The Motion of the Observer in Celestial Navigation

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Abstract Conventional approaches to celestial navigation are based on the geometry of a stationary observer. Any motion of the observer during the time observations are taken must be compensated for before a fix can be determined. Several methods have been developed that account for the observer's motion and allow a fix to be determined. These methods are summarized and reviewed.

Key Words celestial navigation, celestial fix, motion of observer

Introduction

The object of celestial navigation is the determination of the latitude and longitude of a vessel at a specific time, through the use of observations of the altitudes of celestial bodies. Each observation defines a circle of position on the surface of the Earth, and the small segment of the circle that passes near the observer's estimated position is represented as a line of position (LOP). A position fix is located at the intersection of two or more LOPs. This construction works for a fixed observer or simultaneous observations. However, if the observer is moving, the LOPs from two consecutive observations do not necessarily intersect at a point corresponding to the observer's position at any time; if three or more observations are involved, there may be no common intersection. Since celestial navigation normally involves a single observer on a moving ship, something has to be done to account for the change in the observer's position during the time required to take a series of observations. This report reviews the methods used to deal with the observer's motion. A basic familiarity with the procedures, terminology, and notation of celestial navigation is assumed.

The fundamental principle involved is that each point on an LOP represents a possible true location of the vessel at the time of the observation, and should therefore move with the vessel's course and speed (Bowditch [1], pp. 129–130). Since the estimated position of the vessel also moves with the vessel's course and speed, an equivalent principle is that the difference between a vessel's true position and its estimated position—the error in position—remains constant, in two coordinates, as the vessel moves. The two coordinates are usually taken to be azimuth and distance. We have not distinguished here between the vessel's actual course and speed and its assumed course and speed. For the present, we will consider the vessel's course and speed to be known exactly, or at least well enough that any resulting errors are negligible compared to the errors of observation.

Chart-Based Approach

In the chart-based approach to celestial navigation, the principle that an LOP moves with the vessel's course and speed can be directly applied. The procedure is called *advancing* an LOP (to a later time) or *retiring* and LOP (to an earlier time). The plotting is done on a Mercator chart, where rhumb-line tracks are straight lines.

Consider a single celestial observation consisting of a sextant altitude, hs, of a known body made at time t from estimated position p. We assume that hs is appropriately corrected for instrumental error, dip, refraction, etc., yielding the observed altitude Ho. The observed body's computed altitude and azimuth, Hc and Zn, are obtained in the usual way for time t and position p, the altitude intercept $a =$ Ho $-$ Hc evaluated, and the LOP drawn.

Now suppose we have a group of such observations, taken from a moving vessel, each made at a different time and position. A fix is to be determined from these observations for some time t_0 when the vessel is at estimated position p_0. We will assume that t_0 is within the period of time spanned by the observations (often, t_0 will be the time of one of the observations). For each observation, the interval between the time of observation and the time of the fix is $\Delta t = t_0 - t$. If the vessel's course, C, and speed, S, are constant, in the interval between a given observation and the time of the fix the vessel's track is a rhumb line of length $S\Delta t$ in the direction C, a line that connects points p and p_0. (We are assuming that C and S have been adjusted for the current's set and drift.) In the chart-based approach to celestial navigation, each observation is advanced (if Δt is positive) or retired (if Δt is negative) to the time of the fix by simply moving its LOP on a Mercator chart by the amount $S\Delta t$ in the direction C. Each LOP's azimuth is held constant during this process, that is, the relocated LOP is drawn parallel to the original LOP. The distance of the relocated LOP from point p_0 is the same as the distance of the original LOP from point p. For details of the plotting procedure, see Bowditch [1], pp. 129–132.

If each observation's LOP is properly advanced (or retired) in this way, the LOPs should intersect (to within observational error) at a point near p_0. This intersection defines the fix for time t_0.

Mathematical Approaches

For mathematical approaches to sight reduction, there are several algorithms that account for the change in the observer's position. These are all based on the principle that the difference between the true and estimated positions does not change significantly as the vessel moves. That is, the error in position remains essentially constant in two coordinates.

