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MODELING WIND WAVES USING WAVENUMBER-DIRECTION SPECTRA AND A VARIABLE WAVENUMBER GRID*

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The use of the wavenumber-direction spectrum in wind wave models results in an effective loss of model resolution for waves traveling from deep to shallow water and in additional numerical disadvantages, when a conventional invariant spectral grid is used. In this paper we present a theoretical study of how the effects of variable depths and currents may be incorporated in a spatially varying wavenumber grid. It is shown that effects of currents cannot be efficiently incorporated in the grid. Effects of variable depths are incorporated in a wavenumber grid which is equivalent to a spatially invariant frequency grid. The resulting equations are nearly identical to the conventional equations for the frequency-direction spectrum, but include a more elegant way to address effects of temporal variations of the water depth. Furthermore, the technique employed to derive the equations for the variable grid approach closely resembles that used in the conversion between different spectral descriptions. The formulations presented in this paper may therefore serve as a basis for discussion of the selection of spectral descriptions.

Keywords: Wind waves; wind wave modeling

1. INTRODUCTION

Wind waves in shelf seas and oceans are historically described using their surface elevation spectrum as a function of frequency f, as can be obtained from a time series analysis of the water level elevation at a single point. A

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similar approach was adopted in most early spectral wave models, which usually solve an energy balance equation for the two-dimensional spectrum $F(f, \theta)$, where θ represents the direction of the spectral component (see, for instance, SWAMP Group 1985). From a theoretical point of view, however, the wavenumber spectrum $F(\mathbf{k})$ or $F(k, \theta)$ has been considered more appropriate for modeling wind waves due to its invariance properties with respect to the water depth for the physics of wave growth and decay (*e.g.*, Kitaigorodskii, 1962, 1983; Kitaigorodskii *et al.*, 1975; Bouws *et al.*, 1985). For this reason, the wavenumber-direction spectrum has been used to describe the wave field in several recent wave models (*e.g.*, Abdalla and Ozhan, 1993; Jansen *et al.*, 1993; Van Vledder and Dee, 1994). For such models the wavenumber-direction spectrum $F(k, \theta)$ is generally preferable to the vector wavenumber spectrum $F(\mathbf{k})$, because discretization of the latter spectrum leads to a directional anisotropy of the spectral resolution.

When a component of a wave field travels from deep to shallow water ('shoaling'), its wavenumber undergoes large kinematic variations. This is illustrated in Figure 1, which shows characteristics in wavenumber-depth $(k-d)$ space for waves propagating in water with varying but steady depths without mean currents (solid lines). In such conditions, the frequency f remains invariant, and k follows directly from d and the dispersion relation

$$\sigma^2 = gk \tanh kd, \tag{1}$$

where a $\sigma = 2\pi f$ is the radian (intrinsic) frequency. The characteristics in Figure 1 correspond to a set of discrete frequencies, representative for the spectral discretization in ocean wave models. The discrete wavenumber grid of a numerical wave model is usually kept constant throughout the model (illustrated by the dotted lines in Fig. 1). Shoaling then leads to two numerical disadvantages.

First, the kinematic wavenumber variations compress the wave field in k-space for decreasing depths, resulting in an effective loss of resolution in shallow water. This is quantified in Table I, which presents the resolution of an invariant k-grid relative to that of an invariant σ-grid for several wave periods and water depths, assuming identical resolutions in deep water (upper left corner of table). For intermediate depths, the resolution of the wavenumber grid actually improves somewhat (up to 20% for $kd \approx 1.2$). For shallow water, however, (lower right side of table), the relative resolution of the wavenumber grid deteriorates dramatically (by more than 50% for $kd < 0.27$). Particularly swell energy is generally poorly resolved by commonly used spectral resolutions. To assure that the spectral

