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A DEFINITION FOR VECTOR CORRELATION AND ITS APPLICATION TO MARINE SURFACE WINDS

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L. C. Breaker, and W. H. Gemmill, NWS, NMC

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This is an unreviewed manuscript, primarily intended for informal exchange of information among NMC staff members

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TECHNICAL NOTE

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1. INTRODUCTION

The search for a vector correlation coefficient was motivated by a requirement to verify various forecast schemes for ocean surface winds. The most common method used to correlate vector quantities (i.e. winds, wind stress, or currents) has been to apply standard linear correlation techniques to the scalar components of a vector, i.e. its magnitude and direction or its orthogonal u and v components (e.g.,Charles, 1959; Buell, 1971). However, the most common measures of correlation do not incorporate relationships of both speed and direction (or u and v) simultaneously. A vector is represented by both a magnitude and direction , and thus can not be used in the standard definition of linear correlation. In fact, because wind direction is a circular function, the standard definition of linear correlation for direction only can not be used.

Over the past 30 years, only rarely has a "true" two-dimensional vector correlation coefficient been used in studying relationships between vector quantities in meteorology or oceanography (e.g., Lamberth and Armendarz, 1966). Also, it has become evident that there is no universally accepted definition for vector correlation. For the various proposed definitions of vector correlation, each has its own formulation, which is usually presented with primary emphasis on the interpretation of the coefficients with little or no concern for its basis or its statistical properties.

In this note we give a short history of the various definitions for two-dimensional vector correlation which have appeared in the meteorological and oceanographic literature. We indicate the properties that are desirable for a vector correlation coefficient. We also investigate the properties of some of the previous definitions. Finally, we propose a definition from the statistical literature which we believe should be universally adopted, at least for a certain class of problems. The properties of this new definition are presented as well as simulation studies related its statistical properties. Finally, we give an example of its application to real data.

BASIC DEFINITIONS

First, some basic definitions for vector quantities are presented. These definitions are presented for a two-dimensional vector (W) with orthogonal components u and v. The definitions of the symbols used in this paper are given in appendix 1.

1) The magnitude (speed) of a vector W is given in terms of its u and v components as:

$$|W| = (u^2 + v^2)^{0.5}.$$

2) The direction of a vector θ is given by

$$\theta = \tan^{-1}(v/u).$$

Let W_i (i=1,...n) be a set of n vectors.

3) The mean resultant vector is then

$$\bar{W}= (1/n)\sum_{i=1}^{n} W_i$$

The magnitude of this vector is

$$|\bar{W}| = (((1/n)\sum_{i=1}^{n} u_i)^2 + ((1/n)\sum_{i=1}^{n} v_i)^2)^{0.5}.$$

4) The mean vector speed is given by

$$\overline{|W|} = (1/n)\sum_{i=1}^{n} |W|_i.$$

Let Σ_w be the covariance matrix of the vector W. If we have a sample of vectors W_i (i=1 to n), let S_w be the sample covariance matrix. That is

$$S_w=(1/(n-1))\sum_{i=1}^{n} (W-\bar{W})(W-\bar{W})^T = \begin{bmatrix} s_u^2 & s_{uv} \\ s_{vu} & s_v^2 \end{bmatrix} \qquad (1.1)$$

where the sum is taken over a sample of size n and T the matrix transpose. The variance of a vector can be defined as

$$TR(\Sigma_W)= \sigma_u^2 + \sigma_v^2,$$

where TR( ) represents the trace of the matrix. This, of course, is the sum of the variances of the individual components u and v. A more standard definition is the generalized variance given by $|\Sigma|$, the determinant of the covariance matrix. For a discussion of the generalized variance and its meaning see Anderson (1984).

There have been a number of definitions of vector correlation presented in the oceanographic and meteorological literature. Most of these definitions appear to have been adopted because they could be interpreted geometrically. In most cases there has been little or no attention given to ascertaining the statistical properties associated with these definitions.

Most of the definitions have been an attempt to generalize the definition of the standard one dimensional linear correlation coefficient. In order to clarify and motivate the discussions of vector correlation we next review some of the properties of the standard product moment correlation coefficient, ρ.

Given two random variables **u** and **v**, with standard deviations σ_u, σ_v ,and covariance σ_{uv}, the correlation coefficient is defined as

$$\rho = \sigma_{uv}^{0.5}/(\sigma_u \sigma_v)$$

Given a sample of **u** and **v**, the sample correlation coefficient, r, is defined as

$$r = s_{uv}^{0.5}/(s_u s_v).$$

This correlation coefficient has the following properties:

1) $-1 \leq \rho \leq 1$.

2) If $x=a+bu$, and $y=c+dv$, then

$$\rho_{UV}=\rho_{XY}.$$

Thus ρ is invariant under linear transformations of u and v.

3) If u and v are independent, then $\rho=0$.

4) $\rho=1$ if and only if $u=a+bv$ for some a and b.

A vector correlation coefficient should have the vector equivalent of these properties.

Except for property 3, similar properties hold for the sample correlation coefficient. That is, even if u and v are independent the sample correlation coefficient will not be equal to zero.

The sample correlation coefficient is related to least squares regression. If, from a sample of u and v, the least squares regression line of u on v is found

$$\hat{u}=a+bv,$$

then

$$b=r(s_u / s_v),$$

and

$$r^2=(\Sigma(\hat{u}-\bar{u})^2)/(\Sigma(u-\bar{u})^2.$$

The quantity r^2 is referred to as the proportion of explained variance or the coefficient of determination.

In order to present the history and develop other definitions of vector correlation , we provide additional background. Let W_1 and W_2 be two two-dimensional random vectors. Next, let

$$X = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

be a four dimensional vector. Further, let

$$\Sigma_X = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma^2(u_1,u_1) & \sigma(u_1,v_1) & \sigma(u_1,u_2) & \sigma(u_1,v_2) \\ \sigma(v_1,u_1) & \sigma^2(v_1,v_1) & \sigma(v_1,u_2) & \sigma(v_1,v_2) \\ \sigma(u_2,u_1) & \sigma(u_2,v_1) & \sigma^2(u_2,u_2) & \sigma(u_2,v_2) \\ \sigma(v_2,u_1) & \sigma(v_2,v_1) & \sigma(v_2,u_2) & \sigma^2(v_2,v_2) \end{bmatrix} \qquad (1.2)$$

be the four by four covariance matrix of the vector X. In equation (1.2) Σ_{11} is the covariance matrix of W_1, Σ_{22} is the covariance matrix of W_2, Σ_{12} is the cross-covariance matrix of W_1 and W_2 and Σ_{21} is the cross-covariance matrix of W_2 and W_1. If the population covariance matrix Σ_X is not known then it is replaced by the sample covariance matrix S_X, where S_X is defined in the usual way. This is similar to the definition in equation (1.1).

Much of the history given here of early definitions of vector correlation is based on Court (1958). We have put each of the definitions in terms of the population parameters and in matrix notation. Most of the early papers give the definitions in scalar notation and in terms of the sample parameters. A very early definition was given by Detzius (1916) as,

$$\rho_D^2 = TR(\Sigma_{12})^2/(TR(\Sigma_{11})TR(\Sigma_{22}))$$

This is sometimes referred to as the "stretch" correlation coefficient.

A later definition which involves both the "stretch" and the "turn" of a vector was given by Sverdrup (1917),

$$\rho_s^2 = (TR(\Sigma_{12})^2 + (\sigma(u_1,v_2) - \sigma(u_2,v_1))/(TR(\Sigma_{11})TR(\Sigma_{22})).$$

This is probably the most frequently used definition in meteorology and oceanography. It has been used by British meteorologists during the 1950's. See, for example, Durst (1957). Durst (1957) develops this definition in terms of "stretch" and "turn" coefficients. "Stretch" relates the difference in magnitude between two sets of vectors through a constant (k), and "turn" relates the rotation through a constant angle (θ) of one set of vectors to another. It is also related to the complex correlation coefficient. This is defined as

$$\rho_c = (TR(\Sigma_{12}) + i(\sigma(u_1,v_2)-\sigma(u_2,v_1)))/(TR(\Sigma_{11})TR(\Sigma_{22}))^{0.5},$$

where i is the square root of -1. See Kundu (1976) for a geometric interpretation of this parameter. The square of the absolute value of the complex vector correlation coefficient is the same as the definition given by Sverdrup. See equation (1.3)

Hotelling (1936) presented a definition given as

$$\rho_h^2 = |((\Sigma_{11})^{-1}\Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{21})|.$$

Hotelling derives many of the statistical properties of the sample statistic for this parameter. We return to this definition in the next section.

A definition proposed by Court (1958) was based on a generalization of the concept of explained variance.His definition was

$$\rho^2 = TR(\Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{21})/TR(\Sigma_{11}).$$

In this definition, W_1 plays the role of the dependent variable. The definition is not symmetric in W_1 and W_2 and it is not invariant under changes in scale. If each of the variables u_1, v_1, u_2, v_2 are always measured with the same units then the fact that it is not invariant under changes in scale becomes unimportant.

Since 1960, a series of papers on the correlation of directional data or angular association have appeared. For a history and discussion of these papers see Breckling (1989).

Here, we consider vector correlations where the vectors are given in term of u and v. The definition we propose is

$$\rho_v^2 = TR((\Sigma_{11})^{-1}\Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{12}). \tag{1.5}$$

This is the definition given by Jupp and Mardia (1980). In the next section, we show that this definition is a generalization of the standard scalar correlation coefficient. Unlike some of its predecessors, however, which have been restricted to the unit circle (e.g., Mardia and Puri, 1978; Stephens, 1979) this definition includes both direction and magnitude. For applied problems in oceanography and meteorology, this is a very important distinction. It has what we consider a complete set of desirable properties. In addition, it has the very important property that the distribution of a simple function of its sample value is asymptotically robust. That is, if W_1 and W_2 are independent, then the asymptotic distribution of the statistic does not depend on the distributions of W_1 and W_2. Since the distributions of the W_i may be unknown or difficult to express in a closed standard form this property is very significant for possible applications of the parameter, such as hypothesis testing.

2. THEORETICAL DEVELOPMENT

As we explained in the introduction there have been a number of definitions of vector correlation. In this section we explain and prove the properties of the vector correlation defined by Jupp and Mardia (1980). All of the properties stated or proven in this section are contained in their paper. However, their results are presented in a very general context. Many of the details presented in their paper may not be clear to the nonspecialist. We give the results for vectors of two dimensions in ordinary Euclidean space. To generalize the results to three and four-dimensional vectors is conceptually straightforward.

The properties of this new definition are the following: It is a generalization of the simple one-dimensional correlation coefficient. When the vectors are independent, its asymptotic distribution is known, hence it can be used for hypothesis testing. It is symmetric in the arguments. It has a simple interpretation in terms of canonical correlation. It is invariant under transformations of the coordinate axes, including rotations and changes in scale. It is equal to zero when the vectors are independent and obtains its maximum value if and only if they are linearly dependent.

THEORY IN TWO DIMENSIONS

Let W_1 and W_2 be two two-dimensional random vectors. Then we define the vector X , its covariance matrix Σ_X and the submatrices of Σ, exactly as in section 1, equation (1.2). Here, we always assume that Σ_{11} and Σ_{22} are nonsingular. Thus the vectors W_1 and W_2 are nondegenerate or $u_i \neq a_i + b_i v_i (i=1,2)$ for some a_i and b_i. We also assume that all moments of the vector X exist.

