# Topological quantum numbers

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**Hans Dehmelt's question:**

How can precise measurements be made with poorly characterized devices such as semiconductor inversion layers or Josephson junctions?

Josephson junctions are used as secondary voltage standards, are consistent with one another to parts in $10^{17}$, and have led to a revision of accepted values of fundamental constants.

The quantum Hall effect provides a secondary resistance standard that is far more reliable than its predecessors, and different devices agree to parts in $10^{10}$.

The answer is related to **topological quantum numbers**, which may relate a physically observable quantity to a counting process in a way that is robust to changes of the details of a system.

It is not as simple as that, because the topological quantum number is not usually the quantity that is of direct physical interest.

## Symmetry based quantum numbers and topological quantum numbers

Familiar quantum numbers like angular momentum and isospin are based on symmetries of dynamical system — rotational invariance or charge independence.

**Nöther's theorem, symmetry groups and Lie algebras.**

If, for example, the system is unchanged by arbitrary rotations about a center of symmetry, then there is a conserved quantity, angular momentum in this case

Breaking of the symmetry leads to mixing and eventual disappearance of such quantum numbers.

Topological quantum numbers, such as circulation in superfluid $^4$He, magnetic flux in superconductors, Hall conductance in semiconductor inversion layers (two-dimensional electron systems), are insensitive to the symmetry of the system and to changes in the details of the structure.

In some cases they can be determined with very high precision, but not always.

**Homotopy groups and winding numbers.**

A central concern of homotopy theory is the classification of loops in spaces. Remember Ampère's law, which says that the integral of magnetic field round a loop is equal to the current threading the loop.

I will mostly talk about how the phase angle of a wave function changes round a loop. This is what comes into quantization of circulation and flux in superfluids and superconductors.

Something similar happens for the quantum Hall effect, but it is more complicated.

## Bose-Einstein condensates

Einstein (1924, 1925) showed that at sufficiently low temperatures noninteracting bosons will collapse into the lowest energy state.

Fritz London (1938) suggested that this could be an explanation for the peculiar properties of superfluid helium below 2.17 K; specific heat singularity, flow without viscosity, film flow, etc.

Bogoliubov (1947) showed that repulsion between the atoms stabilizes the condensate, the multiply occupied wave function. We now believe this repulsive interaction is essential for superfluidity, but liquid helium is very far from Bogoliubov's weakly interacting gas.

Cornell, Wieman and collaborators (1995) cooled trapped alkali metal atoms well below 1 $\mu$K and found that they formed a Bose-Einstein condensate.

Ketterle and collaborators (2000) rotated a Na atom trap and showed that the rotation creates an array of quantized vortices (**Abrikosov lattice**).

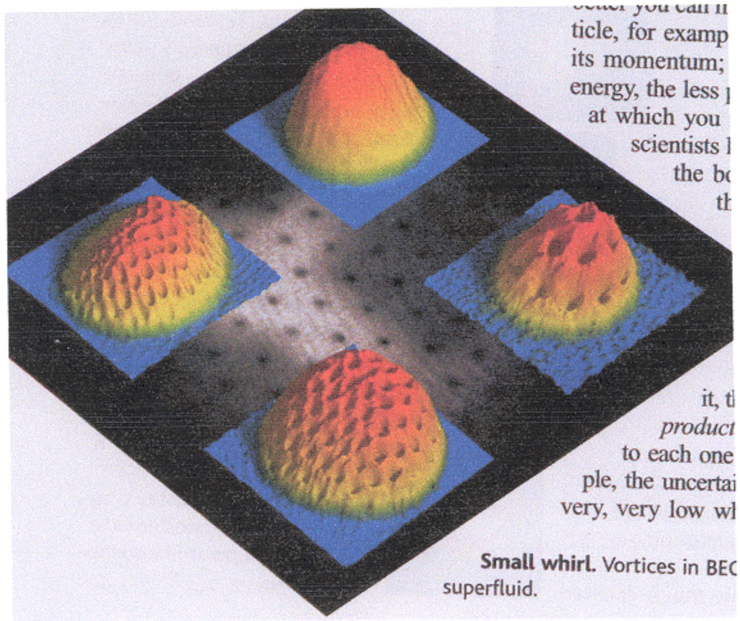


Figure 1: Quantized vortices in rotating sodium atom clusters, as shown by Ketterle et al., 2000

