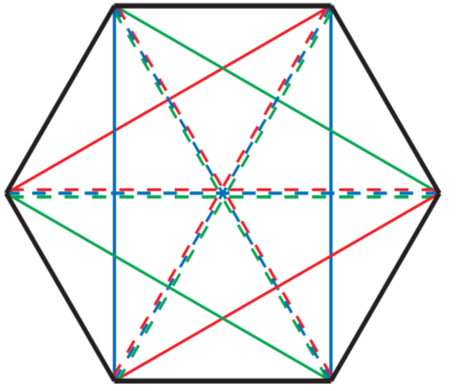
Scattering in N=4 super-Yang-Mills theory and the multi-Regge-Limit



Lance Dixon (SLAC)

with

J. Drummond and J. Henn, arXiv:1108.4461, 1111.1704

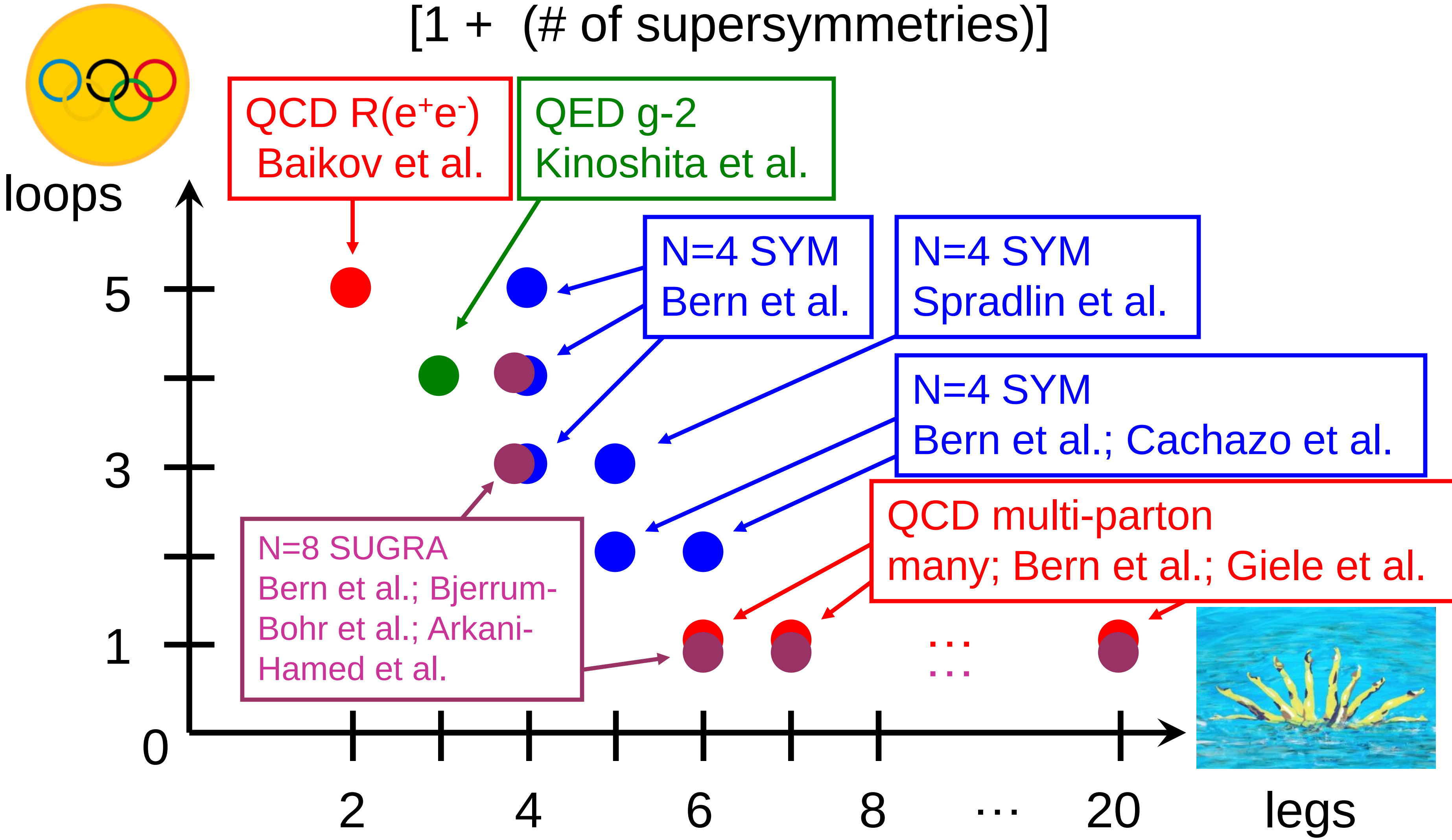
C. Duhr and J. Pennington, arXiv:1207.0186

also: J. Pennington, arXiv:1209.5357

Fermilab Theory Seminar

January 31, 2013

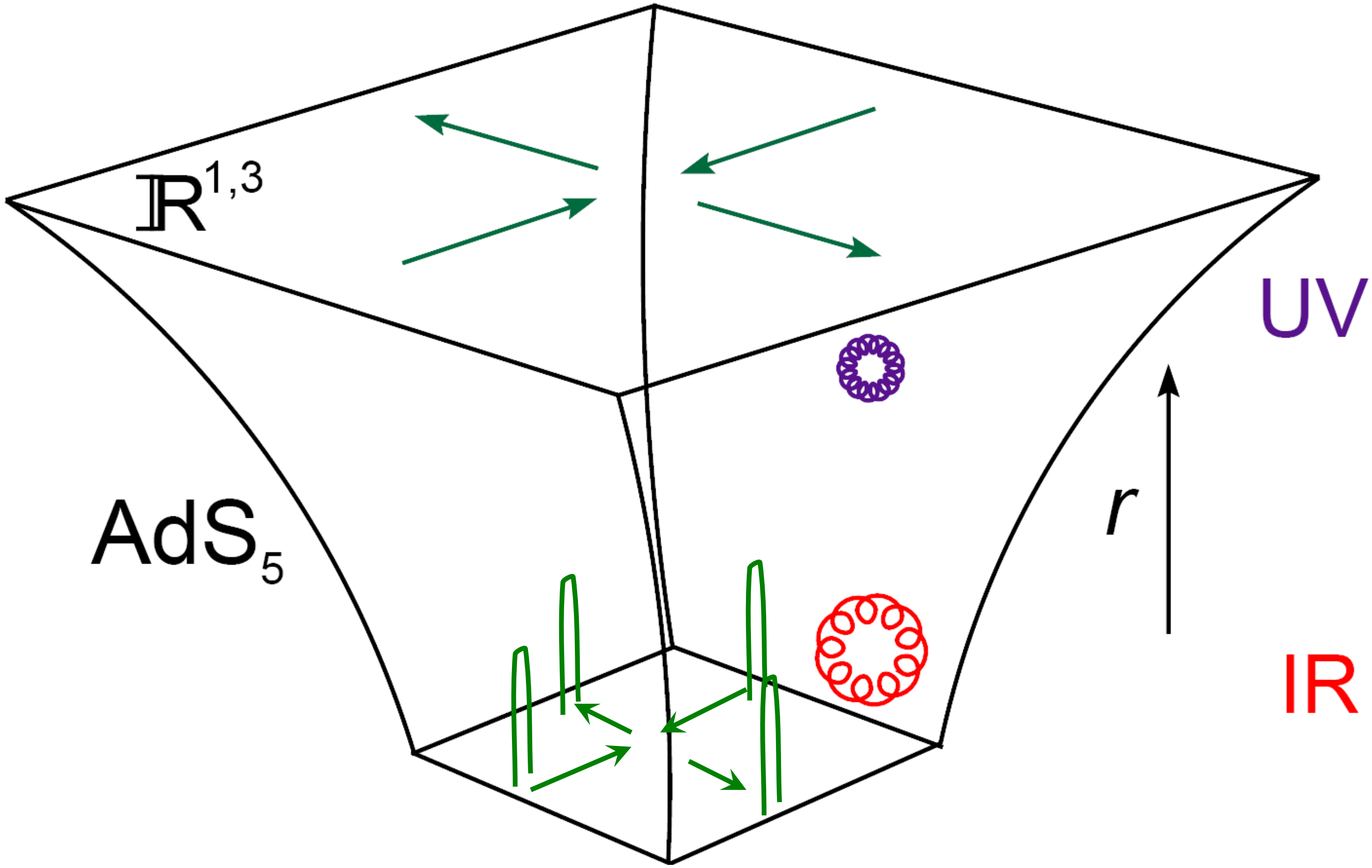
$$\text{Difficulty} = \frac{[\#\text{ of loops}] \times [\#\text{ of legs}]}{[1 + (\#\text{ of supersymmetries})]} \times [\text{style factor}]$$



Scattering amplitudes in planar N=4 Super-Yang-Mills

- Planar (large N_c) N=4 SYM is a 4-dimensional analog of QCD, (potentially) solvable to all orders in $\lambda = g^2 N_c$
- It can teach us what types of mathematical structures will enter multi-loop QCD amplitudes
- Its amplitudes have remarkable hidden symmetries
- In strong-coupling, large λ limit, AdS/CFT duality maps the problem into weakly-coupled gravity/semi-classical strings moving on $AdS_5 \times S^5$

AdS/CFT in one picture



Strong coupling and soap bubbles

Alday, Maldacena, 0705.0303

- Use AdS/CFT to compute scattering amplitude
- High energy scattering in string theory semi-classical:

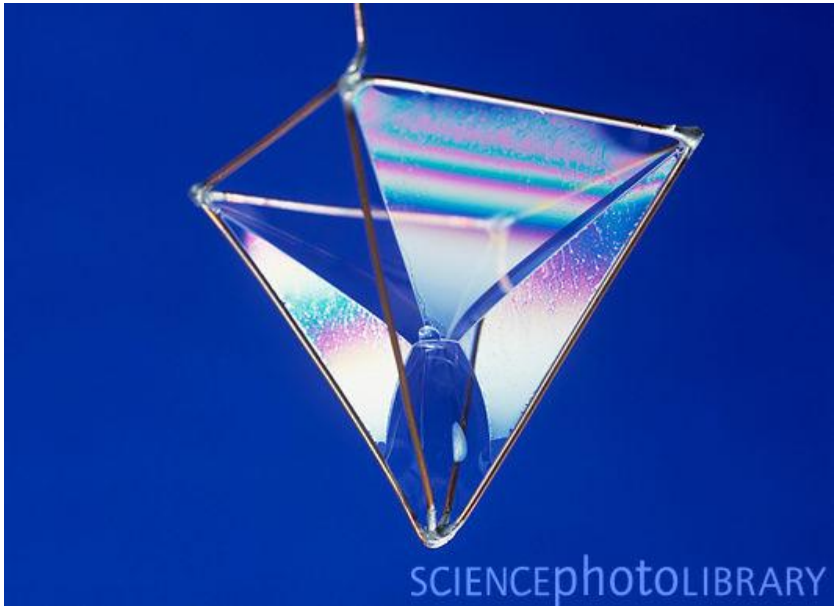
2-d string world-sheet is stretched tightly;
classical solution minimizes area