Then the definition of the vector correlation coefficient between W_1 and W_2 is

$$\rho_v^2 = TR((\Sigma_{11})^{-1}\Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{21}), \tag{2.1}$$

where the Σ_{ij} are as in equation (1.2).In terms of the u's and v's this definition is given by

$$\rho_v^2 = f/g,$$

where

$$\begin{aligned}
f=&\sigma^2(u_1,u_1)(\sigma^2(u_2,u_2)(\sigma(v_1,v_2))^2 + \sigma^2(v_2,v_2)(\sigma(v_1,u_2))^2) + \\
&\sigma^2(v_1,v_1)(\sigma^2(u_2,u_2)(\sigma(u_1,v_2))^2 + \sigma^2(v_2,v_2)(\sigma(u_1,u_2))^2) + \\
&2(\sigma(u_1,v_1)\sigma(u_1,v_2)\sigma(v_1,u_2)\sigma(u_2,v_2)) + \\
&2(\sigma(u_1,v_1)\sigma(u_1,u_2)\sigma(v_1,v_2)\sigma(u_2,v_2)) - \\
&2(\sigma^2(u_1,u_1)\sigma(v_1,u_2)\sigma(v_1,v_2)\sigma(u_2,v_2)) - \\
&2(\sigma^2(v_1,v_1)\sigma(u_1,u_2)\sigma(u_1,v_2)\sigma(u_2,v_2)) - \\
&2(\sigma^2(u_2,u_2)\sigma(u_1,v_1)\sigma(u_1,v_2)\sigma(v_1,v_2)) - \\
&2(\sigma^2(v_2,v_2)\sigma(u_1,v_1)\sigma(u_1,u_2)\sigma(v_1,u_2))
\end{aligned}$$

and

$$g=[\sigma^2(u_1,u_1)\sigma^2(v_1,v_1)-(\sigma(u_1,v_1))^2][\sigma^2(u_2,u_2)\sigma^2(v_2,v_2)-(\sigma(u_2,v_2))^2].$$

The long scalar form presented directly above is the form of the definition used in the program given in appendix 2.

It is easily seen that the definition given in equation (2.1) is a generalization of the square of the standard Pearson correlation coefficient for two, one-dimensional random variables. For the one-dimensional case with random variables u and v, the square of the correlation coefficient is defined as

$$\rho^2 = (\sigma(u,v))^2/\sigma^2(u)\sigma^2(v). \tag{2.2}$$

Equation (2.2) can be rewritten as

$$\rho^2 = (\sigma^2(u))^{-1}(\sigma(u,v)(\sigma^2(v))^{-1}\sigma(v,u)) \tag{2.3}$$

Since all the expressions in equation (2.3) are scalar, it follows that

$$\rho^2 = TR(\sigma^2(u))^{-1}\sigma(u,v)(\sigma^2(v))^{-1}\sigma(v,u)),$$

which is of the same form as the definition given in equation (2.1).

An intuitive justification for this definition of vector correlation is based on canonical correlation. Let W_1 and W_2 be two, two-dimensional random vectors. The canonical correlations are defined in the following manner: linear combinations of u_1 and v_1 and of u_2 and v_2 are formed

$$z_{11} = a_{11}u_1 + b_{11}v_1$$
$$z_{12} = a_{12}u_2 + b_{12}v_2$$

such that for all such linear combinations the standard one-dimensional correlation coefficient,

$$\rho_1 = \mathrm{corr}(z_{11}, z_{12})$$

between z_{11} and z_{12} is a maximum. Note that $\rho_1 \geq 0$. The parameter ρ_1 is the first canonical correlation. The variables z_{11} and z_{12} are called the first canonical variables. Then a second set of variables

$$z_{21} = a_{21}u_1 + b_{21}v_1$$
$$z_{22} = a_{22}u_2 + b_{22}v_2$$

is found such that

$$\mathrm{corr}(z_{11}, z_{21}) = \mathrm{corr}(z_{11}, z_{22}) = \mathrm{corr}(z_{12}, z_{21}) = \mathrm{corr}(z_{12}, z_{22}) = 0$$

and

$$\rho_2 = \mathrm{corr}(z_{21}, z_{22})$$

is a maximum. The parameter ρ_2 is called the second canonical correlation and is nonnegative. The vector correlation coefficient given by equation (2.3) is the sum of the squares of the two canonical correlations. That is

$$\rho_v^2 = \rho_1^2 + \rho_2^2$$

For the definition of Hotelling (1936), given in equation (1.4)

$$\rho_h^2 = \rho_1^2\rho_2^2.$$

That is, this vector correlation coefficient is the product of the canonical correlations. This means, for example, if the correlation between u_1 and u_2 is one, and the correlation between v_1 and v_2 is zero, then this vector correlation coefficient will be equal to zero. We believe that this is an undesirable property and for this reason we do not recommend its use.

PROPERTIES OF ρ_v^2

We will need the following results: If **M** and **N** are square matrices, then

$$TR(\mathbf{MN}) = TR(\mathbf{NM}). \tag{2.4}$$

If **M** and **N** are nonsingular, then

$$(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}, \tag{2.5}$$

Graybill (1969).

Property (1). The coefficient ρ_v^2 is symmetric in W_1 and W_2.Using (2.4) twice, it is seen that

$$\begin{aligned}\rho_v^2(W_1,W_2) &= TR(\Sigma_{11}^{-1}\,\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})\\ &= TR(\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}\,\Sigma_{12})\\ &= \rho_v^2(W_2,W_1).\end{aligned}$$

As discussed in the introduction, this is not true for some alternative definitions of vector correlation.

Property (2). The parameter ρ_v^2 is invariant under transformations of the coordinate axes, including rotations and changes in scale. For translations this property is obvious since the covariance matrix is unchanged by such transformations. The second part of this property can be restated as the following theorem.

Theorem 1. The vector correlation ρ_v^2 is invariant under linear transformations of W_1 and W_2 if the transformations are of rank 2. That is, if a linear transformation of the form

$$\mathbf{L} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{bmatrix}$$

where A and B are nonsingular, is applied to the four-dimensional vector X, then ρ_v^2 is unchanged. To see this, let

$$X^* = \mathbf{L}\ X = \begin{bmatrix} W_1^* \\ W_2^* \end{bmatrix} = \begin{bmatrix} \mathbf{A}\ W_1 \\ \mathbf{B}\ W_2 \end{bmatrix}.$$

The covariance matrix of X^* is given by

$$\begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} \mathbf{A}^T & 0 \\ 0 & \mathbf{B}^T \end{bmatrix}.$$

The covariance of X^* is then equal to

$$\left[\begin{array}{c|c} A\,\Sigma_{11}\,A^T & A\,\Sigma_{12}\,B^T \\ \hline B\,\Sigma_{21}\,A^T & B\,\Sigma_{22}\,B^T \end{array}\right].$$

Then for the new vectors W_1^*, and W_2^*

$$\rho_v^2(W_1^*,W_2^*) = TR((A\Sigma_{11}A^T)^{-1}(A\Sigma_{12}B^T)(B\Sigma_{22}B^T)^{-1}(B\Sigma_{21}A^T)) \qquad (2.6)$$

Using the results in equations (2.4) and (2.5), the right-hand-side of equation (2.6) becomes

$$TR(\Sigma_{11}^{-1}\,\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) = \rho_v^2(W_1,W_2).$$

Property (3). The parameter ρ_v^2 is the sum of the squares of the canonical correlations. This can be shown using property (2). Computing the canonical correlations is equivalent to finding an A and B of equation (2.6) such that the covariance matrix of X^* is equal to

$$\begin{bmatrix} 1 & 0 & \rho_1 & 0 \\ 0 & 1 & 0 & \rho_2 \\ \rho_1 & 0 & 1 & 0 \\ 0 & \rho_2 & 0 & 1 \end{bmatrix}$$

where $\rho_1 \geq \rho_2 \geq 0.0$ and ρ_1 and ρ_2 are maximized. That is, ρ_1 is the first canonical correlation and ρ_2 is the second. See Anderson (1984). Next, let

$$\mathbf{D} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}.$$

Then for W_1^* and W_2^*

$$\rho_v^2 = TR(\mathbf{I}^{-1}\mathbf{D}\mathbf{I}^{-1}\mathbf{D})$$

$$= \rho_1^2 + \rho_2^2, \tag{2.7}$$

which by property (2) is equal to $\rho_v^2(W_1,W_2)$ and where $\mathbf{I}$ is the two by two identity matrix.

Property(4). If W_1 and W_2 are independent then $\rho_v^2 = 0$.
If W_1 and W_2 are independent then,

$$\Sigma_{12} = \Sigma_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

and

$$\rho_v^2 = TR(\Sigma_{11}^{-1}\mathbf{0}\Sigma_{22}^{-1}\mathbf{0}) = 0.$$

Property (5) If $corr(u_1,u_2)$, $corr(u_1,v_2)$, $corr(v_1,u_2)$ and $corr(v_1,v_2)$ are not all 0, then $\rho_v^2 > 0$.

To show this property we note the following set of inequalities

$$\rho_v^2 \geq \rho_1^2 \geq \max(|corr(u_1,u_2)|, |corr(u_1,v_2)|, |corr(v_1,u_2)|, |corr(v_1,v_2)|).$$

Property (6). The random vectors W_1 and W_2 are linearly dependent if and only if $\rho_v^2 = 2$.

Assume W_1 and W_2 are linearly dependent. Then there are nonsingular matrices C and D and a vector A such that

$$CW_1 + DW_2 + A = 0.$$

Here, 0 represents a 2 by 1 vector of all 0's. Hence,

$$W_1 = -C^{-1}DW_2 - C^{-1}A. \tag{2.8}$$

This relationship can be written as

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} 0 & -C^{-1}D \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ W_2 \end{bmatrix} + \begin{bmatrix} -C^{-1}A \\ 0 \end{bmatrix}$$

It follows that the covariance matrix of X can be written as

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} C^{-1}D\Sigma_{22}D^T(C^{-1})^T & -C^{-1}D\Sigma_{22} \\ -\Sigma_{22}D^T(C^{-1})^T & \Sigma_{22} \end{bmatrix} \tag{2.9}$$

Then from equation (2.9) , we have

$$\rho_v^2 = \mathrm{TR}((C^{-1}D\Sigma_{22}D^T(C^{-1})^T)^{-1}C^{-1}D\Sigma_{22}(\Sigma_{22})^{-1}\Sigma_{22}D^T(C^{-1})^T) = \mathrm{TR}(I)=2.$$

Assume

$$\rho_v^2 = 2.$$

Then the canonical correlations ρ_1 and ρ_2 both are equal to 1.
This is obvious because

$$\rho_v^2 = \rho_1^2 + \rho_2^2$$

and

$$0 \le \rho_2 \le \rho_1 \le 1.$$

Then as in the proof of property (3), there are nonsingular matrices **A** and **B** such that

$$X^* = \begin{bmatrix} W_1^* \\ W_2^* \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \tag{2.10}$$

and the covariance matrix of X^* is of the form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Hence, the correlation between u_1^* and u_2^* is 1 and the correlation between v_1^* and v_2^* is one. Since these are ordinary correlation coefficients, this implies that

$$u_1^* = c_0 + c_1 u_2^*$$

and

$$v_1^* = d_0 + d_1 v_2^*.$$

Hence, it follows that

$$W_1^* = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} W_2^* + \begin{bmatrix} c_0 \\ d_0 \end{bmatrix} \tag{2.11}$$

From equation (2.10), we have

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} W_1^* \\ W_2^* \end{bmatrix} \tag{2.12}$$

From equations (2.11) and (2.12) it follows that

$$W_1 = \mathbf{A}^{-1} W_1^* = \mathbf{A}^{-1} \begin{bmatrix} c_1 & 0 \\ 0 & d_2 \end{bmatrix} W_2^* + \mathbf{A}^{-1} \begin{bmatrix} c_0 \\ d_0 \end{bmatrix}$$

$$= \mathbf{A}^{-1} \begin{bmatrix} c_1 & 0 \\ 0 & d_2 \end{bmatrix} \mathbf{B}\, W_2^* + \mathbf{A} \begin{bmatrix} c_0 \\ d_0 \end{bmatrix},$$

which proves the assertion.

THE SAMPLING DISTRIBUTION OF ρ_v^2

If the covariance matrix Σ_x of X is estimated in the usual way from a sample of size n by

$$S_X = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = (1/(n-1))\ \Sigma(X-\bar{X})(X-\bar{X})^T$$

Then ρ_v^2 is estimated by

$$\hat{\rho}_v^2 = TR(S_{11}^{-1}S_{12}S_{22}^{-1}S_{21}). \quad (2.13)$$

The statistic defined by equation (2.13) will have several of the properties of ρ_v^2. That is: It is symmetric in W_1 and W_2. It is invariant under transformations of the coordinate axes. It is equal to the sum of the squares of the sample canonical correlations. If W_1 and W_2 are linearly dependent, then the probability that $\hat{\rho}_V^2=2.0$ is one.

