X \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} A \\ I \end{bmatrix}$$

$$\bar{P} = \begin{bmatrix} \underset{(n\times n)}{P} & 0 \\ 0 & \underset{(u_1\times u_1)}{P_x} \end{bmatrix}$$

and

$$\hat{\bar{X}} = (\bar{A}'\bar{P}\bar{A})^{-1}\,\bar{A}'\bar{P}\bar{L}.$$

Finally,

$$\Sigma_{\hat{X}_C} = \hat{\sigma}^2_{0_C}\,(\bar{A}'\bar{P}\bar{A})^{-1} = \hat{\sigma}^2_{0_C}\,(A'PA + P_X)^{-1}.$$

These estimators are further described in table 3.

TABLE 3.—Computational analysis of estimators

Model	Equations used	Conditions	Estimator
1	(1)	$\Sigma_x = 0$, P known, $\sigma^2_{0_L}$ to be estimated	$\hat{\sigma}^2_{0_L}$
2	(1) and (2)	Σ_x and P known, $\sigma^2_{0_L}$ to be estimated	$\hat{\sigma}^2_{0_T}$
3	(1) and (2)	P and P_x known, $\sigma^2_{0_C}$ to be estimated	$\hat{\sigma}^2_{0_C}$

6. Description of the Computer Program

The geodetic horizontal adjustment program, TRAV10, was chosen to implement the techniques discussed in this report and to generate conventionally computed comparison data. This program is ideally suited for several reasons: It takes advantage of the natural sparseness of geodetic network normal equations by automatically reordering the unknowns to minimize storage and computing requirements. Bulk data are stored on disks when not in use, further minimizing internal storage requirements. Inverses of partial normal equation matrices can be generated efficiently. In addition, the modular construction of the program makes it easier to understand, modify, and supplement.

For our purposes TRAV10 was modified to batch run a series of problems using one set of observations and a variable set of station coordinate constraints. For any given set of constraints TRAV10 first computed a free adjustment, using all given directions, distances, and azimuths. The program "fixed" the last pair of station position coordinates by detecting numerical singularities and filling the last two rows of the reduced normals with zeros. Following the free adjustment, two complete adjustments were made for each value of station coordinate constraints in the set. The first was a conventional adjustment, using the reciprocals of the input coordinate variances as weights. The second adjustment employed Theil's procedure, as described in section 5. Because our geodetic application was nonlinear, Theil's equations required the usual modifications used for non-linear problems. Each adjustment was iterated once for a total of two solution passes.

7. Conclusions

These conclusions pertain only to the computation and analysis of the estimators given in table 3. In most scientific least-squares applications $\hat{\sigma}^2_{0_C}$ is determined and used to compute the final estimates for the unknown variances (i.e., to scale the final covariance matrix). Practitioners assume implicitly that the structure (except for the unknown scalar) of the observational weight matrix P, and the weight matrix for the parameters P_X, are known a priori. Generally this assumption is quite reasonable.

It is usually assumed that only one scalar quantity, viz., $\hat{\sigma}^2_{0_C}$, is required to scale the final estimated covariance matrix of the parameters in the following manner:

$$\Sigma_{\hat{X}_C} = \hat{\sigma}^2_{0_C}\,(A'PA + P_X)^{-1}.$$

This *may* be a reasonable assumption depending on the relationship between the relative dispersion of the observations, L, and the observations, L_X. If Σ_X is known to be significantly better than Σ_L, or vice versa, then it would not be appropriate to scale the entire matrix $(A'PA + P_X)^{-1}$ by the common scalar $\hat{\sigma}^2_{0_C}$. If it is assumed, however, that $P_X = \Sigma_X^{-1}$, i.e., the variances or weights for the parameters are *known* a priori and do not need to be scaled after the adjustment, the appropriate application of Theil's estimator, $\hat{\sigma}^2_{0_T}$, is probably the most reasonable choice.

Figure 3 shows the plotted root mean square (rms) for each of the final variance-covariance matrices for both estimators $\Sigma_{\hat{X}_T}$ and $\Sigma_{\hat{X}_C}$. The rms is defined as:

$$\text{rms}\Sigma_{\hat{X}} = \left[\frac{\text{trace }\Sigma_{\hat{X}}}{u}\right]^{1/2}$$

where u is the number of unknown parameters (weighted or unweighted). An examination of figure 3 reveals several comforting facts. First, the separation between the lines is small when compared to the total range of $\text{rms}\Sigma_{\hat{X}}$. This indicates that our conventional estimates are consistent with Theil's. Second, near the more reasonable values of $\Sigma_X^{1/2}$ (≈ 0.30 m) the values are almost identical. For small values of $\Sigma_x^{1/2}$ (large weights), the conventional estimates are conservative. This is most fortunate because geodesists often use unreasonably large weights to hold parameters fixed and would prefer a more conservative estimate.