Gross, Mende (1987,1988)

Classical action imaginary
→ exponentially suppressed
tunnelling configuration

$$A_n \sim \exp[-\sqrt{\lambda} S^{\rm E}_{\rm cl}]$$

Same "wire frame" also
at weak coupling
(polygonal Wilson loop)



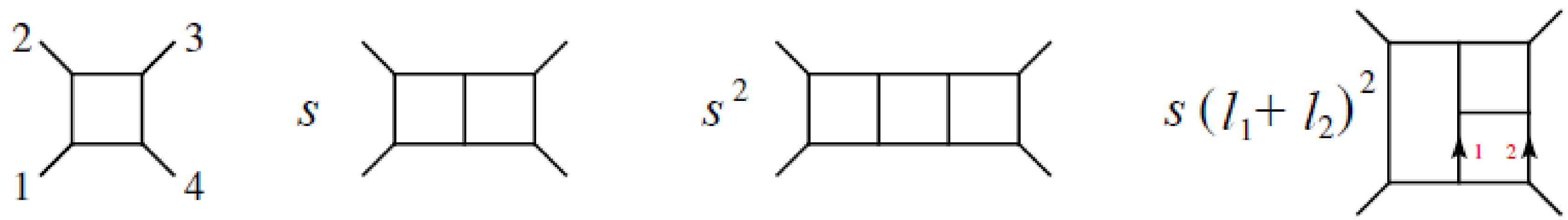
Solving planar N=4 SYM scattering

- Exact exponentiation of 4 & 5 gluon amplitudes
- Dual (super)conformal invariance
- Amplitudes equivalent to Wilson loops
- “Soap bubbles” for strong coupling limit

Can these structures be used to solve exactly in coupling for all planar N=4 SYM amplitudes? What is the first nontrivial case to solve?

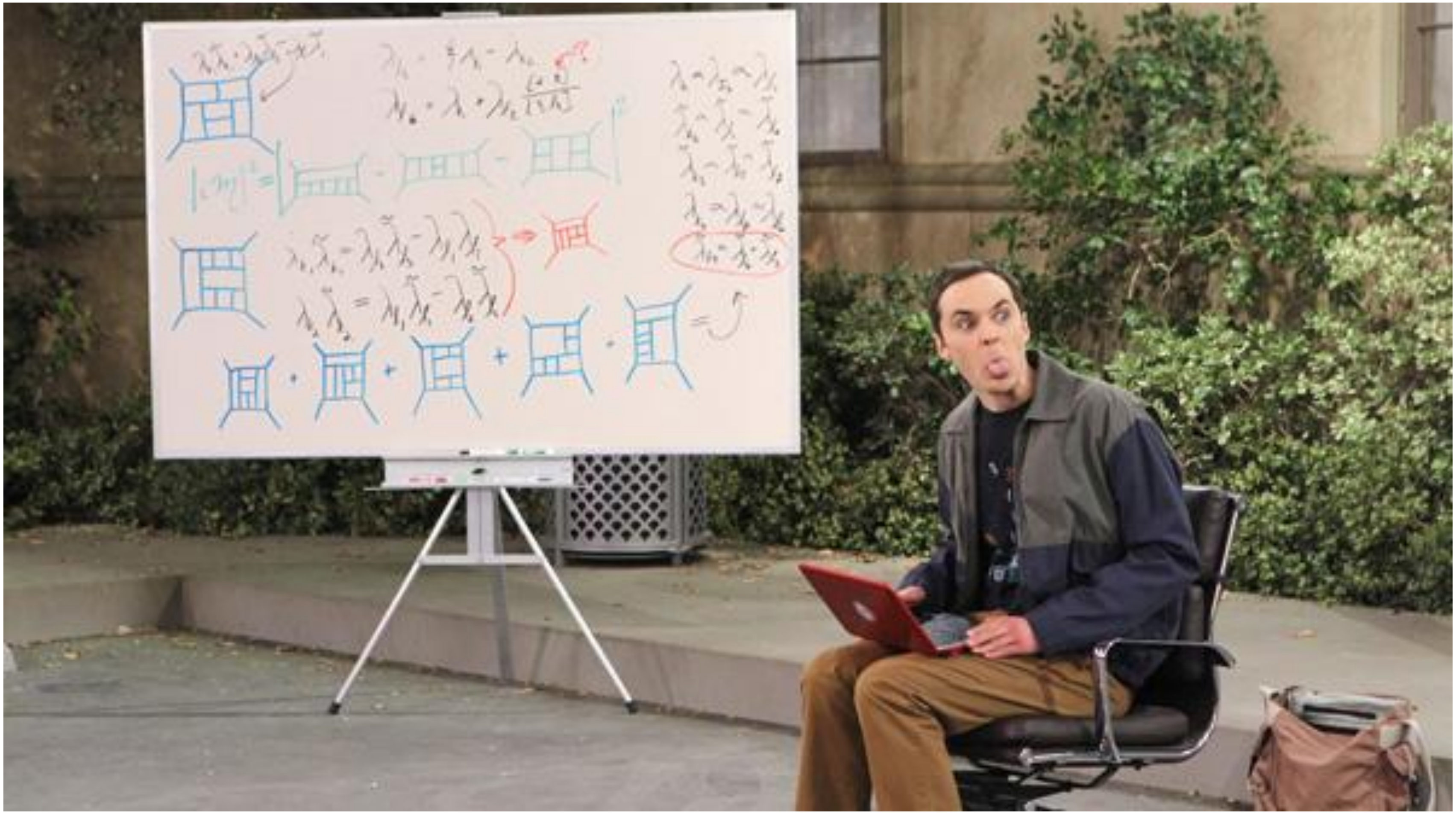
Integrands

- Using unitarity and other techniques, one can construct loop integrands for planar (or nonplanar) N=4 SYM amplitudes **very efficiently** without ever evaluating a single Feynman diagram.
- Planar 4-point amplitude especially simple.
- **1, 2 and 3 loop** integrands built out of:



(1) (2) (3)a (3)b

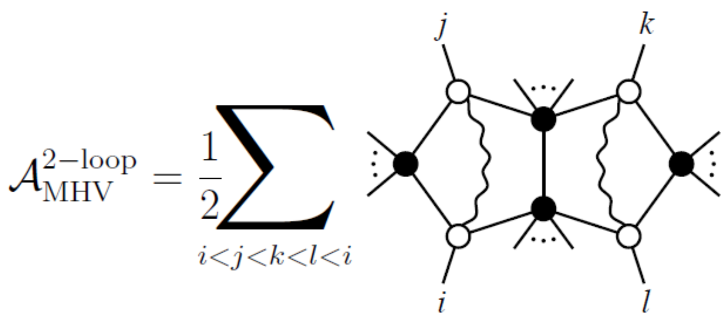
Even Sheldon Cooper can do it

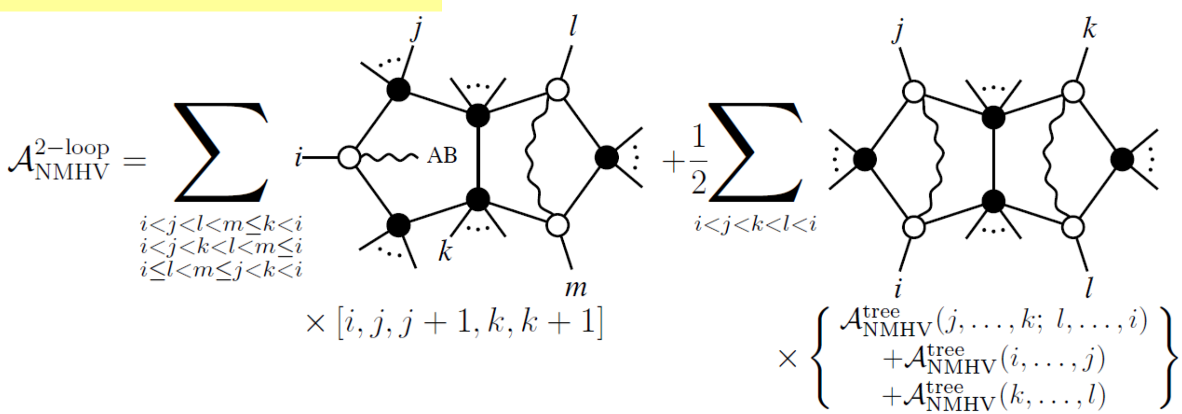


All planar N=4 SYM integrands

Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka, 1008.2958, 1012.6032

- All-loop BCFW recursion relation for integrand ☺
- Manifest Yangian invariance (huge group containing dual conformal symmetry).
- Multi-loop integrands written in terms of "momentum-twistors".
- Still have to do integrals over the loop momentum ☹





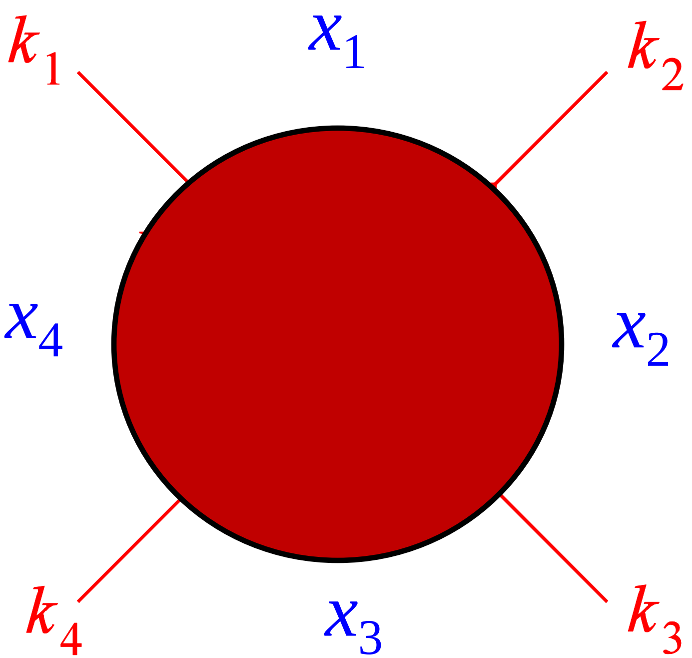
Do we actually need integrands?