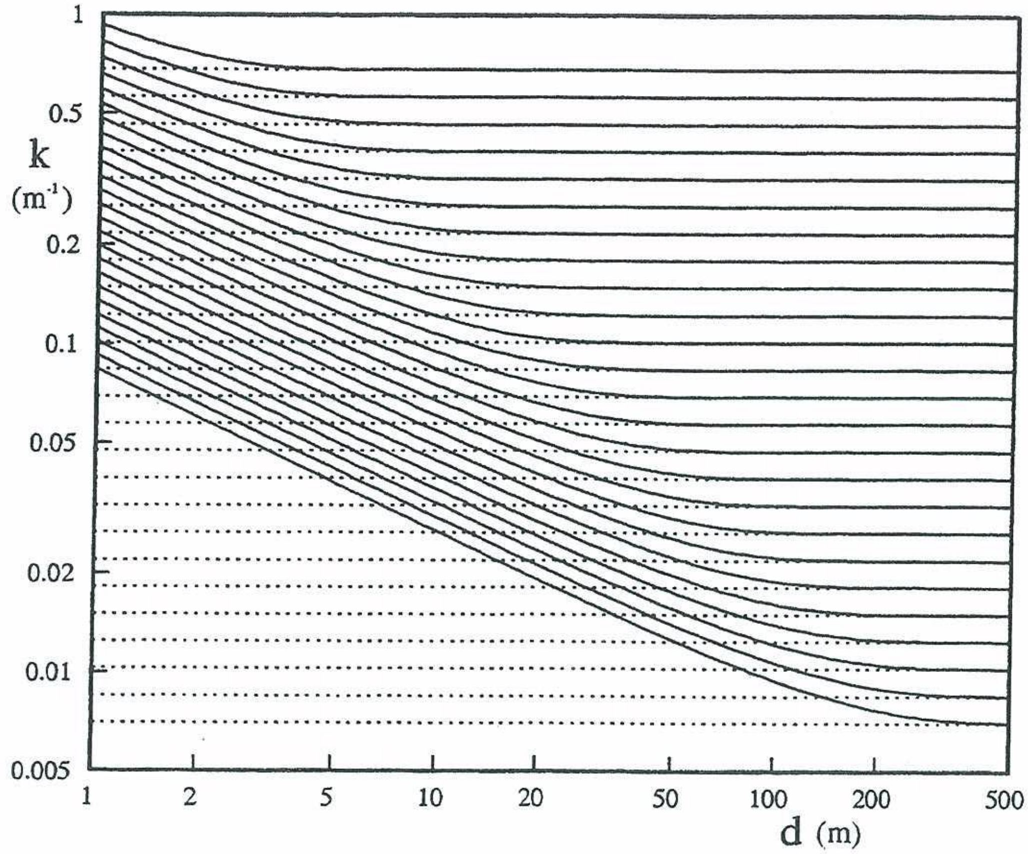

FIGURE 1 Characteristics (solid lines) in wavenumber-depth ($k-d$) space for quasi-steady water depth variations in conditions without mean currents. The dotted lines represent an exponential wavenumber grid corresponding to a frequency grid as representative for numerical wave models ($f_{i+1} = 1.1\, f_i$, with f ranging from 0.041 Hz to 0.41 Hz).

TABLE I Resolution of an invariant wavenumber grid relative to an invariant relative frequency grid for several wave periods T and depths d. Relative resolution defined as inverse ratio of grid increments Δk or $\Delta\sigma$. Identical deep-water resolutions

$T(s)$	$d(m)$					
	80	40	20	10	5	2.5
5	1.00	1.00	1.02	1.15	1.19	1.03
10	1.02	1.15	1.19	1.03	0.81	0.60
20	1.19	1.03	0.81	0.60	0.44	0.31

resolution for target swell periods of the k-grid is comparable to that of a σ-grid, the number of discrete wavenumbers has to be increased by the inverse of the normalized resolutions presented in Table I. This will obviously have a significant impact on memory and run-time constraints of a practical model.

Secondly, shoaling corresponds to significant propagation in the k-space of the spectrum, which invites numerical errors, and might impose stability constraints on a model. Numerical errors depend directly on the schemes used, and can therefore not be addressed objectively without selecting specific numerical schemes. Stability (and accuracy), however, can be assessed objectively by considering CFL (Courant-Friedrichs-Lewy) limitations of a numerical model. Consider, for example, a numerical model with a spatial resolution of 25 km and a logarithmic distribution of discrete wavenumbers with $\Delta k = 0.21k$ (which in deep water corresponds to the common $\Delta\sigma = 0.1\sigma$). Assume that the longest spectral components have a deepwater period of 25 s, and that CFL < 1 in all spaces for reasons of stability or accuracy. Spatial propagation then allows a maximum time step of approximately 900 s. In a shelf sea, where the change in depth between grid points is typically smaller than the local depth, propagation in k-space generally will allow larger time steps, unless extremely shallow water is encountered (depth less than 5 m). At the edge of a continental shelf, however, the depth may drop from the order of 100 m to 1000 m between grid points. In such conditions, the required time step for propagation in k-space becomes an order of magnitude smaller than the required time step for spatial propagation. For most wave models this implies a serious increase of computational effort, or requires modifications like smoothing of the bathymetry or filtering of propagation velocities in k-space.