These conclusions are reassuring for users who are presently computing $\hat{\sigma}^2_{0_C}$ because, in most cases, estimates of accuracies are too optimistic. Furthermore, the computation of Theil's estimator is complicated and expensive. We concluded it is more conservative and economical (for computer costs) to compute $\hat{\sigma}^2_{0_C}$, using the derived value to obtain a final estimated variance-covariance matrix for the parameters. Because this has been the usual procedure, our results confirmed past practice.

It is reasonable to ask whether this concept can be extended to adjustments containing more than two components of variance, one (or more) of which may be assumed known. It appears that a method for a general model II (random effects) analysis of variance can be based on equations similar to those used in this paper. This provides a relatively cheap alternative to MINQUE, described in Roa and Mitra (1971), or other existing methods for ANOVA II.

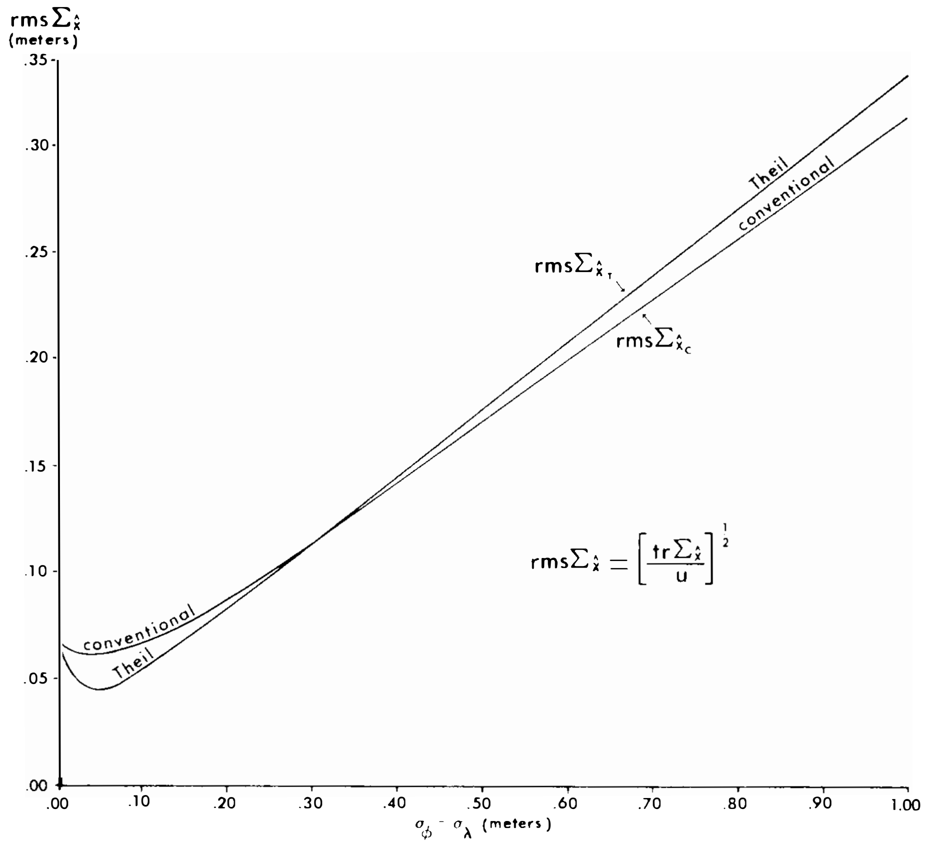

FIGURE 3.—A posteriori root-mean-square errors of station coordinates as functions of the a priori standard errors σ_ϕ and σ_λ.

Acknowledgment

The authors wish to thank Allen Pope for various discussions concerning this work and its connection to the analysis of variance problem.

References

Blaha, G., 1974: Influence de l'incertitude des paramètres tenus fixes dans une compensation sur la propagation des variances-covariances. *Bulletin Géodésique,* No. 113, 307-315.

Bossler, J. D. (Department of Geodetic Science, The Ohio State University, Columbus, Ohio) 1972: Bayesian inference in geodesy. Ph.D. Dissertation, 79 pp. National Oceanic and Atmospheric Administration, National Ocean Survey/National Geodetic Survey, Rockville, Md., revised and reprinted, 1976.

Rao, C. R., 1965: *Linear Statistical Inference and Its Applications.* John Wiley and Sons, New York, N.Y., 522 pp.

Rao, C. R. and Mitra, S. K., 1971: *Generalized Inverse of Matrices and Its Applications.* John Wiley and Sons, Inc., New York, N.Y. 240 pp.

Schmid, H. and Schmid, E., 1965: A generalized least-squares solution for hybrid measuring systems. *Canadian Surveyor,* IX (1), 27-41.

