Topological Quantum Information Theory

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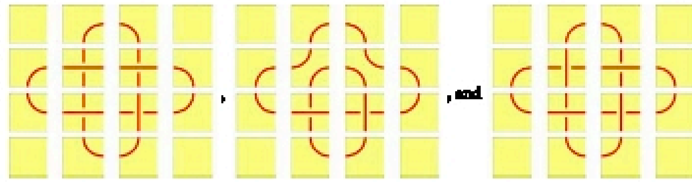
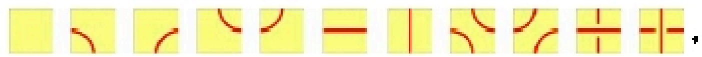
Quantum knots and mosaics

with
Sam
Lomonaco

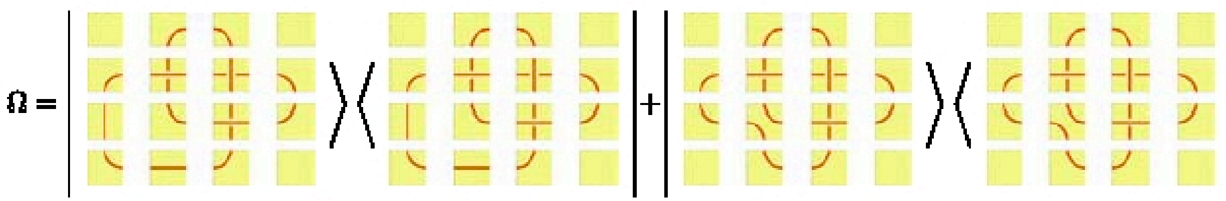


Each of these knot mosaics is a string made up of the following 11 symbols



called *mosaic tiles*.

Each mosaic is a tensor product of elementary tiles.



This observable is a quantum knot invariant for 4x4 tile space. Knots have characteristic invariants in nxn tile space.

Unitary Representations of the Braid Group and Topological Quantum Computing

Spin Networks and Anyonic Topological Computing

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quant-ph/0603131 and quant-ph/0606114

arXiv:0805.0339 **Quantum Knots and Mosaics**

arXiv:0804.4304 The Fibonacci Model and the Temperley-Lieb Algebra

arXiv:0706.0020 A 3-Stranded Quantum Algorithm for the Jones Polynomial

arXiv:0909.1080 **Title:** NMR Quantum Calculations of the Jones Polynomial
Authors: Raimund Marx, Amr Fahmy, Louis Kauffman, Samuel Lomonaco, Andreas Spörl, Nikolas Pomplun, John Myers, Steffen J. Glaser

arXiv:******** Anyonic topological quantum computation and the virtual braid group. H. Dye and LK.

Untying Knots by NMR: first experimental implementation of a quantum algorithm for approximating the Jones polynomial

Raimund Marx[1], Andreas Spörl[1], Amr F. Fahmy[2], John M. Myers[3], Louis H. Kauffman[4], Samuel J. Lomonaco, Jr.[5], Thomas-Schulte-Herbrüggen[1], and Steffen J. Glaser[1]

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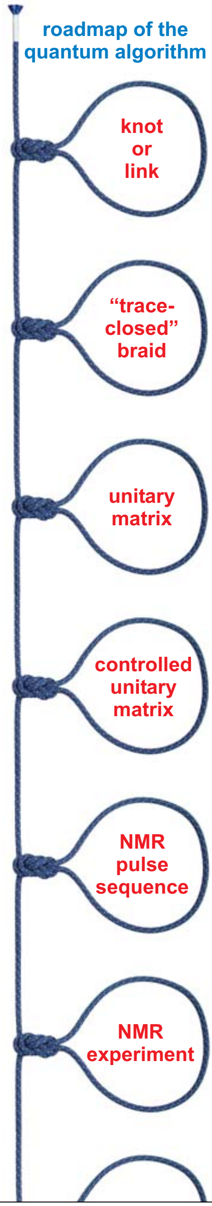
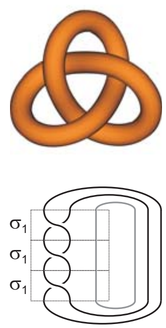
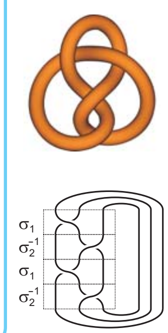
roadmap of the quantum algorithm



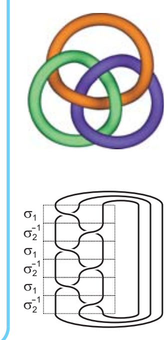
example #1 Trefoil



example #2 Figure-Eight

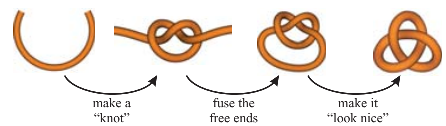


example #3 Borromean rings



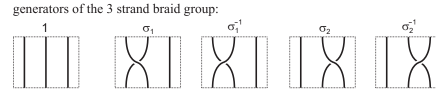
A knot is defined as a closed, non-self-intersecting curve that is embedded in three dimensions.

example: "construction" of the Trefoil knot:



J. W. Alexander proved, that any knot can be represented as a closed braid (polynomial time algorithm)

generators of the 3 strand braid group:



$$U_{Trefoil} = \left(U_1\right)^3 \qquad U_{Figure-Eight} = \left(U_2^{-1} \cdot U_1\right)^2 \qquad U_{Borrom.R.} = \left(U_2^{-1} \cdot U_1\right)^3$$

$$U_1 = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & -e^{i\theta}\dfrac{\sin(4\theta)}{\sin(2\theta)} + e^{-i\theta} \end{pmatrix}$$

$$U_2 = \begin{pmatrix} -e^{i\theta}\dfrac{\sin(6\theta)}{\sin(4\theta)} + e^{-i\theta} & -e^{i\theta}\dfrac{\sqrt{\sin(6\theta)\sin(2\theta)}}{\sin(4\theta)} \\ -e^{i\theta}\dfrac{\sqrt{\sin(6\theta)\sin(2\theta)}}{\sin(4\theta)} & -e^{i\theta}\dfrac{\sin(2\theta)}{\sin(4\theta)} + e^{-i\theta} \end{pmatrix}$$

It is well known in knot theory, how to obtain the unitary matrix representation of all generators of a given braid goup (see "Temperley-Lieb algebra" and "path model representation"). The unitary matrices U_1 and U_2, corresponding to the generators σ_1 and σ_2 of the 3 strand braid group are shown on the left, where the variable "θ" is related to the variable "A" of the Jones polynomial by: $A = e^{-i\theta_{rad}}$. The unitary matrix representations of σ_1^{-1} and σ_2^{-1} are given by U_1^{-1} and U_2^{-1}.

The knot or link that was expressed as a product of braid group generators can therefore also be expressed as a product of the corresponding unitary matrices.

Step #1:
from the 2x2 matrix U to the 4x4 matrix cU:

$$cU = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U \end{pmatrix}$$

Step #2:
application of cU on the NMR product operator I_{1x} :

$$cU\ I_{1x}\ cU^\dagger = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U \end{pmatrix}^\dagger = \frac{1}{2}\begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix}$$

Step #3:
measurement of I_{1x} and I_{1y} :

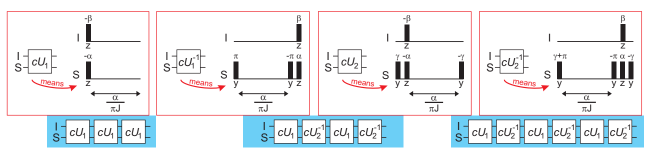
$$tr\left\{ I_{1x}\ \frac{1}{2}\begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix} \right\} = \frac{1}{2}\Re\left(tr\{U\}\right)$$

$$tr\left\{ I_{1y}\ \frac{1}{2}\begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix} \right\} = \frac{1}{2}\Im\left(tr\{U\}\right)$$

Instead of applying the unitary matrix U, we apply it's controlled variant cU. This matrix is especially suited for NMR quantum computers [4] and other thermal state expectation value quantum computers: you only have to apply cU to the NMR product operator I_{1x} and measure I_{1x} and I_{1y} in order to obtain the trace of the original matrix U.

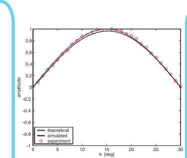
Independent of the dimension of matrix U you only need ONE extra qubit for the implementation of cU as compared to the implementation of U itself.

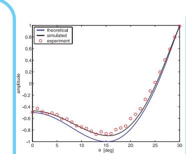
The measurement of I_{1x} and I_{1y} can be accomplished in one single-scan experiment.

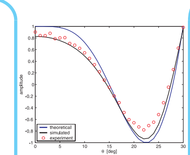


All knots and links can be expressed as a product of braid group generators (see above). Hence the corresponding NMR pulse sequence can also be expressed as a sequence of NMR pulse sequence blocks, where each block corresponds to the controlled unitary matrix cU of one braid group generator.

This modular approach allows for an easy optimization of the NMR pulse sequences: only a small and limited number of pulse sequence blocks have to be optimized.







Comparison of experimental results, theoretical predictions, and simulated experiments, where realisitic inperfections like relaxation, B_1 field inhomogeneity, and finite length of the pulses are included.

For each data point, four single-scan NMR experiments have been performed: measurement of I_{1x}, measurement of I_{1y}, reference for I_{1x}, and reference for I_{1y}. If necessary each data point can also be obtained in one single-scan experiment by measuring amplitude and phase in a referenced setting.

Jones Polynomial "Trefoil":

Jones Polynomial "Figure-Eight":

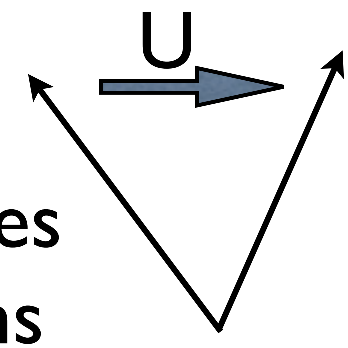
Jones Polynomial "Borromean rings":

The Jones Polynomials can be reconstructed out of the NMR experiments by:

$$V_L(A) = (-A^3)^{-w(L)}\left(tr\{U\} + A^{I(L)}\left[(-A^2 - A^{-2})^2 - 2\right]\right)$$

Quantum Mechanics in a Nutshell

0. A state of a physical system corresponds to a unit vector |S> in a complex vector space.



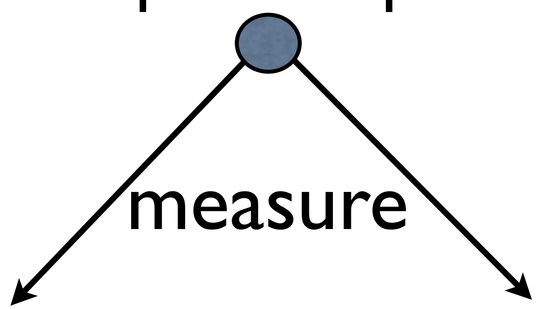
1. (measurement free) Physical processes are modeled by unitary transformations applied to the state vector: |S> -----> U|S>

2. If |S> = z1|e1> + z2|e2> + ... + zn|en> in a measurement basis {e1,e2,...,en}, then measurement of |S> yields |ei> with probability |zi|^2.

Qubit

A qubit is the quantum version of a classical bit of information.

a|0> + b|1>



|0>
prob = |a|^2

|1>
prob = |b|^2

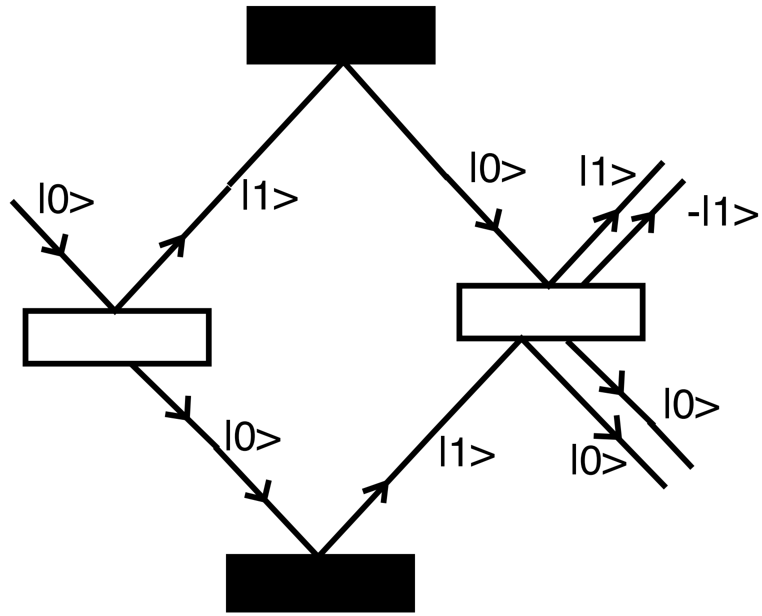
Quantum Gates

are unitary transformations enlisted for the purpose of computation.