Linearized LOPs A mathematical approach to celestial navigation is presented in [2] that is based on the plane geometry and straight lines formed by LOPs near the estimated position. A least-squares solution for the fix is used. The method is a direct mathematical translation of chart-based navigation. It was developed independently at the Royal Greenwich Observatory (RGO) and is described in RGO's publication *Compact Data for Navigation and Astronomy* [3]. The algorithm is also briefly presented on page 282 of the *Nautical Almanac*, in the section titled "Position from intercept and azimuth using a calculator." In this method, the equation for each straight-line LOP is developed with respect to the estimated position p at the time t of the observation. In rectangular coordinates (nautical miles east-west and north-south) any point (x, y) on the LOP satisfies

$$a = x \sin \mathrm{Zn} + y \cos \mathrm{Zn} \tag{1}$$

where a is the altitude intercept (in arcminutes). The estimated position p at the time of the observation is used as the basis for the computations as well as the origin of the rectangular coordinate system used. In this construction, advancing or retiring an LOP amounts to simply a change of origin, and the origin can be any point along the vessel's estimated track. Thus, to advance the LOPs to the time of the fix we simply consider that the equations (1) for all the LOPs refer to a common origin at p_0, the estimated position of the vessel at the time of the fix. Then the equations can be solved, using a least-squares procedure, for x and y (in nautical miles). The point (x, y) represents the best estimate (in a least-squares sense) of the intersection point of all the LOPs with respect to p_0. After the solution is computed, the resulting x and y values are converted to corrections to longitude and latitude and applied to point p_0 to form the fix.

This procedure thus uses equation (1) as a conditional equation for a least-squares solution for the error in position. Each conditional equation that enters the solution is computed for the time and estimated position of an individual observation, yet the positional parameters solved for (x and

y) are assumed to be constant offsets that apply to every estimated position in the problem. This is simply a restatement of the principle of constant error described above, with the linear coordinates x and y substituting for azimuth and distance.

For the sight-reduction algorithm described in [4], a nearly identical strategy is suggested to deal with the motion of the observer. This algorithm uses a different conditional equation, which allows the least-squares solution to directly yield corrections to estimated latitude and longitude. Applied to a moving observer, this procedure depends on the assumption that there is a constant error in latitude and a constant error in longitude. This assumption is somewhat different from the principle of constant error that we have been using.

Motion-of-Observer Formula Another way to account for the observer's motion, commonly used, is to adjust each observed altitude for the change in the position of the observer during the time Δt. This motion-of-observer correction, in arcminutes, is

$$\Delta\text{Ho} = \frac{S\Delta t}{60}\cos(\text{Zn} - C) \tag{2}$$

where S is in knots and Δt is in minutes of time; then ΔHo, which is in arcminutes, is added to the observed altitude. The quantity Ho + ΔHo represents the altitude that the observed body would have if it were observed at the same time t but from a different position—a position $S\Delta t$ further along the vessel's track, which is p_0. Essentially, use of this formula holds the geographical position (GP) of the observed body fixed (the GP for time t), but defines a new circle of position for the observation; for positive Δt, the circle has a larger radius if the vessel's course is away from the body or a smaller radius if the vessel's course is toward the body. For the small area on the surface of the Earth near the vessel's track, this is essentially equivalent to advancing (or retiring) the observation's LOP.

Applied to all observations, each with a different Δt, equation (2) yields a set of LOPs that intersect near p_0, defining the fix for time t_0. When equation (2) is used, the Hc and Zn values are computed for the individual observation times t but for the common position p_0. Equation (2) is an approximation, of course, but it works quite well for observations taken within a few minutes of each other. Even out to distances $S\Delta t$ of 25 nmi (typically 1–2 hours of sailing) the error in the formula itself is usually only a few tenths of an arcminute.