In many cases, symmetries and other constraints on the multi-loop planar N=4 SYM amplitude are so powerful that we don't even need to know the integrand at all! ☺

Dual conformal invariance

Broadhurst (1993); Lipatov (1999); Drummond, Henn, Smirnov, Sokatchev, hep-th/0607160

Conformal symmetry acting in momentum space, on dual or sector variables x_i : $k_i = x_i - x_{i+1}$



invariance under inversion:

$$x_i^\mu \rightarrow \frac{x_i^\mu}{x_i^2}$$

$$x_{ij}^2 \rightarrow \frac{x_{ij}^2}{x_i^2 x_j^2}$$

Dual conformal constraints

- Symmetry fixes form of amplitude, up to functions of dual conformally invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

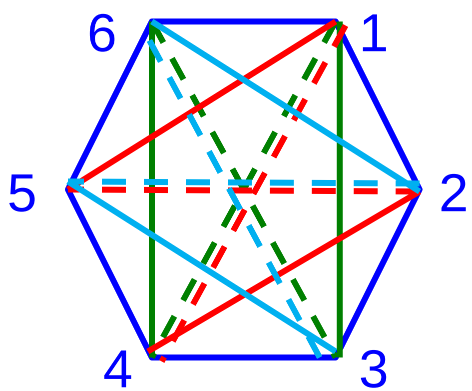
- Because $x_{i-1,i}^2 = k_i^2 = 0$ there are no such variables for n = 4,5
- Amplitude fixed to BDS ansatz:

$$\mathcal{A}_{4,5}(\epsilon; s_{ij}) \;=\; \mathcal{A}_{4,5}^{\text{BDS}}(\epsilon; s_{ij})$$

For $n = 6$, precisely 3 ratios:

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

+ 2 cyclic perm's



$$\mathcal{A}_6(\epsilon; s_{ij}) \;=\; \mathcal{A}_6^{\text{BDS}}(\epsilon; s_{ij}) \exp[R_6(u_1, u_2, u_3)]$$

MHV (--++++)

Formula for $R_6^{(2)}(u_1,u_2,u_3)$

- First found analytically from Wilson loop integrals

Del Duca, Duhr, Smirnov, 0911.5332, 1003.1702

17 pages of "Goncharov polylogarithms"

- Simplified to a few classical polylogarithms using symbology

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

$$L_4(x^+,x^-) = \frac{1}{8!!}\log(x^+x^-)^4$$
$$+\sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-))$$

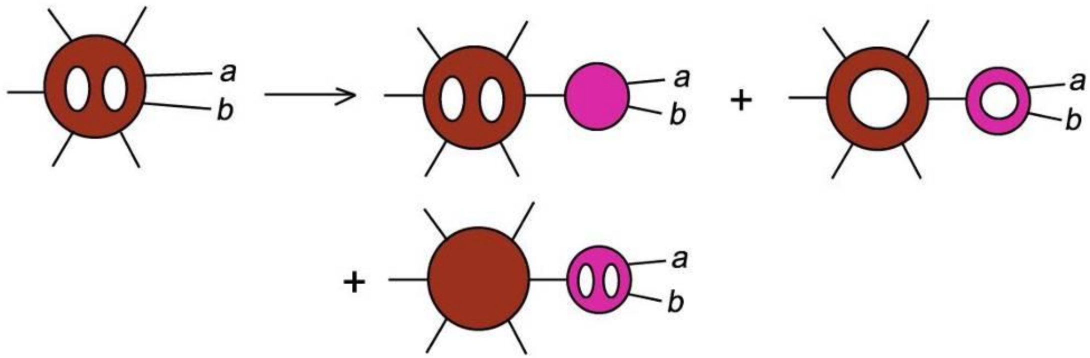
$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\,\mathrm{Li}_n(1/x)\right)$$
$$J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-))$$

$$x_i^\pm = u_i x^\pm, \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3} \qquad \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$$

Wilson loop OPEs

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; GMSV, 1010.5009, 1102.0062

- Remarkably, $R_6^{(2)}(u_1, u_2, u_3)$ can be recovered directly from analytic properties, using “near collinear limits”



- Wilson-loop equivalence → this limit is controlled by an operator product expansion (OPE)
- Now we can go (most of the way) to 3 and 4 loops, by combining the OPE expansion with symbology

Symbology?

• Multi-loop integrals generate complicated transcendental functions, iterated integrals, generalizations of the ordinary polylogarithm:

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \qquad \mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)$$

• Symbol $\mathcal{S}[\,f\,]$ of function f remembers "important" properties of f: derivatives and locations of branch cuts. It forgets other properties, like precise integration contours and numerical values; reconstruct them later.

• Trivializes complicated polylogarithmic identities.

Iterated differentiation

- A pure function $f^{(k)}$ of transcendental degree k is a linear combination of k-fold iterated integrals, with constant (rational) coefficients.
- Can also add terms like $\zeta(p) \times f^{(k-p)}$
- Derivatives of $f^{(k)}$ can be written as

$$d\, f^{(k)} = \sum_r f_r^{(k-1)}\, d \log \phi_r$$

for a finite set of algebraic functions ϕ_r

- Define the symbol $\mathcal{S}$ [Goncharov, 0908.2238] recursively in k:

$$\mathcal{S}(f^{(k)}) = \sum_r \mathcal{S}(f_r^{(k-1)}) \otimes \phi_r$$

Polylog examples

- By definition, $\mathcal{S}[\ln x] = x \qquad \mathcal{S}[\ln(1-x)] = 1-x$
- If derivative is known, symbol is known:

$$\frac{d}{dx}\mathrm{Li}_2(x) = -\frac{\ln(1-x)}{x} \quad \Rightarrow \quad \mathcal{S}[\mathrm{Li}_2(x)] = -[(1-x)\otimes x]$$

$$\frac{d}{dx}\mathrm{Li}_n(x) = \frac{\mathrm{Li}_{n-1}(x)}{x} \quad \Rightarrow \quad \mathcal{S}[\mathrm{Li}_n(x)] = -[(1-x)\otimes \underbrace{x\otimes\ldots\otimes x}_{n-1}]$$

- Symbols of products are mergings of symbols of factors:

$$\mathcal{S}[\ln(x)\,\ln(1-x)] = x\otimes(1-x) \;+\; (1-x)\otimes x$$

$$\begin{aligned}
&\mathcal{S}[\mathrm{Li}_2(x)\,\mathrm{Li}_2(y)] \\
&= (1-x)\otimes x\otimes(1-y)\otimes y \;+\; (1-x)\otimes(1-y)\otimes x\otimes y \\
&+ (1-x)\otimes(1-y)\otimes y\otimes x \;+\; (1-y)\otimes(1-x)\otimes x\otimes y \\
&+ (1-y)\otimes(1-x)\otimes y\otimes x \;+\; (1-y)\otimes y\otimes(1-x)\otimes x
\end{aligned}$$

Polylog identities at symbol level

- A well-known identity:

$$\text{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x \ln(1-x) - \text{Li}_2(x)$$

- Take symbol of it:

$$\mathcal{S}[\text{Li}_2(1-x)] = \mathcal{S}[\pi^2/6] - \mathcal{S}[\ln(x)\ \ln(1-x)] - \mathcal{S}[\text{Li}_2(x)]$$

$$-x \otimes (1-x) = 0 \quad - \ x \otimes (1-x) - (1-x) \otimes x \quad + \ (1-x) \otimes x$$

- Biggest virtue of symbol: Transforms all identities between multi-variable transcendental functions into simple algebraic identities

Elementary symbol properties

- **Factorization:**

$$\ldots \otimes xy \otimes \ldots \;=\; \ldots \otimes x \otimes \ldots \;+\; \ldots \otimes y \otimes \ldots$$

- **Integrability:**

Not every (multi-variable) symbol is a function

$$\mathcal{S}[\ln(x)\ln(y)] = x \otimes y \;+\; y \otimes x$$

but no function has symbol $\quad x \otimes y \;-\; y \otimes x$

- Integrability test [Goncharov; GMSV, 1102.0062] :

$$\phi_1 \otimes \ldots \otimes \phi_i \otimes \phi_{i+1} \otimes \ldots \otimes \phi_k$$

$$\rightarrow \; d\ln\phi_i \wedge d\ln\phi_{i+1} \;\; \phi_1 \otimes \ldots \otimes \ldots \otimes \phi_k$$

$\Rightarrow \; 0$ for symbols of functions

Symbol entries for $R_6^{(L)}(u_1,u_2,u_3)$

• Based on $R_6^{(2)}$, we assume entries can all be drawn from this set:

$$\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

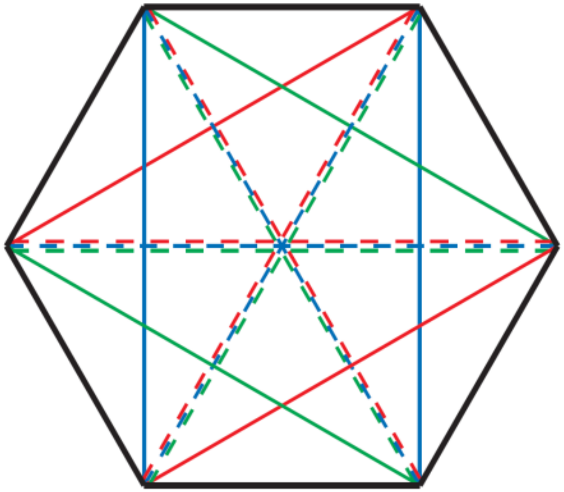
with

$$y_u \equiv \frac{u - z_+}{u - z_-} \quad + \quad \text{perms}$$

$$z_\pm = \frac{1}{2}\left[-1 + u + v + w \pm \sqrt{\Delta}\right]$$

$$\Delta = (1 - u - v - w)^2 - 4uvw$$

y_i depend on u_i via square roots



$\mathcal{S}[\ R_6^{(2)}(u,v,w)\ ]$ in these variables

GSVV, 1006.5703

$$
\begin{aligned}
-8\,\mathcal{S}[R_6^{(2)}] \;=\;& u \otimes (1-u) \otimes \frac{u}{(1-u)^2} \otimes \frac{u}{1-u} \\
&+ 2\,(u \otimes v + v \otimes u) \otimes \frac{w}{1-v} \otimes \frac{u}{1-u} \\
&+ 2\,v \otimes \frac{w}{1-v} \otimes u \otimes \frac{u}{1-u} \\
&+ u \otimes (1-u) \otimes y_u y_v y_w \otimes y_u y_v y_w \\
&- 2\,u \otimes v \otimes y_w \otimes y_u y_v y_w
\end{aligned}
$$

$$+\ 5 \text{ permutations of } (u, v, w)$$

First entry

• Always drawn from $\{u, v, w\}$ GMSV, 1102.0062

because first entry controls branch-cut location

• Only massless particles

→ all cuts start at origin in $s_{i,i+1}$, $s_{i,i+1,i+2}$

→ Branch cuts all start from 0 or ∞ in

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}^2 s_{45}^2}{s_{123}^2 s_{345}^2}$$

Final entry

- Always drawn from $\left\{ \frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w \right\}$