If W_1 and W_2 are independent, it does not follow that

$$S_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and hence $\hat{\rho}_v^2 \neq 0$. However, as the sample size increases, S_{12} will approach the zero matrix and $\hat{\rho}_v^2$ will approach 0.

Property (7). If W_1 and W_2 are independent then $n\hat{\rho}_v^2$ is distributed asymptotically as a chi-square variable with four degrees of freedom. This asymptotic distribution is valid for any marginal distribution of the W_1 and the W_2 vectors.

Since

$$\hat{\rho}_v^2 = TR(S_{11}^{-1}S_{12}S_{22}^{-1}S_{21})$$

is invariant under linear transformations of the vectors W_1 and W_2, it follows that the sampling distribution of

$$nTR(S_{11}^{-1}S_{12}S_{22}^{-1}S_{21}) \quad = TR(S_{11}^{-1}n^{1/2}S_{12}S_{22}^{-1}n^{1/2}S_{21}) \qquad (2.14)$$

will be invariant under such transformations. Let μ_X be the mean vector of X and let $\Sigma_{11}^{-1/2}$ and $\Sigma_{22}^{-1/2}$ be the unique positive definite square roots of the matrices Σ_{11}^{-1} and Σ_{22}^{-1} respectively.

We then transform the vector X by

$$X^* = \begin{bmatrix} \Sigma_{11}^{-1/2} & 0 \\ 0 & \Sigma_{22}^{-1/2} \end{bmatrix} (X-\mu_X).$$

Under the assumption that W_1 and W_2 are independent, the covariance matrix of X is given by

$$\Sigma_X = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}$$

and hence, as in the proof of theorem 1, the covariance matrix of X^* is given by

$$\Sigma_{X^*} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

and the mean of X^* is 0.

By the above, we only need to consider the sampling distribution of a vector with mean 0 and covariance matrix I. In order to simplify the notation we will assume that the vector X has these properties.

We now consider the sampling distribution of the statistic given by equation (2.14). By the weak law of large numbers it follows that

$$\lim_{n\Rightarrow\infty} S_{11}^{-1} = \Sigma_{11}^{-1} = I \text{ and } \lim_{n\Rightarrow\infty} S_{22}^{-1} = \Sigma_{22}^{-1} = I.$$

We now consider an individual element of the matrix $n^{1/2}S_{12}$. The first element of $n^{1/2}S_{12}$ is equal to

$$n^{1/2}(\sum_{i=1}^{n}(u_{1i}u_{2i})/(n - 1) - ((\sum_{i=1}^{n}u_{1i})(\sum_{i=1}^{n}u_{2i})/(n-1)n)).$$

By the weak law of large numbers

$$\lim_{n\Rightarrow\infty} n^{1/2}((\sum_{i=1}^{n}u_{1i})(\sum_{i=1}^{n}u_{2i})/(n-1)n) = 0.$$

Hence for large n this element will be approximately

$$n^{1/2}(\sum_{i=1}^{n}(u_{1i}u_{2i})/(n-1)). \tag{2.15}$$

It is a well-known result in mutivariate analysis that the statistic given in equation (2.15) will have asymptotically a standard normal distribution. Similar results hold for the other elements of $n^{1/2}S_{12}$. It is also the case that the asymptotic covariance matrix of the vector consisting of these elements is equal to the identity matrix I.

Combining the above results we find that for large n

$$nTR(S_{11}^{-1}S_{12}S_{22}^{-1}S_{21}) = n(\sum_{i=1}^{n} u_{1i}u_{2i}/n)^2 + n(\sum_{i=1}^{n} u_{1i}v_{2i}/n)^2 + n(\sum_{i=1}^{n} v_{1i}u_{2i}/n)^2$$
$$+ n(\sum_{i=1}^{n} v_{1i}v_{2i}/n)^2 + \text{terms which go to 0 with increasing n.} \qquad (2.16)$$

The right-hand side of equation (2.16) will be asymptotically chi-square with four degrees of freedom.

The importance of this property can not be over emphasized. This means that for large samples, the statistic $\hat{\rho}_v^2$ can be used to carry out standard statistical procedures such as hypothesis testing even if the distributions of the W_i are not known. It is one of the primary reasons we are suggesting that this definition be adopted. In many applications in meteorology and oceanography the form of distribution of the vectors may not be known. A study of some of the small sample properties of $\hat{\rho}_v^2$ follows.

SMALL SAMPLE DISTRIBUTION OF $\hat{\rho}_v^2$

For the case where W_1 and W_2 are independent, the statistic $n\hat{\rho}_v^2$ is distributed asymptotically as chi-square with four degrees of freedom. In practice, however, sample sizes which are too small to use this asymptotic result (i.e. $n<<64$) are frequently encountered. Thus, we seek to extend the results presented by Jupp and Mardia to small sample sizes by estimating the small sample distributions using Monte Carlo techniques. In particular, a random number generator (Press et al., 1986) was used to generate normally-distributed (0,1) u and v vector components for two-component vectors for sample sizes of 8, 12, 32, and 64. Values of $\hat{\rho}_v^2$ were calculated for 1,000,000 cases for each sample size. One million runs were required for each sample size in order to achieve reasonable accuracy (i.e., to the second decimal place) over the tails of the distributions that were generated.

The resulting empirical cumulative frequency distributions are shown together with the theoretical chi-square distribution with four degrees of freedom in Figure 1. There is a significant departure from the theoretical chi-square distribution for small sample sizes. The reason for the crossover of the curves at approximately constant values of $n\hat{\rho}_v^2$ is not known. For sample sizes greater than 64, the form of the distribution closely approximates chi-square with four degrees of freedom; for samples smaller than 8, the general form of the distribution breaks down, no longer resembling chi-square. These curves are used in a subsequent section to estimate levels of significance.

The density functions of the cumulative distributions are shown in Figure 2. The results are given for sample sizes of 12 and 32. As the sample size increases, the distributions become (1) more positively skewed and (2), more peaked.

A curve derived from the samples used to construct Figure 1 shows the 95% level for $n\hat{\rho}_v^2$ as a function of sample size (Figure 3). The mean values ± 1 sigma are plotted for each sample size. This curve show a smooth, well-behaved relation . Hence, interpolating for samples sizes other than those given should provide reasonably accurate results for sample sizes of 8, or greater.

In cases where the variables are normally distributed the theory for testing the independence of sets of variables is well developed. For a discussion of this theory see Morrison (1990).

INTERPRETING $\hat{\rho}_v^2$

In order to provide more insight into the types of vectors (i.e., vector sequences) that may lead to relatively high values of the sample values of this parameter, we consider four natural situations which lead to perfect correlation (i.e. ρ_v^2=2.).

Four cases that lead to perfect correlation are shown in figure 4. The first or trivial case, arises when the vectors are identical (Fig.4a). The second case which produces perfect correlation arises when the magnitudes of the vectors in the second sequence are multiplied by a constant.(i.e. magnification; Fig. 4b). A third case of perfect correlation arises when the directions of the vectors in the second sequence are each rotated by a constant angle (Fig. 4c). The fourth case arises when the second sequence is derived from the first by a combining both magnification and rotation (Fig. 4d).

From the above, it becomes apparent that we can generalize these results to include any situation where one vector sequence can be expressed as a linear combination of the other (i.e., any case where two vector sequences are linearly dependent). In vector notation if there exists a nonsingular matrix **A** and a vector B such that

$$W_{1i} = \mathbf{A}W_{2i} + B$$

then the vector correlation between the series W_{1i} and W_{2i} will be perfect. This is a restatement of property 5.

Next, we consider the situation when there is zero correlation between two vector sequences. It is, of course, the case that if two vectors are independent then their vector correlation will be zero. In the cases where the vectors are normally distributed their vector correlation will be zero, if and only if they are independent. Using the random number generator described above, we generated independent vector sequences with normally-distributed vector components for sample sizes of 10, 100, 1000, 10,000 and 100,000 and computed $\hat{\rho}_v^2$ for each sample size. This experiment was repeated 50 times for each sample size. As theory predicted, $\hat{\rho}_v^2$ clearly approaches zero for increasing sample size (Fig. 5). For a sample of size 100,000, for example, the averaged value of $\hat{\rho}_v^2$ is approximately 0.006. These results also demonstrate that relatively high correlations (e.g., >0.6) can be obtained solely by chance for small sample sizes (e.g. 10).

In the interpretation of $\hat{\rho}_v^2$, it is also important to consider the proper choice of sample size when the vectors W_1 and W_2 are not independent. This will especially be of significance for vector time series. The optimum choice will depend in part on time scales over which the vectors vary significantly. For example, if a sample size is chosen which is too small to encompass significant variability within the vector sequences, the resulting values of $\hat{\rho}_v^2$ may not be meaningful. Help in identifying this problem may be obtained by examining the trace of the sample covariance matrix from which $\hat{\rho}_v^2$ is calculated. The trace of the covariance matrix may provide a measure of the "signal-like" character of the calculated values of $\hat{\rho}_v^2$. Thus, for some threshold, values of the trace which exceed this threshold will have corresponding vector correlations which are meaningful. In this regard, calculations performed in the next section indicate, although not clearly, that there is a tendency for larger values of the trace to correspond to higher values of $\hat{\rho}_v^2$ (i.e., they are positively correlated).

Another potentially useful diagnostic tool in helping to interpret the vector correlation coefficient of Jupp and Mardia is the determinant of the covariance matrix used to compute $\hat{\rho}_v^2$. For cases where the determinant of this matrix approaches zero, it may be difficult to obtain a meaningful solution. In our experience, the value of the determinant has a wide dynamic range, often spanning six orders of magnitude, and so may prove to be a sensitive indicator for interpreting the vector correlation coefficient obtained from the matrix.

APPLICATIONS

An obvious application of the previously defined vector correlation coefficient is to conventional oceanographic and meteorological time (and spatial) series. However, it is important to recognize that this correlation technique can be applied to other data constructs as well. It could, for example, be applied to wind observations at all of the reporting sites within a given geographic domain (e.g., a state), at different times. It could also be applied to ocean surface temperature gradients at selected locations over the Gulf Stream for different seasons.

In the example which follows, we apply the vector correlation technique of Jupp and Mardia to surface wind observations (i.e., time series) from two NDBC environmental data buoys in the NE Atlantic located at 40.5^{0}N, 69.5^{0}W (buoy number 44008) and 34.9^{0}N, 72.9^{0}W (buoy number 41001; Fig. 6). These buoys are approximately 700km apart, close enough so that synoptic-scale disturbances that typically pass through this region as indicated by the typical winter storm track which has been included (Klein, 1957) will, in most cases, influence the winds at both locations. The observations, taken hourly, extend from 1 December 1987 to 4 February 1988, a period of 65 days. As winter low-pressure systems leave the east coast of the U. S., they often deepen over the Gulf Stream and expand as they propagate to the NE along the expected storm tracks. Thus, the winds at both buoys are expected to be strongly influenced by the passage of these low pressure systems which pass through the area during the winter months.

Autocorrelation analysis of the u and v wind components indicated correlation time scales on the order of half a day; consequently the original data have been subsampled every 12th observation to produce series with observations which are approximately independent. Vector correlations have been calculated for four sample sizes, 8, 16, 24, and the entire series (i.e., 130). The sample window was stepped forward one data interval at a time for each sample size, producing partially redundant values of $\hat{\rho}_v^2$. To more fully interpret the vector correlations, we have also calculated the trace and the determinant of the associated 4 x 4 matrix from which $\hat{\rho}_v^2$ is obtained, plus the determinants of each ofthe 2 x 2 submatrices (S_{11}, S_{12}, S_{21}, S_{22}). Since the primary matrix is symmetric, we only present the determinants for the three unique submatrices. Separate figures for each sample size include the above information (Figs. 7-10). Also, the 95^{th} percentiles have been included to determine whether or not the individual values of $\hat{\rho}_v^2$ are statistically significant.