# Onsager–Feynman argument for quantized circulation

Bose-Einstein condensation involves a finite proportion of bosons in system sharing a common single particle state

$$\Psi(\mathbf{r}) = |\Psi(\mathbf{r})| \exp[iS(\mathbf{r})] \ ,$$

a single-valued function of position.

Superfluid velocity is $\mathbf{v}_s = \hbar \mathbf{grad} S/m_4$, where $m_4$ is helium atom mass.

The phase need not be single valued, but can change by a multiple of $2\pi$ on a closed path that goes round either an obstacle, such as a wire, or when it goes round a mathematical line singularity on which $|\Psi|$ vanishes. The circulation of the superfluid velocity round a path is given by

$$\oint \mathbf{v}_s \cdot d\mathbf{r} = \frac{\hbar}{m_4} \oint \mathbf{grad} S \cdot d\mathbf{r} = n \frac{h}{m_4} \, .$$

The number $n$ of quanta of circulation $\kappa_0 = h/m_4$ is given by the **winding number** of the phase of the condensate wave function.

1. The order parameter $\Psi$ of the superfluid, the condensate wave function, is the feature that allows the topological properties of its phase to be defined and studied.

2. The phase $S$ satisfies the Laplace equation, since $-i\hbar\mathbf{grad}S$ represents a conserved current.

3. Controlling equation $\nabla^2 S = 0$ leads to significant boundary effects on measurements.

4. Superfluid velocity is not directly measured; superfluid momentum density is measured.

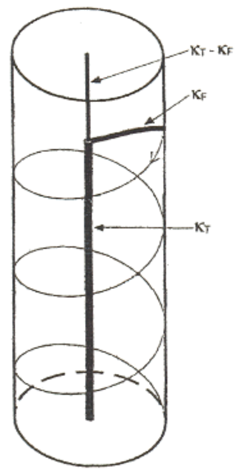


Figure 2: Vinen vibrating wire apparatus, as shown by Zieve et al., 1993

## Vinen **experiment** (1961)

Cylinder with stretched wire running down middle is filled with helium, which can be made to circulate round the wire.

**Magnus force:** $\mathbf{F}_M = \rho_s \mathbf{k}_s \times (\mathbf{v}_w - \mathbf{v}_s)$ splits vibrational modes of wire by $\Delta\nu = \rho_s \kappa / 2\pi w$, where $w$ is mass density of wire, $\kappa$ circulation round it.

Rotating apparatus was cooled through superfluid transition, then brought to a stop, leaving fluid rotating around wire.

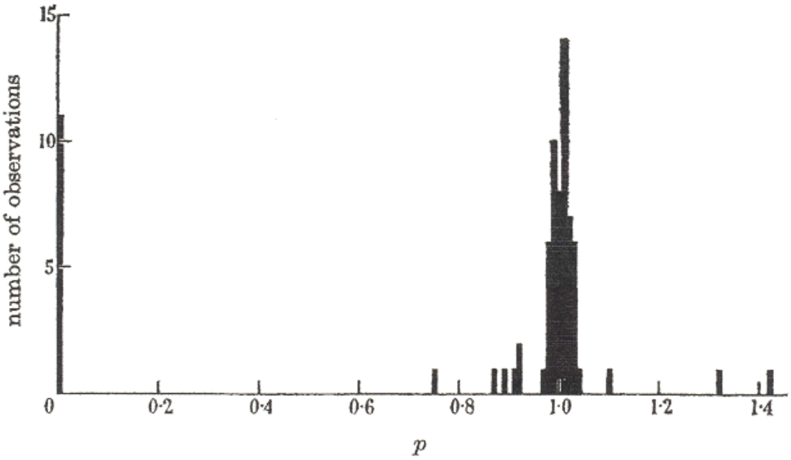


Figure 3: Histogram of measured circulation $p$, in units of $h/m_4$, from Vinen's experiment.