Because the frequency of wave components is invariant with respect to shoaling (for steady depths), both disadvantages of using wavenumber spectra do not occur when using frequency spectra. Because kinematic wavenumber variations due to shoaling are reversible, a natural way of getting around these disadvantages of the wavenumber spectra is to include kinematic effects in the discrete grid, in other words, to use the characteristics in $k - d$ space (solid lines in Fig. 1) to generate a spatially varying wavenumber grid to replace the spatially invariant grid (dotted lines in Fig. 1). The present paper discusses the application of such variable wavenumber grids to a conventional action balance equation. In Section 2 a spectral balance equation for arbitrary grids is discussed. In Section 3 practical grids are discussed. It is shown that effects of variable depths can be incorporated in the grid, but that effects of currents should not be included in the grid. In Section 4 the equations for a grid incorporating effects of depth variations are discussed. The corresponding equations closely resemble the balance equations for a spectrum defined in terms of the intrinsic frequency, except for the treatment of temporal variations of the water depth. A final discussion is presented in Section 5.

2. VARIABLE WAVENUMBER GRIDS

The surface elevation spectrum $F(k, \theta)$ adequately describes ocean surface waves. The action spectrum $N(k, \theta) \equiv F(k, \theta)/\sigma$, however, is more suitable for the use in numerical wave models due to its invariance characteristics for propagation over slowly varying depths and currents [$\mathbf{U} = (U_x, U_y)$] (*e.g.*, Bretherthon and Garrett, 1968; Whitham, 1974). The governing equations for the evolution of this spectrum can be written as (*e.g.*, Hasselmann *et al.*, 1973; Willebrand, 1974)

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x}\dot{x}N + \frac{\partial}{\partial y}\dot{y}N + \frac{\partial}{\partial k}\dot{k}N + \frac{\partial}{\partial \theta}\dot{\theta}N = S, \tag{2}$$

$$\dot{x} = c_g \cos\theta + U_x, \tag{3}$$

$$\dot{y} = c_g \sin\theta + U_y, \tag{4}$$

$$\dot{k} = -\frac{\partial \sigma}{\partial d}\frac{\partial d}{\partial s} - \mathbf{k} \cdot \frac{\partial \mathbf{U}}{\partial s}, \tag{5}$$

$$\dot{\theta} = -\frac{1}{k}\left[\frac{\partial \sigma}{\partial d}\frac{\partial d}{\partial m} - \mathbf{k} \cdot \frac{\partial \mathbf{U}}{\partial m}\right], \tag{6}$$

where $c_g \equiv \partial\sigma/\partial k$ is the group velocity, s is a coordinate in the direction θ, m is a coordinate perpendicular to s and S represents the net source term for wave generation and dissipation. Formally N and S depend on k, θ, the spatial coordinates (x, y) and time t; $N = N(k, \theta; x, y, t)$ and $S \equiv S(k, \theta; x, y, t)$. For convenience of notation, however, the spectrum and source terms are simply denoted as N and S.

Equations (2) through (6) are usually solved on a spatially and temporally invariant $k - \theta$-grid and a temporally invariant spatial grid. To facilitate its transfer to an arbitrary grid, Eq. (2) is rewritten as

$$\frac{\partial}{\partial x_i}(v_i N) = S, \tag{7}$$

where the summation convention is used and where $\mathbf{x} = (t, x, y, k, \theta)$ represents the conventional grid. The characteristics $\dot{x}_i = v_i$ are given by Eqs. (3) through (6) and by $\dot{t} \equiv 1$. Let $\mathbf{y}$ be a second grid where $y_i = f(\mathbf{x})$. Its characteristics $\dot{y}_i = w_i$ have to coincide with the characteristics in $\mathbf{x}$, so that

$$\dot{y}_i = w_i = \frac{\partial y_i}{\partial x_j} v_j, \tag{8}$$

After some algebra (see Appendix), the equations on the new grid become

$$\frac{\partial}{\partial y_i}\left(w_i \frac{N}{J}\right) = \frac{S}{J}, \tag{9}$$

where J is the Jacobean of the grid transformation, *i.e.*, the determinant of $\partial y_i/\partial x_j$. As expected, this equation implies that the transformed spectrum $\mathcal{N}(\mathbf{y}) = N(\mathbf{x})/J$ is subject to a similar spectral conservation equation, as long as the Jacobean $J \neq 0$.