Our choices of sample size are based primarily on the synoptic time scales of variation in the surface wind fields. The winds shown in Figs. 7-10 indicate time scales of variation (i.e., "event" time scales) on the order of 2-4 days. Sample sizes of 8 (4 days), 16 (8 days) and 24 (12 days) clearly encompass these time scales. It is important to recognize that the sample size must be sufficient to include significant variation in the vector sequences being correlated. For sample sizes that are too small in this respect, spurious correlations may arise.

The results for a sample size of 8 (Fig. 7) indicate that significant variation in $\hat{\rho}_v^2$ itself occurs over the length of the series. The sample parameter $\hat{\rho}_v^2$ exceeds the 95th percentile slightly less than 50% of the time. Relatively high values ($\hat{\rho}_v^2$=1.5 or greater) tend to occur where major changes in surface wind (particularly noticeable in wind direction) are similar at both locations. Relatively low values of $\hat{\rho}_v^2$ (less than about 0.4) tend to occur throughout the record, but we find no obvious explanation for their occurrence. The trace and the determinant sequences do not provide consistent indications that reflect the behavior of $\hat{\rho}_v^2$ in this case; however, upon occasion relatively high values of the trace and/or the S_{12} determinant do tend to coincide with high values of $\hat{\rho}_v^2$.

As sample size increases from 8 to 16 and from 16 to 24, the correlations tend to be statistically significant in most cases (Figs. 8 and 9) but the changes in $\hat{\rho}_v^2$ tend to reflect to a lesser extent the major 2-4 day event-scale changes in surface wind. It becomes increasingly difficult to relate the values of $\hat{\rho}_v^2$ to individual events in the wind field. In the limit, when N equals 130, we obtain a single value for $\hat{\rho}_v^2$ that represents the correlation between the surface wind fields at the two locations over the entire record. In this case (Fig. 10) $\hat{\rho}_v^2$ is equal to 0.54 and is clearly statistically significant.

CONCLUSIONS

A need exists within the oceanographic and meteorological communities to agree upon a single definition for vector correlation. Because of the desirable statistical properties associated with the definition of vector correlation given by Jupp and Mardia (1980), we propose that this definition be adopted.

The primary emphasis of this report has been to present the definition of vector correlation of Jupp and Mardia with preliminary guidance on its use and interpretation. Consequently, considerably more effort should be devoted to the application of this technique to the practical problems that frequently arise in oceanography and meteorology in comparing vector quantities.

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APPENDIX 1

Definitions of symbols used in this paper.

Scalars are represented by small letters; u, v.

Vectors are represented by capital letters; W, X.

Matrices are represented by bold face capital letters; **S**, **A**.

The theoretical covariance of a vector W is represented by Σ_w.

TR(**A**) is the trace of a matrix.

|**A**| is the determinant of a matrix.

$\mathbf{A}^T$ indicates the transpose of a matrix.

$\mathbf{A}^{-1}$ indicates the inverse of a matrix.

I is an identity matrix.

0 is a matrix of zeros.

Greek letters are used to indicate population parameters (i.e. Σ, σ, ρ_v^2) and Roman letters for the corresponding sample parameters (i.e. S, s, r_v^2). The sample parameters may be represented by Greek letters with a ^ (i.e. $\hat{\sigma}$, $\hat{\rho}_v^2$).

APPENDIX 2

```
C
C    PROGRAM IN FORTRAN 77 TO CALCULATE THE VECTOR CORRELATION
C    COEFFICIENT OF JUPP AND MARDIA(BIOMETRIKA,1980).
C
C    THE FOLLOWING PROGRAM CALCULATES THE VECTOR CORRELATION BETWEEN
C    SEQUENCES OF TWO-DIMENSIONAL VECTORS. THESE VECTORS MUST BE
C    SPECIFIED IN TERMS OF THEIR U(X1,Y1) AND V(X2,Y2) COMPONENTS.
C    FOR TWO-DIMENSIONAL VECTORS, THE VECTOR CORRELATION COEFFICIENT
C    WILL VARY BETWEEN 0.0(UNCORRELATED) AND 2.0(COMPLETELY CORRELATED).
C
C    VECTOR CORRELATION COEFFICIENTS CALCULATED ACCORDING TO THE
C    FOLLOWING PROGRAM SHOULD BE ACCURATE TO AT LEAST THREE DECIMAL
C    PLACES USING SINGLE PRECISION ARITHMETIC.
C
C    NS IS THE SAMPLE SIZE - SAMPLE SIZES LESS THAN 8 ARE NOT
C    RECOMMENDED BECAUSE THE ASYMPTOTIC CHI-SQUARE DISTRIBUTION
C    BREAKS DOWN FOR SMALLER VALUES.
C
      PARAMETER (NS = SAMPLE SIZE)
      REAL*4 VAR1,VAR2,VAR3,VAR4,COV12,COV13,COV14,COV23,COV24,
     *COV34,X1(NS),Y1(NS),X2(NS),Y2(NS),R
C
C    ENTER INPUT VECTOR COMPONENTS X1(NS),Y1(NS),X2(NS) AND Y2(NS)
C
      WRITE(6,50)
   50 FORMAT(12X,' INPUT ',//)
      WRITE(6,100)
  100 FORMAT(5X,'X1          Y1          X2          Y2',//)
      DO 76 LK = 1,NS
        WRITE(6,70) X1(LK),Y1(LK),X2(LK),Y2(LK)
   76 CONTINUE
   70 FORMAT(4X,4(2X,F7.4))
      WRITE(6,700)
  700 FORMAT(2X,///)
         CALL STAT1(X1,Y1,X2,Y2,NS,VAR1,VAR2,COV12)
         CALL STAT2(X1,Y1,X2,Y2,NS,VAR1,VAR3,COV13)
         CALL STAT3(X1,Y1,X2,Y2,NS,VAR1,VAR4,COV14)
         CALL STAT4(X1,Y1,X2,Y2,NS,VAR2,VAR3,COV23)
         CALL STAT5(X1,Y1,X2,Y2,NS,VAR2,VAR4,COV24)
         CALL STAT6(X1,Y1,X2,Y2,NS,VAR3,VAR4,COV34)
         CALL RSQ(VAR1,VAR2,VAR3,VAR4,COV12,COV13,COV14,COV23,
     *   COV24,COV34,R)
       WRITE(6,200) R
  200 FORMAT(4X,'VECTOR CORRELATION COEFFICIENT = ',F8.5,//)
       STOP
       END
       SUBROUTINE STAT1(X1,Y1,X2,Y2,NS,VAR1,VAR2,COV12)
       REAL*4 VAR1,VAR2,COV12,SUMX1,SUMY1,SX1SQ,SY1SQ,SX1Y1,
     *X1(*),Y1(*),X2(*),Y2(*)
       SUMX1 = 0.0
       SUMY1 = 0.0
       SX1SQ = 0.0
       SY1SQ = 0.0
       SX1Y1 = 0.0
       DO 100 I = 1,NS
         SUMX1 = SUMX1 + X1(I)
         SUMY1 = SUMY1 + Y1(I)
         SX1SQ = SX1SQ + X1(I)*X1(I)
         SY1SQ = SY1SQ + Y1(I)*Y1(I)
         SX1Y1 = SX1Y1 + X1(I)*Y1(I)
  100 CONTINUE
       VAR1 = (FLOAT(NS) * SX1SQ - SUMX1 * SUMX1) / FLOAT(NS * (NS - 1))
       VAR2 = (FLOAT(NS) * SY1SQ - SUMY1 * SUMY1) / FLOAT(NS * (NS - 1))
       COV12 = (FLOAT(NS) * SX1Y1 - SUMX1 * SUMY1) / FLOAT(NS * (NS - 1))
       RETURN
       END
       SUBROUTINE STAT2(X1,Y1,X2,Y2,NS,VAR1,VAR3,COV13)
       REAL*4 VAR1,VAR3,COV13,SUMX1,SUMX2,SX1SQ,SX2SQ,SX1X2,
     *X1(*),Y1(*),X2(*),Y2(*)
       SUMX1 = 0.0
       SUMX2 = 0.0
       SX1SQ = 0.0
       SX2SQ = 0.0
       SX1X2 = 0.0
```

```
      DO 100 I = 1,NS
         SUMX1 = SUMX1 + X1(I)
         SUMX2 = SUMX2 + X2(I)
         SX1SQ = SX1SQ + X1(I)*X1(I)
         SX2SQ = SX2SQ + X2(I)*X2(I)
         SX1X2 = SX1X2 + X1(I)*X2(I)
  100 CONTINUE
      VAR1 = (FLOAT(NS) * SX1SQ - SUMX1 * SUMX1) / FLOAT(NS * (NS - 1))
      VAR3 = (FLOAT(NS) * SX2SQ - SUMX2 * SUMX2) / FLOAT(NS * (NS - 1))
      COV13 = (FLOAT(NS) * SX1X2 - SUMX1 * SUMX2) / FLOAT(NS * (NS - 1))
      RETURN
      END

      REAL*4 VAR1,VAR4,COV14,SUMX1,SUMY2,SX1SQ,SY2SQ,SX1Y2,
     *X1(*),Y1(*),X2(*),Y2(*)
      SUMX1 = 0.0
      SUMY2 = 0.0
      SX1SQ = 0.0
      SY2SQ = 0.0
      SX1Y2 = 0.0
      DO 100 I = 1,NS
         SUMX1 = SUMX1 + X1(I)
         SUMY2 = SUMY2 + Y2(I)
         SX1SQ = SX1SQ + X1(I)*X1(I)
         SY2SQ = SY2SQ + Y2(I)*Y2(I)
         SX1Y2 = SX1Y2 + X1(I)*Y2(I)
  100 CONTINUE
      VAR1 = (FLOAT(NS) * SX1SQ - SUMX1 * SUMX1) / FLOAT(NS * (NS - 1))
      VAR4 = (FLOAT(NS) * SY2SQ - SUMY2 * SUMY2) / FLOAT(NS * (NS - 1))
      COV14 = (FLOAT(NS) * SX1Y2 - SUMX1 * SUMY2) / FLOAT(NS * (NS - 1))
      RETURN
      END
      SUBROUTINE STAT4(X1,Y1,X2,Y2,NS,VAR2,VAR3,COV23)
      REAL*4 VAR2,VAR3,COV23,SUMY1,SUMX2,SY1SQ,SX2SQ,SY1X2,
     *X1(*),Y1(*),X2(*),Y2(*)
      SUMY1 = 0.0
      SUMX2 = 0.0
      SY1SQ = 0.0
      SX2SQ = 0.0
      SY1X2 = 0.0
      DO 100 I = 1,NS
         SUMY1 = SUMY1 + Y1(I)
         SUMX2 = SUMX2 + X2(I)
         SY1SQ = SY1SQ + Y1(I)*Y1(I)
         SX2SQ = SX2SQ + X2(I)*X2(I)
         SY1X2 = SY1X2 + Y1(I)*X2(I)
  100 CONTINUE
      VAR2 = (FLOAT(NS) * SY1SQ - SUMY1 * SUMY1) / FLOAT(NS * (NS - 1))
      VAR3 = (FLOAT(NS) * SX2SQ - SUMX2 * SUMX2) / FLOAT(NS * (NS - 1))
      COV23 = (FLOAT(NS) * SY1X2 - SUMY1 * SUMX2) / FLOAT(NS * (NS - 1))
      RETURN
      END
      SUBROUTINE STAT5(X1,Y1,X2,Y2,NS,VAR2,VAR4,COV24)
      REAL*4 VAR2,VAR4,COV24,SUMY1,SUMY2,SY1SQ,SY2SQ,SY1Y2,
     *X1(*),Y1(*),X2(*),Y2(*)
      SUMY1 = 0.0
      SUMY2 = 0.0
      SY1SQ = 0.0
      SY2SQ = 0.0
      SY1Y2 = 0.0
      DO 100 I = 1,NS
         SUMY1 = SUMY1 + Y1(I)
         SUMY2 = SUMY2 + Y2(I)
         SY1SQ = SY1SQ + Y1(I)*Y1(I)
         SY2SQ = SY2SQ + Y2(I)*Y2(I)
         SY1Y2 = SY1Y2 + Y1(I)*Y2(I)
  100 CONTINUE
      VAR2 = (FLOAT(NS) * SY1SQ - SUMY1 * SUMY1) / FLOAT(NS * (NS - 1))
      VAR4 = (FLOAT(NS) * SY2SQ - SUMY2 * SUMY2) / FLOAT(NS * (NS - 1))
      COV24 = (FLOAT(NS) * SY1Y2 - SUMY1 * SUMY2) / FLOAT(NS * (NS - 1))
      RETURN
      END
```

```
      SUBROUTINE STAT6(X1,Y1,X2,Y2,NS,VAR3,VAR4,COV34)
      REAL*4 VAR3,VAR4,COV34,SUMX2,SUMY2,SX2SQ,SY2SQ,SX2Y2,
     *X1(*),Y1(*),X2(*),Y2(*)
      SUMX2 = 0.0
      SUMY2 = 0.0
      SX2SQ = 0.0
      SY2SQ = 0.0
      SX2Y2 = 0.0
      DO 100 I = 1,NS
        SUMX2 = SUMX2 + X2(I)
        SUMY2 = SUMY2 + Y2(I)
        SX2SQ = SX2SQ + X2(I)*X2(I)
        SY2SQ = SY2SQ + Y2(I)*Y2(I)
        SX2Y2 = SX2Y2 + X2(I)*Y2(I)
100   CONTINUE
      VAR3 = (FLOAT(NS) * SX2SQ - SUMX2 * SUMX2) / FLOAT(NS * (NS - 1))
      VAR4 = (FLOAT(NS) * SY2SQ - SUMY2 * SUMY2) / FLOAT(NS * (NS - 1))
      COV34 = (FLOAT(NS) * SX2Y2 - SUMX2 * SUMY2) / FLOAT(NS * (NS - 1))
      RETURN
      END
      SUBROUTINE RSQ(VAR1,VAR2,VAR3,VAR4,COV12,COV13,COV14,
     *COV23,COV24,COV34,R)
      REAL*4 R,R2,VAR1,VAR2,VAR3,VAR4,COV12,COV13,COV14,COV23,
     *COV24,COV34,A1,A2,A3,A4,A5,A6,A7,A8,B1,B2,B3,B4,B5,B6,
     *B7,B8,C,D,E,F,G,H,I,J,M,N,T1,T2,U1,U2,T,U,P
      A1 = VAR2 * COV13
      A2 = COV12 * COV23
      A3 = VAR4 * COV13
      A4 = COV34 * COV14
      A5 = VAR2 * COV14
      A6 = COV12 * COV24
      A7 = COV34 * COV13
      A8 = VAR3 * COV14
      B1 = COV12 * COV13
      B2 = VAR1 * COV23
      B3 = VAR4 * COV23
      B4 = COV34 * COV24
      B5 = COV12 * COV14
      B6 = VAR1 * COV24
      B7 = COV34 * COV23
      B8 = VAR3 * COV24
      C = A1 - A2
      D = A3 - A4
      E = A5 - A6
      F = A8 - A7
      G = B2 - B1
      H = B3 - B4
      I = B6 - B5
      J = B8 - B7
      M = C*D + E*F
      N = G*H + I*J
      T1 = VAR1 * VAR2
      T2 = COV12 * COV12
      U1 = VAR3 * VAR4
      U2 = COV34 * COV34
      T = T1 - T2
      U = U1 - U2
      P = T * U
      R2 = (M + N)/P
      R = SQRT(R2)
      RETURN
      END
```