- Restriction characteristic of many Feynman integrals

Arkani-Hamed et al., 1108.2958; Drummond, Henn, Trnka 1010.3679; LD, Drummond, Henn, 1104.2787, V. Del Duca et al., 1105.2011

- Same condition also found via dual superconformal anomaly equation for supersymmetric Wilson loops

Caron-Huot, 1105.5606; Caron-Huot, He, 1112.1060

Ansatz for $\mathcal{S}[\, R_6^{(3)}(u,v,w) \,]$

$$
\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}
\otimes
\begin{Bmatrix} u \\ v \\ w \\ 1-u \\ 1-v \\ 1-w \\ y_u \\ y_v \\ y_w \end{Bmatrix}
\otimes
\begin{Bmatrix} u \\ v \\ w \\ 1-u \\ 1-v \\ 1-w \\ y_u \\ y_v \\ y_w \end{Bmatrix}
\otimes
\begin{Bmatrix} u \\ v \\ w \\ 1-u \\ 1-v \\ 1-w \\ y_u \\ y_v \\ y_w \end{Bmatrix}
\otimes
\begin{Bmatrix} u \\ v \\ w \\ 1-u \\ 1-v \\ 1-w \\ y_u \\ y_v \\ y_w \end{Bmatrix}
\otimes
\begin{Bmatrix} \frac{u}{1-u} \\ \frac{v}{1-v} \\ \frac{w}{1-w} \\ y_u \\ y_v \\ y_w \end{Bmatrix}
$$

$3 \times 9^4 \times 6 = 118098$ parameters before imposing any constraints

Generic Constraints

- **Integrability** (immediately forbids y_u, y_v, y_w from second entry)
- S_3 permutation **symmetry** in $\{u, v, w\}$
- Even under "**parity**": every term must have an even number of y_i – 0, 2 or 4

$$i\sqrt{\Delta} \leftrightarrow -i\sqrt{\Delta}$$
$$z_+ \leftrightarrow z_-$$
$$y_i \leftrightarrow 1/y_i$$

- Vanishing in **collinear** limit $v \to 0$

$$y_u \to \frac{u}{1-w} \qquad y_v \to \frac{v(1-u)(1-w)}{(1-u-w)^2} \qquad y_w \to \frac{w}{1-u}$$

followed by $w \to 1-u$

- These 4 constraints reduce 118,098 → 35 free parameters

OPE Constraint

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; GMSV, 1010.5009; 1102.0062

- $R_6^{(L)}(u,v,w)$ vanishes in the collinear limit,

$$v = 1/\cosh^2\tau \rightarrow 0 \qquad \tau \rightarrow \infty$$

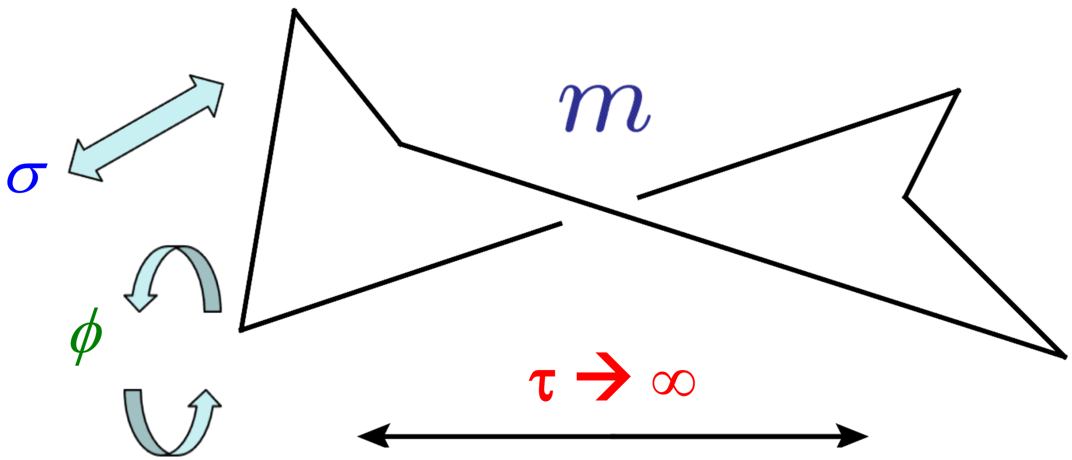
In the **near**-collinear limit, its behavior is described by an Operator Product Expansion, with generic form

$$R_6^{(L)}(u,v,w) = R_6^{(L)}(\tau,\sigma,\phi) \sim \int dm\ C_m(g)\ \exp[-E_m(g)\tau]$$

$$u = \frac{e^{\sigma}\sinh\tau\tanh\tau}{2(\cosh\sigma\ \cosh\tau + \cos\phi)}$$

$$v = \frac{1}{\cosh^2\tau}$$

$$w = u\,e^{-2\sigma}$$



OPE Constraint (cont.)

- As $\tau \rightarrow \infty$, $v = 1/\cosh^2\tau$ $\rightarrow$ $\tau^{L-1} \sim [\ln v]^{L-1}$
- **Leading τ^{L-1} dependence of $R_6^{(L)}$ needs only one-loop anomalous dimension** $E_n^{(1)} \sim \gamma_m(p)$
- Extract from symbol: only terms with L-1 leading v entries

$$\underbrace{v \otimes \ldots \otimes v}_{\text{clip } L-1 \text{ entries}} \otimes \underbrace{\ldots}_{\text{keep } L+1 \text{ entries}}$$

$$\Delta_v^{L-1} R_6^{(L)} \propto \int dp e^{-ip\sigma} \Big[\sum_{m=1}^{\infty} \frac{[\gamma_{m+2}(p)]^{L-1} \cos(m\phi)}{p^2+m^2} + \sum_{m=2}^{\infty} \frac{[\gamma_{m-2}(p)]^{L-1} \cos((m-2)\phi)}{p^2+(m-2)^2} \Big] \times \mathcal{C}_m(p) \mathcal{F}_{m/2,p}(\tau)$$

where $\gamma_m(p) = \psi(\frac{m+ip}{2}) + \psi(\frac{m-ip}{2}) - 2\,\psi(1)$

Basso 1010.5237

OPE constraint on symbol

- $\Delta_v^2 R_6^{(3)}$ still complicated. Simplify by acting with 2 different differential operators (easily applied to symbol):

1) $$\mathcal{S}[\,\mathcal{D}_+\mathcal{D}_-\Delta_v^2 R_6^{(3)}(u,v,w)\,] = 0$$

annihilators of conformal blocks are [GMSV, 1102.0062]:

$$\begin{aligned}\mathcal{D}_\pm = \frac{4}{1-v}\Big[&-z_\pm u\partial_u - (1-v)v\partial_v - z_\pm w\partial_w \\ &+ (1-u)vu\partial_u u\partial_u + (1-v)^2 v\partial_v v\partial_v + (1-w)vw\partial_w w\partial_w \\ &+ (-1+u-v+w)((1-v)u\partial_u v\partial_v - vu\partial_u w\partial_w + (1-v)v\partial_v w\partial_w)\Big]\end{aligned}$$

2) $$\mathcal{S}[\Box\Delta_w^2\Delta_v^2 R_6^{(3)}(u,v,w)\,] \propto \mathcal{S}[\Box\Delta_w\Delta_v R_6^{(2)}(u,v,w)\,] = \frac{w(1-u+v-w)}{(1-v)(1-w)}$$

$$\begin{aligned}\Box &= -(\partial_\sigma^2 + \partial_\phi^2) \\ &= \frac{4uw}{1-v}[u\partial_u + w\partial_w - (1-u)\partial_u u\partial_u - (1-w)\partial_w w\partial_w \\ &\qquad + (1-u-v-w+2uw)\partial_u\partial_w]\end{aligned}$$

Solution to Constraints

• OPE constraints mutually consistent, reduce symbol ansatz to just 2 parameters:

$$\mathcal{S}[R_6^{(3)}] = \mathcal{S}[X] + \alpha_1\,\mathcal{S}[f_1] + \alpha_2\,\mathcal{S}[f_2]$$

• Later Caron-Huot, He [1112.1060] found

$$\alpha_1 = -\frac{3}{8} \qquad \alpha_2 = \frac{7}{32}$$

Reconstructing functions

- $\mathcal{S}[f_1]$ is only made from $\{u, v, w, 1-u, 1-v, 1-w\}$ and is so simple we can integrate it in terms of [harmonic] polylogarithms of a single variable:

$$f_1(u,v,w) = h(u)h(v) + h(u)h(w) + h(v)h(w) + k(u) + k(v) + k(w)$$

$$h(u) = \frac{1}{3}\ln^3 u + \ln u \, \mathrm{Li}_2(1-u) - \mathrm{Li}_3(1-u) - 2\,\mathrm{Li}_3(1-1/u)$$

$$\begin{aligned} k(u) = & -\ln^3 u \, H_3 + \frac{3}{2}\ln^2 u \, (H_4 - H_{2,2} - 4\,H_{3,1}) \\ & - \log u \, (H_{2,3} - 6\,H_{4,1} + H_{2,1,2} + 6\,H_{2,2,1} + 18\,H_{3,1,1}) \\ & + 3\,H_{2,4} + 4\,H_{3,3} + 3\,H_{4,2} + H_{2,1,3} - H_{2,2,2} - 2\,H_{2,3,1} \\ & - 2\,H_{3,1,2} + 9\,H_{4,1,1} - 2\,H_{2,1,2,1} - 9\,H_{2,2,1,1} - 24\,H_{3,1,1,1} \end{aligned}$$

Reconstructing functions (cont.)

- Terms in $\mathcal{S}[f_2]$ can contain y_i in the form

$$a_1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes y_1 \otimes y_2$$

with

$$a_i \in \{u, v, w, 1-u, 1-v, 1-w\}$$
$$y_i \in \{y_u, y_v, y_w\}$$

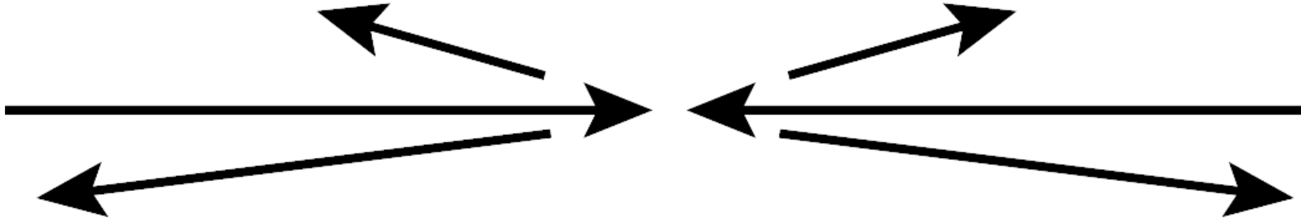
- f_2 is not classical polylog, but a 1-d integral over them
- Terms in $\mathcal{S}[X]$ can have up to four y_i

$$a_1 \otimes a_2 \otimes y_1 \otimes y_2 \otimes y_3 \otimes y_4$$

X can be broken up into functions which are at worst 2-d integrals over classical polylogs (in progress).