Schwarz, C., 1978: The TRAV10 horizontal network adjustment program. *NOAA Technical Memorandum* NOS NGS-12, 52 pp. (Available from National Technical Information Service, Springfield, Va., accession no. PB283087.)

Theil, H., 1963: On the use of incomplete prior information in regression analysis. *Journal of the American Statistical Association,* 58, 401-414.

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NOS NGS-3 Adjustment of geodetic field data using a sequential method. Marvin C. Whiting and Allen J. Pope, March 1976, 11 pp (PB253967). A sequential adjustment is adopted for use by NGS field parties.

NOS NGS-4 Reducing the profile of sparse symmetric matrices. Richard A. Snay, June 1976, 24 pp (PB-258476). An algorithm for improving the profile of a sparse symmetric matrix is introduced and tested against the widely used reverse Cuthill-McKee algorithm.

NOS NGS-5 National Geodetic Survey data: availability, explanation, and application. Joseph F. Dracup, Revised January 1979, 45 pp (PB80 118615). The summary gives data and services available from NGS, accuracy of surveys, and uses of specific data.

NOS NGS-6 Determination of North American Datum 1983 coordinates of map corners. T. Vincenty, October 1976, 8 pp (PB262442). Predictions of changes in coordinates of map corners are detailed.

NOS NGS-7 Recent elevation change in Southern California. S.R. Holdahl, February 1977, 19 pp (PB265-940). Velocities of elevation change were determined from Southern Calif. leveling data for 1906-62 and 1959-76 epochs.

NOS NGS-8 Establishment of calibration base lines. Joseph F. Dracup, Charles J. Fronczek, and Raymond W. Tomlinson, August 1977, 22 pp (PB277130). Specifications are given for establishing calibration base lines.

NOS NGS-9 National Geodetic Survey publications on surveying and geodesy 1976. September 1977, 17 pp (PB275181). Compilation lists publications authored by NGS staff in 1976, source availability for out-of-print Coast and Geodetic Survey publications, and subscription information on the Geodetic Control Data Automatic Mailing List.

NOS NGS-10 Use of calibration base lines. Charles J. Fronczek, December 1977, 38 pp (PB279574). Detailed explanation allows the user to evaluate electromagnetic distance measuring instruments.

NOS NGS-11 Applicability of array algebra. Richard A. Snay, February 1978, 22 pp (PB281196). Conditions required for the transformation from matrix equations into computationally more efficient array equations are considered.

NOS NGS-12 The TRAV-10 horizontal network adjustment program. Charles R. Schwarz, April 1978, 52 pp (PB283087). The design, objectives, and specifications of the horizontal control adjustment program are presented.

NOS NGS-13 Application of three-dimensional geodesy to adjustments of horizontal networks. T. Vincenty and B. R. Bowring, June 1978, 7 pp (PB286672). A method is given for adjusting measurements in three-dimensional space without reducing them to any computational surface.

NOS NGS-14 Solvability analysis of geodetic networks using logical geometry. Richard A. Snay, October 1978, 29 pp (PB291286). No algorithm based solely on logical geometry has been found that can unerringly distinguish between solvable and unsolvable horizontal networks. For leveling networks such an algorithm is well known.

NOS NGS-15 Goldstone validation survey - phase 1. William E. Carter and James E. Pettey, November 1978, 44 pp (PB292310). Results are given for a space system validation study conducted at the Goldstone, Calif., Deep Space Communication Complex.

NOS NGS-16 Determination of North American Datum 1983 coordinates of map corners (second prediction). T. Vincenty, April 1979, 6 pp (PB297245). New predictions of changes in coordinates of of map corners are given.

NOS NGS-17 The HAVAGO three-dimensional adjustment program. T. Vincenty, May 1979, 18 pp (PB297069). The HAVAGO computer program adjusts numerous kinds of geodetic observations for high precision special surveys and ordinary surveys.

NOS NGS-18 Determination of astronomic positions for California-Nevada boundary monuments near Lake Tahoe. James E. Pettey, March, 1979, 22 pp (PB301264). Astronomic observations of the 120th meridian were made at the request of the Calif. State Lands Commission.

NOS NGS-19 HOACOS: A program for adjusting horizontal networks in three dimensions. T. Vincenty, July 1979, 18 pp (PB301351). Horizontal networks are adjusted simply and efficiently in the height-controlled spatial system without reducing observations to the ellipsoid.

NOS NGS-20 Geodetic leveling and the sea level slope along the California coast. Emery I. Balazs and Bruce C. Douglas, September 1979, 23 pp (PB80 120611). Heights of four local mean sea levels for the 1941-59 epoch in California are determined and compared from five geodetic level lines observed (leveled) between 1968-78.

NOS NGS-21 Haystack-Westford Survey. W. E. Carter, C. J. Fronczek, and J. E. Pettey, September 1979, 57 pp. A special purpose survey was conducted for VLBI test comparison.