FIGURE CAPTIONS

Figure 1. Cumulative frequency distributions for two-dimensional vectors for sample sizes of 8, 12, 32, 64, and for the theoretical chi-square distribution with four degrees of freedom. See text for details.

Figure 2. Probability density functions (PDFs) corresponding to the cumulative frequency distributions shown in Figure 1 for sample sizes of 12 and 32. The PDF for the theoretical chi-square distribution is also included.

Figure 3. Nr^2 versus sample size for a 95% level of confidence. Mean values of nr^2 $\pm$ one sigma are plotted for each sample size.

Figure 4. Examples of vector sequences which produce perfect correlation (i.e., =2.0). The first case, (a), arises when the vectors are identical; the second case, (b), arises when the magnitudes of the vectors in the second sequence are multiplied by a constant; the third case, (c), arises when the directions of the vectors in the second sequence are each rotated by a constant angle; and the fourth case, (d), arises when the second sequence is both multiplied by a constant, and rotated by a constant angle.

Figure 5. The case leading to zero correlation between two vector sequences. In this case, the vectors in each sequence are generated randomly and the results averaged over 50 realizations and then plotted for sample sizes of 10, 100, 1000, 10000 and 100000.

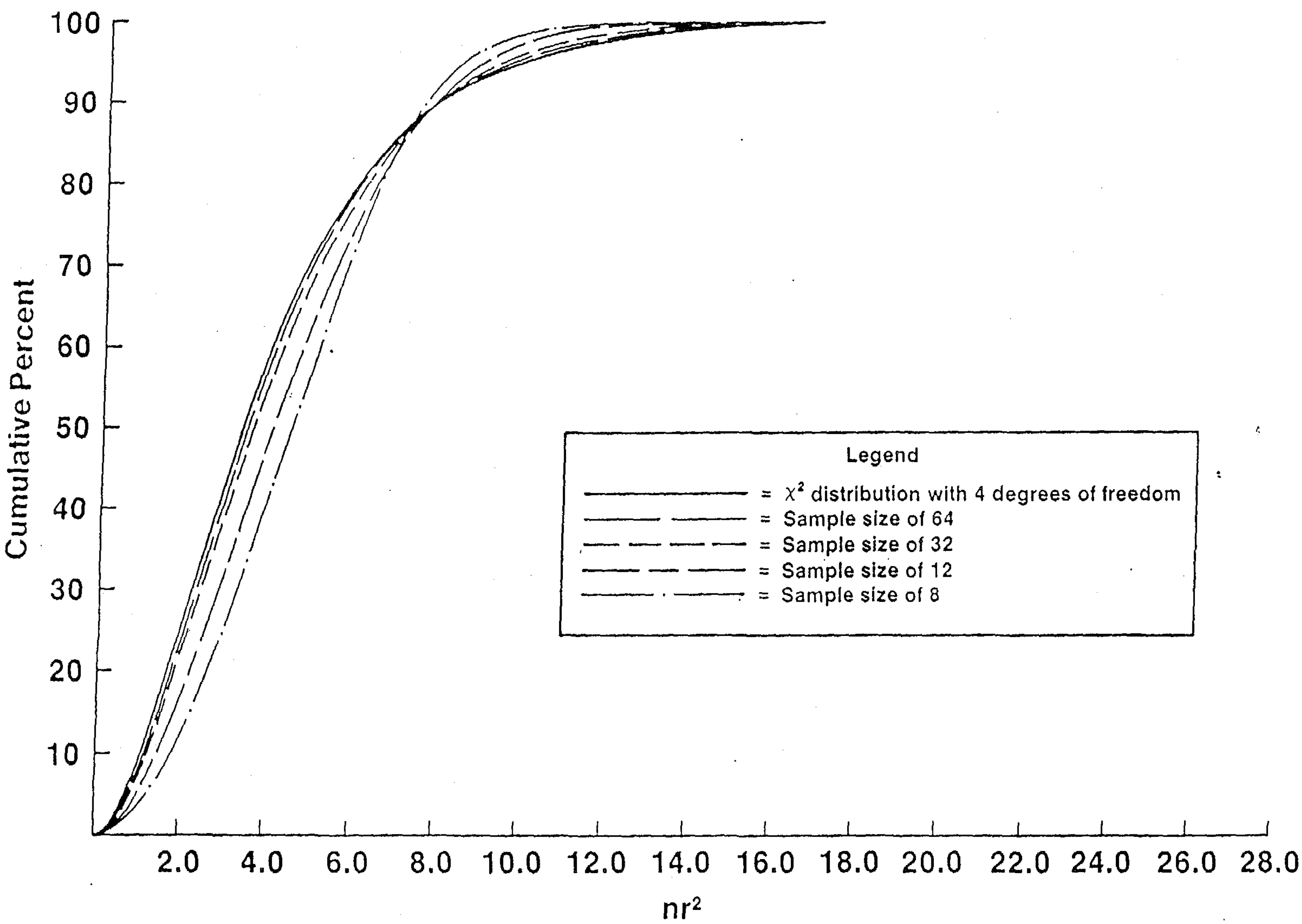
Figure 6. Locations of the two NDBC environmental data buoys from which time-series surface winds were extracted. Period covers 1 December 1987 to 4 February 1988. Typical winter storm track has been included (Klein, 1957).

Figure 7. Vector correlation and related sequences for buoy winds for a sample size of 8. Stick diagrams for the winds at each buoy are shown in the upper two plots. The vector correlation coefficient plus the 95% level of confidence follows. Next, the trace of the vector correlation matrix is shown, followed by the determinants of the A_{11}, A_{12} (A_{21}), A_{22} submatrices, and the full 4 x 4 matrix.

Figure 8. Same as Figure 7 but for a sample size of 16.

Figure 9. Same as Figure 7 but for a sample size of 24.

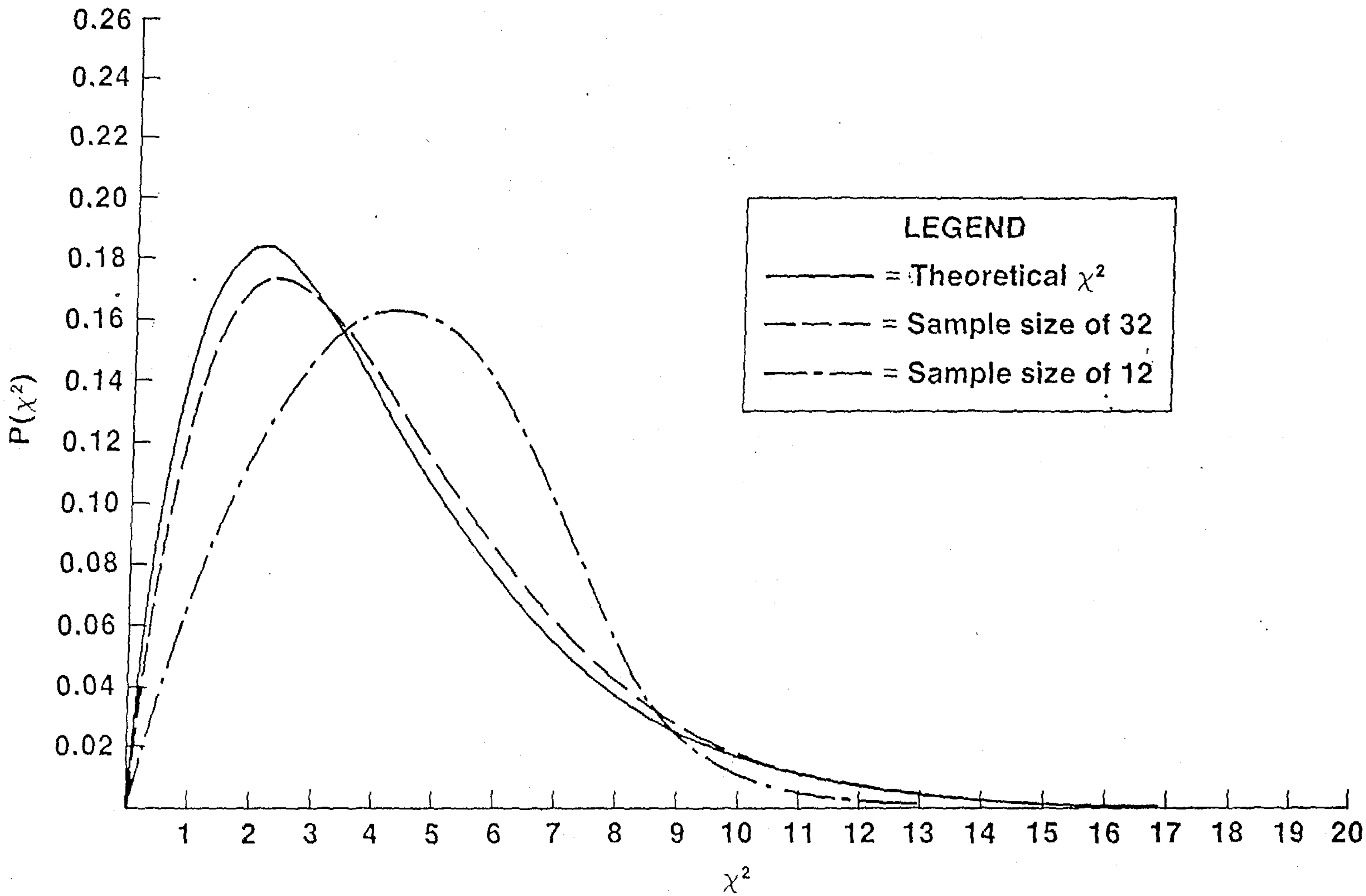
Figure 10. Same as Figure 7 but for a sample size of 130 (i.e., the entire sequence).



