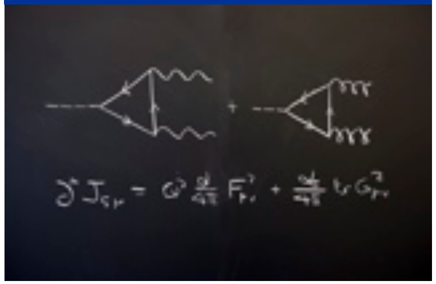

$\pi^0 \rightarrow \gamma\gamma$ on the lattice

Based on Xu Feng et al. [JLQCD collaboration], arXiv:1206.1375, to appear in PRL.

Shoji Hashimoto (KEK)

@ Fermilab Theory Seminar, Oct 4, 2012.

計算基礎科学連携拠点
Joint Institute for Computational Fundamental Science
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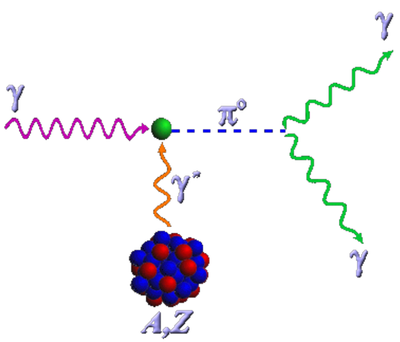
The famous, $\pi^0 \rightarrow \gamma\gamma$

- Occurs through chiral anomaly (the ABJ anomaly)
  - Essentially quantum effect.
  - One-loop is exact (Adler-Bardeen (1969)).
  - One of the evidences for N_c=3.

$$\Gamma(\pi^0 \to \gamma\gamma) = \left(\frac{m_\pi}{4\pi}\right)^3 \left(\frac{\alpha}{F_\pi}\right)^2 = 7.76 \text{ eV}$$

- Renewed interest.
  - Precise measurement: PrimEX at JLab.
    - 2.8% (2011) → 1.4% (goal)
  - May play an unique role in Muon g-2
    - Dominant contribution to light-by-light



Lattice calculation?

Non-trivial, because

1. Chiral symmetry is the key for axial anomaly.
   - Chiral symmetry is non-trivial for lattice fermions
     - Or, even prohibited by Nielsen-Ninomiya theorem (?)
   - Want to maintain, but have to reproduce anomaly.

2. Final state is not a QCD eigenstate.
   - If we do naively, we get $\pi^0 \rightarrow \rho\rho$.

Plan

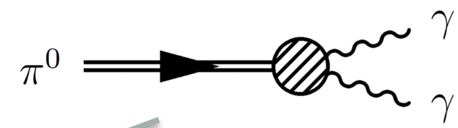
1. Preliminaries
   - $\pi^0 \to \gamma\gamma$ and axial anomaly
   - Overlap fermion and anomaly
2. Method for photon external states
   - How to treat non-QCD final states
3. Calculation setup
   - Dynamical overlap fermion (not in detail)
   - All-to-all propagator
4. Results
   - Analysis steps
5. Discussions

1. Preliminaries

- $\pi^0 \to \gamma\gamma$ and axial anomaly
- overlap fermion and anomaly
- Status beside lattice

Standard argument

- $\pi^0 \rightarrow \gamma\gamma$ comes from axial anomaly.



$$\Gamma_{\mu\nu}(k_1,k_2,q) = e^2 \int d^4z d^4y \, e^{ik_1z+ik_2y} \langle 0 | T\left(J_\mu^{em}(z) J_\nu^{em}(y)\right) | \pi^0(q) \rangle$$

$$\Gamma_{\mu\nu\lambda}(k_1,k_2,q) = e^2 \int d^4x d^4y \, e^{ik_2y-iqx} \langle 0 | T\left(J_\mu^{em}(0) J_\nu^{em}(y) A_\lambda^3(x)\right) | 0 \rangle$$

$$q^\lambda \Gamma_{\mu\nu\lambda}(k_1,k_2,q) = -ie^2 \int d^4x d^4y \, e^{ik_2y-iqx} \langle 0 | T\left(J_\mu^{em}(0) J_\nu^{em}(y) \partial^\lambda A_\lambda^3(x)\right) | 0 \rangle$$

$= f_\pi m_\pi^2 \phi^3(x)$

vanish in the q→0 limit.

$$= \frac{f_\pi m_\pi^2}{m_\pi^2 - q^2} \Gamma_{\mu\nu}(k_1,k_2,q) \quad - \frac{ie^2}{12\pi^2} \varepsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma$$

  - Sutherland, Veltman (1967): vanish in the soft pion limit.
  - Not the case, because

$$\partial^\lambda A_\lambda(x) = 2mP(x) - \frac{e^2}{2\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x)$$

Adler-Bell-Jackiw anomaly

Lattice gauge theory

- The way that axial anomaly realizes depends on the fermion formulation:
  - Wilson fermions:
    - Chiral symmetry is explicitly violated.
    - No conserved current. $\partial_\mu A_\mu$-$2mP$ ($\neq 0$) coincides with the chiral anomaly in the continuum limit (Karsten-Smit (1981)).
  - Ginsparg-Wilson fermions:
    - Modified chiral symmetry exists:

$$\delta\bar{\psi} = i\alpha\bar{\psi}\left(1-\frac{a}{2\rho}D\right)\gamma_5,\ \delta\psi = i\alpha\gamma_5\left(1-\frac{a}{2\rho}D\right)\psi$$

    - Axial-current conserves, classically. In the quantum theory, the fermion measure produces Jacobian, which leads to the correct anomaly (as in Fujikawa's analysis).
    - Shown in perturbation theory (Kikukawa-Yamada (1998)).

Overlap fermion formulation

Neuberger (1998)

- Dirac operator has the form

$$D = \frac{\rho}{a}\left[1 + \frac{X}{\sqrt{X^\dagger X}}\right], \quad X \equiv D_W - \frac{\rho}{a}$$

  - Constructed using Wilson-Dirac operator D_W as a building block.
  - Singular when $(X^\dagger X)$ has a zero eigenvalue. Well-defined? --- next page.

- Satisfies the Ginsparg-Wilson relation

$$D\gamma_5 + \gamma_5 D = \frac{a}{\rho} D\gamma_5 D$$

  - so has the modified chiral symmetry.