NOS NGS-22 Gravimetric tidal loading computed from integrated Green's functions. C. C. Goad, October 1979, 15 pp. Tidal loading is computed using integrated Green's functions.

NOS NGS-23 Use of auxiliary ellipsoids in height-controlled spatial adjustments. B. R. Bowring and T. Vincenty, November 1979, 6 pp. Auxiliary ellipsoids are used in adjustments of networks in the height-controlled three-dimensional system for controlling heights and simplifying transformation of coordinates.

NOS NGS-24 Determination of the geopotential from satellite-to-satellite tracking data. B. C. Douglas, C. C. Goad, and F. F. Morrison, January 1980, 32 pp. The capability of determining the geopotential from satellite-to-satellite tracking is analyzed.

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NOS 65 NGS 1 The statistics of residuals and the detection of outliers. Allen J. Pope, May 1976, 133 pp (PB258428). A criterion for rejection of bad geodetic data is derived on the basis of residuals from a simultaneous least-squares adjustment. Subroutine TAURE is included.

NOS 66 NGS 2 Effect of Geoceiver observations upon the classical triangulation network. R. E. Moose and S. W. Henriksen, June 1976, 65 pp (PB260921). The use of Geoceiver observations is investigated as a means of improving triangulation network adjustment results.

NOS 67 NGS 3 Algorithms for computing the geopotential using a simple-layer density model. Foster Morrison, March 1977, 41 pp (PB266967). Several algorithms are developed for computing with high accuracy the gravitational attraction of a simple-density layer at arbitrary altitudes. Computer program is included.

NOS 68 NGS 4 Test results of first-order class III leveling. Charles T. Whalen and Emery Balazs, November 1976, 30 pp (GPO# 003-017-00393-1) (PB265421). Specifications for releveling the National vertical control net were tested and the results published.

NOS 70 NGS 5 Selenocentric geodetic reference system. Frederick J. Doyle, Atef A. Elassal, and James R. Lucas, February 1977, 53 pp (PB266046). Reference system was established by simultaneous adjustment of 1,233 metric-camera photographs of the lunar surface from which 2,662 terrain points were positioned.

NOS 71 NGS 6 Application of digital filtering to satellite geodesy. C. C. Goad, May 1977, 73 pp (PB-270192). Variations in the orbit of GEOS-3 were analyzed for M_2 tidal harmonic coefficient values that perturb the orbits of artificial satellites and the Moon.

NOS 72 NGS 7 Systems for the determination of polar motion. Soren W. Henriksen, May 1977, 55 pp (PB274698). Methods for determining polar motion are described and their advantages and disadvantages compared.

NOS 73 NGS 8 Control leveling. Charles T. Whalen, May 1978, 23 pp (GPO# 003-017-00422-8) (PB286838). The history of the National network of geodetic control, from its origin in 1878, is presented, and the latest observation and computation procedures are explained.

NOS 74 NGS 9 Survey of the McDonald Observatory radial line scheme by relative lateration techniques. William E. Carter and T. Vincenty, June 1978, 33 pp (PB287427). Results of experimental application of the "ratio method" of electromagnetic distance measurements are given for high resolution crustal deformation studies in the vicinity of the McDonald Lunar Laser Ranging and Harvard Radio Astronomy Stations.

NOS 75 NGS 10 An algorithm to compute the eigenvectors of a symmetric matrix. E. Schmid, August 1978, 5 pp (PB287923). Method describes computations for eigenvalues and eigenvectors of a symmetric matrix.

NOS 76 NGS 11 The application of multiquadric equations and point mass anomaly models to crustal movement studies. Rolland L. Hardy, November 1978, 63 pp (PB293544). Multiquadric equations, both harmonic and nonharmonic, are suitable as geometric prediction functions for surface deformation and have potentiality for usage in analysis of subsurface mass redistribution associated with crustal movements.

NOS 79 NGS 12 Optimization of horizontal control networks by nonlinear programing. Dennis G. Milbert, August 1979, 44 pp (PB80 117948). Several horizontal geodetic control networks are optimized at minimum cost while maintaining desired accuracy standards.

NOS 82 NGS 13 Feasibility study of the conjugate gradient method for solving large sparse equation sets. Lothar Grundig, February 1980, 19 pp. This method is well suited for constrained adjustments but not for free adjustments.

NOS 83 NGS 14 Tidal corrections to geodetic quantities. Peter Vaníček, February 1980, 30 pp. Corrections for tidal force are formulated and tidal aspects relating to geodesy discussed.

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NOS NGS 1 Geodetic bench marks. Lt. Richard P. Floyd, September 1978, 56 pp (GPO#003-017-00442-2) (PB296427). Reference guide provides specifications for highly stable bench marks, including installation procedures, vertical instability, and site selection considerations.

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