The multi-Regge limit

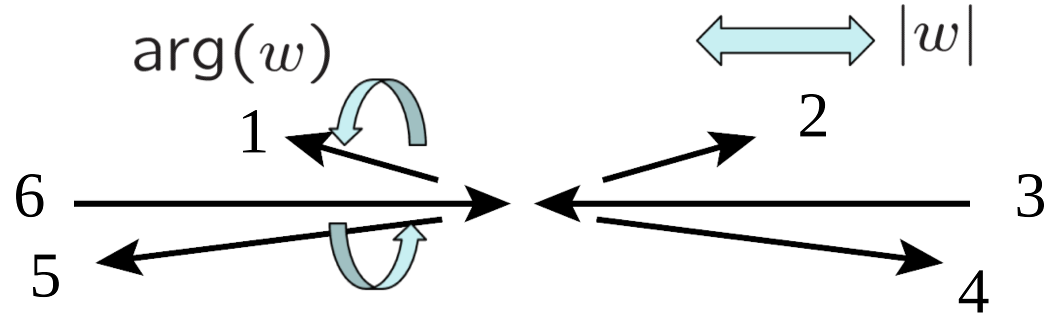
• To simplify the problem enough to go to very high loop order, we take the limit of multi-Regge kinematics (MRK): large rapidity separations between the 4 final-state gluons:



• Properties of planar N=4 SYM amplitude in this limit studied extensively already:

Bartels, Lipatov, Sabio Vera, 0802.2065, 0807.0894; Lipatov, 1008.1015;
Lipatov, Prygarin, 1008.1016, 1011.2673;
Bartels, Lipatov, Prygarin, 1012.3178, 1104.4709;
LD, Drummond, Henn, 1108.4461; Fadin, Lipatov, 1111.0782

Multi-Regge kinematics



$$u_1 = \frac{s_{12}^2 s_{45}^2}{s_{123}^2 s_{345}^2} \to 1$$

$$\frac{u_2}{1-u_1} \to x$$

$$\frac{u_3}{1-u_1} \to y$$

$$y_1 \to 1$$

$$y_2 \to \frac{1+w^*}{1+w}$$

$$y_3 \to \frac{(1+w)w^*}{w(1+w^*)}$$

A very nice change of variables [LP, 1011.2673] is to (w, w^*) :

$$x = \frac{1}{(1+w)(1+w^*)}$$

$$y = \frac{ww^*}{(1+w)(1+w^*)}$$

2 symmetries: conjugation $w \leftrightarrow w^*$
and inversion $w \leftrightarrow 1/w, \ \ w^* \leftrightarrow 1/w^*$

Physical 2→4 multi-Regge limit

- To get a nonzero result, for the physical region, one must first let $u_1 \to u_1 e^{-2\pi i}$, and extract one or two discontinuities → factors of $-2\pi i$.
- Then let $u_1 \to 1$. Bartels, Lipatov, Sabio Vera, 0802.2065, …

$$R_6^{(L)} \to (2\pi i) \sum_{n=0}^{L-1} \ln^n(1-u_1) \, [g_n^{(L)}(w,w^*) + 2\pi i \, h_n^{(L)}(w,w^*)]$$

imaginary part, from single discontinuity

real part, from double discontinuity

Simpler pure functions

$$R_6^{(L)} \to (2\pi i) \sum_{n=0}^{L-1} \ln^n(1-u_1) \, [g_n^{(L)}(w,w^*) + 2\pi i \, h_n^{(L)}(w,w^*)]$$

weight $2L$-n-1 $2L$-n-2

Symbol entries → $r_i \in \{w, 1+w, w^*, 1+w^*\}$

- Single-valued in $(w,w^*) = (-z,-\bar{z})$ plane

Precise class of functions defined by Brown:
F.C.S. Brown, C. R. Acad. Sci. Paris, Ser. I 338 (2004)

SVHPLs: $\mathcal{L}_{m_1,\ldots,m_w}(z,\bar{z}) \sim \sum_{i,j} c_{i,j} H_{\vec{m}_i}(z) H_{\vec{m}_j}(\bar{z})$

H = ordinary harmonic polylogarithms

Remiddi, Vermaseren, hep-ph/9905237

Harmonic Polylogarithms (HPLs)

Remiddi, Vermaseren, hep-ph/9905237

w = word formed from noncommuting letters x_0, x_1

$$H_{x_0 w}(z) = \int_0^z dz' \, \frac{H_w(z')}{z'} \qquad H_{x_1 w} = \int_0^z dz' \, \frac{H_w(z')}{1-z'}$$

Special cases:

$$H_e(z) = 1 \qquad H_{x_0^n}(z) = \frac{1}{n!} \log^n z$$

Shuffle identity:

$$H_{w_1}(z) \, H_{w_2}(z) = \sum_{w \in w_1 \sqcup\!\sqcup \, w_2} H_w(z)$$

Shorthand example:

$$H_w(z) = H_{x_0 x_0 x_1 x_0 x_1}(z) = H_{0,0,1,0,1}(z) = H_{3,2}(z)$$

Brown construction of SVHPLs

$$\frac{\partial}{\partial z}\mathcal{L}_{x_0 w}(z,\bar{z}) = \frac{\mathcal{L}_w(z,\bar{z})}{z} \qquad \frac{\partial}{\partial z}\mathcal{L}_{x_1 w}(z,\bar{z}) = \frac{\mathcal{L}_w(z,\bar{z})}{1-z}$$

Special cases: $\mathcal{L}_e = 1 \qquad \mathcal{L}_{x_0^n} = \frac{1}{n!}\log^n|z|^2$

Shuffle identity: $\mathcal{L}_{w_1}\mathcal{L}_{w_2} = \sum_{w\in w_1 ш w_2} \mathcal{L}_w$

Main formula:

$$\mathcal{L}(z,\bar{z}) = L_X(z)\tilde{L}_Y(\bar{z}) \equiv \sum_{w\in X^*}\mathcal{L}_w(z,\bar{z})w$$

$$L_X(z) = \sum_{w\in X^*} H_w(z)w \qquad \tilde{L}_Y(\bar{z}) = \sum_{w\in Y^*} H_{\phi(w)}(\bar{z})\tilde{w}$$

word reversal operator " ~ " $\qquad \phi$ renames y to x

The y alphabet

- Related to the x alphabet using the Drinfel'd associator:

$$Z(x_0, x_1) = \sum_{w \in X^*} \zeta(w) w$$

and definitions

$$y_0 = x_0$$
$$\tilde{Z}(y_0, y_1) y_1 \tilde{Z}(y_0, y_1)^{-1} = Z(x_0, x_1)^{-1} x_1 Z(x_0, x_1)$$

$$\rightarrow \quad \begin{aligned} y_1 = \; & x_1 + \zeta_3 (2x_0x_0x_1x_1 - 4x_0x_1x_0x_1 + 2x_0x_1x_1x_1 \\ & + 4x_1x_0x_1x_0 - 6x_1x_0x_1x_1 - 2x_1x_1x_0x_0 \\ & + 6x_1x_1x_0x_1 - 2x_1x_1x_1x_0) \\ & + \ldots \end{aligned}$$

Example: $$\mathcal{L}_{0,0,1,1}(z, \bar{z}) = H_{0,0,1,1} + \overline{H}_{1,1,0,0} + H_{0,0,1} \overline{H}_1 + H_0 \overline{H}_{1,1,0} + H_{0,0} \overline{H}_{1,1} - 2\zeta_3 \overline{H}_1$$

Z_2 x Z_2 symmetry

- $z \leftrightarrow \bar{z}$

$$L_w(z) = \frac{1}{2}\left(\mathcal{L}_w(z) - (-1)^{|w|}\,\mathcal{L}_w(\bar{z})\right)$$

$$\overline{L}_w(z) = \frac{1}{2}\left(\mathcal{L}_w(z) + (-1)^{|w|}\mathcal{L}_w(\bar{z})\right)$$

reducible to products of lower weight

- $z \leftrightarrow 1/z$

$$L_w(z) = (-1)^{|w|+d_w} L_w(\tfrac{1}{z})$$

reducible to products of lower weight

$$L_w^{\pm}(z) \equiv \frac{1}{2}\left[L_w(z) \pm L_w(\tfrac{1}{z})\right]$$

Keep the irreducible one

MRK Master Formula:
factorization in moment space

Fadin, Lipatov, 1111.0782

$$e^{R+i\pi\delta}|_{\rm MRK} = \cos\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,|w|^{2i\nu}\,\Phi_{\rm Reg}(\nu,n)$$
$$\times\exp\left[-\omega(\nu,n)\left(\log(1-u_1)+i\pi+\frac{1}{2}\log\frac{|w|^2}{|1+w|^4}\right)\right]$$

BFKL eigenvalue

MHV impact factor

$$\omega(\nu,n) = -a(E_{\nu,n} + a\,E_{\nu,n}^{(1)} + a^2\,E_{\nu,n}^{(2)} + \cdots)$$

$$\Phi_{\rm Reg}(\nu,n) = 1 + a\Phi_{\nu,n}^{(1)} + a^2\Phi_{\nu,n}^{(2)} + a^3\Phi_{\nu,n}^{(3)} + \cdots$$

LL NLL NNLL NNNLL

Formula may get corrections beyond NLL

Evaluating the master formula

- Every $g_n^{(L)}(w,w^*)$ and $h_n^{(L)}(w,w^*)$

is a linear combination of a finite basis of SVHPLs.

- Evaluate ν integral by residues

  → master formula leads to double sum.