Luscher (1998)

Overlap fermion formulation

- Dirac operator is singular. Well-defined?

$$D = \frac{\rho}{a}\left[1 + \frac{X}{\sqrt{X^\dagger X}}\right],\ X \equiv D_W - \frac{\rho}{a}$$

  - Operator is local when the background gauge field is sufficiently smooth (Hernandez-Jansen-Luscher (1999)).
  - Operator tr(γ_5D) plays the role of topological charge density. It satisfies a (kind of) index theorem.

$$\sum_x \frac{1}{2}\mathrm{Tr}[\gamma_5 D] = n_R^{(0)} - n_L^{(0)}$$

  - On sufficiently smooth gauge config, one can show that

$$\frac{1}{2}\mathrm{Tr}[\gamma_5 D] = \frac{1}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

= Chiral anomaly.
Indeed, LHS is what you find in the Jacobian.

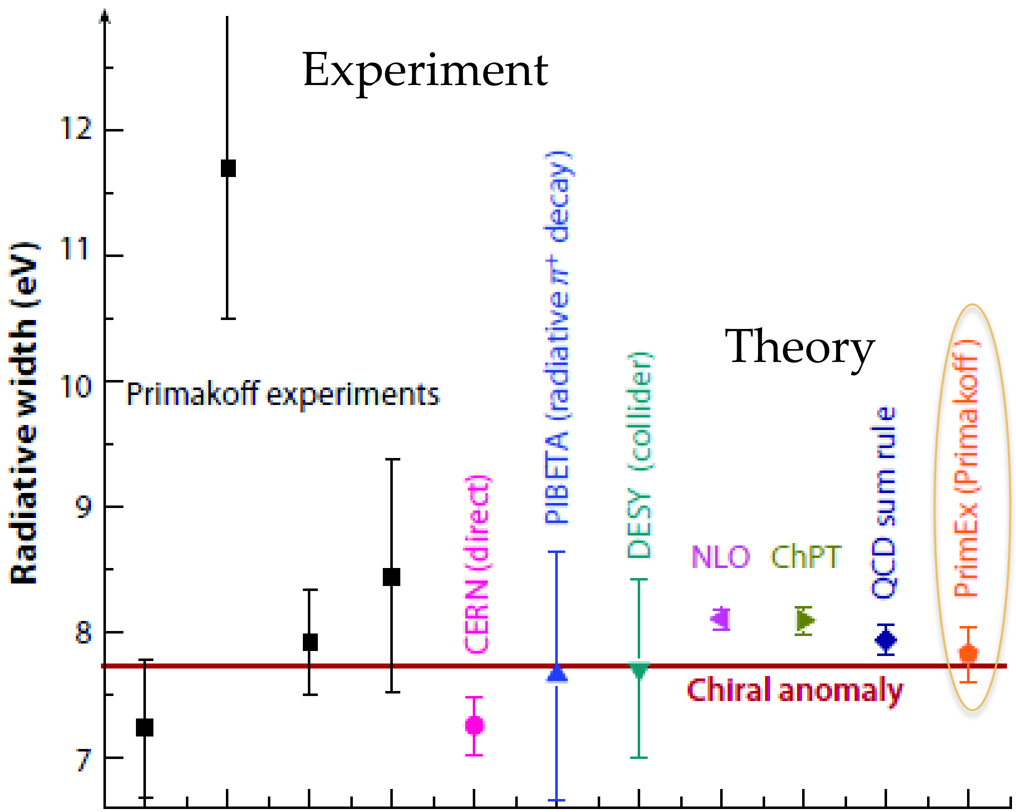
Motivations

1. Test if the overlap fermion formulation really produces the (theoretically required) chiral anomaly in the physical process. Non-trivial when background gauge field is not very smooth, say $a \sim 0.1$ fm.

2. Calculate the $\pi^0 \to \gamma\gamma$ amplitude away from the limit of massless pion and soft photons. Only some phenomenological analyses (χPT, etc) are available so far. Maybe tiny, but relevant at the 1% precision.

Status besides lattice

- PrimEX is improving a lot.



From Miskimen,
Chiral Dynamics 2012

Entering the stage of precisely testing the chiral anomaly and corrections to that limit.

2. Method for photon external states

- How to treat non-QCD final states

Non-QCD final state

- First guess: $\pi^0 \to \gamma\gamma$ is extracted from a PVV 3-point func.
  - Correct. But, doesn't work if too naïve.
  - At large (Euclidean) time separations, PVV corresponds to $\pi^0 \to \rho\rho$, though unphysical.
- More careful derivation:

Ji, Jung, PRL 86 (2001) 208; Dudek, Edwards, PRL 97 (2006) 172001.

  - LSZ reduction formula

$$\langle \gamma(p_1,\lambda_1)\gamma(p_2,\lambda_2)|\pi(q)\rangle = -\lim_{\substack{p_1'\to p_1\\ p_2'\to p_2}} \varepsilon_\mu^*(p_1,\lambda_1)\varepsilon_\nu^*(p_2,\lambda_2)p_1'^2 p_2'^2$$

$$\times \int d^4x\, d^4y\, e^{ip_1'y+ip_2'x}\langle 0|T\left\{A^\mu(y)A^\nu(x)\right\}|\pi(q)\rangle$$

photon fields

Non-QCD final state

- Perturbative expansion for QED

$$\langle \gamma(p_1,\lambda_1)\gamma(p_2,\lambda_2)|\pi(q)\rangle = -e^2 \lim_{\substack{p_1'\to p_1 \\ p_2'\to p_2}} \varepsilon_\mu^*(p_1,\lambda_1)\varepsilon_\nu^*(p_2,\lambda_2) p_1'^2 p_2'^2$$

$$\times \int d^4x\, d^4y\, d^4w\, d^4z\; e^{ip_1' y + ip_2' x} D^{\mu\rho}(y,z) D^{\nu\sigma}(x,w) \langle 0|T\{j_\rho(z) j_\sigma(w)\}|\pi(q)\rangle$$

photon propagator

$$D^{\mu\nu}(0,z) = -ig^{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ikz}}{k^2+i\varepsilon}$$

vector current $j_\mu(x) = \bar{q}(x)\gamma_\mu q(x)$

canceling $p_1'^2 p_2'^2$

  - essentially gives

$$\int d^4x\, e^{ip_1 x}\langle 0|T\{j^\mu(x) j^\nu(0)\}|\pi(q)\rangle$$

As you can see, the integral comes from the photon propagator.