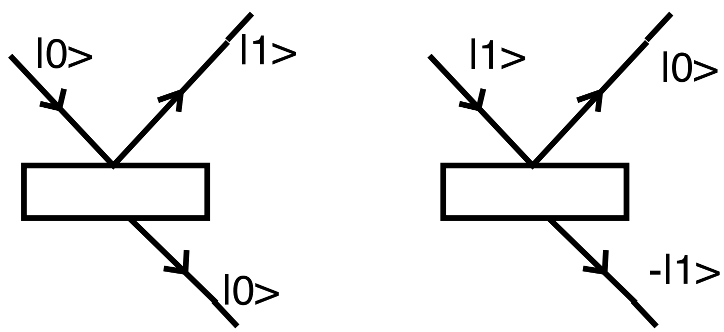
CNOT =

1	0	0	0
0	1	0	0
0	0	0	1
0	0	1	0

CNOT|00> = |00>
CNOT|01> = |01>
CNOT|10> = |11>
CNOT|11> = |10>



Mach-Zender Interferometer



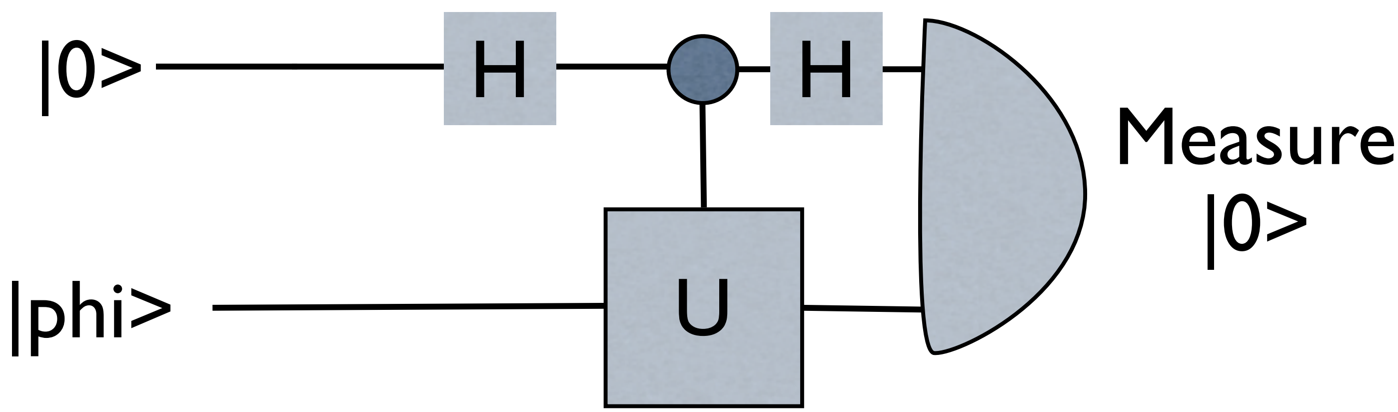
Hadamard Matrix

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} / \mathrm{Sqrt}(2) \qquad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$HMH = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Quantum Computation of the Trace of a Unitary Matrix

Hadamard Test



|0> occurs with probability
1/2 + Re[<phi|U|phi>]/2

Universal Gates

A *two-qubit gate* G is a unitary linear mapping

$$G : V \otimes V \longrightarrow V \otimes V$$ where V is

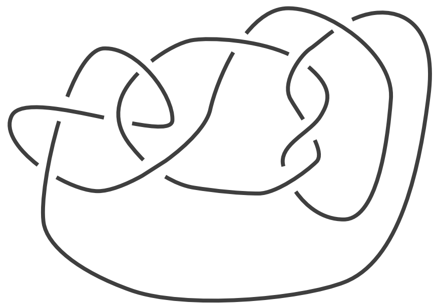
a two complex dimensional vector space. We say that the gate G is *universal for quantum computation* (or just *universal*) if G together with local unitary transformations (unitary transformations from V to V) generates all unitary transformations of the complex vector space of dimension 2^n to itself. It is well-known [44] that $CNOT$ is a universal gate.

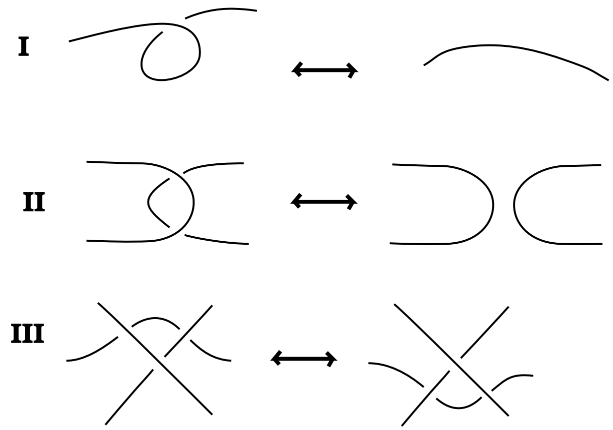
Local Unitaries are generated (up to density) by a small number of gates.

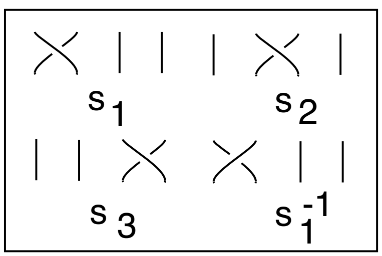
Explicit gate realization in the basis $\{|0\rangle, |1\rangle\}$:

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

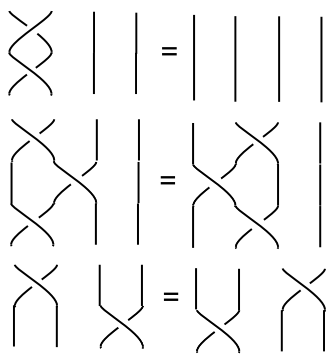
Knot Theory







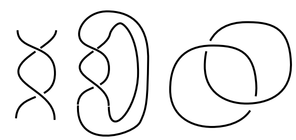
Braid Generators



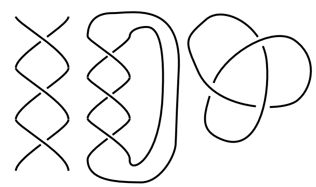
$s_1^{-1} s_1 = 1$

$s_1 s_2 s_1 = s_2 s_1 s_2$

$s_1 s_3 = s_3 s_1$



Hopf Link



Trefoil Knot

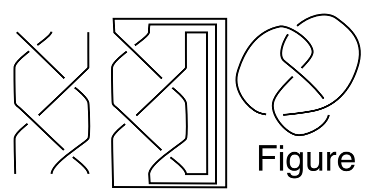


Figure Eight Knot

Representative Examples of Unitary Solutions to the Yang-Baxter Equation that are Universal Gates.

$$R = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix}$$

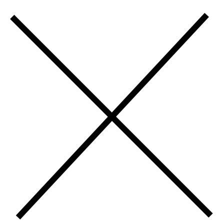
Bell Basis Change Matrix

R + R* = Sqrt[2]I

Corresponding Link Invariant is Special Case of Homfly Poly.

$$R' = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \qquad R'' = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ d & 0 & 0 & 0 \end{pmatrix}$$

$$R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



(virtual crossing corresponds to swap gate.)

Swap Gate with Phase

See paper by Heather Dye for classification of 2 x2 Yang-Baxter gates.

Quantum Hall Effect

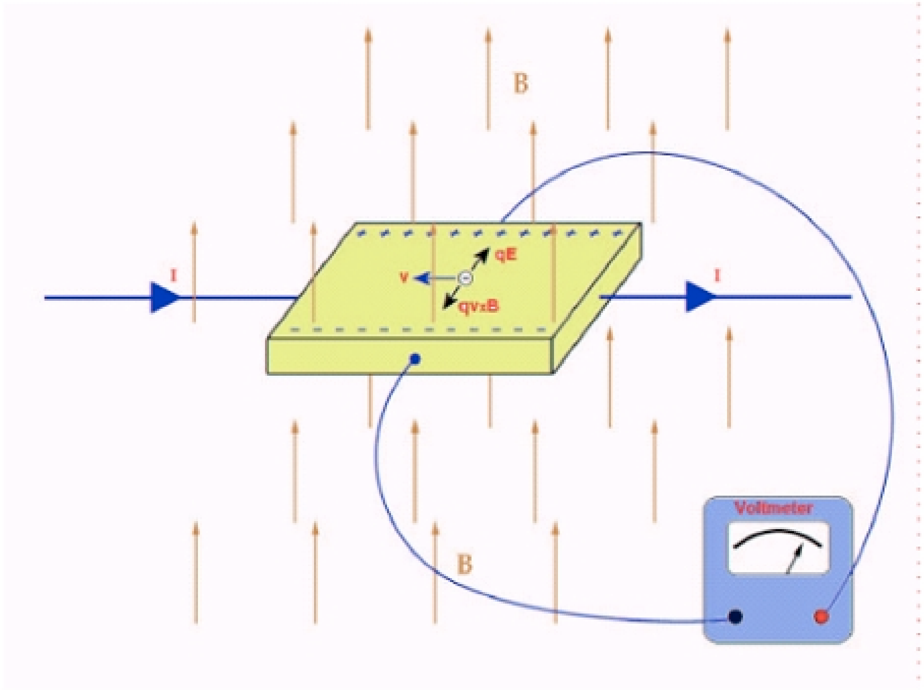


Figure 1: A schematics of the experimental setup of the Hall effect. A current driven through the conductor, drawn as a prism, leads to the emergence of voltage in the perpendicular direction. This is the Hall voltage, which Maxwell erroneously predicted to be zero.

There are two main theories of the FQHE:

- **Fractionally-charged quasiparticles**. This theory, proposed by Laughlin, hides the interactions by constructing a set of quasiparticles with charge $e^{*}=e/q$, where the fraction is p/q as above.
- **Composite Fermions**. This theory was proposed by Jain, and Halperin, Lee and Read. In order to hide the interactions, it attaches two (or, in general, an even number) flux quanta h/e to each electron, forming integer-charged quasiparticles called Composite Fermions. The fractional states are mapped to the Integer QHE. This makes electrons at a filling factor 1/3, for example, behave in the same way as at filing factor 1. A remarkable result is that filling factor 1/2 corresponds to zero magnetic field. Experiments support this.

A quasi-particle theory connected with Chern-Simons Theory explains the FQHE on the basis of "anyons": particles that have non-trivial (not +1 or -1) phase change when they exchange places in the plane.

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NONABELIONS IN THE FRACTIONAL QUANTUM HALL EFFECT

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Applications of conformal field theory to the theory of fractional quantum Hall systems are discussed. In particular, Laughlin's wave function and its cousins are interpreted as conformal blocks in certain rational conformal field theories. Using this point of view a hamiltonian is constructed for electrons for which the ground state is known exactly and whose quasihole excitations have nonabelian statistics; we term these objects "nonabelions". It is argued that universality classes of fractional quantum Hall systems can be characterized by the quantum numbers and statistics of their excitations. The relation between the order parameter in the fractional quantum Hall effect and the chiral algebra in rational conformal field theory is stressed, and new order parameters for several states are given.

1. Introduction

The past few years have seen a great deal of interest in two-dimensional many particle and $(2+1)$-dimensional field-theoretic systems from several motivations. These include the fractional quantum Hall effect, high-temperature superconductivity and the anyon gas, conformal field theory in $1+1$ dimensions and its relation to $2+1$ Chern–Simons–Witten (CSW) theories, knot invariants, exactly soluble statistical mechanical models in $1+1$ dimensions, and general investigations of

3. Electron wave functions as conformal blocks: Laughlin states and the hierarchy

Let us return to the Laughlin state in the disc geometry:

$$\Psi_{\text{Laughlin}}(z_1,\dots,z_N) = \prod_{i<j}(z_i - z_j)^q \exp\left[-\tfrac{1}{4}\sum |z_i|^2\right], \tag{3.1}$$

where q is an odd integer [3]. In the thermodynamic limit this state $|0_\text{L}; N\rangle$ describes a fluid ground state with a uniform number density $\rho_0 \equiv \nu/2\pi = 1/2\pi q$ inside a radius of order $\sqrt{2N}$. The GL description of this limit for a normalized fluid state $|\alpha\rangle$ of slowly varying density involves a gauge field

$$i\mathscr{A}(z) \sim \int \frac{\langle\alpha|\psi^\dagger\psi(z')|\alpha\rangle}{z - z'}\,\mathrm{d}^2 z'. \tag{3.2}$$

In the GL description [4] this gauge field couples to the order parameter (which has charge q; we set the charge of the electron to 1 from now on) and also enters with a Chern–Simons term

$$\frac{q}{4\pi}\int \mathscr{A}\,\mathrm{d}\mathscr{A} \tag{3.3}$$

in the action. If we are interested primarily in statistics of excitations we may expect such topological terms in the action to play a dominant role – since they dominate all other terms at long distances and low energies. On the other hand, it is now well known that CSW theory (i.e. (2 + 1)-dimensional gauge theory with only a CS term in the action) for an abelian gauge field is closely connected to the (1 + 1)-dimensional conformal field theory known as the "rational torus" [1,5]. The rational torus theory is characterized by a "level" N and is denoted by $\text{U}(1)_N$*. The level N can be determined in terms of q by comparing the abelian

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(54) **SYSTEMS AND METHODS FOR QUANTUM BRAIDING**

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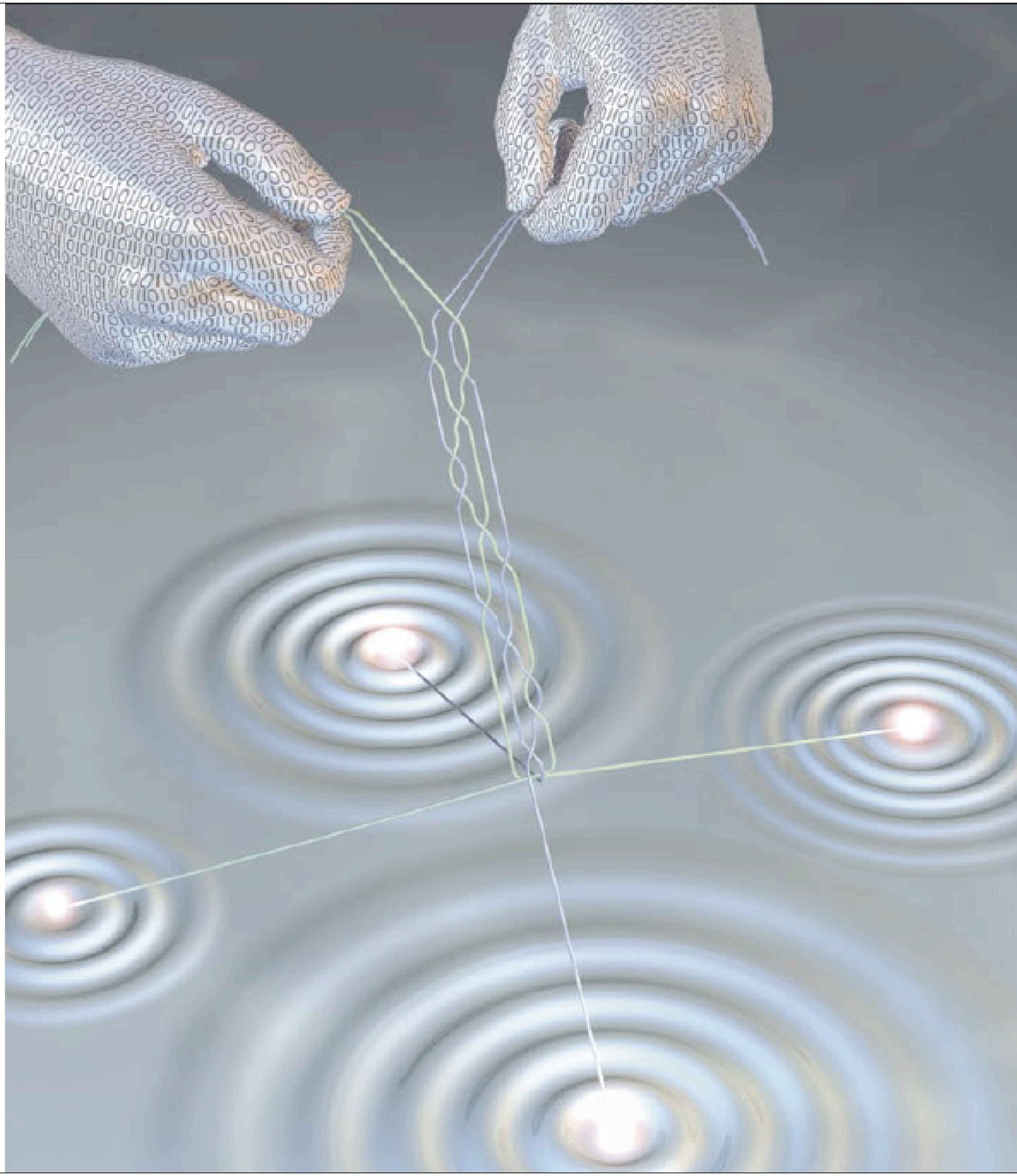
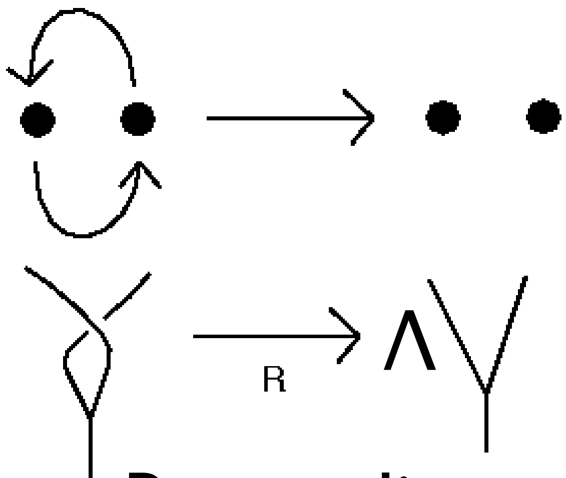
(52) **U.S. Cl.** **257/9**; 257/14