- Truncating double sum ←→ truncating power series in $(w,w^*) = (-z,-\bar{z})$ around origin.
- Match the two series to determine the coefficients in the linear combination.
- LL and NLL ω and Φ known Fadin, Lipatov 1111.0782

MHV LLA $g_{L\text{-}1}{}^{(L)}$ through 5 loops

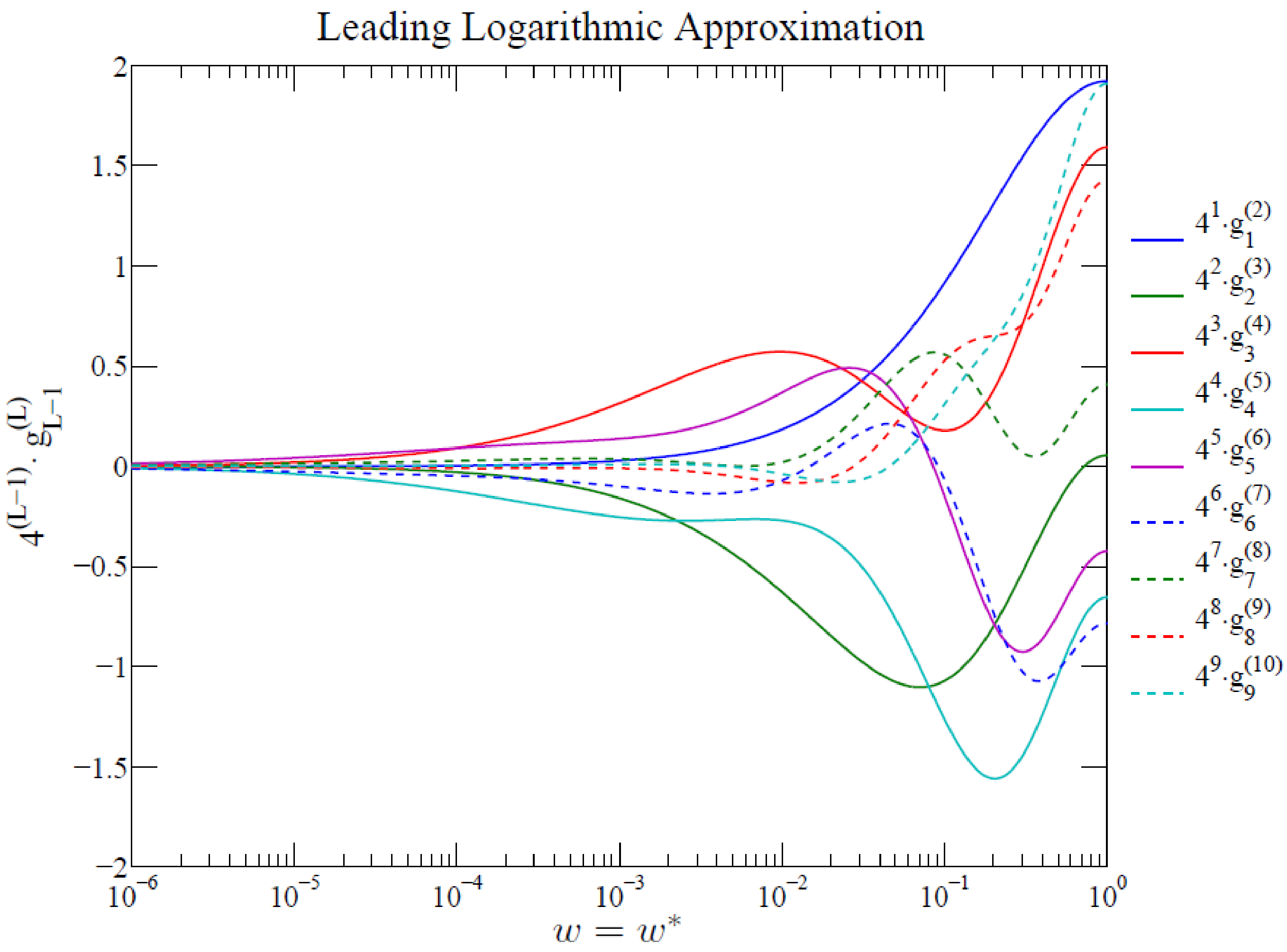
$$g_1^{(2)}(w,w^*) = \frac{1}{4}[L_1^+]^2 - \frac{1}{16}[L_0^-]^2 = \frac{1}{4}\ln|1+w|^2 \ln\frac{|1+w|^2}{|w|^2}$$

$$g_2^{(3)}(w,w^*) = -\frac{1}{8}L_3^+ + \frac{1}{12}[L_1^+]^3 \;=\; -\frac{1}{8}\Big[\mathrm{Li}_3(-w) + \mathrm{Li}_3(-w^*) - \frac{1}{2}\ln|w|^2(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)) + \ln|1+w|^2(\frac{2}{3}\ln^2|1+w|^2 - \ln|w|^2\ln|1+w|^2 + \frac{1}{4}\ln^2|w|^2)\Big]$$

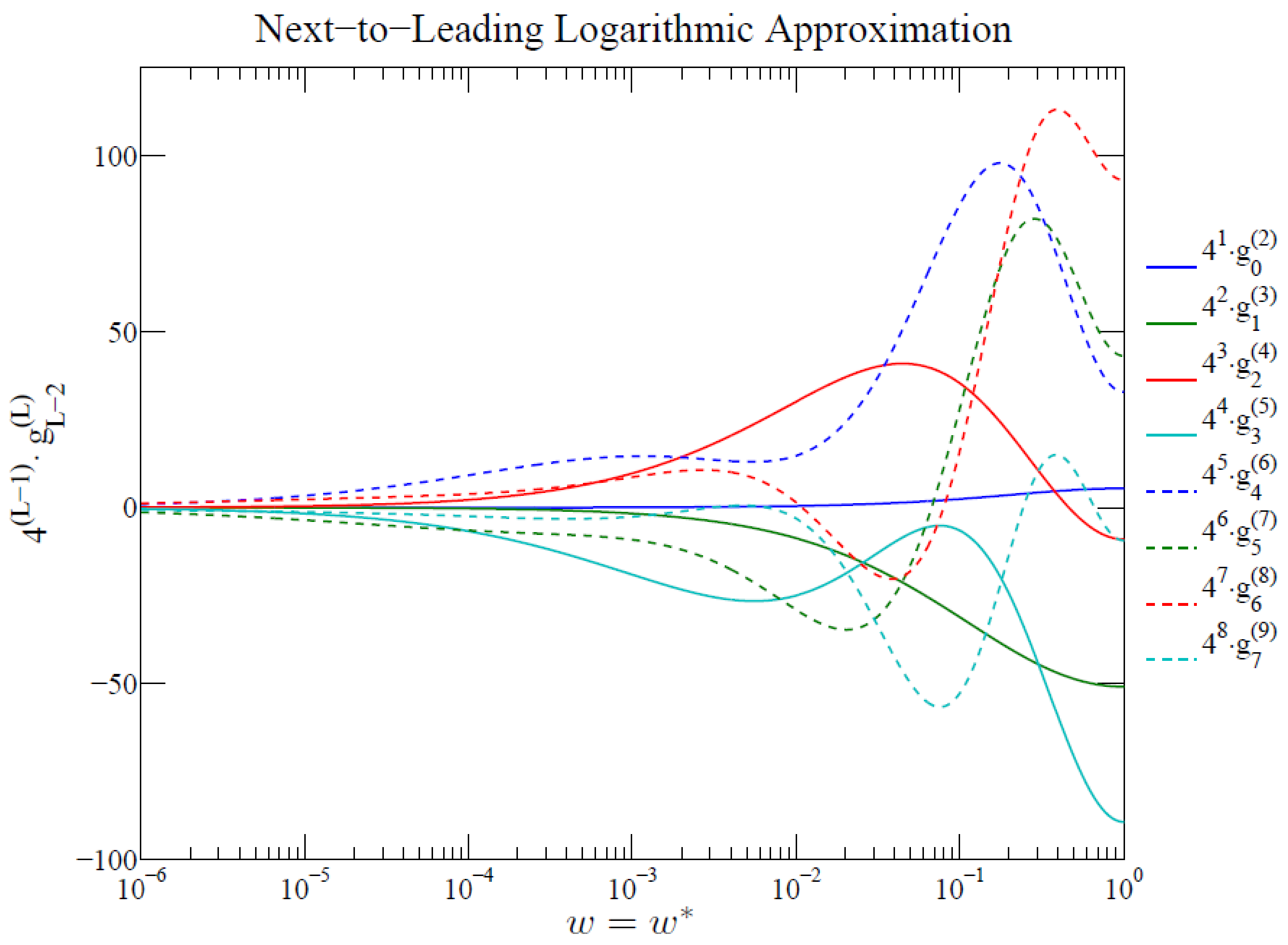
$$g_3^{(4)}(w,w^*) \;=\; \frac{1}{48}[L_2^-]^2 + \frac{1}{48}[L_0^-]^2[L_1^+]^2 + \frac{7}{2304}[L_0^-]^4 + \frac{1}{48}[L_1^+]^4 - \frac{1}{16}L_0^- L_{2,1}^- - \frac{5}{48}L_1^+ L_3^+ - \frac{1}{8}L_1^+\zeta_3\,,$$

$$g_4^{(5)}(w,w^*) \;=\; \frac{1}{96}[L_0^-]^2[L_1^+]^3 + \frac{17}{9216}L_1^+[L_0^-]^4 - \frac{5}{384}L_3^+[L_0^-]^2 + \frac{1}{24}[L_0^-]^2\zeta_3 - \frac{1}{12}[L_1^+]^2\zeta_3 + \frac{1}{240}[L_1^+]^5 - \frac{1}{24}L_0^- L_{2,1}^- L_1^+ + \frac{43}{384}L_5^+ + \frac{1}{8}L_{3,1,1}^+ + \frac{1}{12}L_{2,2,1}^+ - \frac{1}{24}L_3^+[L_1^+]^2\,,$$

MHV LLA $g_{L\text{-}1}^{(L)}$ through 10 loops



MHV NLLA $g_{L\text{-}2}^{(L)}$ through 9 loops



LLA to all orders

Pennington, 1209.5357

$$\eta = a \log(1 - u_1)$$

$$\rho(w) \equiv \mathcal{L}_w$$

$$R_6^{\rm MHV}\big|_{\rm LLA} = \frac{2\pi i}{\log(1-u_1)}\,\rho\Big(\mathcal{X}\,\mathcal{Z}^{\rm MHV} - \frac{1}{2}\,x_1\eta\Big)$$

$$\mathcal{X} = e^{\frac{1}{2}x_0\eta}\left[1 - x_1\left(\frac{e^{x_0\eta}-1}{x_0}\right)\right]^{-1},$$

$$\mathcal{Z}^{\rm MHV} = \frac{1}{2}\sum_{k=1}^{\infty}\left(x_1\sum_{n=0}^{k-1}(-1)^n x_0^{k-n-1}\sum_{m=0}^{n}\frac{2^{2m-k+1}}{(k-m-1)!}\,\mathfrak{Z}(n,m)\right)\eta^k$$

$$\exp\left[y\sum_{k=1}^{\infty}\zeta_{2k+1}x^{2k+1}\right] \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathfrak{Z}(n,m)\,x^n y^m$$

Matches series expansion through L = 14. All orders proof?