Non-QCD final state

- Analytic continuation

$$\int d^4x\, e^{ip_1 x} \langle 0 | T\left\{ j^\mu(x) j^\nu(0) \right\} | \pi(q) \rangle$$

$$\Rightarrow \int dt\, e^{\omega t} \int d^3\vec{x}\, e^{-i\vec{p}_1\cdot\vec{x}} \langle 0 | T_E\left\{ j_\mu(x) j_\nu(0) \right\} | \pi(q) \rangle$$

  - Possible only when p_1^2 does not develop singularities (cut, pole) = photon does not mix with other QCD states.
  - Necessary to ensure that the photon field correlator is saturated by the photon state.
  - We are restricted in the region p_1^2, $p_2^2 < m_\rho^2$ (or $< 4E_\pi^2$).

Non-QCD final state

- On the lattice we calculate

$$M_{\mu\nu}(p_1,p_2) = \lim_{t_{1,2}-t_\pi\to\infty} \frac{1}{\frac{\phi_{\pi,\vec{q}}}{2E_{\pi,\vec{q}}} e^{-E_{\pi,\vec{q}}(t_2-t_\pi)}} \int dt_1\, e^{\omega(t_1-t_2)} C_{\mu\nu}(t_1,t_2,t_\pi),$$

$$C_{\mu\nu}(t_1,t_2,t_\pi) \equiv \int d^3\vec{x}\, e^{-i\vec{p}_1\cdot\vec{x}} \int d^3\vec{z}\, e^{i\vec{q}\cdot\vec{z}} \left\langle 0 \middle| T_E\left\{ j_\mu(\vec{x},t_1) j_\nu(\vec{y},t_2) P(\vec{z},t_\pi)\right\} \middle| 0 \right\rangle$$

  - ω is an arbitrary parameter as far as $p_1 = (\omega, \vec{p}_1)$ satisfies $p_1{}^2 < m_\rho{}^2$.
  - p_2 is given by momentum conservation: $p_2 = (E_{\pi,\vec{q}} - \omega, \vec{q} - \vec{p}_1)$

  - The factor $e^{\omega(t_1-t_2)}$ seems divergent when $|t_1-t_2|$ is large. Still well-defined? Yes. The lowest-lying vector states survives and gives a stronger suppression factor $e^{-E_\rho(t_1-t_2)}$.

Form factor to be obtained

- Then, we obtain

$$M_{\mu\nu}(p_1,p_2) = i\int d^4x\, e^{ip_1x} \langle 0 | T\{ j_\mu(x) j_\nu(0)\} | \pi^0(q)\rangle$$
$$= \varepsilon_{\mu\nu\alpha\beta} p_1^\alpha p_2^\beta F_{\pi^0\gamma\gamma}(m_\pi^2; p_1^2, p_2^2)$$

- $F_{\pi\gamma\gamma}$ is the form factor of interest.
  - In the soft pion limit, we should reproduce the ABJ anomaly.

$$F_{\pi^0\gamma\gamma}(0;0,0) = \frac{1}{4\pi^2 F_0}$$

  - Away from the chiral limit, it gives the correction due to finite quark mass.
  - For off-shell photons $p_{1,2}{}^2 \neq 0$, it gives a form factor relevant for other process like $\gamma^*\pi^0 \rightarrow \gamma$.

3. Calculation setup

- Dynamical overlap fermion (not in detail)
- All-to-all propagator

Dynamical overlap fermion

- Ensemble generation (JLQCD collaboration, 2006~)
  - 2+1 flavors represented by overlap fermions
  - Lattice spacing a = 0.11 fm (Iwasaki gauge action).
  - Lattice volume: 16^3x48, 24^3x48.
  - Quark mass: 4 points covering m_π = 290 – 540 MeV.
  - Gauge field topology fixed. (Necessary to run dynamical overlap reasonably fast.) A source of finite volume effect of O(1/V).
  - Was successful to calculate the Dirac spectrum and to extract chiral condensate Σ. Several other applications.

All-to-all propagator

- Quark propagator from any y to any x.

$$D^{-1}(x,y) = \sum_{k=1}^{N_{ev}} \frac{1}{\lambda^{(k)}} u^{(k)}(x) u^{(k)\dagger}(y) + \sum_{d=1}^{N_d} \left[D_{high}^{-1} \eta^{(d)} \right](x) \eta^{(d)}(y)$$

Low mode contribution

$$Du^{(k)}(x) = \lambda^{(k)} u^{(k)}(x)$$

High mode contribution
From a random noise

Random noise

  - Low-lying eigenvalues/eigenvectors are calculated and stored.
  - Also used to accelerate the fermion inversion (exact deflation).
  - High-mode contribution is calculated and stored. N_d times = each color/spinor, (every other) time-slice.

Foley et al., Comp. Phys. Comm. 172 (2005) 145.
and references therein.

All-to-all propagator

- Crucial to calculate

$$M_{\mu\nu}(p_1,p_2) = \lim_{t_{1,2}-t_\pi\to\infty} \frac{1}{\frac{\phi_{\pi,\vec{q}}}{2E_{\pi,\vec{q}}} e^{-E_{\pi,\vec{q}}(t_2-t_\pi)}} \int dt_1\, e^{\omega(t_1-t_2)} C_{\mu\nu}(t_1,t_2,t_\pi),$$

$$C_{\mu\nu}(t_1,t_2,t_\pi) \equiv \int d^3\vec{x}\, e^{-i\vec{p}_1\cdot\vec{x}} \int d^3\vec{z}\, e^{i\vec{q}\cdot\vec{z}} \left\langle 0 \middle| T_E\left\{ j_\mu(\vec{x},t_1) j_\nu(\vec{y},t_2) P(\vec{z},t_\pi) \right\} \middle| 0 \right\rangle$$

  - 3pt functions with all *x*, *y*, *z* points have to be summed. Different time-slices with different weight depending on ω.