Cumulative Percent
100
90
80
70
60
50
40
30
20
10
2.0 4.0 6.0 8.0 10.0 12.0 14.0 16.0 18.0 20.0 22.0 24.0 26.0 28.0
nr^2
Legend
= χ^2 distribution with 4 degrees of freedom
= Sample size of 64
= Sample size of 32
= Sample size of 12
= Sample size of 8

Fig. 1

Fig. 2



LEGEND
= Theoretical χ^2
= Sample size of 32
= Sample size of 12
$P(\chi^2)$
0.26
0.24
0.22
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
χ^2

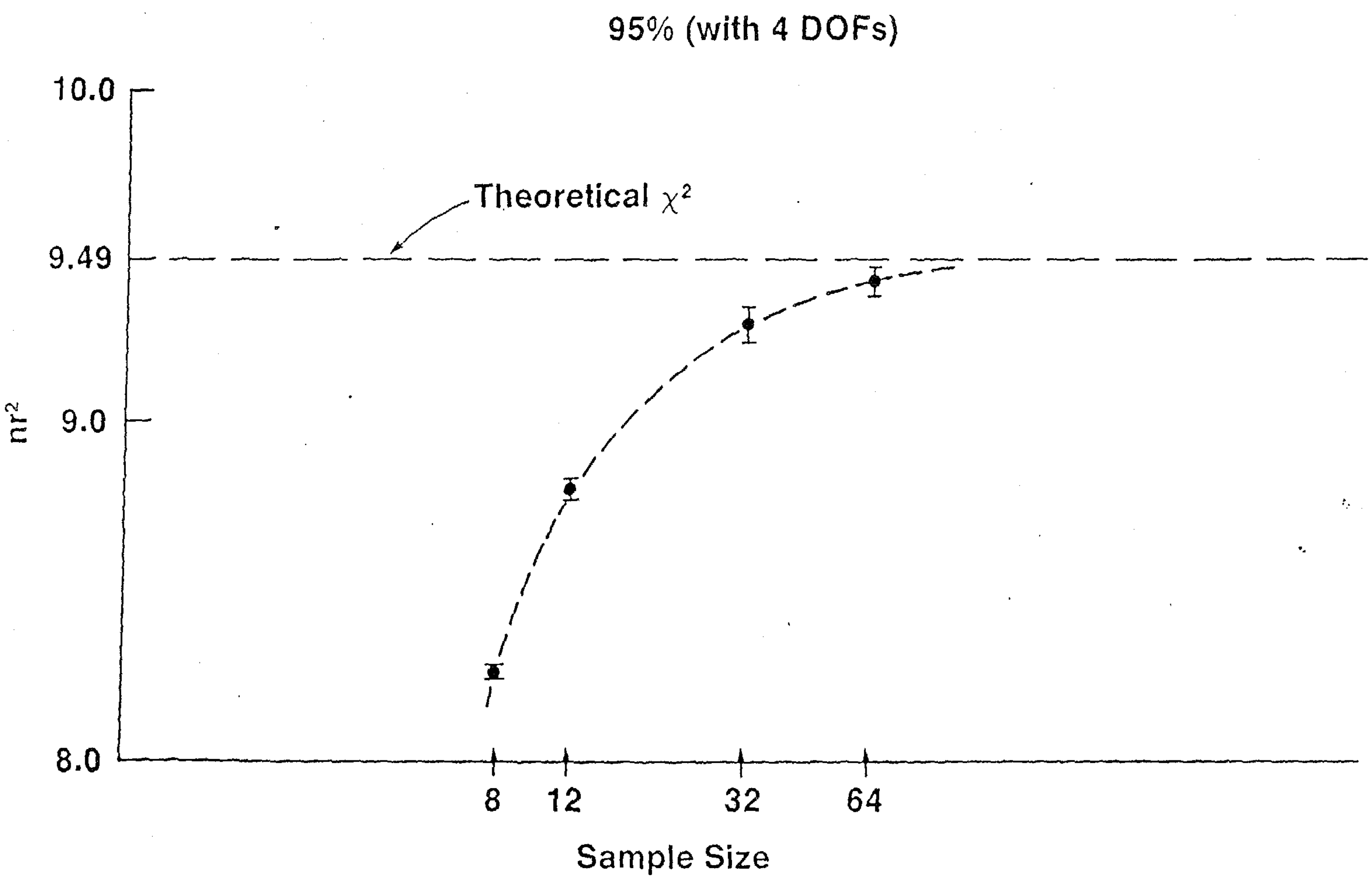

95% (with 4 DOFs)
Theoretical χ^2
10.0
9.49
9.0
8.0
nr^2
8
12
32
64
Sample Size

Fig. 3

CASES FOR PERFECT CORRELATION

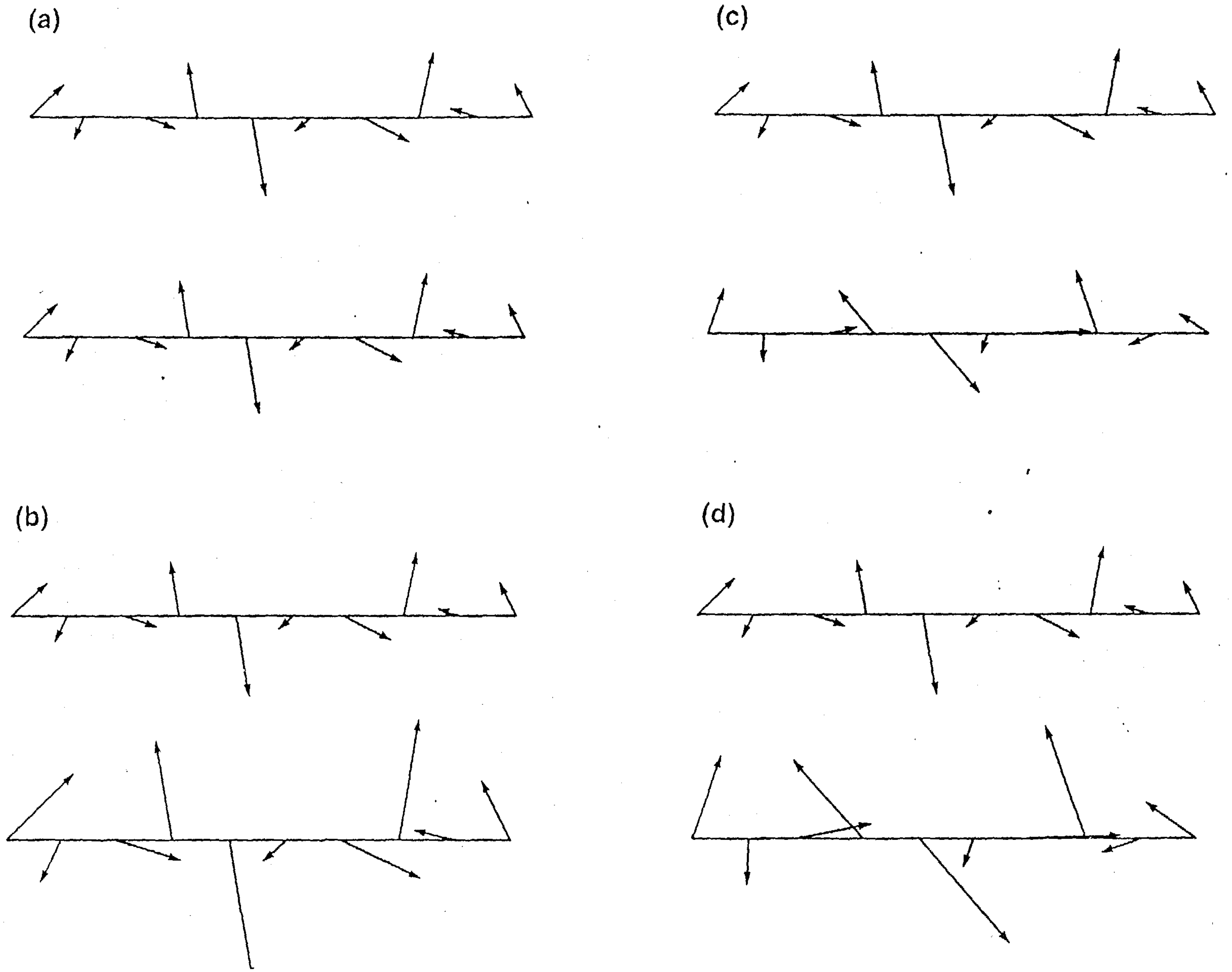


Fig. 4

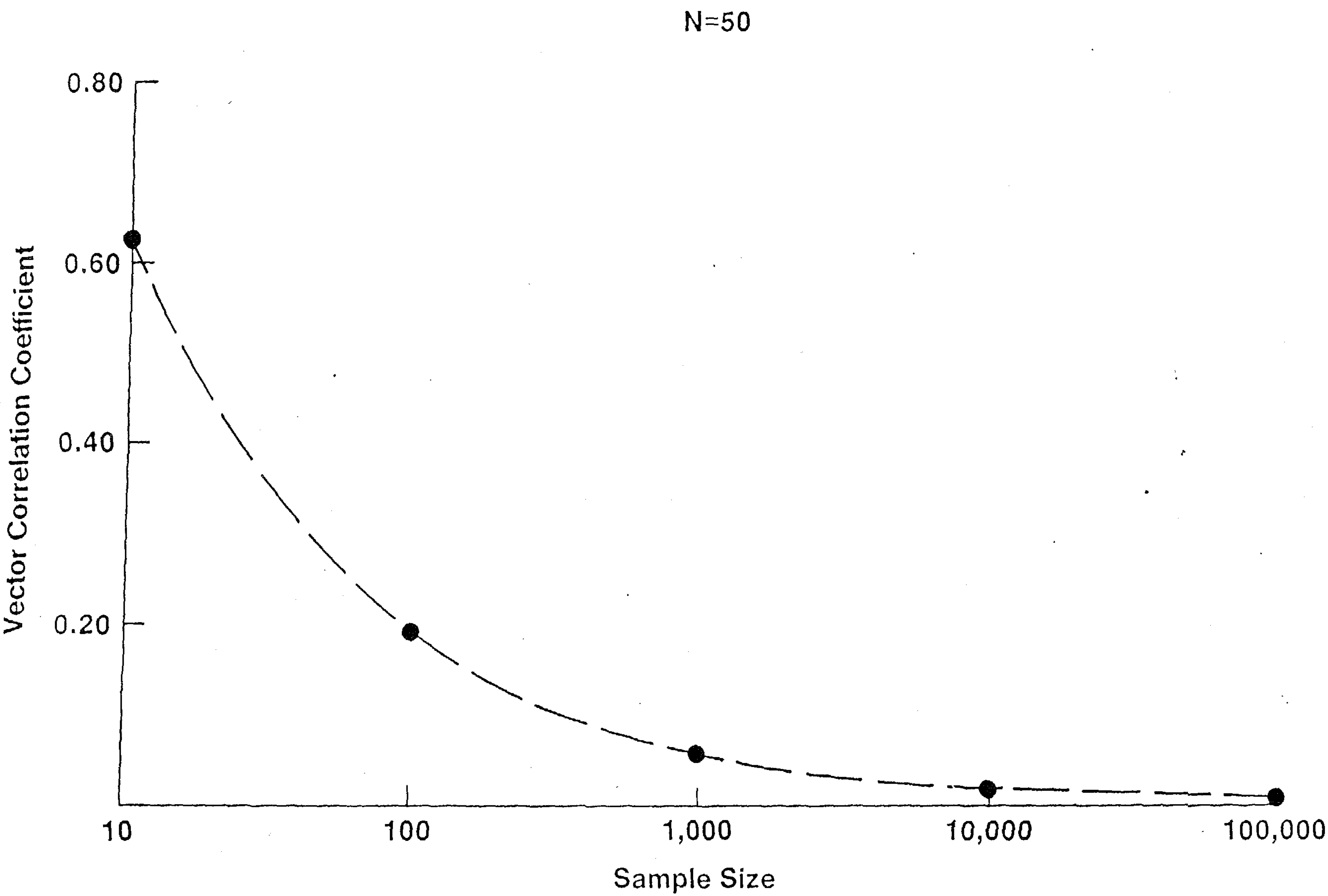

N=50
Vector Correlation Coefficient
0.80
0.60
0.40
0.20
10
100
1,000
10,000
100,000
Sample Size

Fig. 5

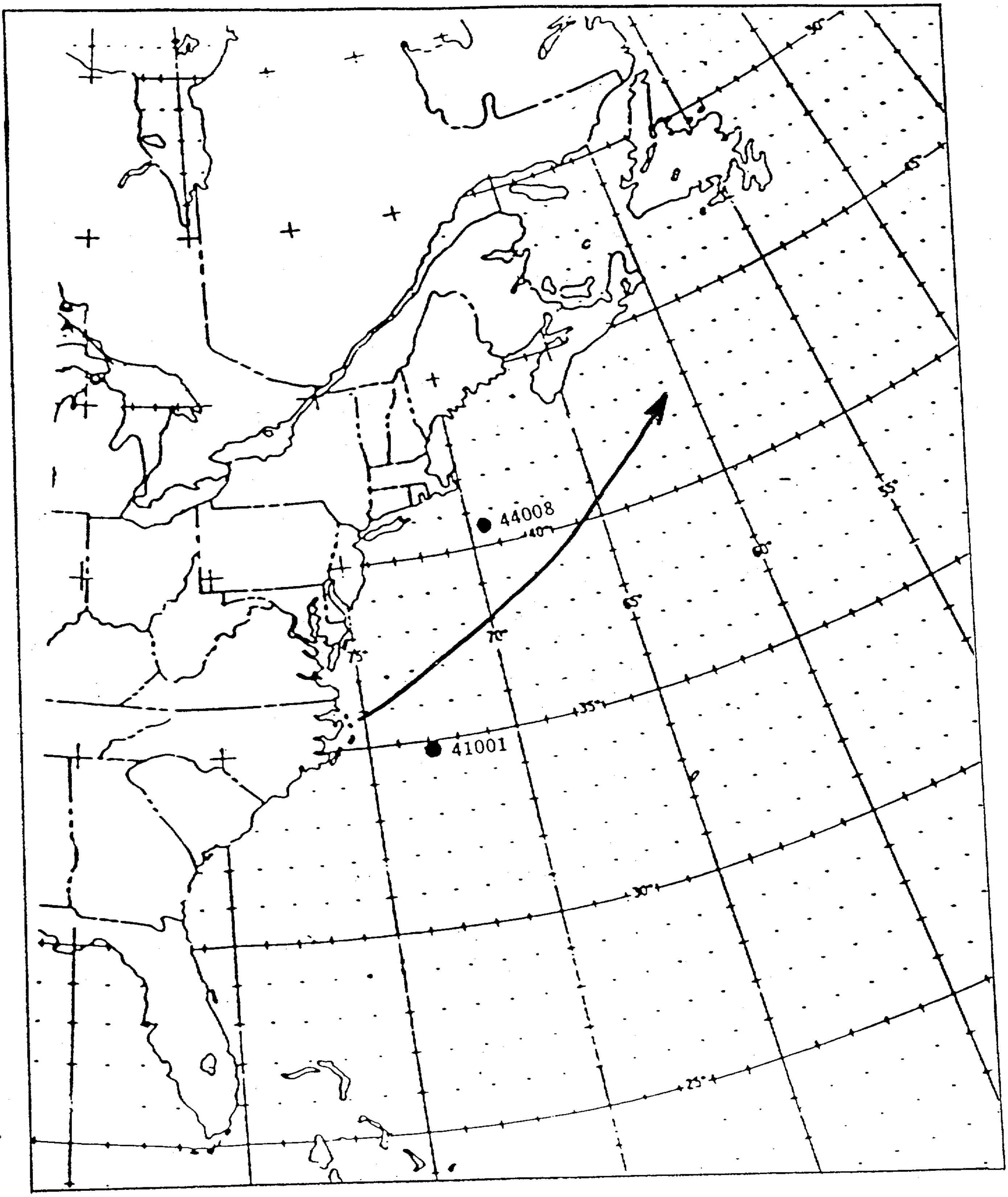


Fig. 6

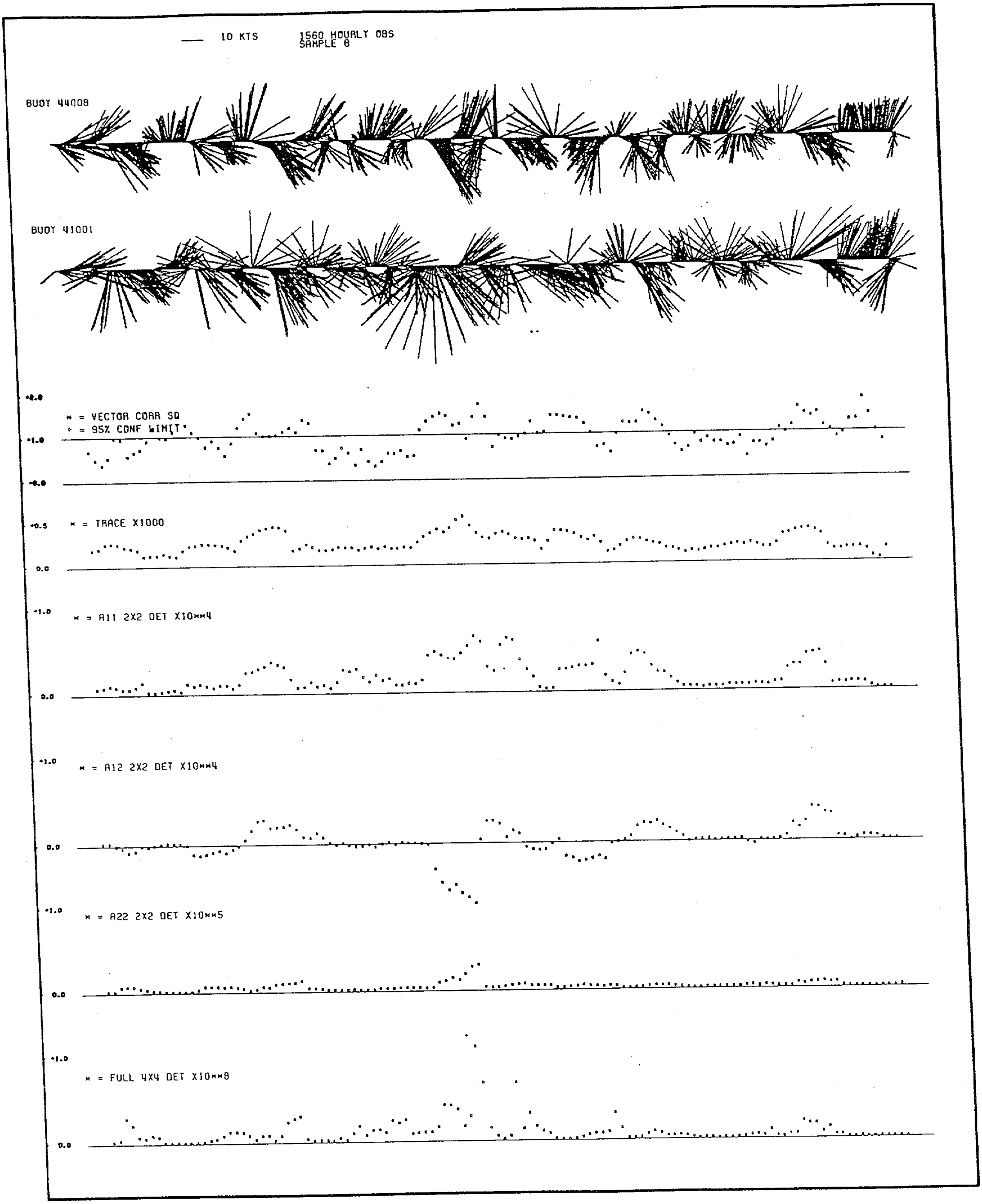
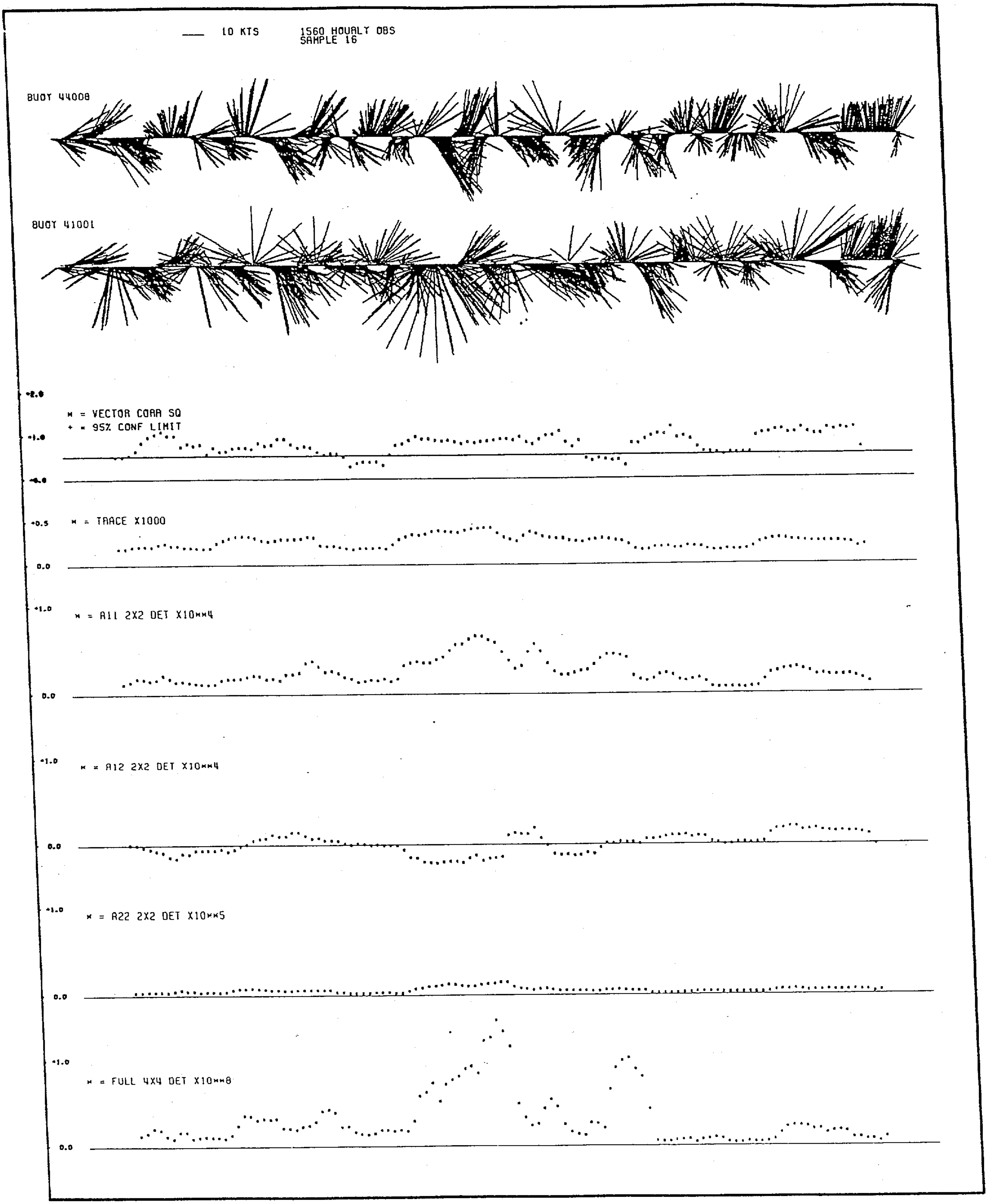
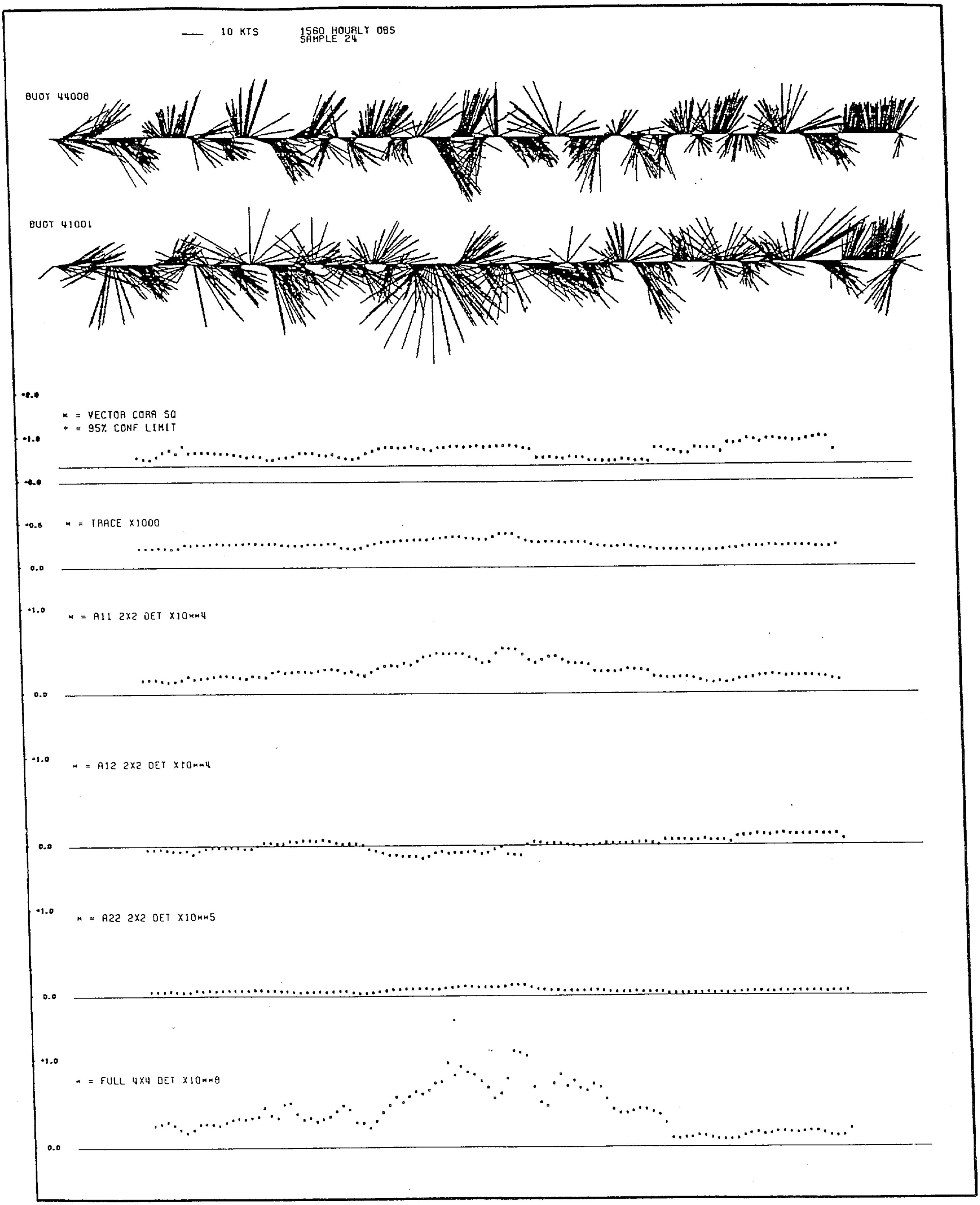