Initially the vortex is often on only part of the wire, and the rest of it goes through the fluid. Repeated shaking of the wire usually gets rid of the free end. This leaves all the vortex on the wire, and a quantized circulation is measured.

Vinen found that measured circulations were clustered around 0 and $h/m_4$, with about $\pm 3\%$ precision.

Packard group (1993) has confirmed $\kappa_0 = h/2m_3$ for B phase of superfluid $^3$He, where condensed object is a pair of $^3$He atoms.

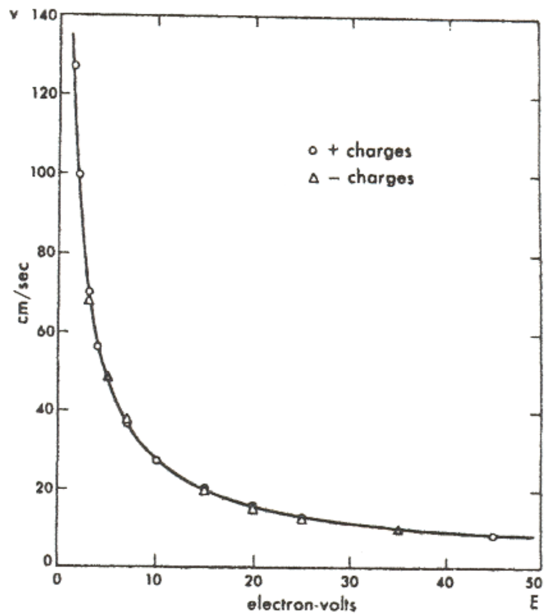


Figure 4: Rayfield and Reif (1963) measurements of speed versus energy for ions trapped on vortex rings, compared with theory.

Beautiful studies of vortex rings by Rayfield and Reif (1963) measure energy–velocity relation for ions trapped on vortex rings.

For vortex ring of radius $R$, core radius $a$, momentum $P$ and energy $E$ are

$$P = \pi(R-a)^2 \rho_s \kappa_0 \ , \quad E \approx \frac{1}{2}\kappa_0^2 \rho_s R \ln \frac{R}{a} \ ,$$

so speed is inversely proportional to energy.

## Flux quantization in superconductors

Superconductor is superfluid in which condensate consists of electron pairs.

In Hamiltonian mechanics for particles of charge $q$ the relation between momentum $\mathbf{p}$ and velocity $\mathbf{v}$ is

$$\mathbf{v} = \frac{1}{m}(\mathbf{p} - q\mathbf{A}) ,$$

where $\mathbf{A}$ is the vector potential whose curl gives the magnetic field $\mathbf{B}$.

So, in quantum theory the current density is

$$\mathbf{j} = -(\frac{e\hbar}{m}\mathbf{grad}S + \frac{2e^2}{m}\mathbf{A})|\Psi|^2 .$$

$\Psi$ represents condensate wave function for electron pairs, so factor $-2e$ is put in front of vector potential $\mathbf{A}$ to allow for charge $-2e$, mass $2m$.

Curl of this equation gives London equation

$$\nabla^2\mathbf{B} = \frac{e^2\mu_0}{m}n_s\mathbf{B} , \implies \nabla^2\mathbf{j} = \frac{e^2\mu_0}{m}n_s\mathbf{j} ,$$

so magnetic field and current density fall off exponentially inside superconductor or away from vortex core.

In interior region where current density vanishes,

$$\oint \mathbf{A} \cdot d\mathbf{r} = -\frac{\hbar}{2e} \oint \mathbf{grad} S \cdot d\mathbf{r} = n\frac{h}{2e} \ .$$

Since integral of the vector potential round a ring gives the flux enclosed by the ring, this shows that the flux trapped by a superconductor is equal to $n$ times $h/2e$, where again $n$ is the winding number of the phase of the condensate wave function.

Path enclosing quantized flux has to be in a region free of current density. It may either surround regions in which there is no superconducting material, where the flux is concentrated, or, for a type II superconductor in a weak magnetic field, it may surround flux lines where the superconducting order parameter has singularities.

Because London equation gives **exponential decay** of current density, corrections to flux quantization may be made exponentially small by increasing length scale of system.

## Josephson effects

Josephson (1962)showed that phase of superconducting order parameter is directly detectable when two superconductors, otherwise isolated from one another, are connected at one or more points by a thin insulating layer.

If the phase difference $S_1 - S_2$ is constant between the two layers there is a steady current

$$J = J_0 \sin(S_1 - S_2)$$

between the two layers This is the dc Josephson effect. It is used in the SQUID magnetometer, which gives a very accurate way of detecting changes in magnetic fields.

If a potential difference $V$ is applied between the two superconductors, the phases of the two change at a rate which differs by $2eV/\hbar$. This produces an alternating current $J_0 \sin(2eVt/\hbar)$. This is the ac Josephson effect. The Josephson frequency–voltage relation $V = h\nu/2e$ provides by far the most precise and reproducible way of measuring voltages.

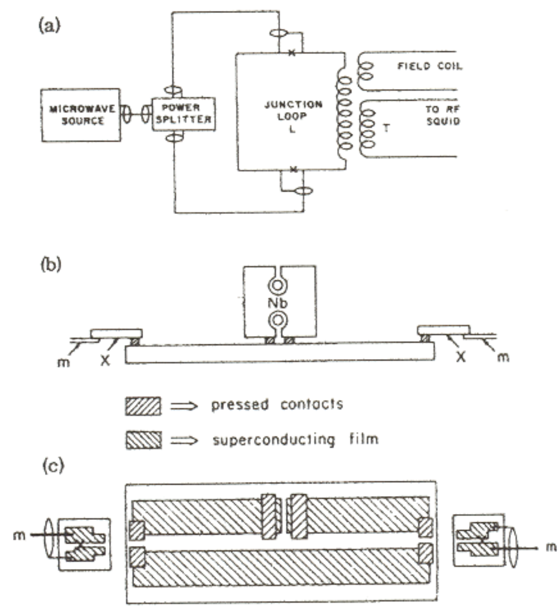


Figure 5: Apparatus for measuring balance between two voltages generated by the same microwave frequency at two different types of Josephson junction, by Tsai, Jain and Lukens (1983)

John Clarke (1968) developed high precision method of balancing the voltages generated by the same constant frequency source (microwave in later experiments) on two different junctions made of different materials.

A small difference in the dc voltage would generate a slowly rising current in the inductive loop.

Tsai et al. experiment shows difference less than 2 parts in $10^{16}$.

## Quantum Hall effect

Experiments done by Klitzing (1980) on two-dimensional electron systems at low temperatures in high magnetic field (MOSFETs) showed very precise Hall voltages (voltage transverse to the current) where longitudinal voltage was negligible (no Ohmic dissipation).

At these plateaus,

$$\frac{I}{V_H} = \frac{ne^2}{h}$$

with very high precision — initially better than one part in $10^5$, soon shown to be more precise than any other resistance measurement.

In the fractional quantum Hall effect, found by Tsui, Störmer and Gossard (1982), integer $n$ is replaced by simple exact fraction.

Yes its origin is topological, as shown by Laughlin, but the story of how a winding number appears is a little more complicated.

at gate voltage close to the left side of the plateau. In Fig. 2, this minimum is relatively shallow and has a value of 6 452.87 Ω at $V_g$ = 23.30 V.

In order to demonstrate the insensitivity of the Hall resistance on the geometry of the device, measurements on two samples with a length-to-width ratio of $L/W$=0.65 and $L/W$=25, respectively, are plotted in Fig. 3. The gate-voltage scale

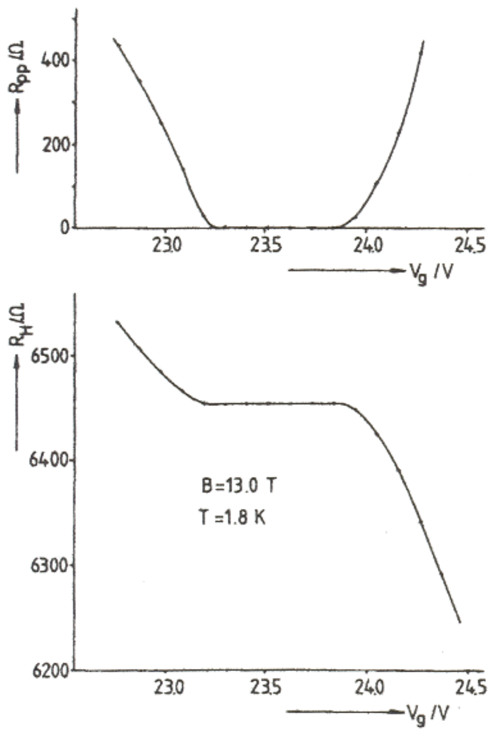


FIG. 2. Hall resistance $R_H$, and device resistance, $R_{pp}$, between the potential probes as a function of the gate voltage $V_g$ in a region of gate voltage corresponding to a fully occupied, lowest ($n$ = 0) Landau level. The plateau in $R_H$ has a value of 6453.3 ± 0.1 Ω. The geometry of the device was $L$ = 400 μm, $W$ = 50 μm, and $L_{pp}$ = 130 μm; $B$ = 13 T.

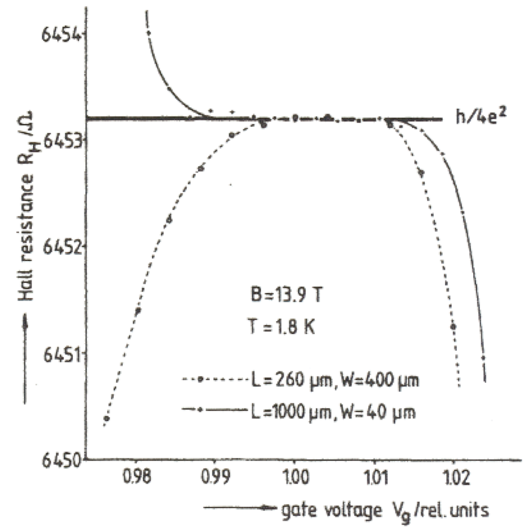


FIG. 3. Hall resistance $R_H$ for two samples with different geometry in a gate-voltage region $V_g$ where the $n$ = 0 Landau level is fully occupied. The recommended value $h/4e^2$ is given as 6453.204 Ω.

Figure 6: Original measurement of quantum Hall effect by Klitzing Dorda and Pepper (1980)

**Conclusions.**

1. Topological quantum numbers seem to allow high precision to be obtained from not very well determined macroscopic systems.

2. High precision of topological quantum numbers requires that thermal and quantum fluctuations do not carry order parameter between different topological states at an appreciable rate.

3. I do not know if neutral superfluids have circulation quantized with high precision. Is this a difference between the Helmholtz equation in superconductors and the Laplace equation for neutral superfluids?

4. Dirac (1931) gave a topological argument for the high precision of the quantization of electric charge.

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