- Momentum configuration taken:
  - #1: p_1 = (0,0,0), q = (0,0,1)
  - #2: p_1 = (0,0,1), q = (0,0,0)

in unit of 2π/L

4. Results

- Analysis steps

Analysis steps

- A bit different from conventional lattice analysis:

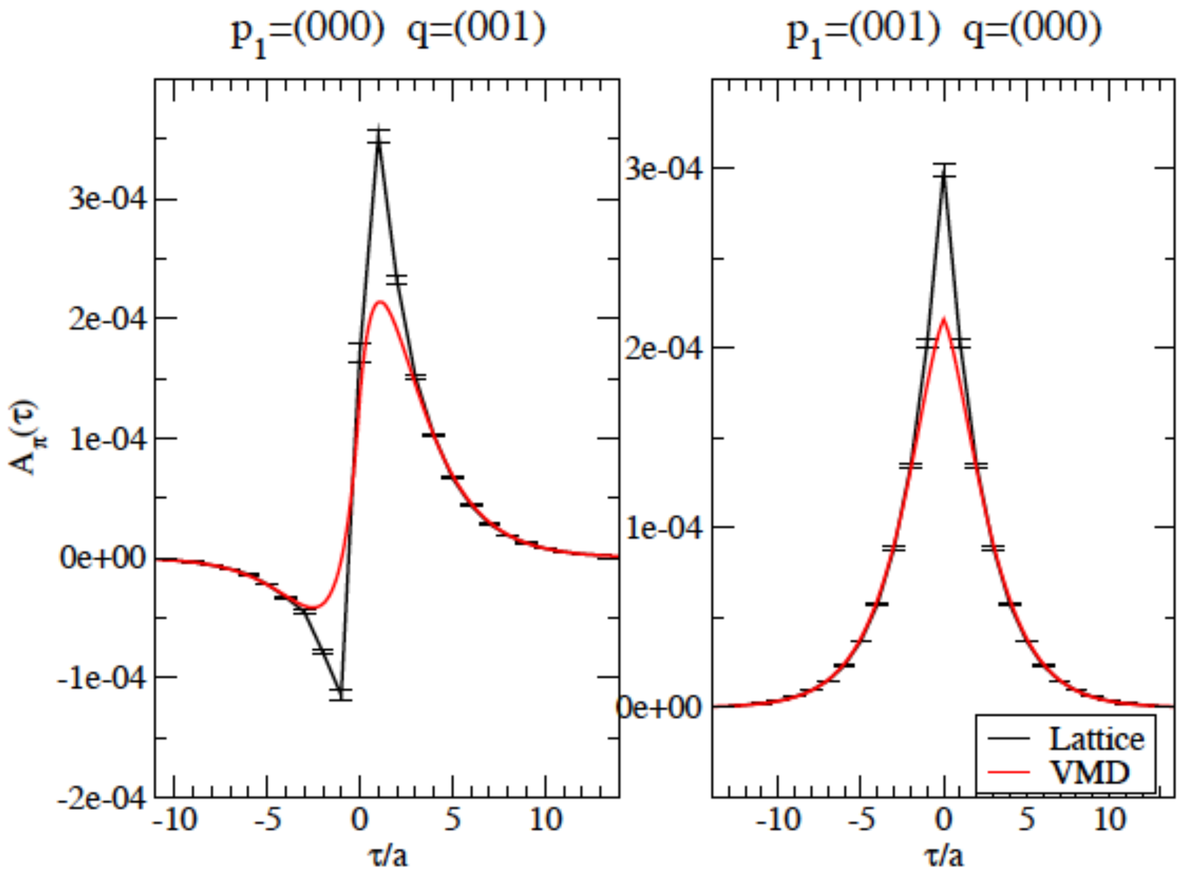
$$M_{\mu\nu}(p_1,p_2) = \lim_{t_{1,2}-t_\pi\to\infty} \frac{1}{\frac{\phi_{\pi,\vec{q}}}{2E_{\pi,\vec{q}}} e^{-E_{\pi,\vec{q}}(t_2-t_\pi)}} \int dt_1\, e^{\omega(t_1-t_2)} C_{\mu\nu}(t_1,t_2,t_\pi),$$

$$C_{\mu\nu}(t_1,t_2,t_\pi) \equiv \int d^3\vec{x}\, e^{-i\vec{p}_1\cdot\vec{x}} \int d^3\vec{z}\, e^{i\vec{q}\cdot\vec{z}} \left\langle 0 \middle| T_E\left\{ j_\mu(\vec{x},t_1) j_\nu(\vec{y},t_2) P(\vec{z},t_\pi)\right\} \middle| 0 \right\rangle$$

  - Need an integral over t_1.
  - First, we want to see how $C_{\mu\nu}(t_1,t_2,t_\pi)$ looks like.
  - Define

$$A_\pi(\tau) \equiv \lim_{t-t_\pi\to\infty} \frac{C_{\mu\nu}(t_1,t_2,t_\pi)}{e^{-E_{\pi,\vec{q}}(t-t_\pi)}}, \quad \tau = t_1 - t_2,\ t = \min\{t_1,t_2\}$$

  which is the integrand.

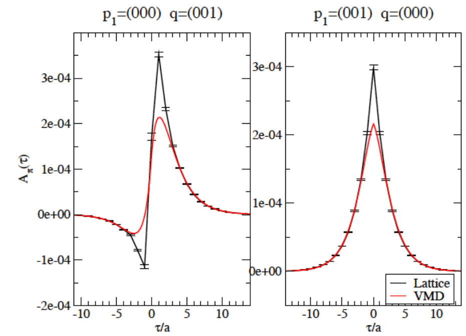


- As a cross check, a VMD curve is also shown.

$$F^{VMD}_{\pi^0\gamma\gamma}(m_\pi^2; p_1^2, p_2^2) = c_V G_V(p_1^2) G_V(p_2^2), \quad G_V(p^2) = \frac{m_V^2}{m_V^2 - p^2}$$

- When $|t_1 - t_2|$ is large, the lowest vector state saturates.