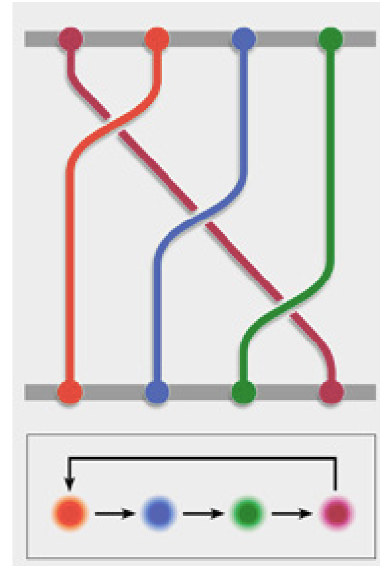
(57) **ABSTRACT**

Apparatus and methods for performing quantum computations are disclosed. Such apparatus and methods may include identifying a first quantum state of a lattice having a system of quasi-particles disposed thereon, moving the quasi-particles within the lattice according to at least one predefined rule, identifying a second quantum state of the lattice after the quasi-particles have been moved, and determining a computational result based on the second quantum state of the lattice. A topological quantum computer encodes information in the configurations of different braids. The computer physically weaves braids in the 2D+1 space-time of the lattice, and uses this braiding to carry out calculations. A pair of quasi-particles, such as non-abelian anyons, can be moved around each other in a braid-like path. The quasi-particles can be moved as a result of a magnetic or optical field being applied to them, for example. When the pair of quasi-particles are brought together, they may annihilate each other or create a new anyon. A result is that an anyon may be present or not, which can be thought of as a "one" or "zero," respectively. Such ones and zeros can be interpreted to provide information.

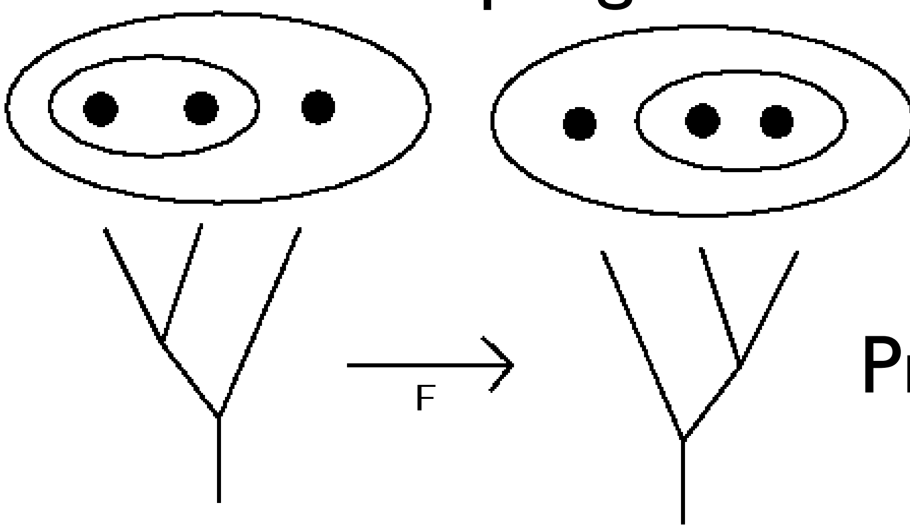


Braiding Anyons



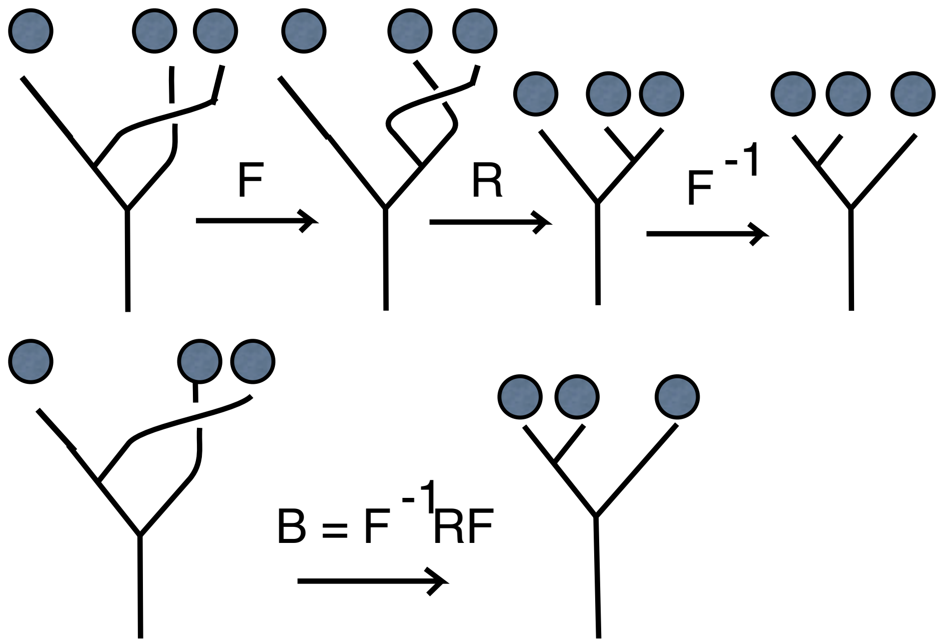


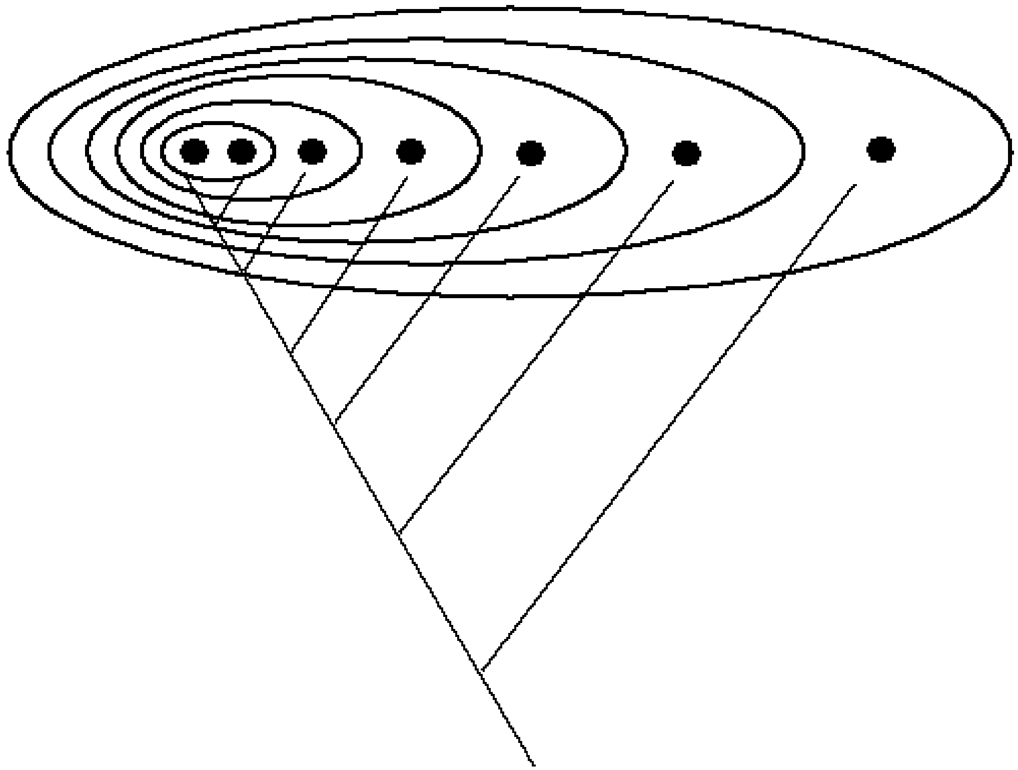
Recoupling



Process Spaces

Non-Local Braiding is Induced via Recoupling

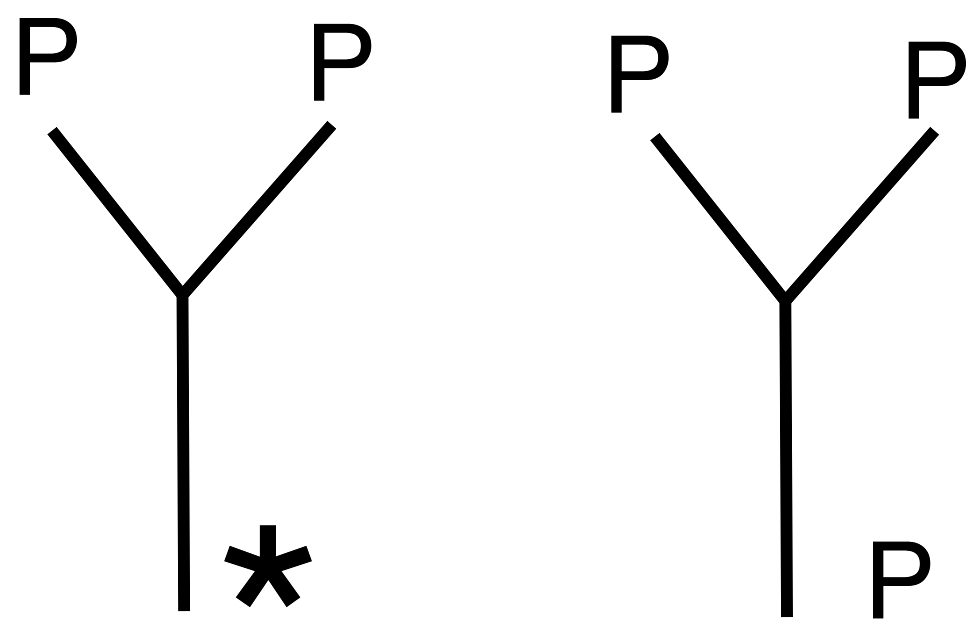


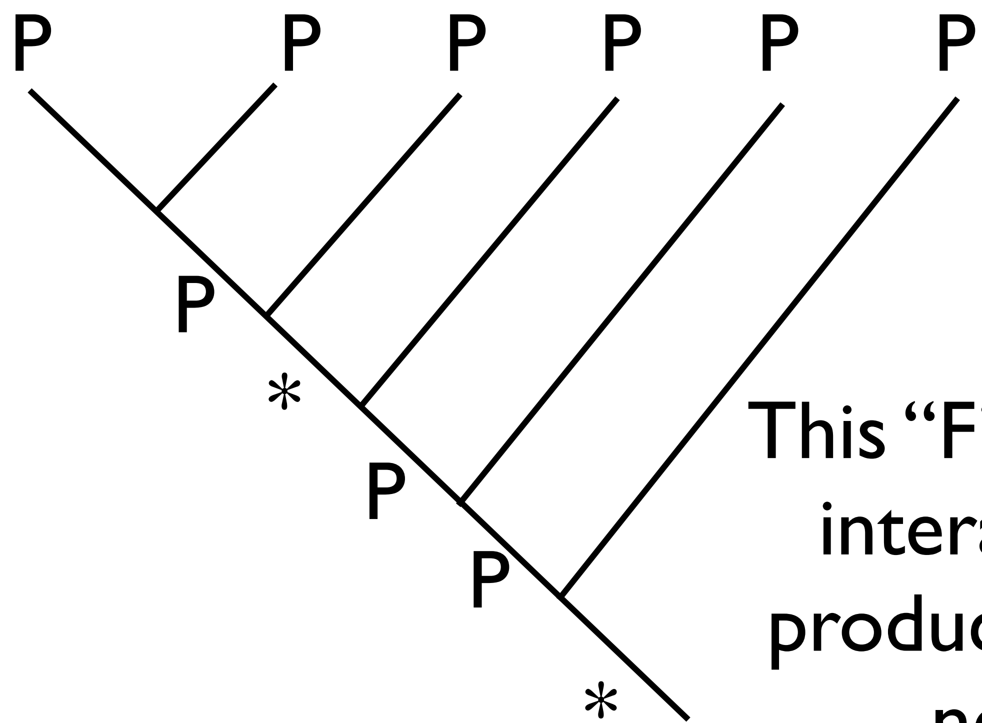


Process Spaces Can be Abitrarily Large.
With a coherent recoupling theory, all transformations are in the representation of one braid group.