10 KTS
1560 HOURLY OBS
SAMPLE 8
BUOY 44008
BUOY 41001
* = VECTOR CORR SQ
+ = 95% CONF LIMIT
* = TRACE X1000
* = A11 2X2 DET X10**4
* = A12 2X2 DET X10**4
* = A22 2X2 DET X10**5
* = FULL 4X4 DET X10**8

Fig. 7



10 KTS
1560 HOURLY OBS
SAMPLE 16
BUOY 44008
BUOY 41001
+2.0
* = VECTOR CORR SQ
+ = 95% CONF LIMIT
+1.0
+0.0
+0.5
* = TRACE X1000
0.0
+1.0
* = A11 2X2 DET X10**4
0.0
+1.0
* = A12 2X2 DET X10**4
0.0
+1.0
* = A22 2X2 DET X10**5
0.0
+1.0
* = FULL 4X4 DET X10**8
0.0

Fig. 8



10 KTS
1560 HOURLY OBS
SAMPLE 24
BUOY 44008
BUOY 41001
+2.0
* = VECTOR CORR SQ
+ = 95% CONF LIMIT
+1.0
+0.0
+0.5
* = TRACE X1000
0.0
+1.0
* = A11 2X2 DET X10**4
0.0
+1.0
* = A12 2X2 DET X10**4
0.0
+1.0
* = A22 2X2 DET X10**5
0.0
+1.0
* = FULL 4X4 DET X10**8
0.0

Fig. 9

37.	OPC Product Rev. Summary.	L.D. BURROUGHS	TECH. NOTE OFFICE NOTE #359	
38.	Compendium of Marine Meteorological & Oceanographic Products of the Ocean Products Center. (rev.1)	D.M. FEIT	NOAA TECH. MEMO. NWS NMC 68	6/89
39.	Directional Wave Spectra for the Labrador Extreme Wave Experimental (Lewex).	D.C. ESTEVA Y.Y. CHAO	APL TECH. REPORT in press	
40.	An Analysis of Monthly Sea Surface Temperature Anomalies in Waters off the U. S. East and West Coasts.	L.C. Breaker W.B. Campbell	Fisheries Bulletin	8/89
41.	A Definition for Vector Correlation and its Application to Marine Surface Winds.	D.S. Crosby L.C. Breaker W.H. Gemmill	TECH NOTE OFFICE NOTE #365	
42.	Expert System for Quality Control and Marine Forecasting Guidance.	D.M. Feit W.S. Richardson		
43.	OPC Unified Marine Database Verification System.	V.M. Gerald	TECH NOTE NMC OFFICE NOTE #361	
44.	Sea Ice Edge Forecast Verification Program for the Bering Sea.	G.Wohl	TECH NOTE NMC OFFICE NOTE #370	

DEPARTMENT OF COMMERCE

UNITED STATES OF AMERICA

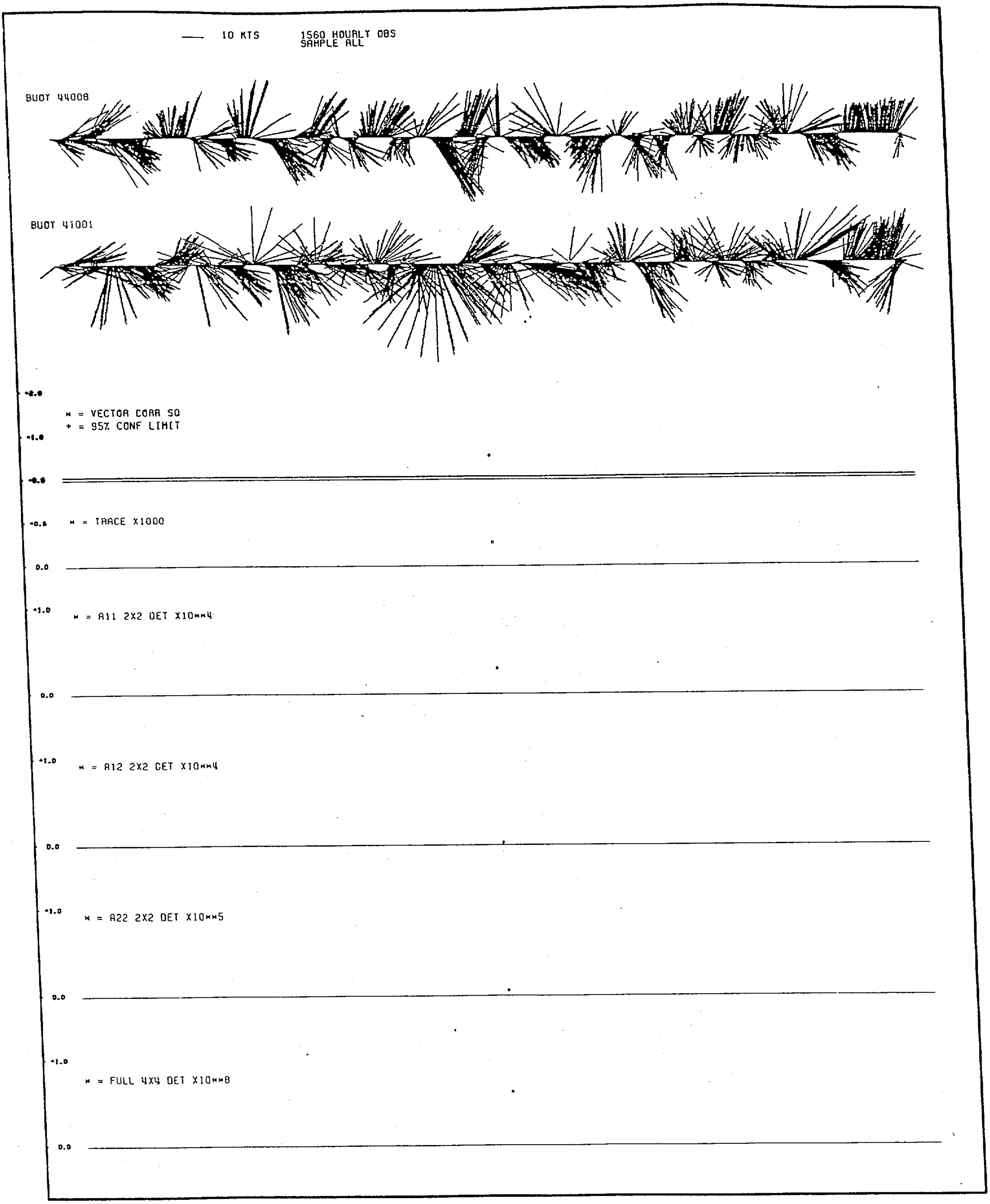

10 KTS
1560 HOURLY OBS
SAMPLE ALL
BUOY 44008
BUOY 41001
+2.0
* = VECTOR CORR SQ
+ = 95% CONF LIMIT
+1.0
+0.0
+0.5
* = TRACE X1000
0.0
+1.0
* = A11 2X2 DET X10**4
0.0
+1.0
* = A12 2X2 DET X10**4
0.0
+1.0
* = A22 2X2 DET X10**5
0.0
+1.0
* = FULL 4X4 DET X10**8
0.0

Fig. 10