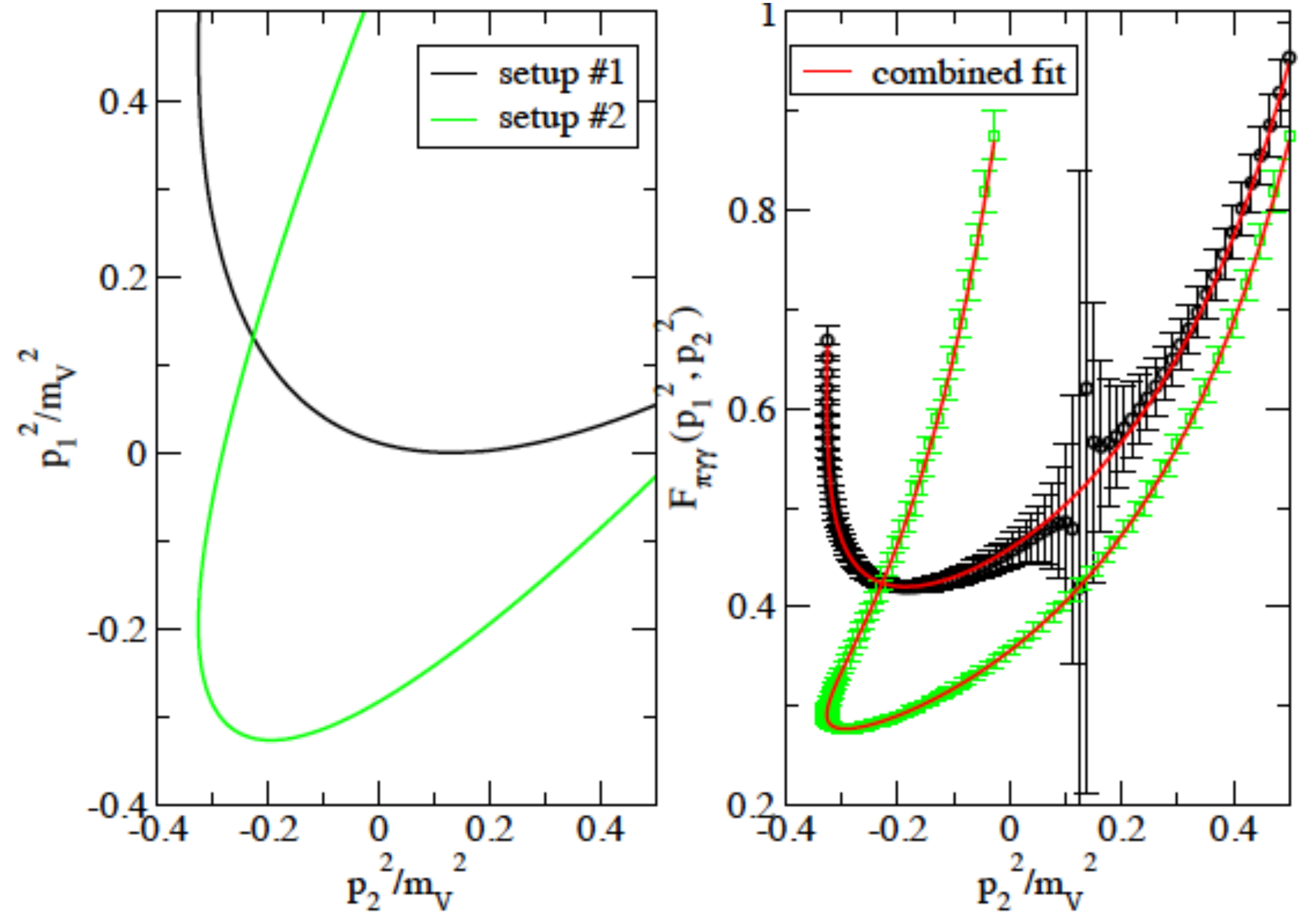
Integral



- When integrate over t_1,
  - We sum over the lattice data at short distances.
  - Long distances are replaced by exp($-M_V|\tau|$) and summed.

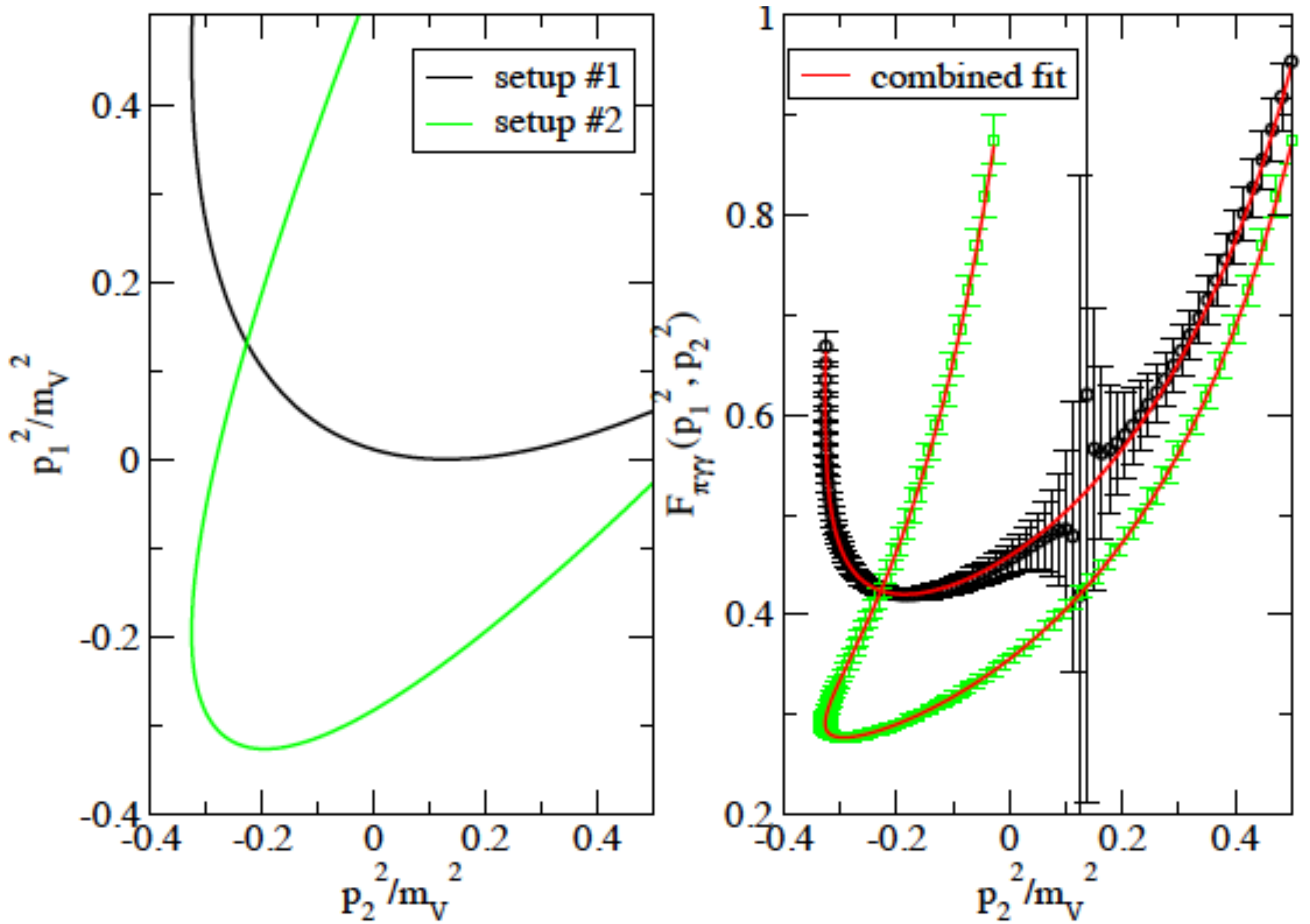
  - Worry about discretization effects? Checked with the VMD curve: the difference between sum and integral is only 0.05%. Expect larger for the real thing, but would be ~0.1%. Tiny.