$\mathrm{N^kDLLA}$ limit to all orders

Pennington, 1209.5357

Collinear-Regge limit as $|w| \to 0$

$$R_6^{\rm MHV}|_{\rm LLA,\,coll.} = \frac{2\pi i}{\log(1-u_1)}(w+w^*)\sum_{k=0}^{\infty}\eta^{k+1}\, r_k^{\rm MHV}(\eta\log|w|)$$

$$r_k^{\rm MHV}(x) = \frac{1}{2}\delta_{0,k} + \sum_{n=0}^{k}\sum_{m=0}^{n}\sum_{j=k-m}^{2k-n-m}\frac{(-2)^{2m+j-k-1}}{(m+j-k)!}\,\mathfrak{Z}(n,m)\,x^{m-k+j/2}\,P_j^{(k-j-n,k-j-m)}(0)\,I_j(2\sqrt{x})$$

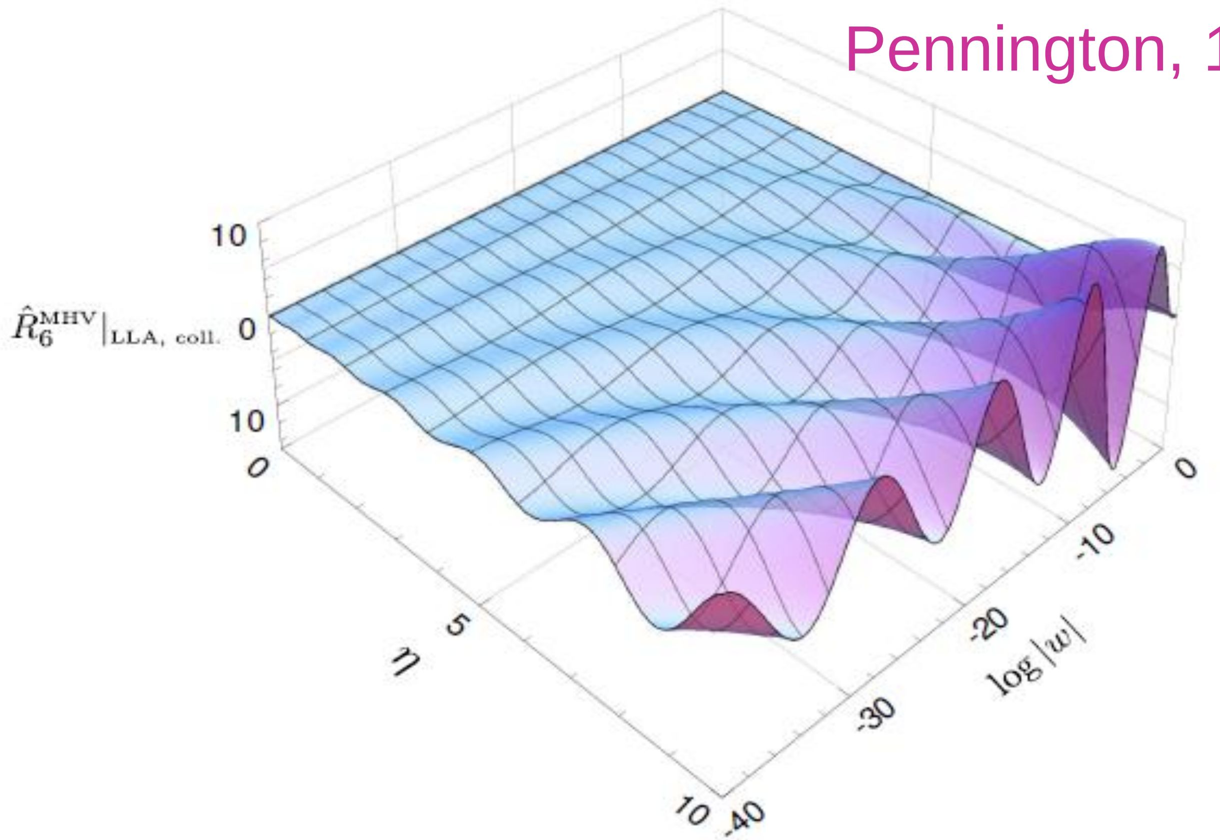
Answer a linear combination of modified Bessel functions I_j

$r_0(x)$ matches known DLLA result

Bartels, Lipatov, Prygarin, 1104.4709

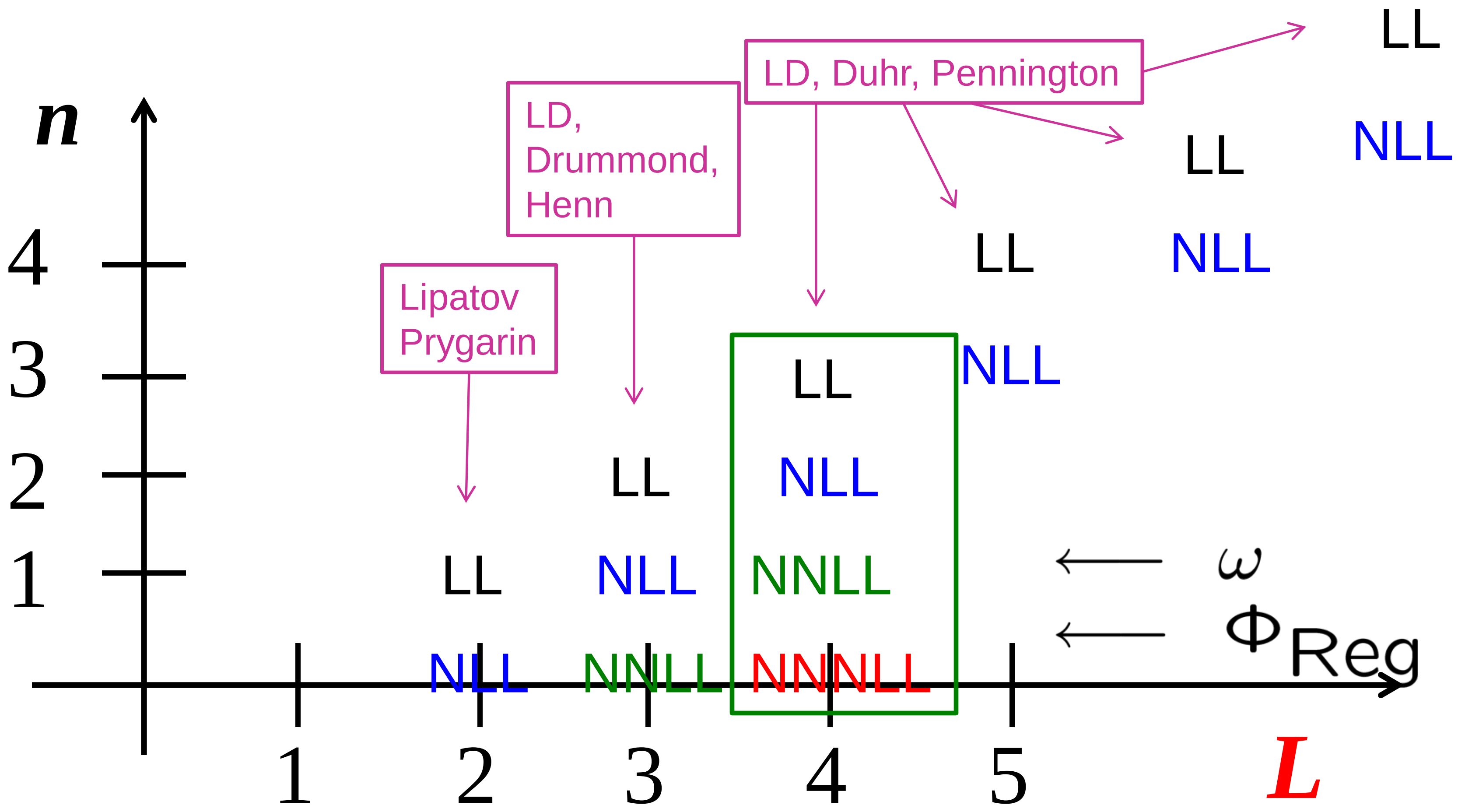
N^kDLLA limit to all orders

Pennington, 1209.5357



Should try to match to strong coupling results
Bartels, Kotanski, Schomerus, 1009.3938

Beyond NLL



(modulo some “beyond-the-symbol” constants starting at NNLL)

BFKL beyond NLL

- Using OPE and other constraints, determine the 4-loop remainder function in MRK, up to some unfixed constants. In particular, we get

$$g_1^{(4)}(w, w^*) \qquad\qquad g_0^{(4)}(w, w^*)$$

- Assuming the master formula (single-Reggeon exchange),

we use this information to compute the NNLL $E^{(2)}_{\nu,n}$

and the NNNLL $\Phi^{(3)}_{\nu,n}$

– after making a dictionary for the Fourier-Mellin transform $(\nu,n) \leftarrow\rightarrow (w,w^*)$

- Only a limited set of (ν,n) functions enter E, Φ: polygamma functions $\psi^{(k)}(x)$ + rational

Building blocks

	weight	$(\nu \leftrightarrow -\nu, n \leftrightarrow -n)$
1	0	$[+,+]$
D_ν	1	$[-,+]$
V	1	$[-,+]$
N	1	$[+,-]$

	weight	$(\nu \leftrightarrow -\nu, n \leftrightarrow -n)$
$E_{\nu,n}$	1	$[+,+]$
$\tilde{F}_4$	3	$[+,-]$
$\tilde{F}_{6a}$	4	$[-,-]$
$\tilde{F}_7$	4	$[-,+]$

$$D_\nu \equiv -i\partial_\nu \equiv -i\,\partial/\partial\nu$$

$$V \equiv -\frac{1}{2}\left[\frac{1}{i\nu+\frac{|n|}{2}} - \frac{1}{-i\nu+\frac{|n|}{2}}\right] = \frac{i\nu}{\nu^2+\frac{|n|^2}{4}}$$

$$N \equiv \mathrm{sgn}(n)\left[\frac{1}{i\nu+\frac{|n|}{2}} + \frac{1}{-i\nu+\frac{|n|}{2}}\right] = \frac{n}{\nu^2+\frac{|n|^2}{4}}$$

$$S_1(n) = \sum_{k=1}^{n}\frac{1}{k} = \psi(n+1) - \psi(1)$$

$$E_{\nu,n} = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi\left(1+i\nu+\frac{|n|}{2}\right) + \psi\left(1-i\nu+\frac{|n|}{2}\right) - 2\psi(1)$$