Adjustment of Celestial Coordinates Equation (2) adjusts the observed altitudes, but alternatively, one can make the corresponding adjustments in the celestial equatorial coordinates—hour angle and declination—of the observed body. An exact solution for a two-body fix is presented in [5], and the paper also contains a thorough explanation and development of formulas for adjusting celestial coordinates for both a change in time and a change in the observer's position. These formulas are meant to be applied over relatively short periods of time (an hour or less) and relatively small changes of position (10 nmi or less). Equation (8) in [5] provides for the change in declination of one body and equation (9) in [5] provides for the change in the difference between the local hour angles of two bodies.

More generally, advancing (or retiring) an LOP in the conventional manner can be accomplished by advancing (or retiring) the position of the object observed (Bowditch [1], p. 130). For celestial LOPs, this is accomplished mathematically by changing the GP of the observed body, which, of course, means adjusting its celestial coordinates. How can the Greenwich hour angle and declination of the observed body be adjusted to correctly advance its LOP? In the chart-based procedure, a section of each observation's LOP is moved by an amount $S\Delta t$ in the direction C. However, this is not, in general, the shift that should be applied to the GP. We want to move the GP in such a way that the error in position—the vector between a vessel's true position and its estimated position—remains constant, in both length and orientation, as the vessel moves. So if we have an altitude observation made from estimated position p, and we want to use that observation to correct an estimated position p_0, then we assume that the altitude intercept and azimuth computed for position p also apply to p_0. Essentially, we imagine a celestial body, observed from position p_0, with the same altitude and azimuth as the real celestial body observed from position p. For this construction, Ho is left unadjusted; and to maintain the same Hc and Zn, the Greenwich hour angle and declination of the imaginary body must be

$$\begin{aligned} \mathrm{GHA} &= -\lambda_0 \pm \arccos\left(\frac{\sin \mathrm{Hc} - \sin\phi_0 \sin \mathrm{d}}{\cos\phi_0 \cos \mathrm{d}}\right) \qquad \begin{cases} + & \text{if } 180° \le \mathrm{Zn} \le 360° \\ - & \text{otherwise} \end{cases} \\ \mathrm{Dec} &= \arctan\left(\frac{\sin \mathrm{d}}{\cos \mathrm{d}}\right) \end{aligned} \tag{3}$$

$$\text{where} \qquad \begin{aligned} \sin \mathrm{d} &= \sin\phi_0 \sin \mathrm{Hc} + \cos\phi_0 \cos \mathrm{Hc} \cos \mathrm{Zn} \\ \cos \mathrm{d} &= \sqrt{1 - \sin^2 \mathrm{d}} \end{aligned}$$

and where ϕ_0 and λ_0 are the latitude (north positive) and longitude (east positive) of position p_0, and Hc and Zn are the computed altitude and azimuth of the real body observed from position p at time t. Note that in the equation for GHA, $\arccos(\cdots)$ will always be positive, leaving a sign ambiguity that is resolved using the azimuth Zn. Equations (3) are simply a reversal of the usual altitude-azimuth formulas, applied to point p_0. Once GHA and Dec have been obtained using these equations, the entire sight reduction process (whatever process is used) can proceed as if the observation were taken from position p_0, with Ho, Hc, and Zn unchanged. The effect is to properly advance the LOP from near p to near p_0.

Note that equations (3) do not involve C, S, or Δt; no assumptions have been made about how the vessel gets from p to p_0, or how long it takes. In this sense this procedure is similar to the linearized LOP scheme, and can, in principle, be applied over extended lengths of time or multiple voyage legs.

Equivalence of the Procedures The three mathematical procedures outlined above should yield virtually identical results for an ordinary round of sights, given the same input data and the same sailing formulas. For example, when applied to the sight-reduction sample case on pp. 282–283 of the 1995 *Nautical Almanac*, the linearized LOP algorithm and the procedure of adjusting celestial coordinates give identical fixes. Use of the motion-of-observer formula yields a fix that is different by only 0.05 arcminute.