- Obtain $M_{\mu\nu}(p_1^2,p_2^2)$ by multiplying $e^{\omega\tau}$ and integrate.
  - Results on a continuous curve on the (p_1^2,p_2^2) plane.



- Not just at a point, but on a curve satisfying

$$(p_1^2, p_2^2) = (\omega^2 - \vec{p}_1^2, (E_{\pi,\vec{q}} - \omega)^2 - (\vec{q} - \vec{p}_1)^2)$$

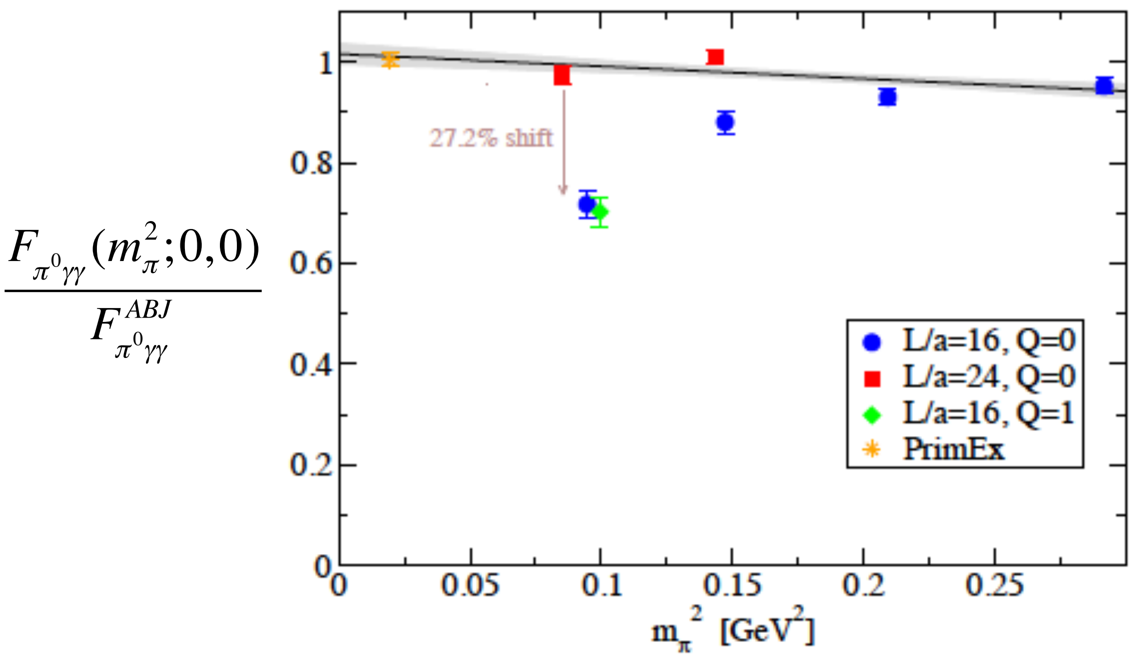


- Fit curve represents a function of the form

$$F_{\pi^0\gamma\gamma}(m_\pi^2; p_1^2, p_2^2) = c_V G_V(p_1^2) G_V(p_2^2) + \sum_m c_m \left((p_2^2)^m G_V(p_1^2) + (p_1^2)^m G_V(p_2^2) \right) + \sum_{m,n} c_{m,n} (p_1^2)^m (p_2^2)^n$$

but only c_V, c_0, $c_{0,0}$ and $c_{0,1}$

Test against ABJ



- Data extrapolated to the on-shell photon limit $p_1^2=p_2^2=0$.
- Data with $m_\pi L > 4$ are consistent with ABJ and exp.

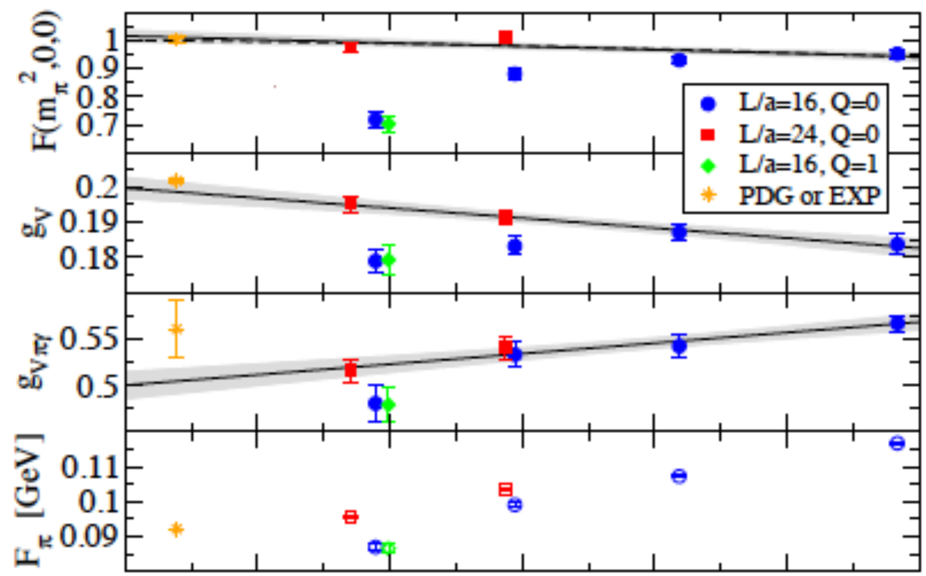
Large finite volume effect?

Finite volume effect?

- JLQCD lattices are too small…
  - So, FVE may be seen in other related modes.
  - Try to see the effect in individual component.
  - Reasonable to add up to the big effect.
  - Correction is attempted based on the data for each component.

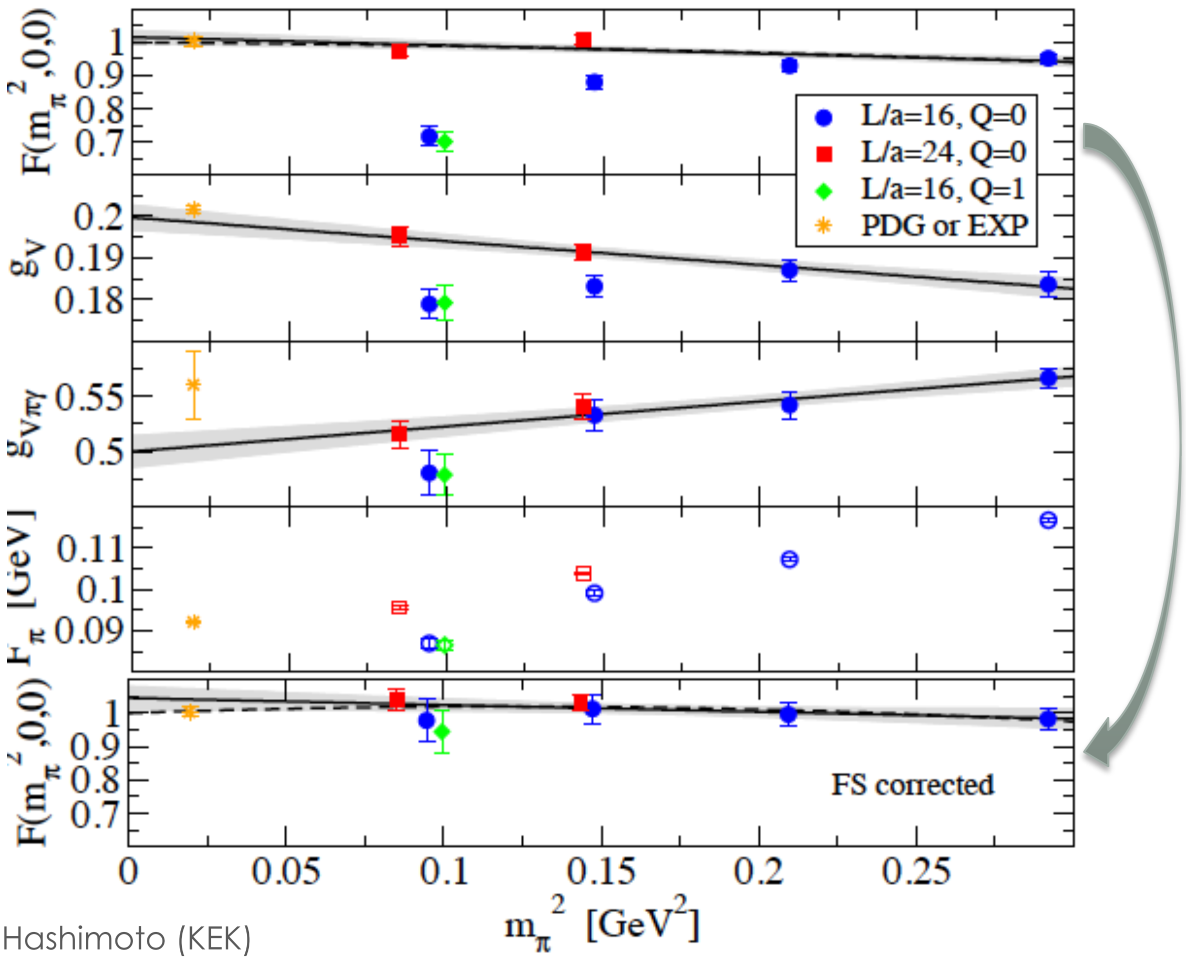
$$\langle j_\mu j_\nu P \rangle \sim \langle 0 | j_\mu | V \rangle \langle V | j_\nu | \pi \rangle \langle \pi | P | 0 \rangle$$

$$\sim g_V \times g_{V\pi\gamma} \times F_\pi$$



Finite volume effect?