NNLL BFKL eigenvalue

$$
\begin{aligned}
E^{(2)}_{\nu,n} =& -E^{(1)}_{\nu,n}\,\Phi^{(1)}_{\rm Reg}(\nu,n) - E_{\nu,n}\,\Phi^{(2)}_{\rm Reg}(\nu,n) + \frac{3}{8}\,D_\nu^2 E_{\nu,n}\,E^2_{\nu,n} + \frac{3}{32}\,N^2\,D_\nu^2 E_{\nu,n} + \frac{1}{8}\,V^2\,D_\nu^2 E_{\nu,n} \\
& - \frac{1}{8}\,V\,D_\nu^3 E_{\nu,n} + \frac{1}{48}\,D_\nu^4 E_{\nu,n} + \frac{\pi^2}{12}\,D_\nu^2 E_{\nu,n} - \frac{3}{4}\,D_\nu E_{\nu,n}\,V\,E^2_{\nu,n} - \frac{5}{16}\,D_\nu E_{\nu,n}\,N^2\,V \\
& - \frac{\pi^2}{4}\,D_\nu E_{\nu,n}\,V + \frac{1}{8}\,E_{\nu,n}\,[D_\nu E_{\nu,n}]^2 + \frac{3}{16}\,N^2\,E^3_{\nu,n} + \frac{61}{4}\,E^2_{\nu,n}\,\zeta_3 + \frac{1}{8}\,E^5_{\nu,n} + \frac{5\pi^2}{6}\,E^3_{\nu,n} \\
& + \frac{19}{128}\,E_{\nu,n}\,N^4 + \frac{5}{16}\,E_{\nu,n}\,N^2\,V^2 + \frac{3\pi^2}{16}\,E_{\nu,n}\,N^2 + \frac{\pi^2}{4}\,E_{\nu,n}\,V^2 + \frac{35}{16}\,N^2\,\zeta_3 + \frac{1}{2}\,V^2\,\zeta_3 \\
& + \frac{11\pi^2}{6}\,\zeta_3 + 10\,\zeta_5 + a_0\,\mathcal{E}_5 + \sum_{i=1}^{5} a_i\,\zeta_2\,\mathcal{E}_{3,i} + a_6\,\zeta_4\,\mathcal{E}_2 + \sum_{i=7}^{8} a_i\,\zeta_3\,\mathcal{E}_{1,i}\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_5 =& \frac{124}{3}\,N^2\,D_\nu^2 E_{\nu,n} + \frac{1210}{3}\,V^2\,D_\nu^2 E_{\nu,n} - \frac{35}{3}\,V\,D_\nu^3 E_{\nu,n} - \frac{31}{6}\,D_\nu^4 E_{\nu,n} - \frac{151}{2}\,D_\nu E_{\nu,n}\,N^2\,V \\
& + \frac{124}{3}\,N^2\,E^3_{\nu,n} - \frac{140}{3}\,V^2\,E^3_{\nu,n} - \frac{31}{2}\,E_{\nu,n}\,N^4 + \frac{10903}{12}\,N^2\,\zeta_3 + \frac{13960}{3}\,V^2\,\zeta_3 \\
& - 62\,D_\nu^2 E_{\nu,n}\,E^2_{\nu,n} + 70\,D_\nu E_{\nu,n}\,V\,E^2_{\nu,n} - 760\,D_\nu E_{\nu,n}\,V^3 + 248\,E_{\nu,n}\,[D_\nu E_{\nu,n}]^2 \\
& + 7431\,E^2_{\nu,n}\,\zeta_3 - 97\,E_{\nu,n}\,N^2\,V^2 + 16072\,\zeta_5\,,
\end{aligned}
$$

1 symbol level ambiguity in $g_1^{(4)}$

$$
\begin{aligned}
\mathcal{E}_{3,1} &= -\frac{3}{4}\,E_{\nu,n}\,N^2 - D_\nu^2 E_{\nu,n} + 5\,E^3_{\nu,n} + 6\,E_{\nu,n}\,V^2 - 2\,E_{\nu,n}\,\pi^2 + 8\,\zeta_3\,, \\
\mathcal{E}_{3,2} &= E^3_{\nu,n}\,, \\
\mathcal{E}_{3,3} &= \frac{3}{4}\,E_{\nu,n}\,N^2 - 3\,D_\nu E_{\nu,n}\,V + 3\,E^3_{\nu,n} + 12\,\zeta_3\,, \\
\mathcal{E}_{3,4} &= -\frac{1}{8}\,D_\nu^2 E_{\nu,n} + \frac{9}{4}\,D_\nu E_{\nu,n}\,V - \frac{3}{4}\,E_{\nu,n}\,N^2 - \frac{3}{2}\,E_{\nu,n}\,V^2 - \frac{25}{2}\,\zeta_3 - 2\,E^3_{\nu,n}\,,
\end{aligned}
$$

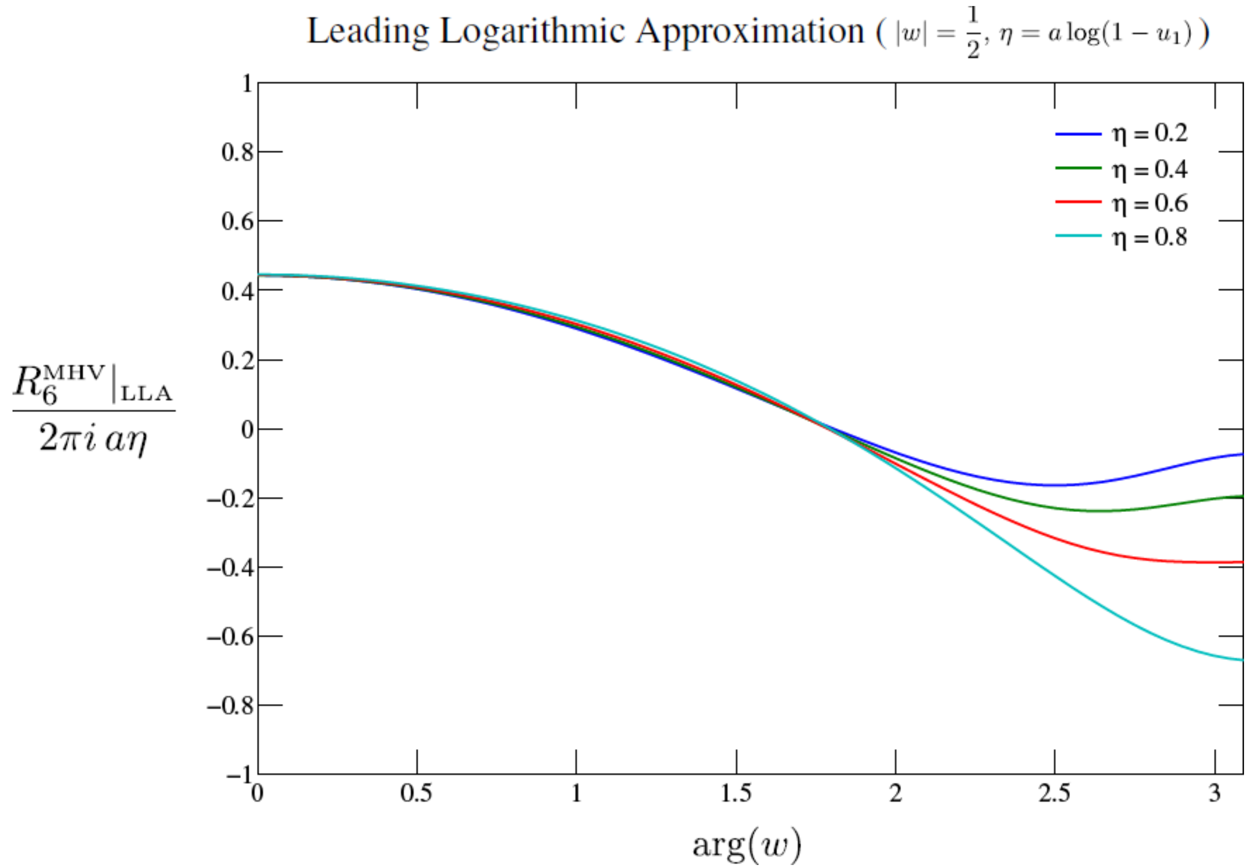
8 beyond symbol level ambiguities in $g_1^{(4)}$

Conclusions

- Planar N=4 SYM is a powerful laboratory for studying 4-d scattering amplitudes, thanks to dual (super)conformal invariance & other properties.
- 6-gluon amplitude is first nontrivial case. Symbol now known through 4 loops (modulo some constants).
- Multi-Regge limit offers even simpler setup to solve first, greatly facilitated by Brown's SVHPLs.
- (NMHV amplitudes in this limit also naturally described by same functions.)
- Multi-Regge limit of 6-gluon amplitude can be solved to all orders in LLA, especially in collinear corner ($|w| \to 0$).
- Bessel functions suggest integrability, localization.
- May be that full multi-Regge limit (i.e. N^kLLA terms) is next to be solved to all orders in the coupling?

Extra Slides

LLA Numerics for fixed |*w*|



Would be interesting to compare with numerical approach of Chachamis, Sabio Vera, 1112.4162, 1206.3140

y alphabet and $\bar{z}$ derivatives

$$\frac{\partial}{\partial z}\mathcal{L}_{x_0 w}(z,\bar{z}) = \frac{\mathcal{L}_w(z,\bar{z})}{z} \qquad \frac{\partial}{\partial z}\mathcal{L}_{x_1 w}(z,\bar{z}) = \frac{\mathcal{L}_w(z,\bar{z})}{1-z}$$

$$\Leftrightarrow \quad \frac{\partial}{\partial z}\mathcal{L}(z,\bar{z}) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right)\mathcal{L}(z)$$

but

$$\frac{\partial}{\partial \bar{z}}\mathcal{L}(z,\bar{z}) = \mathcal{L}(z)\left(\frac{y_0}{\bar{z}} + \frac{y_1}{1-\bar{z}}\right)$$

Need for $R_6^{(2)}(u_1,u_2,u_3)$

- Modification of BDS ansatz for $n = 6$ was suspected, based on:
- A large *n*, strong-coupling limit Alday, Maldacena, 0710.1060
- A 2-loop Wilson-loop calculation

  Drummond, Henn, Korchemsky, Sokatchev, 0712.4138
- A high-energy/Regge limit

  Bartels, Lipatov, Sabio Vera, 0802.2065

- Confirmed by a direct amplitude calculation

  Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

that matched the Wilson loop numerically

Drummond, Henn, Korchemsky, Sokatchev, 0803.1466

OPE Constraints (cont.)

- Using conformal invariance, send one long line to ∞, put other one along x^-
- Dilatations, boosts, azimuthal rotations preserve this configuration.
- σ, ϕ parametrize isometries, so classify conformal primaries by conjugate variables (twist p, spin m)
- Also expand anomalous dimensions in coupling g^2:

$$E_n(g) = E_n^{(0)} + g^2\, E_n^{(1)} + g^4\, E_n^{(2)} + \ldots$$

$$\exp[-E_n(g)\tau]$$

$$= \exp[-E_n^{(0)}\tau] \times \left[1 - g^2\,\tau\, E_n^{(1)} + g^4(\frac{1}{2}\tau^2\,[E_n^{(1)}]^2 - \tau\, E_n^{(2)}) + \ldots\right]$$

- **Leading τ^{L-1} dependence of $R_6^{(L)}$ needs only one-loop anomalous dimension $E_n^{(1)}$**

Professor of symbology at Harvard University, has used these techniques to make a series of important advances: