Lorentz Invariance in Heavy Particle Effective Theories

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Outline

- Motivation: NRQCD/HQET and Reparametrization Invariance (RPI)
- Lorentz Invariance using Wigner's Little Group
- Invariant operator method
- Other application: SCET
- Summary

Motivation: NRQCD/HQET

How to construct an effective theory for a heavy, non-relativistic particle.

Example: NRQCD/NRQED, HQET [Caswell, Lepage '85]

- Describe particle with mass M and four-velocity v^μ by field ψ_v:

$$\not{v}\psi_v = \psi_v$$

(= assume small k in $p^\mu = Mv^\mu + k^\mu$)

- Write down all operators (to order $1/M^n$) compatible with symmetries: gauge symmetry, C, P, ...

$\rightarrow$ Can use field redefinitions to get rid of ∂_t's, except for 1st order kinetic term ("canonical form").

NRQCD/HQET: Lagrangian

Also, remove mass term by $\psi_v \to e^{iM\mathbf{v}\cdot\mathbf{x}}\psi_v$.

- In the particle's rest frame, $v = (1, 0, 0, 0)$
$$\mathcal{L} = \bar{\psi}\left\{ iD^0 + c_2 \frac{\mathbf{D}^2}{2M} + c_F\, g \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2M} + c_D\, g \frac{[\mathbf{D}\cdot\mathbf{E}]}{8M^2} + ic_S\, g \frac{\boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8M^2} + \dots \right\}\psi$$
(Note: This is only rotationally invariant, not Lorentz invariant!)

In an arbitrary frame, fixed $v = (v^0, \vec{v})$

$$\mathcal{L}_v =$$
$$\bar{\psi}_v\left\{ i(v\cdot D) - c_2 \frac{D_\perp^2}{2M} - c_F\, g \frac{\sigma_{\alpha\beta}G^{\alpha\beta}}{4M} - c_D\, g \frac{v^\alpha \left[D_\perp^\beta G_{\alpha\beta}\right]}{8M^2} + ic_S\, g \frac{v_\lambda \sigma_{\alpha\beta}\left\{D_\perp^\alpha, G^{\lambda\beta}\right\}}{8M^2} + \dots \right\}\psi_v$$

where $D_\perp^\mu = D^\mu - v^\mu(v\cdot D)$

- Coefficients c_i can be obtained by matching to full QCD calculations (perturbatively or lattice QCD).

Reparametrization Invariance

- Computations of c_i's require quite some effort/time/...
  Can we simplify somehow further? Are these coefficients related?
- The decomposition of the momentum p^μ is not unique

$$p^\mu = Mv^\mu + k^\mu$$

$\Rightarrow$ Get the same p^μ for

$$(v, k) \to (v + q/M, k - q) \quad \text{with } v^2 = (v + q/M)^2 = 1$$

Reparametrization Invariance (RPI)

The Lagrangian must be invariant under $v \to v + q/M$

Reparametrization Invariance II

How do the fields transform under RPI?

- For our effective fields: $\psi_v \sim e^{iMv\cdot x}\psi$

$\rightarrow$ Fields pick up a phase $e^{iq\cdot x}$ under RPI.

- However, non-scalar fields also have spinor/vector-indices.

General RPI for particle with spin s [Luke,Manohar '92]

$$\Phi_v \rightarrow e^{iq\cdot x}\underbrace{\Lambda(\hat{u}, v+q/M)^{-1}\Lambda(\hat{u}, v)}_{=:X(D)}\Phi_v$$

$\rightarrow$ $\Lambda(w, v)$ such that $w = \Lambda(w, v)v$ and $u = (v + iD/M)$ and $\hat{u} = \frac{u}{|u|}$.

For $s = 1/2$

$$X(D) = 1 + \frac{\not{q}}{2M} + \frac{\sigma_{\alpha\beta}q^\alpha D_\perp^\beta}{2M(M|u| + M + i(v\cdot D))} = 1 + \frac{\not{q}}{2M} + \frac{\sigma_{\alpha\beta}q^\alpha D_\perp^\beta}{4M^2} + \ldots$$

RPI - Relations

- Consequences for the Lagrangian

$$\mathcal{L}_v = \bar{\psi}_v \left\{ iD\cdot v - c_2 \frac{D_\perp^2}{2M} - c_F\, g \frac{\sigma_{\alpha\beta} G^{\alpha\beta}}{4M} + ic_S\, g \frac{v_\lambda \sigma_{\alpha\beta} \left\{ D_\perp^\alpha, G^{\lambda\beta} \right\}}{8M^2} + \dots \right\} \psi_v$$

- We demand invariance under

$$v \to v + q/M$$

$$\psi_v \to e^{iq\cdot x} \left[1 + \frac{\not{q}}{2M} + \frac{\sigma_{\alpha\beta} q^\alpha D_\perp^\beta}{4M^2} + \dots \right] \psi_v$$

$\Rightarrow$ The coefficients must obey

RPI - Relations

$$c_2 = 1$$

$$c_S = 2c_F - 1$$

The origin of RPI

- How does one get the $X(D)$-part of the RPI-transformation?

"In fact, we have not been able to follow the arguments (...) regarding the derivation of the reparameterization transformation step by step. To which extent this is due to our own inabilities, and to which extent the arguments are actually inconclusive or wrong, is not completely clear to us at each point, either. Therefore we decided to investigate the issue on our own along somewhat different lines."

[Finkemeier, Georgi, McIrvin '97]

The origin of RPI - II

There are strong "hints" as to the origin of RPI:

- QCD is Poincaré invariant with generators

$$h = i\partial_t \,,$$
$$\mathbf{p} = -i\boldsymbol{\partial} \,,$$
$$\mathbf{j} = \mathbf{r} \times \mathbf{p} + \boldsymbol{\Sigma} \,,$$
$$\mathbf{k} = \mathbf{r}h - t\mathbf{p} + i\boldsymbol{\Sigma} \,,$$

Poincaré commutation relations (PCR)

$$[h, \mathbf{p}^i] = [h, \mathbf{j}^i] = 0 \qquad [\mathbf{p}^i, \mathbf{p}^j] = 0$$
$$[\mathbf{j}^i, \mathbf{j}^j] = i\epsilon^{ijk}\mathbf{j}^k \qquad [J^i, \mathbf{p}^j] = i\epsilon^{ijk}\mathbf{p}^k$$
$$[h, \mathbf{k}^j] = -i\mathbf{p}^j \qquad [\mathbf{p}^i, \mathbf{k}^j] = -i\delta^{ij}h$$
$$[\mathbf{j}^i, K^j] = i\epsilon^{ijk}\mathbf{k}^k \qquad [\mathbf{k}^i, \mathbf{k}^j] = -i\epsilon^{ijk}\mathbf{j}^k$$

$\Rightarrow$ The conserved charges in NRQCD should also obeys these PCRs.

The origin of RPI - III

- Calculate conserved charges [Brambilla, Gromes, Vairo '03]

$$H = \int d^3x \; \psi^\dagger \left(m - c_2 \frac{\mathbf{D}^2}{2m} - c_F \, g \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2m} - c_D \, g \frac{[\mathbf{D}\cdot,\mathbf{\Pi}]}{8m^2} - i c_S \, g \frac{\boldsymbol{\sigma}\cdot[\mathbf{D}\times,\mathbf{\Pi}]}{8m^2} + \ldots \right) \psi$$

$$\mathbf{P} = \int d^3x \; \left(\psi^\dagger \, (-i\mathbf{D}) \, \psi + \frac{1}{2} [\mathbf{\Pi}^a \times, \mathbf{B}^a] \right),$$

$$\mathbf{J} = \int d^3x \; \left(\psi^\dagger \left(\mathbf{x} \times (-i\mathbf{D}) + \frac{\boldsymbol{\sigma}}{2} \right) \psi \; + \frac{1}{2} \mathbf{x} \times [\mathbf{\Pi}^a \times, \mathbf{B}^a] \right),$$

$$\mathbf{K} = -t \, \mathbf{P} + \int d^3x \, \frac{\{\mathbf{x}, h\}}{2} - k^{(1)} \int d^3x \, \left(\frac{1}{2m} \psi^\dagger \, \frac{\boldsymbol{\sigma}}{2} \times (-i\mathbf{D}) \, \psi \right),$$

($k^{(1)}$ is a coefficient to be determined.)

\+ Quantize canonically:

$$[\mathbf{\Pi}(\mathbf{x}, t), \mathbf{A}(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \;, \; \{\psi_\alpha(\mathbf{x}, t), \psi^\dagger_\beta(\mathbf{y}, t)\} = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \; \ldots$$

\+ Enforce Poincaré algebra:

Relations for coefficients $\sim$ RPI relations

$$k^{(1)} = 1, \qquad \text{and} \qquad c_2 = 1, \quad c_S = 2c_F - 1, \ldots$$

RPI $\sim$ Lorentz ?

Conjecture (Theorem?)

RPI is enforcing Lorentz invariance of the Lagrangian

$\Rightarrow$ Let's forget about NRQCD and RPI for a moment and start from scratch

How can we make a non-relativistic EFT Lorentz invariant?

Wigner's little group in QM

How do QM states transform under Lorentz transformation?

- States are characterized by the value of $p^2 = M^2$ and the sign of p^0.
- Pick a reference vector k^μ, e.g. $k^\mu = (M, 0, 0, 0)$, and a standard Lorentz transformation $L(p)$, s.t. $L(p)k = p$ for any such p.

Define the one particle states

$$|p, m\rangle \sim L(p)\,|k, m\rangle$$

- Under a Lorentz transformation

$$|p, m\rangle \to \Lambda\,|p, m\rangle \sim L(\Lambda p) * \underbrace{L(\Lambda p)^{-1} * \Lambda * L(p)}_{=:W(\Lambda, p)}\,|k, m\rangle$$

- $W(\Lambda, p) \in$ Wigner's little group $\subset$ Lorentz: $W(\Lambda, p)k = k$.

$$\Rightarrow W(\Lambda, p)\,|k, m\rangle = \sum_{m'} D_{m'm}[W(\Lambda, p)]\,|k, m'\rangle$$

Wigner's little group in QM - II

$$|p, m\rangle \to \Lambda |p, m\rangle \sim \sum_{m'} D_{m'm}[W(\Lambda, p)]\ L(\Lambda p) \left|k, m'\right\rangle$$

- The coefficients $D_{m'm}[W(\Lambda, p)]$ form a unitary representation of the little group.

$\Rightarrow$ The little group determines the unitary rep's of the Lorentz group.

Unitary Representations of the Lorentz group [Wigner '39]

(Infinite dim.) Unitary representations of the Lorentz group are given by

$$U(\Lambda) |p, m\rangle \sim \mathcal{D}[W(\Lambda, p)] |\Lambda p, m\rangle$$

where $\mathcal{D}$ is a rep. of the little group element $W(\Lambda, p) = L(\Lambda p)^{-1} \Lambda L(p)$.

- What is the structure of the little group?

$\to$ For a massive particle choose for example $k = (M, 0, 0, 0)$

$\to$ little group is $SO(3)$

$W(\Lambda, p)$ for a massive particle

Let's compute $W(\Lambda, p)$ for $k = Mv$ and $L(p) = \exp[-i\theta \mathcal{J}_{\alpha\beta} \frac{p^\alpha}{M} v^\beta]$.

1. For elements of the Little group, i.e. "rotations" with $\mathcal{R}v = v$,

$$W(\mathcal{R}, p) = \mathcal{R}$$

   This will simplify things a lot!

2. The remaining Lorentz transformations are "boosts" with $\mathcal{B}v = v - q/M$, for infinitesimal q with $q \cdot v = 0$,

$$W(\mathcal{B}, p) = 1 - \frac{i}{2} \left[\frac{q^\alpha p_\perp^\beta - p_\perp^\alpha q^\beta}{M(M + v \cdot p)} \right] \mathcal{J}_{\alpha\beta} + \mathcal{O}(q^2)$$

   with $p_\perp^\alpha = p^\alpha - (v \cdot p) v^\alpha$.

Wigner's little group in field theories

Let's apply this to field theories

Demand invariance under

$$\phi_a(x) \to D[W(\Lambda, i\partial)]_{ab}\phi_b(\Lambda^{-1}x)$$

1. For rotations with $\mathcal{R}v = v$

$$W(\mathcal{R}, i\partial) = \mathcal{R}$$

   No dependence on ∂.

2. For infinitesimal boosts with $\mathcal{B}v = v - q/M$

$$W(\mathcal{B}, i\partial) = 1 - \frac{i}{2}\left[\frac{q^\alpha i\partial_\perp^\beta - i\partial_\perp^\alpha q^\beta}{M(M + v\cdot i\partial)}\right]\mathcal{J}_{\alpha\beta} + \mathcal{O}(q^2).$$

What about Lorentz invariance?

For Simplicity, choose $v = (1, 0, 0, 0)$. The corresponding generators are

Rotations

$$\mathbf{j} = \mathbf{r} \times i\boldsymbol{\partial} + \boldsymbol{\Sigma}$$

Boosts

$$\mathbf{k} = \mathbf{r} i\partial_t - t i\boldsymbol{\partial} + \frac{\boldsymbol{\Sigma} \times i\boldsymbol{\partial}}{M + \sqrt{M^2 - \boldsymbol{\partial}^2}}$$

- With $h = i\partial_t$ and $\mathbf{p} = -i\boldsymbol{\partial}$ these satisfy the PCR on fields which obey

$$i\partial_t \phi = \sqrt{M^2 - \boldsymbol{\partial}^2}\phi \, .$$

$\Rightarrow$ Theory seems to be Lorentz invariant.

BUT: Appearance of ∂ in spin part of **k** violates gauge invariance!

$\Rightarrow$ Consolation: At least the free theory is Lorentz invariant :)

Lorentz invariance of the interacting theory

Appearance of ∂ in $W(\mathcal{B}, i\partial)$ ($=$ spin part of $\mathbf{k}$) violates gauge invariance.

- Obvious fix: Replace ∂ by covariant $D = \partial - igA$.

$\Rightarrow$ $W(\Lambda, iD)\phi_v$ transforms covariantly under gauge transformations.

- This spoils PCRs: Does this still describe a Lorentz invariant theory?

$\rightarrow$ The answer is YES.

Let's prove it by showing

1. The charges in the interacting theory obey the Poincaré algebra.
2. One obtains a Lorentz invariant S matrix.
   $S = lim_{T\to\infty}\Omega(T)^\dagger\Omega(-T)$ with $\Omega(T) = e^{iHT}e^{-iH_0T}$

Lorentz invariance of the interacting theory - II

Condition for a Lorentz invariant theory

Lorentz invariance of the S matrix $\Leftrightarrow$ S commutes with $H_0, \mathbf{P}_0, \mathbf{J}_0$ and $\mathbf{K}_0$.

- Note, the free charges $H_0, \mathbf{P}_0, \mathbf{J}_0, \mathbf{K}_0$ obey the PCR, since the generators $h_0, \mathbf{p}_0, \mathbf{j}_0, \mathbf{h}_0$ do.
- The charges in the interacting theory are $(H, \mathbf{P}, \mathbf{J}, \mathbf{K})$:
  - $H = H_0 + V$, while $\mathbf{P} = \mathbf{P}_0$ and $\mathbf{J} = \mathbf{J}_0$

$\Rightarrow$ Thus $[\mathbf{K}^i, \mathbf{P}^j] = -i\delta^{ij} H$ implies $\mathbf{K} \neq \mathbf{K}_0$!

- How do we find the right form of the generator $\mathbf{k}$?
  I claim we already did.

Lorentz invariance of $\mathcal{L}$

Demand invariance under rotations & $\phi_v(x) \to W(\mathcal{B}, iD)\phi_v(x' = \mathcal{B}^{-1}x)$

$\rightarrow$ Now prove the Lorentz invariance of the S matrix.

Lorentz invariance of the S matrix

Follow the standard proof of Lorentz invariance for the S matrix

[e.g. Weinberg Vol. I]

Need to show two things

1. $[\mathbf{P}_0, V] = [\mathbf{J}_0, V] = 0$
2. $[\mathbf{K}^i, H] = -i\mathbf{P}^i$, with "smooth" $\Delta\mathbf{K} = \mathbf{K} - \mathbf{K}_0$.

Then

- All *free* Lorentz charges commute with the S matrix

$\Rightarrow$ S matrix is Lorentz invariant.

- In addition, charges in the interacting theory are similarity transforms of the free ones, e.g. $\mathbf{K}\Omega = \Omega\mathbf{K}_0$ with $\Omega\Phi_0 = \Phi_{\text{int}}$.

$\Rightarrow$ $(H, \mathbf{P}, \mathbf{J}, \mathbf{K})$ obey the Poincaré algebra.

Lorentz invariance of the S matrix - II

to 1) $[\mathbf{P}_0, V] = 0 : \checkmark$
$[\mathbf{J}_0, V] = 0$: true since $\mathcal{R}v = v$: $\checkmark$

to 2) $\mathbf{k}$ is of the form

$$\mathbf{k} = \mathbf{r} i\partial_t - ti\boldsymbol{\partial} + \frac{\boldsymbol{\Sigma} \times i\mathbf{D}}{M + \sqrt{M^2 - \mathbf{D}^2}} + \mathcal{O}(g)^*$$

Thus the conserved charge has the form

$$\mathbf{K} = -t\mathbf{P} + (...)$$

with no explicit time dependence in the dots.
$\Rightarrow$ In the Heisenberg picture this then yields

$$0 = \frac{d}{dt}\mathbf{K} = \frac{\partial}{\partial t}\mathbf{K} + i[H, \mathbf{K}] = -\mathbf{P} + i[H, \mathbf{K}]$$

hence $[H, K^i] = -iP^i$. $\checkmark$

Lorentz invariance of effective theories

Lorentz invariance

Our procedure using Wigner's little group yields a Lagrangian with a Lorentz invariant S matrix!

Field strength dependent terms

Boost transformation (includes an ordering prescription)

$$W(\mathcal{B}, iD) = 1 - \frac{i}{2}\left[\frac{q^{\alpha} iD_{\perp}^{\beta} - iD_{\perp}^{\alpha} q^{\beta}}{M(M + v \cdot iD)}\right] \mathcal{J}_{\alpha\beta} + \mathcal{O}(g)^{*}$$

* What about the field strength dependent terms?

→ They do not change the free case limit.

→ They do not alter the proof of Lorentz invariance.

⇒ We are free to add such terms if needed:

- Allow us to perform field redefinitions containing g (i.e. also $G_{\mu\nu}$).
- Needed to maintain canonical form of the Lagrangian (i.e. only one ∂_t).
- Will be needed later in the invariant operator method.

How to build a Lorentz invariant theory

For a general theory

$$\mathcal{L} \sim \bar{\phi}_v \left\{ \dots v^\mu \dots D^\mu \dots \gamma^\mu \dots \epsilon^{\mu\nu\alpha\beta} \dots \right\} \phi_v$$

1. For generalized rotations $W(\mathcal{R}, i\partial) = \mathcal{R}$

$$\phi_v(x) \to \mathcal{R}\phi_v(x')$$

$$\partial^\mu \to \mathcal{R}^\mu{}_\nu \partial'^\nu \quad \text{and} \quad A^\mu(x) \to \mathcal{R}^\mu{}_\nu A^\nu(x')$$

$\to$ $\mathcal{L}$ is invariant, since $\mathcal{R}v = v$ and $\gamma^\mu = \mathcal{R}_{\frac{1}{2}} \left(\mathcal{R}^\mu{}_\nu \gamma^\nu\right) \mathcal{R}_{\frac{1}{2}}^{-1}$.

2. For boosts with $\mathcal{B}v = v - q/M + \mathcal{O}(q^2)$

$$\phi_v(x) \to W(\mathcal{B}, iD)\phi_v(x')$$

$$\partial^\mu \to \mathcal{B}^\mu{}_\nu \partial'^\nu \quad \text{and} \quad A^\mu(x) \to \mathcal{B}^\mu{}_\nu A^\nu(x')$$

$\to$ Demand invariance of $\mathcal{L}$ under this.

Connection of Lorentz invariance to RPI

- We can rewrite the transformation law under boosts: Invariance under

$$\phi_v(x) \to W(\mathcal{B}, iD)\phi_v(x'), \quad \partial^\mu \to \mathcal{B}^\mu{}_\nu \partial'^\nu \quad \text{and} \quad A^\mu(x) \to \mathcal{B}^\mu{}_\nu A^\nu(x')$$

is equivalent to invariance under (just using $\mathcal{B}\mathcal{B}^{-1} = 1$)

Lorentz invariance = RPI

$$\phi_v(x) \to B^{-1} W(\mathcal{B}, iD)\phi_v(x), \quad \text{and} \quad v \to w := \mathcal{B}^{-1} v = v + q/M$$

- For a spin 1/2 spinor we obtain:

$$\mathcal{B}^{-1} W(\mathcal{B}, iD) = \left\{1 + \frac{\not{q}}{2M}\right\} \left[1 + \frac{\sigma_{\alpha\beta} q^\alpha D_\perp^\beta}{2M(M + i(v \cdot D))}\right]$$

- Together with $\Phi_v \to e^{iMv\cdot x}\Phi_v$, one obtains the RPI transformation.
- However, we are not changing v!
  We just note an mathematical equivalence.

Obtaining coefficient constraints

- Can implement Lorentz invariance on the field level, and obtain the same relations between coefficients as before, e.g. at order $1/M^3$:

$$\delta\mathcal{L}_3 = \psi^\dagger \Big[\frac{e}{8} c_D [D_t, \mathbf{q}\cdot\mathbf{E}] + \frac{e}{8}(c_F - c_D + 2c_M)\,\mathbf{q}\cdot[\boldsymbol{\partial}\times\mathbf{B}] + \frac{i}{4}(c_2 - c_4)\{\mathbf{q}\cdot\mathbf{D}, \mathbf{D}^2\}$$
$$+ \frac{ie}{8} c_S \{D_t, \boldsymbol{\sigma}\times\mathbf{q}\cdot\mathbf{E}\} + \frac{ie}{8}(c_2 + 2c_F - c_S - 2c_{W1} + 2c_{W2})\{\mathbf{q}\cdot\mathbf{D}, \boldsymbol{\sigma}\cdot\mathbf{B}\}$$
$$+ \frac{ie}{8}(-c_2 + c_F - c_{p'p})\{\boldsymbol{\sigma}\cdot\mathbf{D}, \mathbf{q}\cdot\mathbf{B}\} + \frac{ie}{8}(-c_F + c_S - c_{p'p})\,\mathbf{q}\cdot\boldsymbol{\sigma}(\mathbf{D}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{D}) \Big]\psi\,.$$

+ Lower order relations: $c_2 = 1, \quad c_S = 2c_F - 1$

+ field redefinition to change $W(\mathcal{B}, iD)$:

$$\psi(x) \to e^{-i\mathbf{q}\cdot\mathbf{x}} \left\{ 1 + \frac{i\mathbf{q}\cdot\mathbf{D}}{2M^2} - \frac{\boldsymbol{\sigma}\times\mathbf{q}\cdot\mathbf{D}}{4M^2} + \frac{ic_D}{8M^3} e\mathbf{q}\cdot\mathbf{E} + \frac{c_S}{8M^3} e\mathbf{q}\cdot\boldsymbol{\sigma}\times\mathbf{E} + \dots \right\} \psi(\mathcal{B}^{-1}x)$$

$\Rightarrow$ We get a vanishing variation iff

$$c_4 = 1, \quad 2c_M = c_D - c_F, \quad c_{W2} = c_{W1} - 1, \quad c_{p'p} = c_F - 1$$

$\rightarrow$ Does this work to any order, i.e. is there an invariant Lagrangian to all orders in $1/M$?

Invariant operator method

- We started with a Lagrangian, which was not Lorentz invariant. (Only rotationally invariant before enforcing coefficient relations.)

→ Is there an inherently Lorentz invariant construction?

- We can answer both questions by the same construction: Learn from RPI

Find a redefinition $\Gamma(v, iD)$

$$\Phi_v := \Gamma(v, iD)\phi_v \quad \xrightarrow{\Lambda} \quad \Phi_v$$

- Under a Lorentz transformation (in "RPI version")

$$\Phi_v := \Gamma(v, iD)\phi_v \quad \to \quad \Gamma(\Lambda^{-1}v, iD)\Lambda^{-1}W(\Lambda, iD)\phi_v$$

$$\stackrel{!}{=} \Gamma(v, iD)\phi_v$$

Invariant operator method - II

- Assume we had an operator $\Gamma(v, iD)$ that obeyed

$$\Gamma(\Lambda^{-1} v, iD)\Lambda^{-1} W(\Lambda, iD) = \Gamma(v, iD)\,.$$

then the field $\Phi_v := \Gamma(v, iD)\phi_v$ is invariant under Lorentz.
After $\phi_v \to e^{iMvx}\phi_v$ this becomes (need only consider $\mathcal{B}$)

Invariance Equation

$$\Gamma(v + q/M, iD - q)\mathcal{B}^{-1} W(\mathcal{B}, iD + Mv) = \Gamma(v, iD)\,.$$

and under Lorentz boosts:

$$\Phi_v \to e^{iq\cdot x}\Phi_v:$$

$\Rightarrow$ Build invariants using γ^μ and $\mathcal{V}^\mu = v^\mu + iD^\mu/M$

Solving the invariance equation - free case

Invariance Equation

$$\Gamma(v + q/M, iD - q)\underbrace{\mathcal{B}^{-1}W(\mathcal{B}, iD + Mv)}_{:=X(v,iD)} = \Gamma(v, iD)\,.$$

- Note that in the free case we can write (with $\mathcal{V}^{\mu}_{\text{free}} \equiv v^{\mu} + i\partial^{\mu}/M$)

$$X(v, i\partial) = \Lambda_{\frac{1}{2}}(\hat{\mathcal{V}}_{\text{free}}, v + q/M)^{-1} * \Lambda_{\frac{1}{2}}(\hat{\mathcal{V}}_{\text{free}}, v)$$
$$= 1 + \frac{\not{q}}{2M} + \frac{1}{4M^2}\sigma^{\mu\nu}_{\perp} q_{\mu}\partial_{\nu}\left[1 - \frac{iv\cdot\partial}{M}\right] + \ldots$$

Comparing to the invariance equation one can see:

Closed, all-order solution

$$\Gamma(v, i\partial) = \Lambda_{\frac{1}{2}}(\hat{\mathcal{V}}_{\text{free}}, v) = 1 + \frac{i\not{\partial}_{\perp}}{2M} + \frac{1}{M^2}\left[-\frac{1}{8}(i\partial_{\perp})^2 - \frac{1}{2}i\not{\partial}_{\perp} iv\cdot\partial\right] + \ldots$$

Solution to the invariance equation - interacting case

- Why not proceed as before? Just replace $\partial \to D$

RPI - invariant operator method

$$\Gamma^{\rm naive}(v,iD) = 1 + \frac{i\not{D}_\perp}{2M} + \frac{1}{M^2}\left[-\frac{1}{8}(iD_\perp)^2 - \frac{1}{2}i\not{D}_\perp iv\cdot D\right] + \frac{1}{M^3}\left[\frac{1}{4}(iD_\perp)^2 iv\cdot D + \frac{i\not{D}_\perp}{2}\left(-\frac{3}{8}(iD_\perp)^2 + (iv\cdot D)^2\right)\right] + \mathcal{O}(1/M^4)$$

$\rightarrow$ Unfortunately, the interacting case is more complicated!

- $\Gamma^{\rm naive}(v,iD)$ is NOT a solution to the invariance equation starting at order $1/M^3$.

$\Rightarrow$ The "RPI invariant operator method" starts to fail to build a Lorentz invariant Lagrangian at order $1/M^4$!

One needs to solve the invariance equation explicitly
(pedestrian way or systematically oder-by-order)

Solution to the invariance equation - interacting case II

Invariance Equation

$$\Gamma(v + q/M, iD - q)X(v, iD) = \Gamma(v, iD)\,.$$

- The systematic solution shows, that we may choose
$$X(v, iD) = 1 + \frac{\not{q}}{2M} + \frac{1}{4M^2}\sigma^{\perp}_{\mu\nu} q^{\mu} D^{\nu}_{\perp}\left(1 - \frac{iv\cdot D}{M}\right) + \dots$$
- Expanding Γ in powers of $1/M$
$$\Gamma = 1 + \frac{1}{M}\Gamma^{(1)} + \frac{1}{M^2}\Gamma^{(2)} + \frac{1}{M^3}\Gamma^{(3)} + \dots\,,$$
- Find the solution order by order (remove terms by field redef.)

$$\Gamma^{(1)} = \frac{1}{2} i\not{D}_{\perp}\,, \qquad \Gamma^{(2)} = -\frac{1}{8}(iD_{\perp})^2 - \frac{1}{2} i\not{D}_{\perp} iv\cdot D$$

$$\Gamma^{(3)} = \frac{1}{4}(iD_{\perp})^2 iv\cdot D + \frac{i\not{D}_{\perp}}{2}\left[-\frac{3}{8}(iD_{\perp})^2 + (iv\cdot D)^2\right] - \frac{g}{8} G_{\mu\nu} v^{\mu} D^{\nu}_{\perp} - \frac{g}{16}\sigma^{\mu\nu}_{\perp} G_{\mu\nu} i\not{D}_{\perp}$$

→ Last two terms are essential in solving the invariance equation

Relations for the coefficients

- Since methods differ at $1/M^3$ in $\Gamma(v, iD)$, the relations for the coefficients in the Lagrangian at order $1/M^4$ are different.
- E.g. NRQED at order $1/M^4$ [Hill, Lee, Paz, Solon (in progress)]

$$\mathcal{L} \sim \psi^\dagger \left\{ iD_t + c_F e \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{2M} + c_D e \frac{[\boldsymbol{\partial} \cdot \mathbf{E}]}{8M^2} - c_{A2} e^2 \frac{\mathbf{E}^2}{16M^3} + i c_{X4} e^2 \frac{\{\mathbf{D}^i, [\mathbf{E} \times \mathbf{B}]^i\}}{32M^4} + \ldots \right\} \psi \,.$$

- Enforce Lorentz (field transformations or invariant operator method)

correct

$$c_{X4} = 4c_F^2 + 4c_D - 2c_{A2} - 1 - 4c_F$$

- Using the incorrect RPI invariant operator method

wrong

$$c_{X4} = 4c_F^2 + 4c_D - 2c_{A2} - 1 - 6c_F$$

- Could also be checked using the method of Brambilla et al. (if you are masochistic or want to torture your student)

Generalizations

- Examples all shown for spin 1/2, but method is much more general.
- Can be easily extended to any spin

  $\rightarrow$ e.g. spin 3/2 ψ_v^μ with invariant constraints $\not{v}\psi_v^\mu = \psi_v^\mu$ and $\gamma_\mu \psi_v^\mu = 0$.

  or self-conjugate fields

  $\rightarrow$ e.g. a Majorana spinor $\psi_M = \psi_M^c = \mathcal{C}\psi_M^*$, for an operator $\bar{\psi}_M \mathcal{O}(v) \psi_M$:

$$\mathcal{O}(v) = \mathcal{C}\mathcal{O}(-v)^* \mathcal{C}^\dagger$$

- Less trivial generalization: Other types of little groups, e.g. for massless particles.

For massless particles

Soft-Collinear-Effective theory (SCET)

- Build an effective Lagrangian for massless quarks, separating hard, soft and collinear/anti-collinear degrees of freedom.

$\Rightarrow$ There are two fixed light-like vectors in the Lagrangian:

  - Collinear: $n = (1, \vec{n})$
  - Anti-collinear $\bar{n} = (1, -\vec{n})$

The Little Group for a null vector n

$$\mathbb{E}_2 = \text{Translations and rotations in 2d}$$

$\rightarrow$ Every W can be brought into the form $W = S(\alpha, \beta)R(\theta)$

- $S(\alpha, \beta)$ must act trivially on physical states: $S(\alpha, \beta)\Phi_k = \Phi_k$

$$U(\Lambda)\Phi_k \sim e^{i\theta(\Lambda, n)}\Phi_{\Lambda k}$$

For massless particles: SCET

- Calculate $W(\Lambda, p) = L(\Lambda p)^{-1}\Lambda L(p)$ for:
  1. $[n \to n, \bar{n} \to \bar{n}]$ Rotation: 1 dof,
  2. $[n \to n, \bar{n} \to \bar{n} + s_\perp]$ Parabolic LT: 2 dof,
  3. $[n \to n + t_\perp, \bar{n} \to \bar{n}]$ Parabolic LT: 2 dof,
  4. $[n \to (1+\eta)n, \bar{n} \to (1-\eta)\bar{n}]$ Boost: 1 dof.
- Decompose $W = S(\alpha_{\Lambda,k}, \beta_{\Lambda,k})R(\theta_{\Lambda,k})$

$\rightarrow$ $S(\alpha_{\Lambda,k}, \beta_{\Lambda,k})$ gives a constraint, $R(\theta_{\Lambda,k})$ the transformation.

1. Invariant $W = \Lambda = R(\theta_{\Lambda,k})$
2. Non-trivial $W = S(\alpha_{\Lambda,k}, \beta_{\Lambda,k})R(\theta_{\Lambda,k})$
3. Trivial $W \equiv 1$
4. Trivial $W \equiv 1$

For massless particles: SCET - II

- In the non-trivial case, one obtains the constraints
  - Scalar ϕ: -
  - Spinor ξ^α: $\not{n}\xi = 0 \Leftrightarrow \not{n}\not{\bar{n}}\xi = \xi$
  - Vector A^μ: Does not exist/reduces to scalar!
    As it must, $\exists$ only gauge theories for vectors!
  - $T_{\mu\nu}$: Decomposes into a scalar $T = T^\mu_{\ \mu}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
  - ...
- The corresponding transformation for ξ is

$$\xi \to \left(1 + \frac{1}{2}\not{s}_\perp \frac{1}{\bar{n}\cdot D}\not{D}_\perp\right)\xi$$

This agrees which the one previously obtained from RPI in SCET.
[Manohar, Mehen, Pirjol, Stewart '02]

Recap: How to get a Lorentz invariant effective Lagrangian

To build a Lorentz invariant, effective theory for a heavy particle

- Identify degrees of freedom, e.g. $\not{v} Q_v = Q_v$.
- Determine the Little Group for v^μ and find an $L(p)$.

then

1.) Constructive method

- Write down a Lagrangian that looks formally Lorentz invariant, only constrained by gauge symmetries, C, P, ...
- Make Lagrangian invariant under $W(\Lambda, p)$, where $p^\mu = iD^\mu$.

or

2.) Invariant operator method

- Solve the invariant equation for $\Gamma(v, iD)$ (and $W(\mathcal{B}, p)$)
- Use γ^μ and $\mathcal{V}^\mu = v^\mu + iD^\mu/M$ to build invariant Lagrangian

Conclusion

- Wigner's little group allows the construction of Lorentz invariant EFTs by transformations on the field level.
- It clears up the connection between Lorentz and RPI.
  $\rightarrow$ Shows failure of the "RPI - invariant operator method" at $1/M^4$
- Method can easily be generalized to any spin, self-conjugate fields, massless fields, (space-like case?)...
  $\rightarrow$ Applications to dark matter EFTs, atomic bound states, ...
- We never refered to QCD directly as a UV completion
  $\rightarrow$ No need for a UV theory, e.g. effective theory for protons.

Thanks for your attention

Solving the invariant equation

$$\Gamma(v + q/M, iD - q)\mathcal{B}^{-1}W(\mathcal{B}, iD + Mv) = \Gamma(v, iD)$$

- Expand everything in orders of M

$$X \equiv \mathcal{B}^{-1}W = 1 + q^\mu X_\mu = 1 + q^\mu \left[\frac{1}{M}X_\mu^{(1)} + \frac{1}{M^2}X_\mu^{(2)} + \dots\right],$$

$$\Gamma = 1 + \frac{1}{M}\Gamma^{(1)} + \frac{1}{M^2}\Gamma^{(2)} + \dots .$$

- In Γ, find term linear in q

$$\Gamma(v + q/M, iD - q) - \Gamma(v, iD) = q^\mu \left(-\frac{\partial}{\partial iD^\mu}\Gamma + \frac{1}{M}\frac{\partial}{\partial v^\mu}\Gamma\right) + \dots$$

Need to solve

$$\frac{\partial}{\partial iD^\mu}\Gamma^{(n)} = \frac{\partial}{\partial v^\mu}\Gamma^{(n-1)} + \Gamma^{(n-1)}X_\mu^{(1)} + \Gamma^{(n-2)}X_\mu^{(2)} + \dots + \Gamma^{(0)}X_\mu^{(n)} \equiv Y_\mu^{(n)}$$

- Looks like $\boldsymbol{\nabla}\phi = \mathbf{E}$ in electrodynamics.

Solving the invariant equation - II

Solution for $\Gamma^{(n)}$

$$\Gamma^{(n)} = iD_\perp^\mu Y_\mu^{(n)} - \frac{1}{2!} iD_\perp^\mu iD_\perp^\nu \frac{\partial}{\partial iD^\mu} Y_\nu^{(n)} + \dots$$

- In EM: Need $\boldsymbol{\nabla} \times \mathbf{E} = 0$ to solve $\boldsymbol{\nabla}\phi = \mathbf{E}$.
  Here:

Contraint for $Y^{(n)}$

$$\frac{\partial}{\partial iD^{[\nu}} Y_{\mu]}^{(n)} = 0$$

- Let's solve the constraint for $Y^{(n)}$.

Solving the invariant equation - III

- Plug in definition of Y and solution for lower order Γ's

$\Rightarrow$ $X = \mathcal{B}^{-1} W$ must obey a contraint.

Contraint for $X^{(n)}$

$$\frac{\partial}{\partial iD^{[\nu}} X^{(n)}_{\mu]} = -\frac{\partial}{\partial v^{[\mu}} X^{(n-1)}_{\nu]} + X^{(n-1)}_{[\mu} X^{(1)}_{\nu]} + X^{(n-2)}_{[\mu} X^{(2)}_{\nu]} + \cdots + X^{(1)}_{[\mu} X^{(n-1)}_{\nu]} \equiv Z^{(n)}_{\mu\nu} \, .$$

- Again EM: This looks like $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$.

$\rightarrow$ $\mathbf{B}$ needs to obey $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$.

Contraint for $Z^{(n)}$

$$0 = v_\sigma \epsilon^{\mu\nu\rho\sigma} \frac{\partial}{\partial iD^\rho} Z^{(n)}_{\mu\nu}$$

Solving the invariant equation - IV

- Can show that constraint for $Z^{(n)}$ is obeyed by induction,i.e. if we have $X^{(1)}, \ldots, X^{(n-1)}$ obeying all constraints, then so does $Z^{(n)}$.

$\Rightarrow$ We can "solve" for $X^{(n)}$, but what does that mean?

- Before: we can add terms to $X = \mathcal{B}^{-1} W$ that vanish in the free case.

$\rightarrow$ Constraint for $X^{(n)}$: we must add some such terms to solve for Γ.

Solution for $X^{(n)}$

$$X_{\mu}^{(n)} = \hat{X}_{\mu}^{(n)} + 2 \sum_{m=1}^{n-1} \frac{(-1)^m}{(m+1)!} iD_{\perp}^{\nu_1} \cdots iD_{\perp}^{\nu_m} \frac{\partial}{\partial iD^{\nu_1}} \cdots \frac{\partial}{\partial iD^{\nu_{m-1}}} \left(Z_{\nu_m \mu}^{(n)} - \hat{Z}_{\nu_m \mu}^{(n)} \right)$$

where $\hat{X}^{(n)} =$ naive covariantization of given $X(i\partial)$, and $\hat{Z}$ corresponding to it.

- Now use this to solve for Y and Γ

Solving the invariant equation - V

- Since $Z^{(n)}$ has mass dimension $n-2$, the first field strength dependent terms can appear at $n=4$.

$\Rightarrow$ We find

$$X_\mu^{(1)} = \frac{\gamma_\mu^\perp}{2} = \hat{X}_\mu^{(1)}$$

$$X_\mu^{(2)} = \frac{1}{4}\sigma_{\mu\nu}^\perp D^\nu = \hat{X}_\mu^{(2)}$$

$$X_\mu^{(3)} = -\frac{1}{4}\sigma_{\mu\nu}^\perp D^\nu iv\cdot D = \hat{X}_\mu^{(3)}\,,$$

$$X_\mu^{(4)} = \underbrace{\sigma_{\mu\nu}^\perp D^\nu \left[\frac{1}{4}(iv\cdot D)^2 - \frac{1}{16}(iD_\perp)^2\right]}_{=\hat{X}_\mu^{(4)}} + \frac{g}{32} iD_\perp^\nu \left(-iG_{\mu\nu}^\perp + \sigma_{\mu\sigma}^\perp G_\nu^{\perp\sigma} - \sigma_{\nu\sigma}^\perp G_\mu^{\perp\sigma}\right)$$

$\Rightarrow$ Find extra terms, whichh are needed to obtain the correct $\Gamma(v, iD)$.