To include effects of kinematic variations of the wavenumber in the grid we replace the coordinate k in $\mathbf{x}$ by a variable wavenumber $\kappa(t, x, y, k, \theta)$ while leaving the other coordinates and the definition of N on $\mathbf{x}$ unchanged. The corresponding Jacobean J is given as

$$J = \frac{\partial \kappa}{\partial k}. \tag{10}$$

The balance equation (9) and the velocity $\dot{\kappa}$ according to (8) then become

$$\frac{\partial}{\partial t}\frac{N}{J} + \frac{\partial}{\partial x}\frac{\dot{x}N}{J} + \frac{\partial}{\partial y}\frac{\dot{y}N}{J} + \frac{\partial}{\partial \kappa}\frac{\dot{\kappa}N}{J} + \frac{\partial}{\partial \theta}\frac{\dot{\theta}N}{J} = \frac{S}{J}, \tag{11}$$

$$\dot{\kappa} = \frac{\partial \kappa}{\partial t} + \dot{x}\frac{\partial \kappa}{\partial x} + \dot{y}\frac{\partial \kappa}{\partial y} + \dot{k}\frac{\partial \kappa}{\partial k} + \dot{\theta}\frac{\partial \kappa}{\partial \theta}, \tag{12}$$

It should be noted that the above grid transformation by definition implies that spatial derivatives like $\partial/\partial x$ are calculated along κ planes. The evaluation of such derivatives therefore does not require interpolation in k or κ space at adjacent grid points.

3. PRACTICAL WAVENUMBER GRIDS

In practical wave models balance equations like (2) or (11) are usually not solved directly. Instead, a fractional step method is commonly used (*e.g.*, Yanenko, 1972), where several parts of the equation are solved by applying consecutive solvers to the wave field. Most models thus separate wave propagation from source terms. For the balance equation (11) on the 'κ-grid' the propagation and source term steps read

$$\frac{\partial}{\partial t}\frac{N}{J}+\frac{\partial}{\partial x}\frac{\dot{x}N}{J}+\frac{\partial}{\partial y}\frac{\dot{y}N}{J}+\frac{\partial}{\partial \kappa}\frac{\dot{\kappa}N}{J}+\frac{\partial}{\partial \theta}\frac{\dot{\theta}N}{J}=0, \tag{13}$$

$$\frac{\partial}{\partial t}\frac{N}{J}=\frac{S}{J}. \tag{14}$$

Most effects of the variable κ-grid are incorporated in the propagation equation (13). Incorporation of this grid in the source term Eq. (14) is relatively straightforward. The propagation equation can be split into other sub-steps, depending on requirements of numerical schemes used, or on economy of memory use.

Ideally, the κ-grid incorporates kinematic effects of all variations of the medium (d and $\mathbf{U}$) on the wavenumber k, *i.e.*, follows the characteristics of wave propagation. Then $\dot{\kappa} \equiv 0$, which eliminates a term from the propagation equation, as well as the resolution problem described in the introduction. For simplicity, we will consider a steady medium first. In such conditions, the absolute frequency $\omega = \sigma + \mathbf{k}\cdot\mathbf{U}$ is constant along characteristics (*e.g.*, Mei, 1983, p. 96), so that κ is defined by

$$\omega_g - \sqrt{g\kappa \tanh \kappa d} = \kappa U \cos(\theta - \theta_c) = \kappa U_p, \tag{15}$$

where θ_c is the current direction, U_p is the current velocity in the direction θ, and where ω_g is constant along a κ-plane in (x, y, θ) space. The corresponding characteristics or κ-grid for deep water are illustrated in Figure 2 as a function of the current velocity U_p. Equation (15) and Figure 2 highlight some shortcomings of this grid.

First, the Jacobean $J = \partial\kappa/\partial k = 0$ at the edge of the shaded area in Figure 2. In such 'blocking' conditions (*e.g.*, Peregrine, 1976; Phillips, 1977; Peregrine and Jonsson, 1983; Shyu and Phillips, 1990) an opposing current is sufficiently strong to stop the propagation of a wave group. For stronger currents or shorter waves (inside the shaded area), wave energy is swept downstream by the current. For $J = 0$, Eq. (11) breaks down. Consequently, the κ-grid defined by ω_g can only be used if currents are sufficiently weak for wave blocking to occur outside the discrete spectral domain of the model.

Secondly, this grid varies with the wave direction θ. Hence, the wavenumber k for each discrete spectral component at each spatial grid

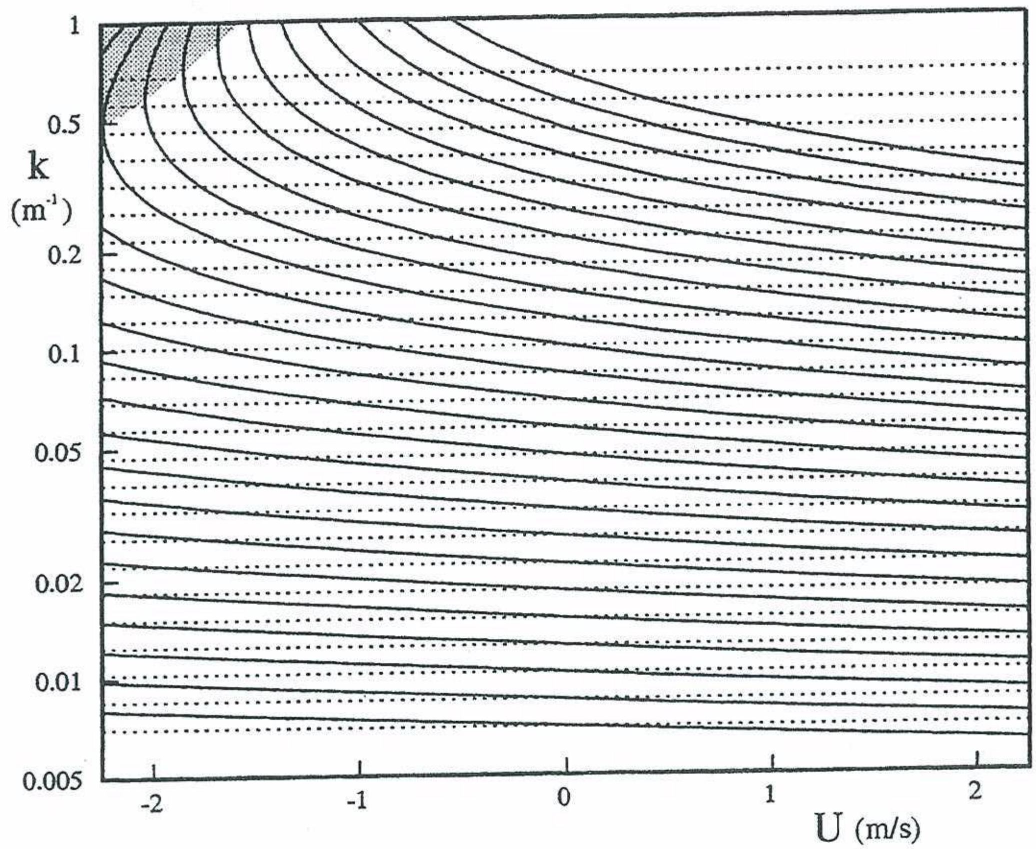

FIGURE 2 Characteristics and ideal κ-grid (solid lines) as a function of the current velocity in the propagation direction of the waves U_p for waves propagating on steady currents in deep water. Dotted lines represent the corresponding invariant grid (as in Fig. 1). The shaded area identifies 'blocked' waves.

point has to be stored or has to be recalculated for every discrete time step. Storing this information roughly doubles the memory requirements of an efficient wave model. Recalculation is potentially expensive, as it requires the iterative solution of Eq. (15) for every discrete spectral component and every discrete time step.

Furthermore, this grid has repercussions for the calculation and integration of source terms, in particular for third-generation wave models which explicitly account for nonlinear wave–wave interactions (*e.g.*, WAM, WAMDI Group, 1988; Komen *et al.*, 1994). Presently, the only feasible method to estimate nonlinear interactions in an operational wave model is the discrete interaction approximation (DIA, Hasselmann *et al.*, 1985). The DIA as implemented in WAM utilizes the regularity of a $\sigma-\theta$ grid to economically estimate nonlinear interactions. The present grid does not provide such regularity.

Considering the above, the wavenumber grid including all effects of depth and current variations has only a limited applicability and several numerical disadvantages (even if only steady depths and currents are considered). Except for special applications (for instance, swell propagation on weak and steady currents), these numerical disadvantages will generally outweigh those of a invariant wavenumber grid. Consequently, its usefulness is limited.

Most of the above disadvantages of the 'ideal' κ-grid are caused by the incorporation of effects of currents in the grid. It is therefore useful to consider a grid which incorporates effects of depth variations only. For simplicity, we will again consider a steady medium first. In such conditions the intrinsic frequency σ remains constant. The κ-grid following such characteristics (as illustrated in Fig. 1) is given by

$$\sigma_g = \sqrt{g\kappa \tanh \kappa d}, \tag{16}$$

where σ_g is constant along the κ-grid. This grid contains none of the disadvantages of the κ-grid defined by ω_g. First, it has a non-zero Jacobeans for any depth (Fig. 1), so that Eq. (11) remains generally applicable. Secondly, the grid is independent of the direction θ, so that the memory required for the storage of the local wavenumbers defining the grid is small compared to the memory required for the storage of the actual spectra. Finally, this grid corresponds by definition to a universal $\sigma-\theta$-grid throughout a model, simplifying the application of the DIA.

The above grid, however, considers steady depths only. If the water depth varies not only in space but also in time (*i.e.*, due to tides and storm surges), the intrinsic frequency of a wave component no longer remains constant during propagation [Eq. (20)], and systematic frequency shifts might occur. It nevertheless appears logical to retain the grid definition in terms of an invariant σ_g. In shallow water this grid will then vary in time as a function of temporal variations of the local water depth.

4. A GRID INCLUDING SHALLOW WATER EFFECTS

As discussed in the previous section, many effects of variable depths can be incorporated in a, κ-grid defined by a set of (globally invariant) discrete frequencies σ_g, the local water depth d, and the dispersion relation (1). The derivation of the governing propagation equations for this grid requires the evaluation of partial derivatives of κ in Eqs. (10), (12) and (13). As κ is a function of σ_g and d,

$$\frac{\partial \kappa}{\partial x_i} = \frac{\partial \kappa}{\partial \sigma_g}\frac{\partial \sigma_g}{\partial x_i} + \frac{\partial \kappa}{\partial d}\frac{\partial d}{\partial x_i}. \tag{17}$$

Furthermore considering that σ_g is independent of t, x, y and θ, and that

$$\frac{\partial \kappa}{\partial d} = \frac{\partial \kappa}{\partial \sigma}\frac{\partial \sigma}{\partial d} = \frac{\partial \kappa}{\partial k}\frac{\partial k}{\partial \sigma}\frac{\partial \sigma}{\partial d} = Jc_g^{-1}\frac{\partial \sigma}{\partial d}, \tag{18}$$

the partial derivatives of κ become

$$\frac{\partial \kappa}{\partial x_i} = Jc_g^{-1}\frac{\partial \sigma}{\partial d}\frac{\partial d}{\partial x_i} \text{ for } x_i = (t, x, y, \theta). \tag{19}$$

Substitution of (19) in (12) then results in

$$\dot{\kappa} J^{-1} c_g = \frac{\partial \sigma}{\partial d}\left(\frac{\partial d}{\partial t} + \mathbf{U}\cdot\nabla d\right) - c_g \mathbf{k}\cdot\frac{\partial \mathbf{U}}{\partial s} = \dot{\sigma}, \tag{20}$$

where ∇ is a gradient operator in physical (x, y) space. As expected the propagation velocity $\dot{\kappa}$ closely resembles the conventional propagation velocity $\dot{\sigma}$. The corresponding spectral balance equation (13) can be expressed in several ways. Amongst others, it can be shown to reproduce the corresponding balance equation of $N(\sigma) = N(k)c_g^{-1}$ exactly, which is not surprising considering the present choice of the κ-grid. Before actual balance equations are discussed, additional splitting of the equations, in particular the treatment of unsteady depths will be considered.

As described above, the κ-grid will vary in time due to temporal variations of the water depth. Equations (2) and (5) show that the wavenumber spectrum is not influenced by temporal variations of the depth. This implies that the modification of the κ-grid due to temporal variations of the depth corresponds to a change of grid without modifying the spectrum. If a separate fractional step is considered to deal with the water level variations, the grid conversion can be performed using a simple interpolation scheme, the conservation properties of which are easily controlled.

In shelf seas away from the coast, depth variations are generally dominated by the spatial variability of the (steady) bathymetry. The contribution of temporal water level variations to such depth variations then is expected to be small. In some shallow coastal areas, temporal water level variations can locally be larger that the spatial variability of the bathymetry. Such areas, however, are generally sufficiently small for wave, depth and current conditions to be quasi-steady on the time scale of wave propagation through the area (*e.g.*, Holthuijsen *et al.*, 1989). This suggests that water levels can be updated sparsely, at intervals dictated by the time scale of the water level variation. The κ-grid thus suggests a simple and elegant way to deal with temporal water level variations.

If effects of the water level variations on the present κ-grid are treated with a sparsely invoked fractional step, the remaining steps become quasi-steady with respect to the water depth (but not with respect to the currents). For numerical economy, and to simplify implementation of accurate numerical schemes, the remaining quasi-steady propagation equation can be solved with separate fractional steps for spatial and intra-spectral propagation. Considering that

$$J = \frac{\partial \kappa}{\partial k} = c_g \frac{\partial \kappa}{\partial \sigma}, \tag{21}$$

where $\partial\kappa/\partial\sigma$ by definition should be considered independent of $\mathbf{x}$, the spatial propagation step can be written as

$$\frac{\partial}{\partial t}\frac{N}{c_g} + \frac{\partial}{\partial x}\frac{\dot{x}N}{c_g} + \frac{\partial}{\partial y}\frac{\dot{y}N}{c_g} = 0, \tag{22}$$

where $\dot{x}$ and $\dot{y}$ are given by Eqs. (3) and (4), respectively. Not surprising, this equation is identical to the conventional propagation equation for $N(\sigma)$. Considering that J is independent of θ and t (in this fractional step), the intra-spectral propagation step becomes

$$\frac{\partial N}{\partial t} + J\frac{\partial}{\partial \kappa}\frac{\dot{\kappa}N}{J} + \frac{\partial}{\partial \theta}\dot{\theta}N = 0, \tag{23}$$

which in turn can be written as

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial k}\dot{k}_\kappa N + \frac{\partial}{\partial \theta}\dot{\theta}N = 0, \tag{24}$$

$$\dot{k}_\kappa = J^{-1}\dot{\kappa} = \frac{\partial \sigma}{\partial d}\frac{\mathbf{U}\cdot\nabla d}{c_g} - \mathbf{k}\cdot\frac{\partial \mathbf{U}}{\partial s}, \tag{25}$$

where $\dot{\theta}$ is given by Eq. (6). By design, $\dot{k}_\kappa = 0$ in cases without currents. In such cases the grid incorporates all effects of shoaling. In cases with currents, the second term in (25) represents the conventional effects of currents as in Eq. (5), and the first term implies that mean currents result in different shoaling behavior.

5. DISCUSSION AND CONCLUSIONS

Ocean wind waves are generally described with variance spectra in terms of the absolute frequency ω, relative freqnency σ, or wavenumber k. Several previous authors have chosen the description of wind waves with the wavenumber spectrum for physical reasons (see introduction). If this spectral description is used, shoaling results in an effective loss of resolution, and in an (unnecessary) propagation term in k-space. In the present paper, an attempt is made to include the kinematic effects of the medium on the wavenumber of spectral components into a spatially varying wavenumber (κ) grid. The resulting equations for grids incorporating effects of shallow water and currents, or of shallow water only, closely resemble the conventional balance equations for the absolute and relative frequency spectrum. The equations and arguments presented here can therefore also be used to address the applicability of the basic equations for the different spectral descriptions.

In a κ-grid that incorporates effects of shallow water and currents, the absolute frequency remains constant along grid lines. The corresponding equations (not presented here) therefore closely resemble the conventional equations for absolute frequency spectra $N(\omega, \theta)$ (*e.g.*, Tolman, 1991). This κ-grid (or spectral description) is not suitable for general applications, as it cannot describe wave components that are blocked by strong opposing currents[1], and as it has economical disadvantages with respect to the solution of the dispersion relation and parameterizations of nonlinear interactions. There are nevertheless applications for which this κ-grid or spectral description might be preferable. Examples are the propagation of long swell over steady currents ($\dot{\kappa} \equiv 0$), where one term drops from the equations, or studies where the effects of unsteadiness of depths and currents are explicitly investigated (see Tolman, 1991).

Effects of shallow water can be included in a κ-grid if the intrinsic frequency σ is kept constant along grid lines, or, in other words, if the equations are solved on a σ-grid. Effects of currents then remain an integral part of the equations [compare Eqs. (25) and (5)]. Temporal water level variations result in a simple grid adjustment, which in most cases can be sparsely invoked. The equations for this κ-grid are practically identical to the conventional equations for the frequency spectrum $N(\sigma, \theta)$. The only

[1] If the absolute frequency spectrum is used to describe the wave field, not only the equation but the entire spectral description breaks down at the blocking point as the spectrum $N(\omega) = (c_g + U_p)^{-1}\ N(k)$ becomes undefined.

exception is the treatment of temporal variations of the water level. As described above, temporal water level variations shift the κ-grid without modifying the corresponding spectrum $N(k)$. The corresponding frequency spectrum $N(\sigma)$, however, is influenced by a temporal change of water depth (the spectrum $N(k)$ is unchanged, and the Jacobean transformation to $N(\sigma)$ is a function of the depth). This complicates sparse updating of the water level in the equations for $N(\sigma)$.

The similarity between the equations for the wavenumber spectrum on a frequency grid and the conventional equations for the frequency spectrum raises the question of the validity of arguments for choosing a wavenumber spectrum to describe the wave field. The main argument appears to be the parameterization of the physics of wave growth and decay. However, as the Jacobean transformation describing the conversion from $N(k, \theta)$ to $N(\sigma, \theta)$ and *vise versa* is well behaved, it can easily be included in the parameterization of the source terms. This argument thus appears irrelevant for numerical wave models. Additional numerical disadvantages of the conventional equations for $N(k, \theta)$ as described in the introduction then suggest that there is no justification for using the conventional equations for $N(k, \theta)$.

From a computational perspective, differences between solving the equations for $N(k, \theta)$ on a frequency grid or solving the conventional equations for $N(\sigma, \theta)$ are small. The advantage of the former approach is the more elegant and therefore simpler treatment of temporal depth variations. The only additional computational effort occurs in the calculation of some Jacobean transformations. When this approach was applied to the third-generation wave model WAVEWATCH-III (Tolman, 1997), the added computational effort proved insignificant.

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APPENDIX

Using Eqs. (8), (7) can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial x_j}(v_jN) - S &= \frac{\partial v_j}{\partial x_j}N + v_j\frac{\partial N}{\partial x_j} - S \\ &= \left\{\frac{\partial v_j}{\partial y_i}\frac{\partial y_i}{\partial x_j}\right\}N + v_j\frac{\partial N}{\partial y_i}\frac{\partial y_i}{\partial x_j} - S \\ &= \left\{\frac{\partial w_i}{\partial y_i} - v_j\frac{\partial}{\partial y_i}\frac{\partial y_i}{\partial x_j}\right\}N + w_i\frac{\partial N}{\partial y_i} - S \\ &= \frac{\partial}{\partial y_i}(w_iN) - S - Nv_j\frac{\partial x_l}{\partial y_i}\frac{\partial^2 y_i}{\partial x_j \partial x_l}. \end{aligned} \tag{A.1}$$

Defining the Jacobean J as the determinant of $\partial y_i/\partial x_l$, the right side of this equation becomes

$$\begin{aligned}\frac{\partial}{\partial y_i}(w_iN) - S - Nv_j\frac{\partial x_l}{\partial y_i}\frac{\partial^2 y_i}{\partial x_j \partial x_l} &= \frac{\partial}{\partial y_i}(w_iN) - S - N\frac{w_i}{J}\frac{\partial J}{\partial y_i} \\ &= J\left\{\frac{1}{J}\frac{\partial}{\partial y_i}(w_iN) + w_iN\frac{\partial}{\partial y_i}\frac{1}{J}\right\} - S \\ &= J\frac{\partial}{\partial y_i}\left(w_i\frac{N}{J}\right) - S. \end{aligned} \tag{A.2}$$

Combination of Eqs. (A.1) and (A.2) results in Eq. (9).