Fibonacci Model

One particle P.
One neutral state *
PP ----> P or
PP ----> *





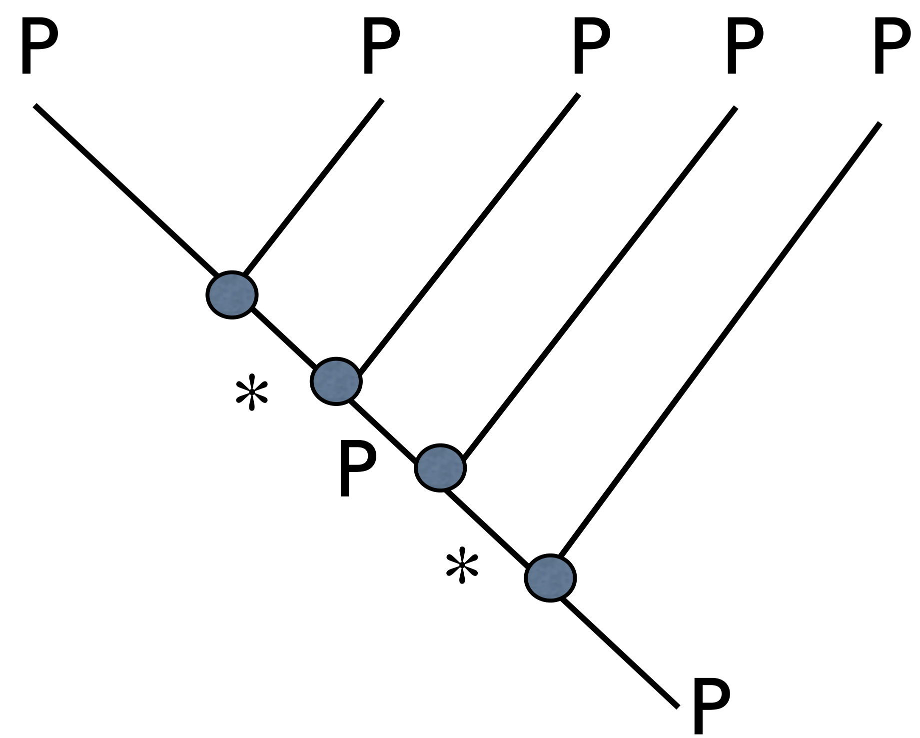
This "Fibonacci Particle" P interacts with itself to produce either itself or a neutral particle *.

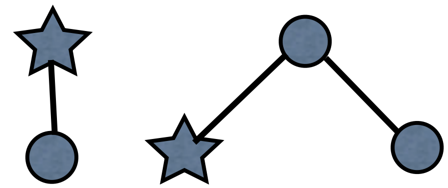
Specific Processes Correspond to Basis Vectors in the Process Space.

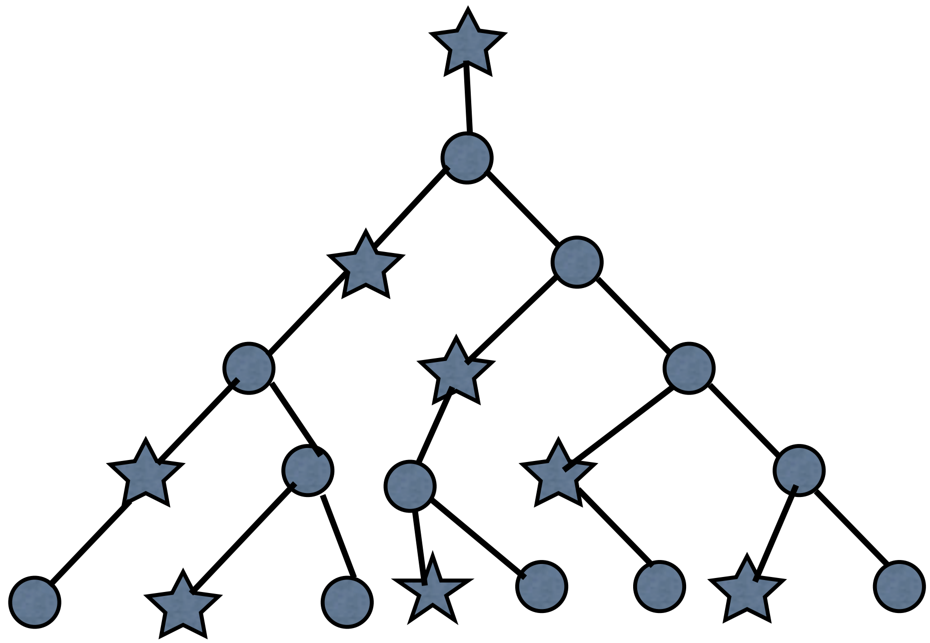
Interaction Sequences of P and * do not admit two *'s in a row.

There are Fibonacci numbers of such sequences.

$|{*}\,P\,{*}\rangle$ in V_P^{PPP}



Fibonacci Tree: 



Admissible Sequences
are the Paths from the Root

The Simple, yet Quantum Universal, Structure of the Fibonacci Model

$$A = e^{3\pi i/5}.$$

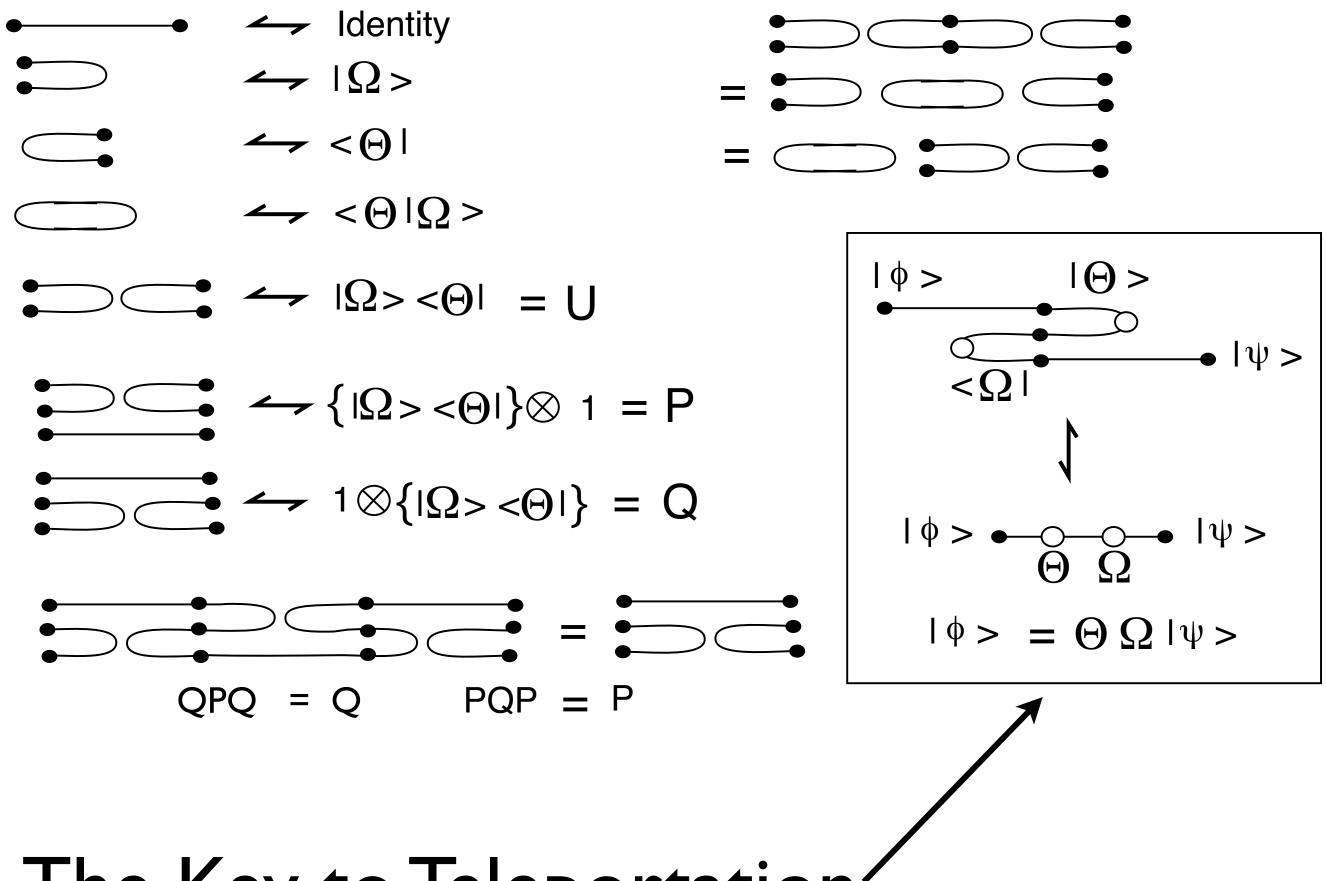
$$\delta = -A^2 - A^{-2}$$

$$\Delta = \delta = (1 + \sqrt{5})/2.$$

$$F = \begin{pmatrix} 1/\Delta & 1/\sqrt{\Delta} \\ 1/\sqrt{\Delta} & -1/\Delta \end{pmatrix} = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$$

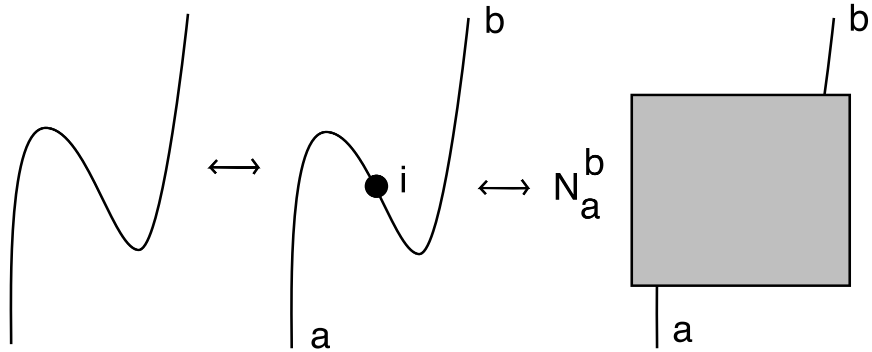
$$R = \begin{pmatrix} -A^4 & 0 \\ 0 & A^8 \end{pmatrix} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix}.$$

Temperley Lieb Category



The Key to Teleportation

Diagrammatic Matrices



$$N_a^b = M_{ai} M^{ib}$$

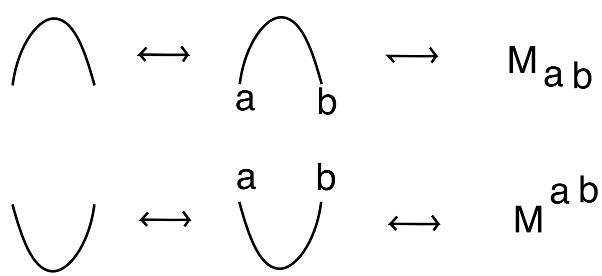
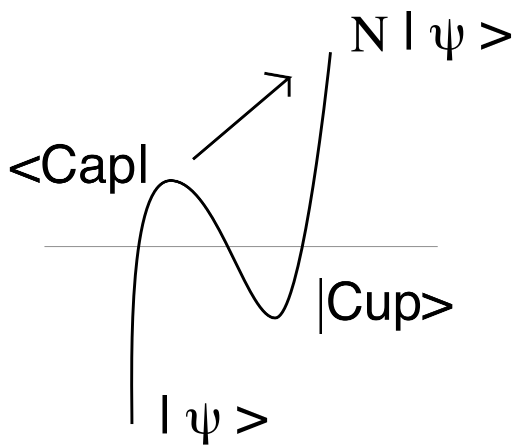
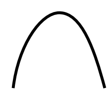
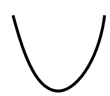


Figure 5 - Matrix Composition

State and Matrix Duality



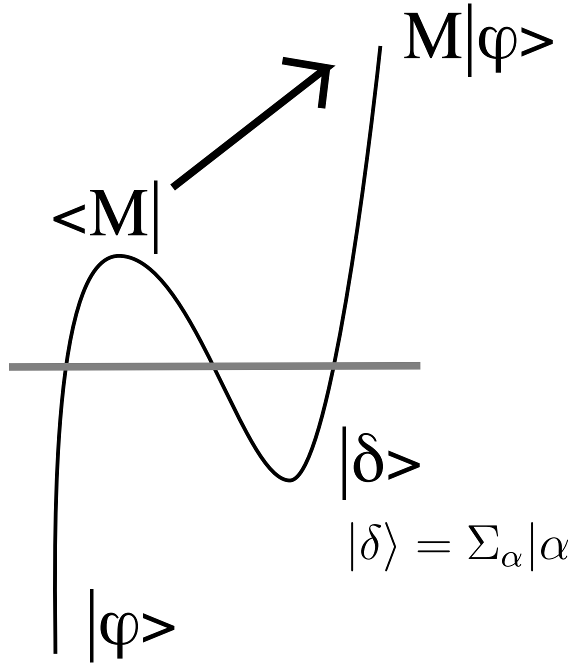
 $\longleftrightarrow \quad \langle Cap| = \sum_{a,i} M_{a,i} \langle a| \langle i|$

 $\longleftrightarrow \quad |Cup\rangle = \sum_{i,b} M^{i,b} |i\rangle |b\rangle$

$$N_a^b = \sum_i M_{ai} M^{ib}$$

The Topology of Teleportation

$$|00\rangle + |11\rangle \quad \longleftarrow\!\!\!-\!-\!-\!\!\!\longrightarrow \quad \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

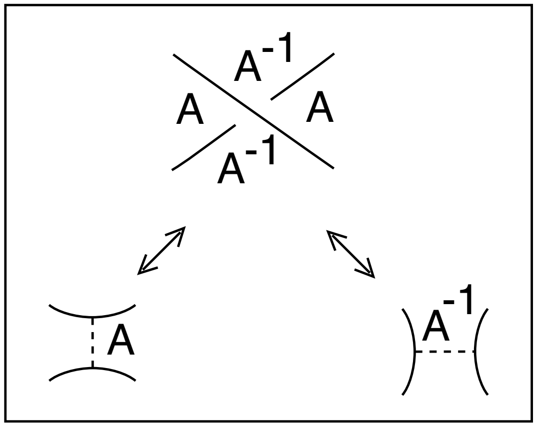


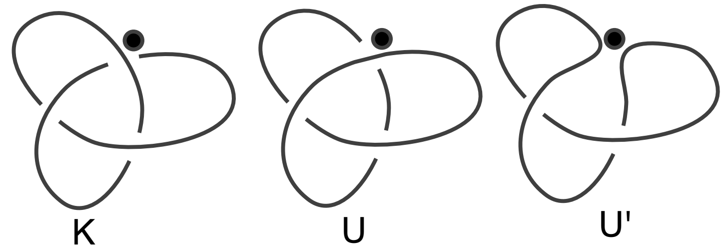
$$|\delta\rangle = \Sigma_{\alpha}|\alpha, \alpha\rangle \in H \otimes H.$$

Mathematical Models for Recoupling Theory with Braiding come from a Combination of Penrose Spin Networks and Knot Theory.

See “Temperley Lieb Recoupling Theory and Invariants of Three-Manifolds” by L. Kauffman and S. Lins, PUP, 1994.

Bracket Polynomial Model for Jones Polynomial



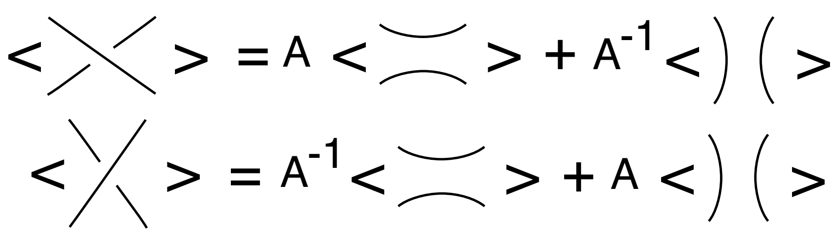


$$A^{-1} < K > - A < U > = (A^{-2} - A^{2}) < U' >$$

$$< U > = -A^{3}$$

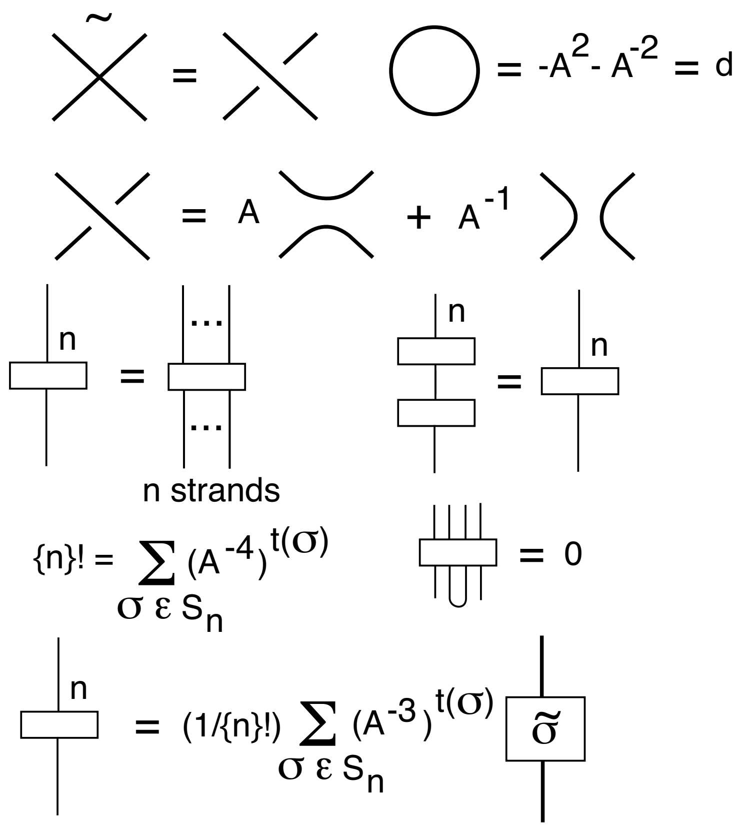
$$< U' > = (-A^{-3})^{2} = A^{-6}$$

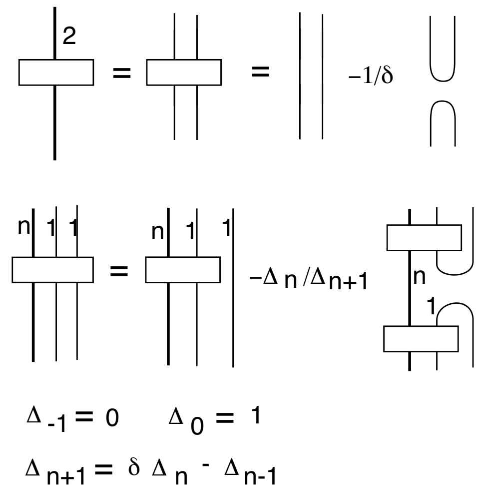
$$< K > = -A^{5} - A^{-3} + A^{-7}.$$

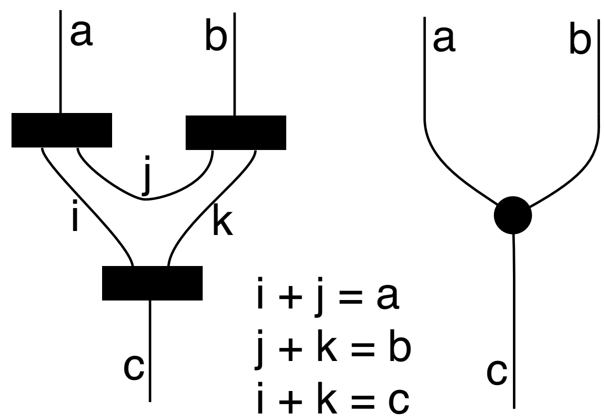


$$< K > = \sum_{S} < K|S > \delta^{||S||-1}.$$

The General Temperley Lieb Model

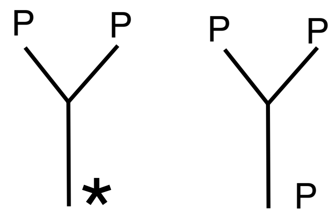




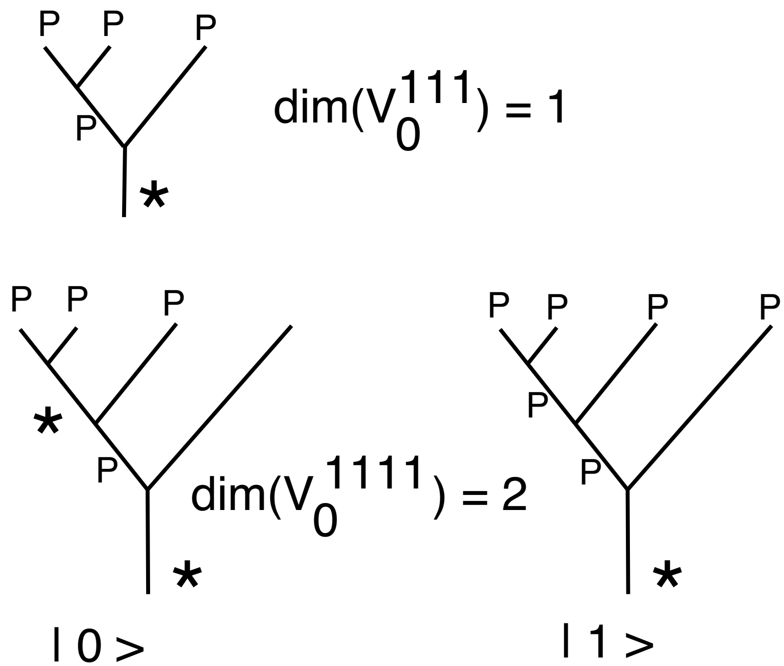


Using the Bracket State Sum Model for the Jones Polynomial

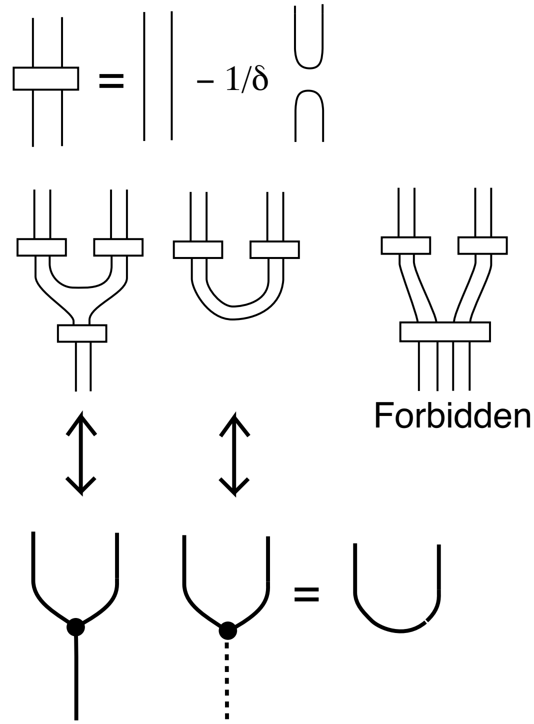
The Jones-Wenzel projectors are the "particles" in this theory.



Fibonacci Model



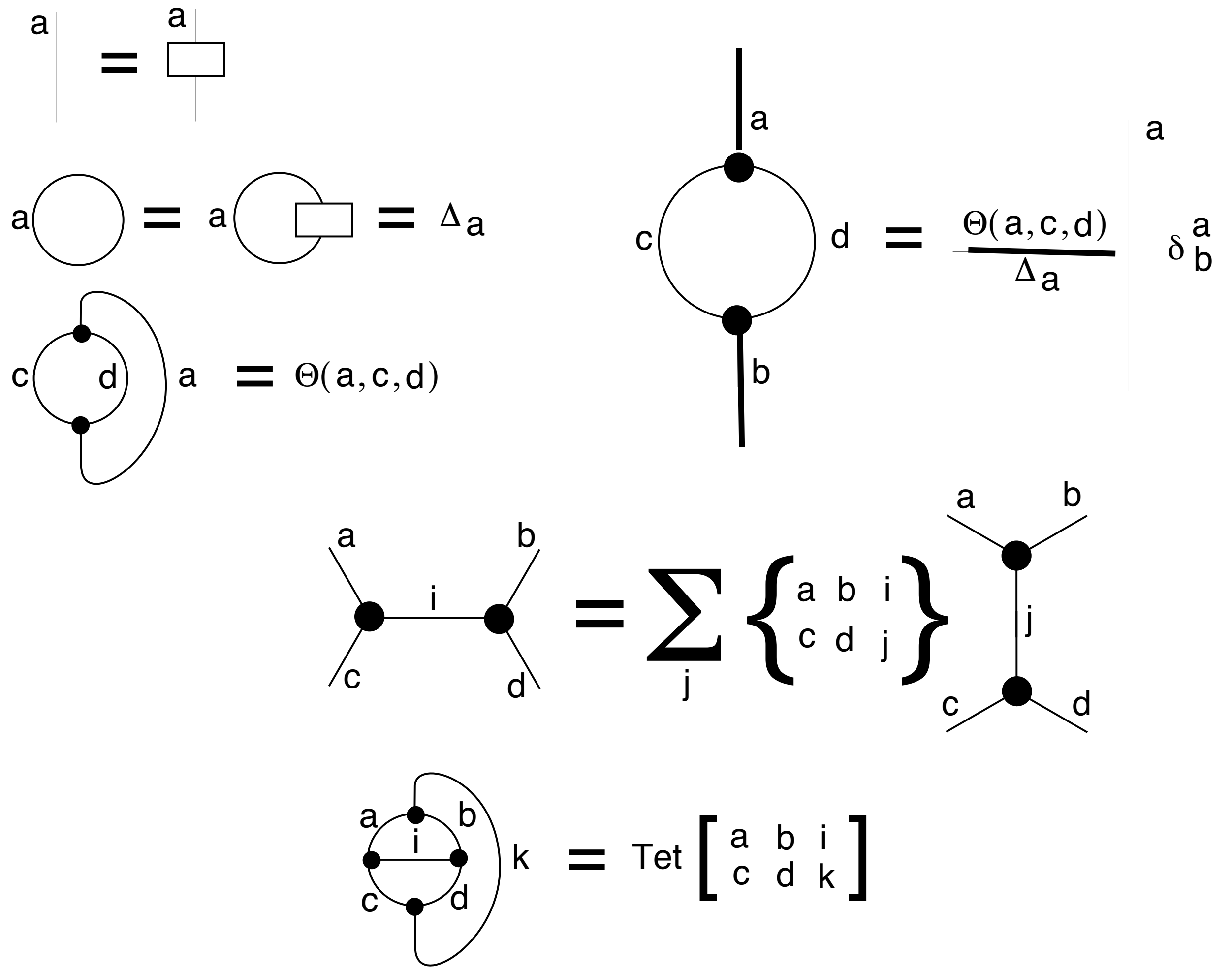
$$A = e^{3\pi i/5}.$$



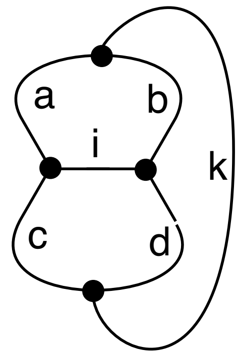
Temperley Lieb Representation of Fibonacci Model

Generalizing the Fibonacci Model

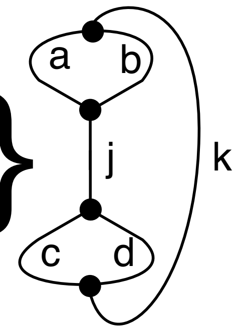
Closure, Bubble and Recoupling



The 6-j Coefficients



$$= \sum_j \begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}$$

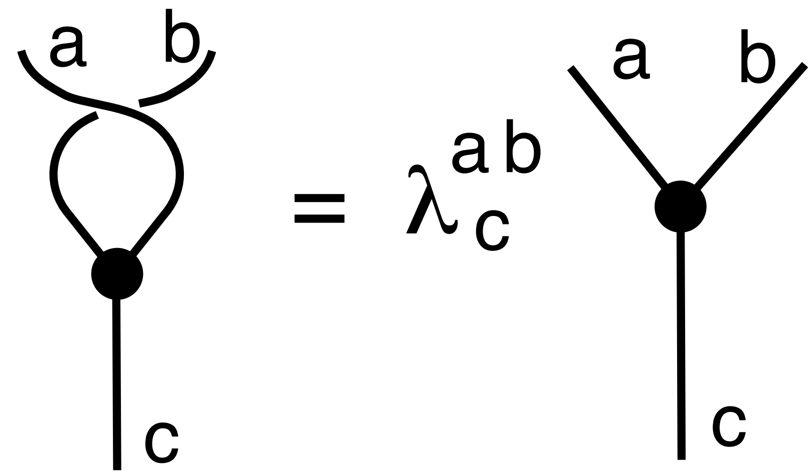


$$= \sum_j \begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix} \frac{\Theta(a,b,j)}{\Delta_j} \frac{\Theta(c,d,j)}{\Delta_j} \Delta_j \, \delta_j^k$$

$$= \begin{Bmatrix} a & b & i \\ c & d & k \end{Bmatrix} \frac{\Theta(a,b,k)\,\Theta(c,d,k)}{\Delta_k}$$

$$\begin{Bmatrix} a & b & i \\ c & d & k \end{Bmatrix} = \frac{\mathrm{Tet}\begin{bmatrix} a & b & i \\ c & d & k \end{bmatrix} \Delta_k}{\Theta(a,b,k)\ \Theta(c,d,k)}$$

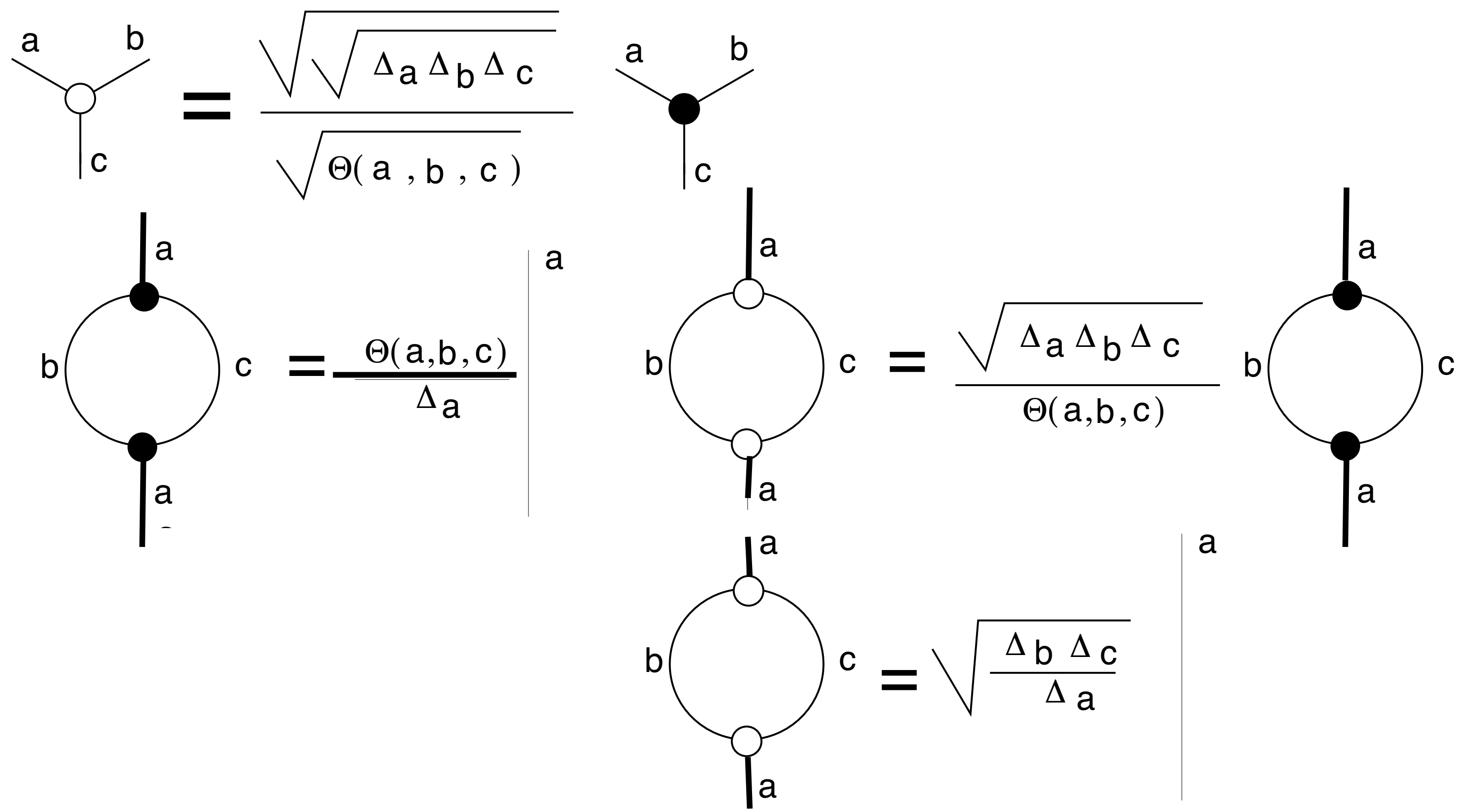
Local Braiding



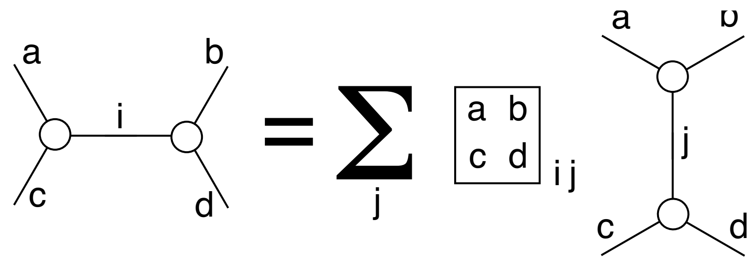
$$\lambda^{ab}_{c} = (-1)^{(a+b-c)/2} A^{(a'+b'-c')/2}$$

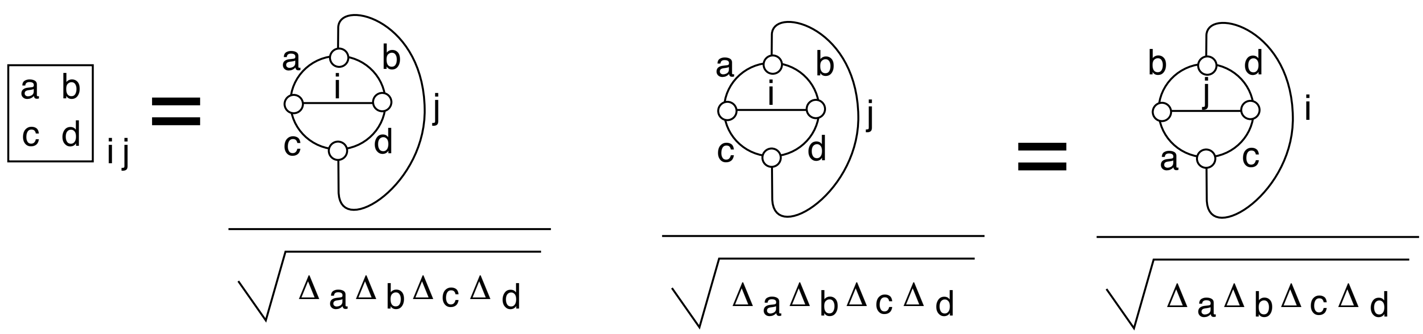
$$x' = x(x+2)$$

Redefining the Vertex is the key to obtaining Unitary Recoupling Transformations.



The Recoupling Matrix is Real Unitary at Roots of Unity.





$$M[a,b,c,d]_{i\,j} = \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}_{\,i\,j}$$

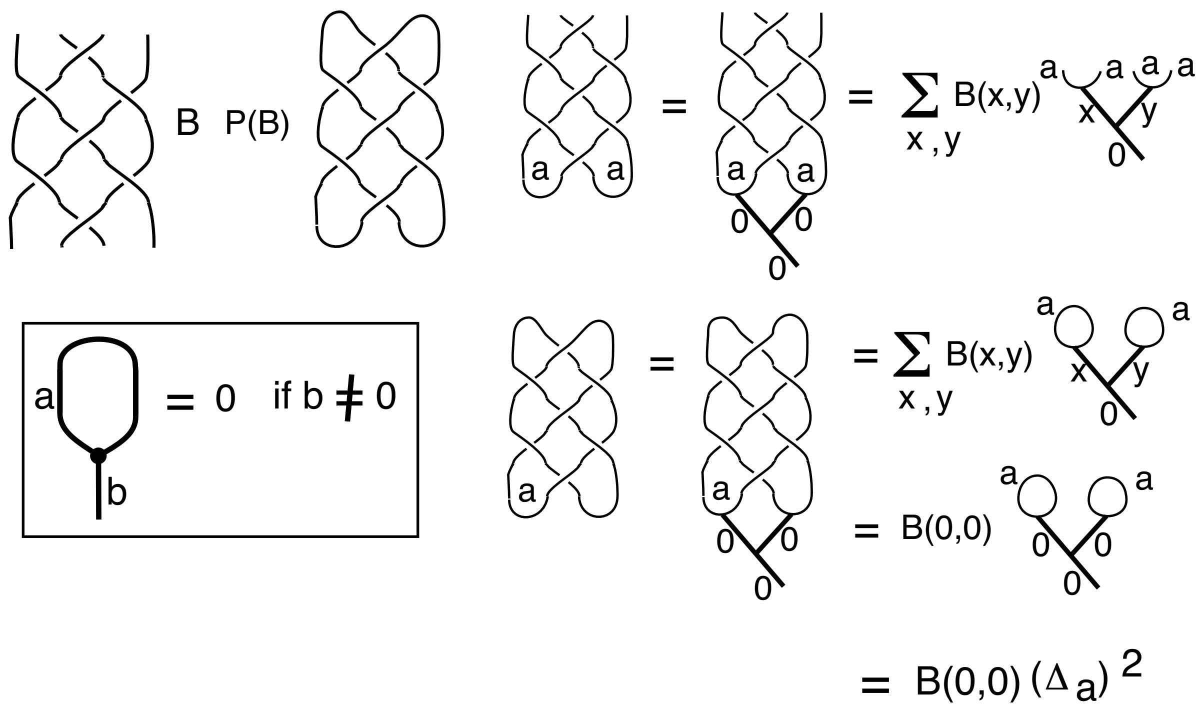
$$\Rightarrow \quad \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}^{T} = \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}^{-1}$$

Theorem. Unitary Representations of the Braid Group come from Temperley Lieb Recoupling Theory at roots of unity.

$$A = e^{i\pi/2r}$$

Sufficient to Produce Enough Unitary Transformations for Quantum Computing.

Quantum Computation of Colored Jones Polynomials and WRT invariants.



Need to compute a diagonal element of a unitary transformation.
Use the Hadamard Test.

Quantum Algorithms for the Jones Polynomial

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and S. J. Lomonaco Jr., UMBC

www.math.uic.edu/~kauffman
<kauffman@uic.edu>

Partial Bibliography

arXiv:0804.4304 The Fibonacci Model and the Temperley-Lieb Algebra

arXiv:0706.0020 A 3-Stranded Quantum Algorithm for the Jones Polynomial

arXiv:0909.1080

Title: NMR Quantum Calculations of the Jones Polynomial

Authors: Raimund Marx, Amr Fahmy, Louis Kauffman, Samuel Lomonaco, Andreas Spörl, Nikolas Pomplun, John Myers, Steffen J. Glaser

Categories: physics.quant-ph Quantum Physics

Untying Knots by NMR: first experimental implementation of a quantum algorithm for approximating the Jones polynomial

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[5]University of Maryland Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250, U.S.A.

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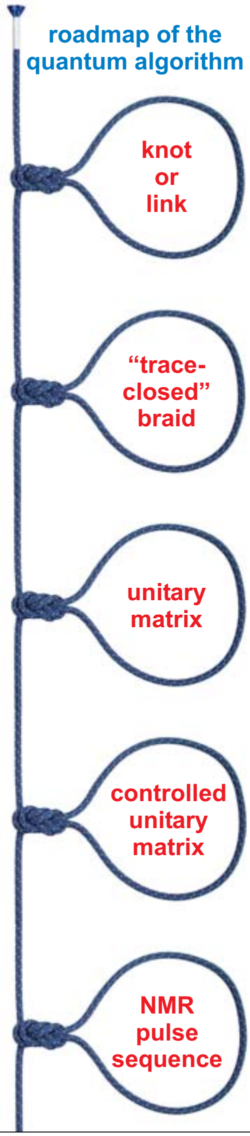
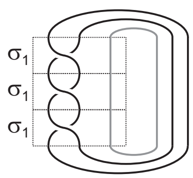
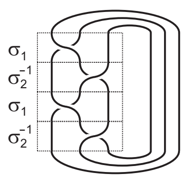
roadmap of the quantum algorithm



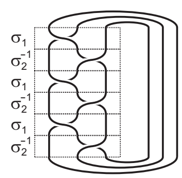
example #1 Trefoil



example #2 Figure-Eight



example #3 Borromean rings



$$U_{Trefoil} = \left(U_1\right)^3 \qquad U_{Figure-Eight} = \left(U_2^{-1} \cdot U_1\right)^2 \qquad U_{Borrom.R.} = \left(U_2^{-1} \cdot U_1\right)^3$$

$$U_1 = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & -e^{i\theta}\frac{\sin(4\theta)}{\sin(2\theta)} + e^{-i\theta} \end{pmatrix}$$

$$U_2 = \begin{pmatrix} -e^{i\theta}\frac{\sin(6\theta)}{\sin(4\theta)} + e^{-i\theta} & -e^{i\theta}\frac{\sqrt{\sin(6\theta)\sin(2\theta)}}{\sin(4\theta)} \\ -e^{i\theta}\frac{\sqrt{\sin(6\theta)\sin(2\theta)}}{\sin(4\theta)} & -e^{i\theta}\frac{\sin(2\theta)}{\sin(4\theta)} + e^{-i\theta} \end{pmatrix}$$

Step #1:
from the 2x2 matrix U
to the 4x4 matrix cU:

$$cU = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U \end{pmatrix}$$

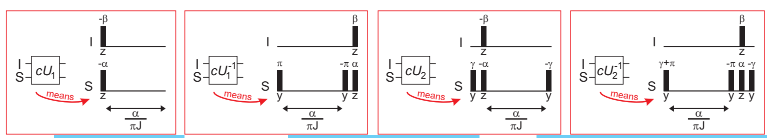
Step #2:
application of cU on the
NMR product operator I_{1x} :

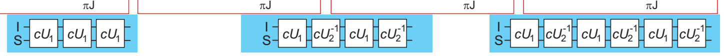
$$cU\ I_{1x}\ cU^{\dagger} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U \end{pmatrix}^{\dagger} = \frac{1}{2} \begin{pmatrix} 0 & U^{\dagger} \\ U & 0 \end{pmatrix}$$

Step #3:
measurement
of I_{1x} and I_{1y} :

$$tr\left\{ I_{1x}\ \frac{1}{2}\begin{pmatrix} 0 & U^{\dagger} \\ U & 0 \end{pmatrix} \right\} = \frac{1}{2}\Re\left(tr\{U\}\right)$$

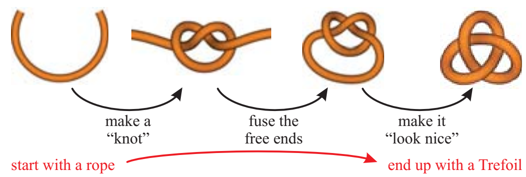
$$tr\left\{ I_{1y}\ \frac{1}{2}\begin{pmatrix} 0 & U^{\dagger} \\ U & 0 \end{pmatrix} \right\} = \frac{1}{2}\Im\left(tr\{U\}\right)$$





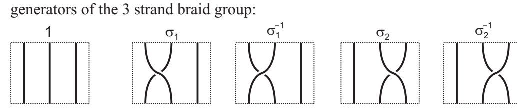
A knot is defined as a closed, non-self-intersecting curve that is embedded in three dimensions.

example: "construction" of the Trefoil knot:



J. W. Alexander proved, that any knot can be represented as a closed braid (polynomial time algorithm)

generators of the 3 strand braid group:



It is well known in knot theory, how to obtain the unitary matrix representation of all generators of a given braid goup (see "Temperley-Lieb algebra" and "path model representation"). The unitary matrices U_1 and U_2, corresponding to the generators σ_1 and σ_2 of the 3 strand braid group are shown on the left, where the variable "θ" is related to the variable "A" of the Jones polynomial by: $A = e^{-i\theta_{rad}}$. The unitary matrix representations of σ_1^{-1} and σ_2^{-1} are given by U_1^{-1} and U_2^{-1}.

The knot or link that was expressed as a product of braid group generators can therefore also be expressed as a product of the corresponding unitary matrices.

Instead of applying the unitary matrix *U*, we apply it's controlled variant *cU*. This matrix is especially suited for NMR quantum computers [4] and other thermal state expectation value quantum computers: you only have to apply *cU* to the NMR product operator I_{1x} and measure I_{1x} and I_{1y} in order to obtain the trace of the original matrix *U*.

Independent of the dimension of matrix *U* you only need ONE extra qubit for the implementation of *cU* as compared to the implementation of *U* itself.

The measurement of I_{1x} and I_{1y} can be accomplished in one single-scan experiment.

All knots and links can be expressed as a product of braid group generators (see above). Hence the corresponding NMR pulse sequence can also be expressed as a sequence of NMR pulse sequence blocks, where each block corresponds to the controlled unitary matrix *cU* of one braid group generator.

This modular approach allows for an easy optimization of the NMR pulse sequences: only a small and limited number of pulse sequence blocks have to be optimized.

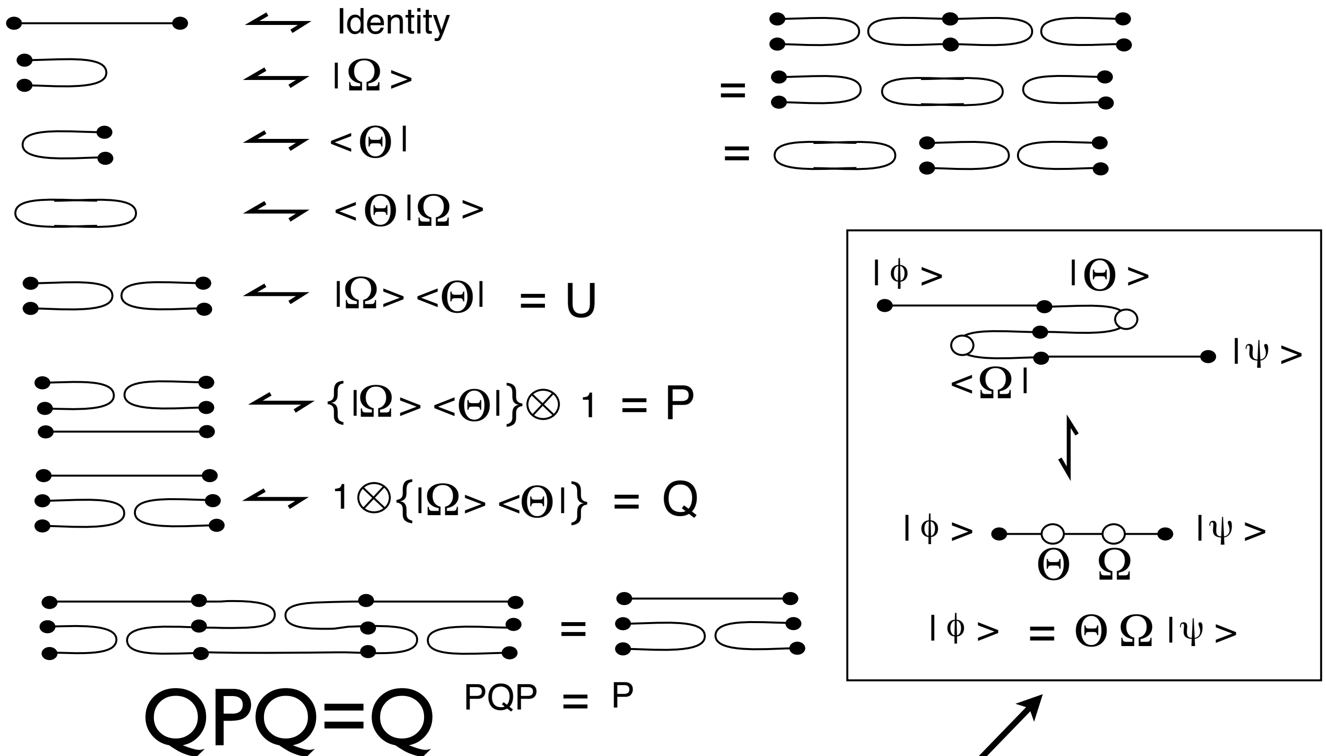
Three Strand and AJL Algorithms

The key idea behind the present quantum algorithms to compute the Jones polynomial is to use unitary representations of the braid group derived from Temperley-Lieb algebra representations that take the form

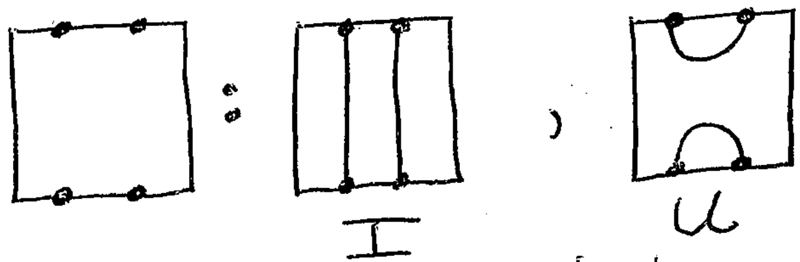
$$\rho(\sigma_i) = AI + A^{-1}U_i$$

where σ_i is a standard generator of the Artin braid group, A is a complex number of unit length, and U_i is a symmetric real matrix that is part of a representation of the Temperley-Lieb algebra. For more details about this strategy and the background information about the Jones

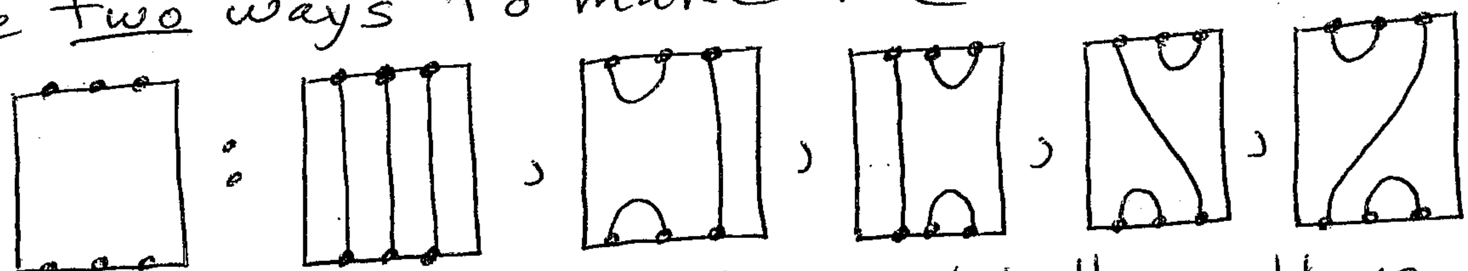
The Temperley-Lieb Category



The Key to Teleportation



With two points at top and bottom, there are <u>two</u> ways to make the connections.

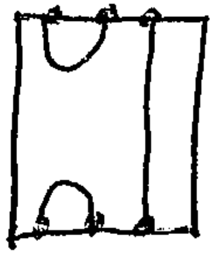
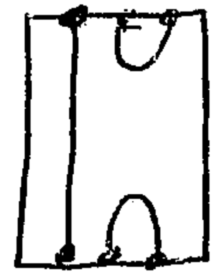


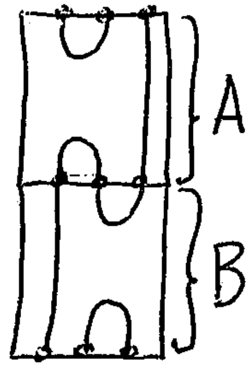
With three points at top and bottom, there are <u>five</u> ways to make the connections.

In general, with n points at top and bottom, there are $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)[n!]^2}$ ways to make the connections.

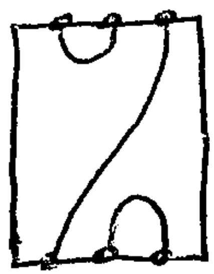
C_n $(n=1,2,3,4,\ldots)$ are the <u>Catalan numbers</u>.

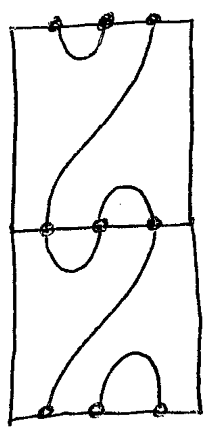
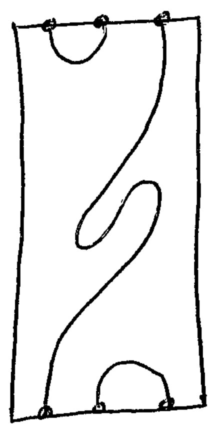
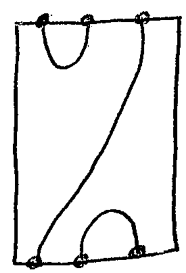
For example, $C_3 = \frac{6!}{4\,(3!)^2} = \frac{6\cdot5\cdot4\cdot3\cdot2}{4\cdot3\cdot2\cdot3\cdot2} = 5.$

$A =$  , $B =$ 

$AB =$ 

$=$ 

$AB =$  $= C.$

$C^2 =$  $=$  $=$  $= C.$

Knot Theory has a Combinatorial Model

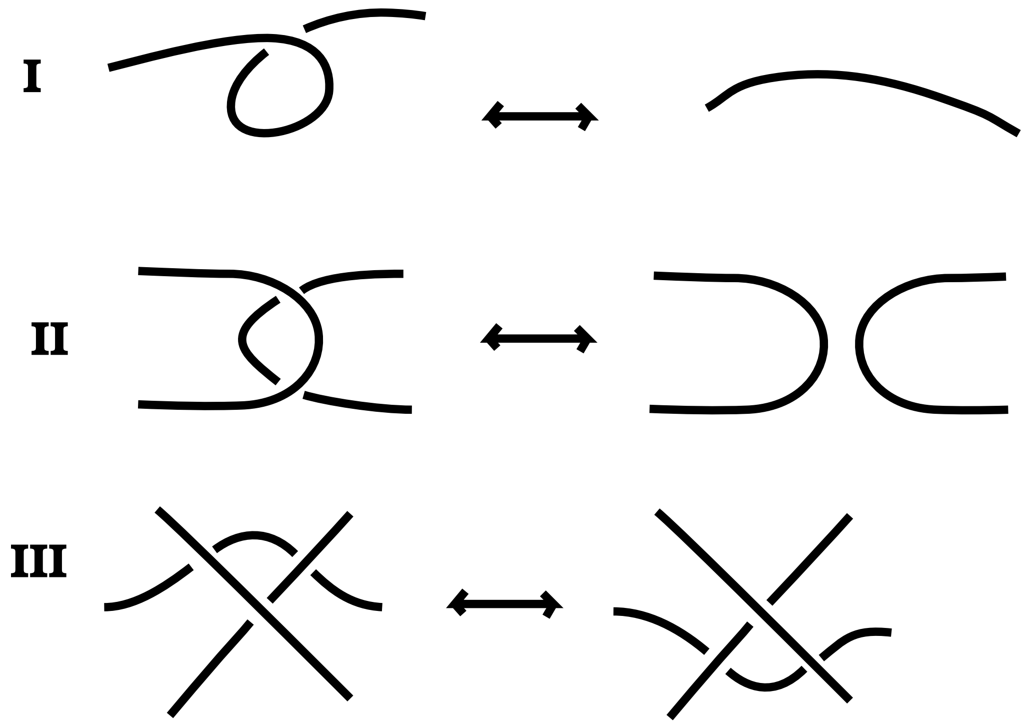
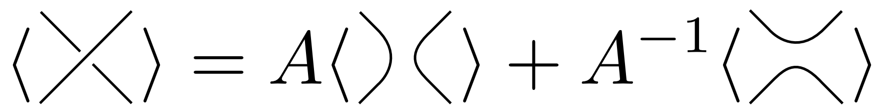
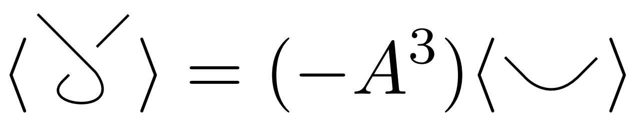


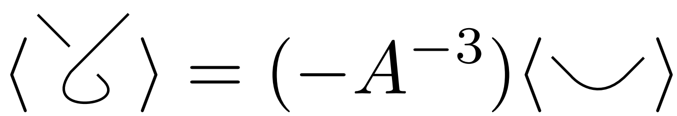
Figure 1. Reidemeister Moves

Bracket Model of Jones Polynomial



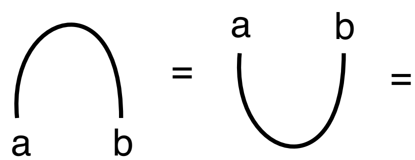
$$\langle K \bigcirc \rangle = (-A^2 - A^{-2})\langle K \rangle$$





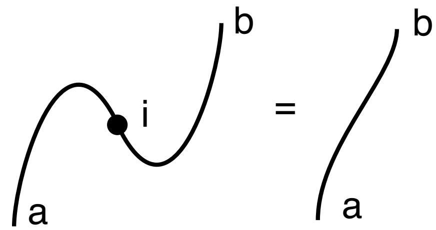
More about Temperley Lieb Representations

Here is the simplest method (but limited for unitarity).

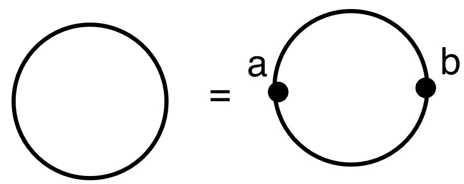


$$= \begin{pmatrix} 0 & iA \\ -iA^{-1} & 0 \end{pmatrix} = M$$

$$M^2 = I$$

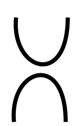


$$\sum M_{a\,i} M^{i\,b} = \delta_a^b$$

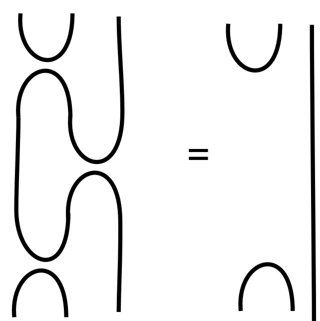


$$= \sum M_{ab} M^{ab} = -A^2 - A^{-2}$$

$$d = -A^2 - A^{-2}$$

$U =$ 

$$U^2 = d\,U$$



$$U_1 U_2 U_1 = U_1$$

It is useful to think of the Temperley Lieb algebra as generated by projections $e_i = U_i/\delta$ so that $e_i^2 = e_i$ and $e_i e_{i\pm 1} e_i = \tau e_i$ where $\tau = \delta^{-2}$ and e_i and e_j commute for $|i-j| > 1$.

With this in mind, consider elementary projectors $e = |A\rangle\langle A|$ and $f = |B\rangle\langle B|$. We assume that $\langle A|A\rangle = \langle B|B\rangle = 1$ so that $e^2 = e$ and $f^2 = f$. Now note that

$$efe = |A\rangle\langle A|B\rangle\langle B|A\rangle\langle A| = \langle A|B\rangle\langle B|A\rangle e = \tau e$$

Thus

$$efe = \tau e$$

where $\tau = \langle A|B\rangle\langle B|A\rangle$.

This algebra of two projectors is the simplest instance of a representation of the Temperley Lieb algebra. In particular, this means that a representation of the three-strand braid group is naturally associated with the algebra of two projectors.

Quite specifically if we let $\langle A| = (a, b)$ and $|A\rangle = (a, b)^T$ the transpose of this row vector, then

$$e = |A\rangle\langle A| = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$

is a standard projector matrix when $a^2 + b^2 = 1$. To obtain a specific representation,

let $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$.

It is easy to check that $e_1 e_2 e_1 = a^2 e_1$ and that $e_2 e_1 e_2 = a^2 e_2$.

We define

$$U_i = \delta e_i$$

for $i = 1, 2$ with $a^2 = \delta^{-2}$. Then we have , for $i = 1, 2$

$$U_i^2 = \delta U_i \, , \, U_1U_2U_1 = U_1, \, U_2U_1U_2 = U_2.$$

Thus we have a representation of the Temperley-Lieb algebra on three strands. See [10] for a discussion of the properties of the Temperley-Lieb algebra.

Now we return to the matrix parameters: Since $a^2 + b^2 = 1$ this means that $\delta^{-2} + b^2 = 1$ whence $b^2 = 1 - \delta^{-2}$. Therefore b is real when δ^2 is greater than or equal to 1.

We are interested in the case where $\delta = -A^2 - A^{-2}$ and *A is a unit complex number*. Under these circumstances the braid group representation

$$\rho(\sigma_i) = AI + A^{-1}U_i$$

will be unitary whenever U_i is a real symmetric matrix. Thus we will obtain a unitary representation of the three-strand braid group B_3 when $\delta^2 \geq 1$.
For any A with $d = -A^2 - A^{-2}$ these formulas define a representation of the braid group. With $A = exp(i\theta)$, we have $d = -2cos(2\theta)$. We find a specific range of angles θ in the following disjoint union of angular intervals

$$\theta \in [0, \pi/6] \sqcup [\pi/3, 2\pi/3] \sqcup [5\pi/6, 7\pi/6] \sqcup [4\pi/3, 5\pi/3] \sqcup [11\pi/6, 2\pi]$$

that give unitary representations of the three-strand braid group. Thus a specialization of a more general representation of the braid group gives rise to a continuous family of unitary representations of the braid group.

ON THE RELATIONSHIP WITH THE AJL ALGORITHM

Here is how the KL (Kauffman-Lomonaco) algorithm described in the previous section becomes a special case of a generalization of the AJL algorithm: Here we use notation from the AJL paper. In that paper, the generators U_i (in our previous notation) for the Temperley-Lieb algebra, are denoted by E_i.

Let $L_k = \lambda_k = sin(k\theta)$. For the time being θ is an arbitrary angle. Let $A = iexp(i\theta/2)$ so that $d = -A^2 - A^{-2} = 2cos(\theta)$.

We need to choose θ so that $sin(k\theta)$ is non-negative for the range of k's we use (these depend on the choice of line graph as in AJL). And we insist that $sin(k\theta)$ is non-zero except for $k = 0$. Then it follows from trigonometry that $(L_{k-1} + L_{k+1})/L_k = d$ for all k.

Recall that the representation of the Temperley-Lieb algebra in AJL is given in terms of E_i such that $E_i^2 = dE_i$ and the E_i satisfy the Temperley-Lieb relations. Each E_i acts non-trivially at the i and $i + 1$ places in the bit-string basis for the space and each E_i is based upon L_{a-1}, L_a, L_{a+1} where $a = z(i)$ is the endpoint of a walk described by the bitstring using only first $(i-1)$ bits. Bitstrings represent walks on a line graph. Thus 1011 represents the walk Right, Left, Right, Right ending at node number 3 in

$$1 - - - - - 2 - - - - - - 3 - - - - - - 4.$$

For $p = 1011$, $z(1) = 1, z(2) = 2, z(3) = 1, z(4) = 1, z(5) = 3$.

More precisely, if we let

$$|v(a)\rangle = [\sqrt{L_{a-1}/L_a}, \sqrt{L_{a+1}/L_a}]^T$$

(i.e. this is a column vector. T denotes transpose.) Then

$$E_i = |v(z(i))\rangle\langle v(z(i))|.$$

Here it is understood that this refers to the action on the bitstrings

$$- - - - - - - - - - 01 - - - - - - - - - -$$

and

$$- - - - - - - - - - 10 - - - - - - - - - -$$

obtained from the given bitstring by modifying the i and $i+1$ places. The basis order is 01 before 10. Conceptually, this is a useful description, but it also helps to have the specific formulas laid out.

Now look at the special case of a line graph with three nodes and two edges:

$$1 - - - - - 2 - - - - - 3.$$

The only admissible binary sequences are $|110\rangle$ and $|101\rangle$, so the space corresponding to this graph is two dimensional, and it is acted on by E_1 with $z(1) = 1$ in both cases (the empty walk terminates in the first node) and E_2 with $z(2) = 2$ for $|110\rangle$ and $z(2) = 2$ for $|101\rangle$. Then we have

$$E_1|110\rangle = 0, E_1|101\rangle = d|101\rangle,$$

$$E_2|xyz\rangle = |v\rangle\langle v|xyz\rangle$$

(xyz $= 101$ or 110) where $v = (\sqrt{1/d}, \sqrt{d - 1/d})^T$.

If one compares this two dimensional representation of the three strand Temperley - Lieb algebra and the corresponding braid group representation, with the representation Kauffman and Lomonaco use in their paper, it is clear that it is the same (up to the convenient replacement of $A = exp(i\theta)$ by $A = iexp(i\theta/2)$). The trace formula of AJL is a variation of the trace formula that Kauffman and Lomonaco use. Note that the AJL algorithm as formulated in [2] does not use the continuous range of angles that are available to the KL algorithm. In the sequel to this paper and in a separate paper on the mathematics, we shall show how the entire AJL algorithm generalizes to continuous angular ranges.

AJL is based on the following projector formalism.

$$v = \begin{pmatrix} \sqrt{\frac{\lambda_-}{\lambda_0}} \\ \sqrt{\frac{\lambda_+}{\lambda_0}} \end{pmatrix} \qquad E = v\, v^T$$

$$E = \begin{pmatrix} \frac{\lambda_-}{\lambda_0} & \frac{\sqrt{\lambda_- \lambda_+}}{\lambda_0} \\ \frac{\sqrt{\lambda_- \lambda_+}}{\lambda_0} & \frac{\lambda_+}{\lambda_0} \end{pmatrix}$$

$$E^2 = \left(\frac{\lambda_+ + \lambda_-}{\lambda_0} \right) E$$

Use this formalism on strings p of binary bits.
Each string is an instruction to walk on a line graph with "1" denoting "go right" and "0" denoting "go left".
Let z(i) = path endpoint(pli)
pli refers to the string from position 0 to position (i-1).

$$E_i(p) \quad \text{<----->} \quad \begin{pmatrix} \sqrt{\dfrac{\lambda_{z(i)-1}}{\lambda_{z(i)}}} \\ \\ \sqrt{\dfrac{\lambda_{z(i)+1}}{\lambda_{z(i)}}} \end{pmatrix}$$

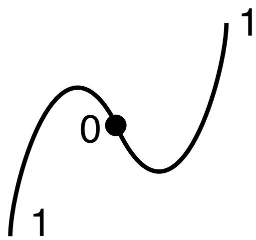
Thus $E_i(p)$ acts on the i and i+1 places in the walk and these places depend upon the walk that is described by the binary string p.

$$E_i(p) \quad \text{<----->} \quad \begin{pmatrix} \sqrt{\dfrac{\lambda_{z(i)-1}}{\lambda_{z(i)}}} \\ \\ \sqrt{\dfrac{\lambda_{z(i)+1}}{\lambda_{z(i)}}} \end{pmatrix}$$

Thus $E_i(p)$ acts on the i and i+1 places in the string.

The operators are indexed by the walk positions.

We need, for all i, $d = \dfrac{\lambda_{i-1} + \lambda_{i+1}}{\lambda_i}$



$$\text{<----->} \quad \sqrt{\frac{\lambda_{z(i)+1}}{\lambda_{z(i)}} \quad \frac{\lambda_{z(i+1)-1}}{\lambda_{z(i+1)}}}$$

$$\updownarrow$$

$$z(i+1) = z(i) + 1$$
$$\lambda_{z(i)} = \lambda_{z(i+1)-1}$$

This shows that the requirement on d (above) is sufficient to obtain the representation of the Termperley Lieb algebra of the space of binary strings.

Let $\lambda_k = \sin(\theta k)$.

Then $$\frac{\lambda_{i-1} + \lambda_{i+1}}{\lambda_i} = 2\cos(\theta)$$

For appropriate range of angles, this gives real symmetric representation of Temperley Lieb algebra on the space of binary strings.

There are continous angular ranges to choose from.

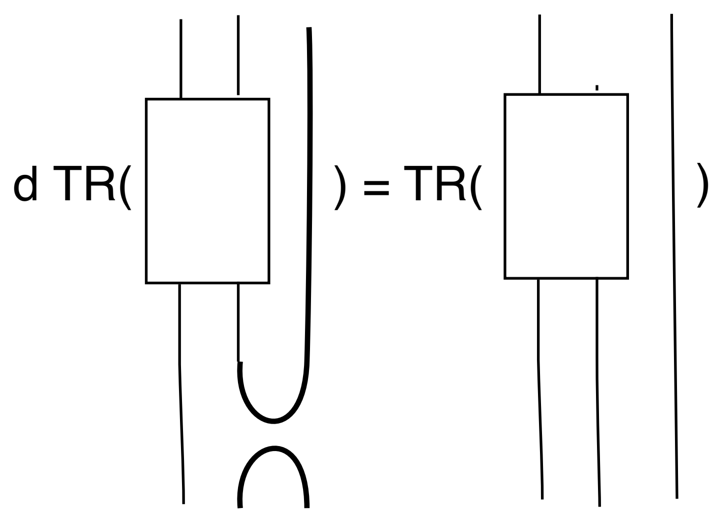
Traces

Let M denote our TL representation on the space of binary strings. Define

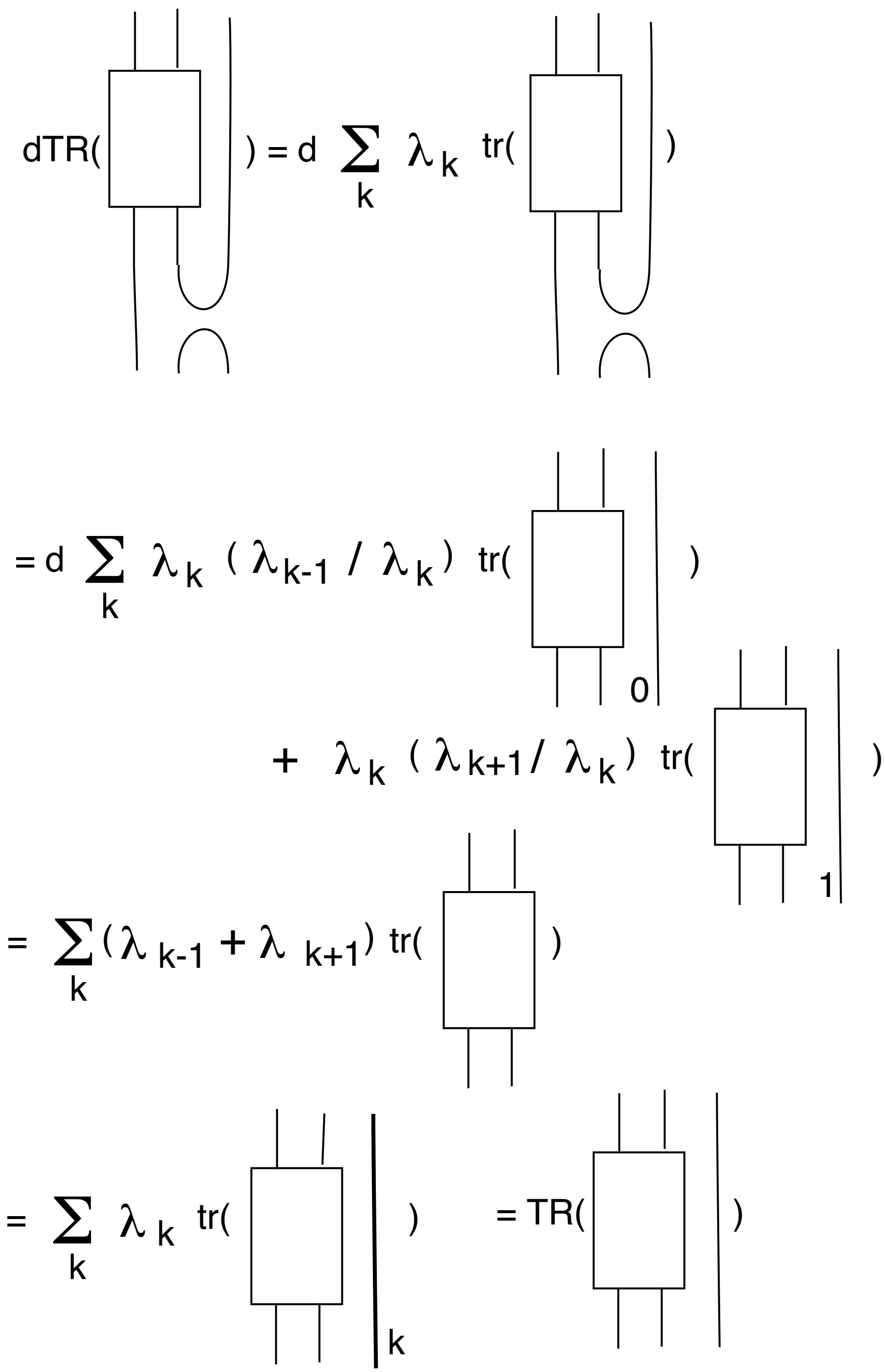
$$TR(M) = \sum_k \lambda_k \, tr(M_k)$$

where M_k denotes M restricted to paths P that end at k.

We will show that



This is the needed Markov trace condition for the link invariant.



dTR( ) = d ∑ k λ k tr( )
= d ∑ k λ k (λ k-1 / λ k) tr( ) 0
+ λ k (λ k+1 / λ k) tr( ) 1
= ∑ k (λ k-1 + λ k+1) tr( )
= ∑ k λ k tr( ) k
= TR( )

This completes a description of
a generalization of the AJL algorithm
that we are using for experiments
with NMR quantum computation.

And this concludes our sketch of
this corner of
topological quantum information theory.

Will topology play a key role
in the future of quantum computation?

Time will tell.