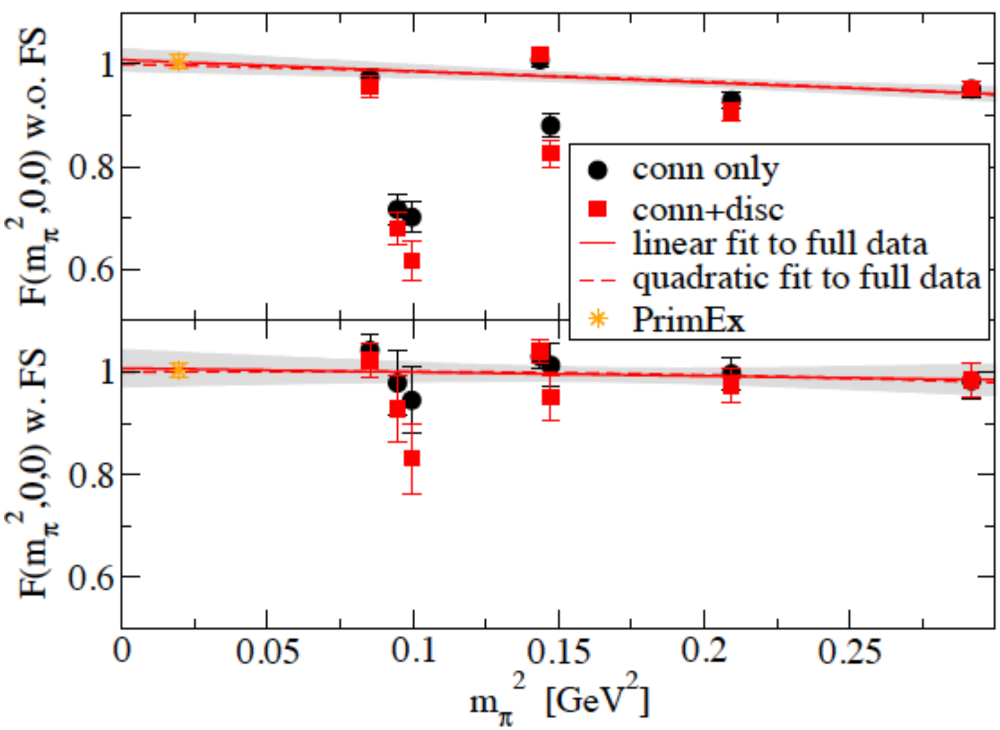
- Assuming that FVE has the form $\exp(-m_\pi L)$.

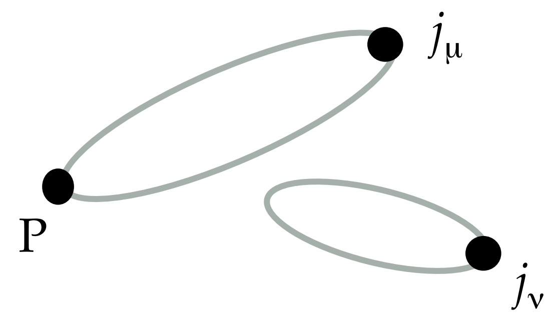


seems to be consistent.

Disconnected contribution?

- Often ignored, but with all-to-all, straightforward to include the disconnected diagrams.





- Vanishes in the SU(3) limit (2/3-1/3-1/3).
- Small as expected. But, will become important at the per cent level.

Numerical results

- After examining the systematic effects, we obtain

$$
\begin{aligned}
F(0,0,0) &= 1.009(22)(29)\ , \\
F(m^2_{\pi,\mathrm{phy}},0,0) &= 1.005(20)(30)\ , \\
\Gamma_{\pi^0\gamma\gamma} &= 7.83(31)(49)\ \mathrm{eV}
\end{aligned}
$$

form factor normalized by ABJ.

  - Consistent with the ABJ anomaly.
  - Quark mass dependence insignificant.

  - Also consistent with PrimEX: 7.82(22) eV.

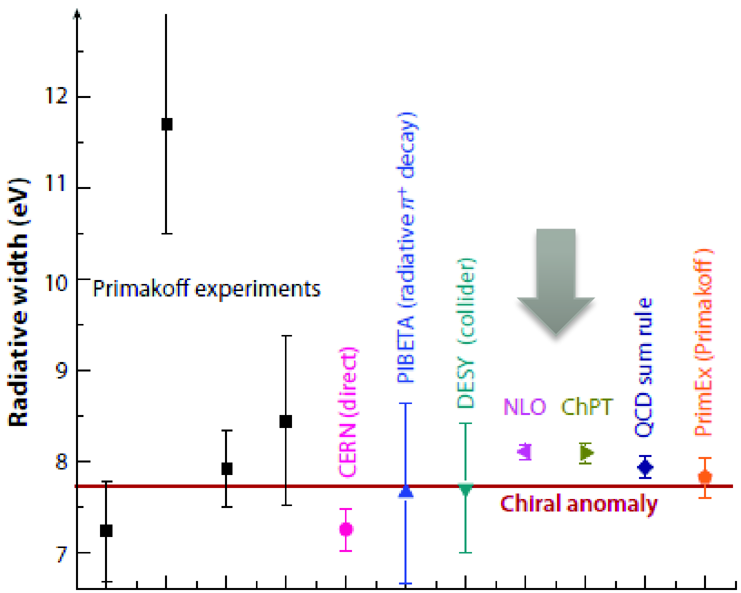
4. Discussions

Consistent with pheno works?

- Several works predicted about +(4—5)% effects. Consistent?

Review article on ph and ex:
Bernstein, Holstein, arXiv:1112.4890.

  - Sum rule, χPT, …
  - Mainly due to a mixing of π^0 with η and η'.
  - So, not inconsistent.



More realistic calculation must take isospin breaking into account. Overall understanding including η and η'.

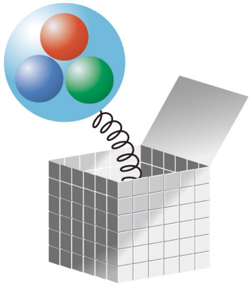
Other applications?

- $\pi^0 \rightarrow \gamma\gamma$ possible. Non-QCD final state can be treated. Other interesting application?
  - $\pi^0 \rightarrow e^+e^-$: through $\pi^0 \rightarrow \gamma^*\gamma^*$. Requires info on the off-shell form factor.
  - Muon g-2: Light-by-light would be dominated by $\gamma\gamma^* \rightarrow \pi^0 \rightarrow \gamma^*\gamma^*$.
  - $\gamma^*\pi^0 \rightarrow \gamma$: Going to higher momentum transfer (possible?). Experimental data available. Eventually connects to the perturbative regime of Brodsky-Lepage.
  - $P^+ \rightarrow l^+\nu\gamma$: P could be π, K, D, B. Interesting as they are related to "light-cone distribution amplitude".

Conclusions

- Chiral lattice fermion works just as expected.
  - Well, more expensive. Yet, theoretically clean formulation is helpful when new applications are considered.
- Non-QCD initial/final state can be treated.
  - Theory was there. Made feasible/realistic by all-to-all.

Fun to have a fresh look at classical problems!



Thank you!