When combining observations taken over a longer span of time, it is important to remember that the motion-of-observer formula is an approximation that degrades as the length of the track over which the observations have been taken increases. The other two procedures do not suffer the same kind of degradation and remain correct and equivalent (given the validity of the basic principles) regardless of the observation span. In any event, the degree of equivalence among these procedures indicates only mathematical precision and should not be mistaken for navigational accuracy.

Limitations of the Procedures

The fundamental principle that is the basis of the chart-based procedure of advancing or retiring LOPs is that each point on an LOP represents a possible location for the vessel at the time of the observation, and should therefore move with the vessel's course and speed. The equivalent principle that is the basis of the mathematical procedures is that the error in the observer's estimated position, in two coordinates, does not significantly change as the vessel moves. These principles and the way they are applied deserve some closer scrutiny.

First, a celestial LOP is actually a circle, and if it is advanced so that each point on it follows a rhumb line defined by the vessel's course and speed, it will not precisely retain its shape. Even a short LOP segment, represented as a straight line, will not, in general, maintain a constant azimuth during such a transformation. Therefore, the usual chart-based construction, in which the advanced LOP is held parallel to the original, is not rigorously correct. One way to visualize the situation is to note that the scale on a Mercator chart is a function of latitude. (At mid latitudes, the scale changes by 1–2% per degree of latitude.) Different points on the original LOP, at different latitudes, should therefore advance different amounts *on the chart* because the scale is slightly different at each point. This means that the advanced LOP should not be drawn precisely parallel to the original LOP. At mid latitudes, the change in the azimuth of the LOP will typically amount to a few tenths of a degree for an advance of 50 nmi.

A similar difficulty arises with the principle of constant error. The notion that the positional error does not change as the vessel moves is contradicted by the mathematics of rhumb lines (except for the trivial case where the positional error is zero). Suppose we have two neighboring points on the surface of the Earth, and we extend a rhumb line from each point at the same azimuth for the same distance. The end points will not, in general, be the same distance from each other as the starting points, nor will their relative azimuths be the same. If we identify one of the starting points with a vessel's true position and the other with its estimated position, we see that the positional error must change as the vessel moves. The change is of order 0.1 nmi for starting points 20 nmi apart and rhumb lines 50 nmi long.

These are small effects, and in applying either of the two basic principles to real navigational situations, we need to be concerned with the resulting errors only if their magnitude approaches or exceeds that of the errors of observation. Clearly, for conventional celestial navigation, where the observational accuracy is of order $\pm 1'$, neither of the mathematical problems described above rises to this level. Much more important in practice is the fact that we do not know the observer's motion exactly. For the normal case of a ship sailing a rhumb-line track, what we are concerned about is how well the course, C, and the speed, S, are known over bottom as a function of time. The accuracy of these quantities is usually limited by our inexact knowledge of the local current. In the reduction of a series of celestial observations, the effect of errors in the ship's assumed motion is systematic: the estimated positions, the computed altitudes, and the altitude intercepts all change. How badly the resulting fix is shifted depends on the magnitude of the errors in course and speed and the accuracy, timing, and geometry of the observations. The usual rule of thumb for sights made with a hand-held sextant is that difficulties may arise for observations spread over more than about half an hour. If automated star trackers or similar high-accuracy devices were used for shipboard celestial navigation, observations spread over only a few minutes might be problematic.

In traditional navigational practice, this problem is minimized by taking a small number of observations within a very short period of time—a round of sights. Nevertheless, navigators frequently have to combine observations made hours apart, especially during the day. In such cases the familiar procedures are followed despite the inherent problems because there has been no other choice.

Another Approach

The observations themselves contain information on the actual track of the vessel, so the possibility exists that the sight-reduction procedure can be made self-correcting. Given enough observations, suitably distributed in time and azimuth, an estimate of the average over-bottom track of the vessel can be obtained as part of the solution for the fix. A development is presented in [6] that includes the observer's motion as an essential part of the mathematics of celestial navigation, rather than as an add-on. This algorithm can recover course and speed information from the observations. The entire problem is thus solved with one mathematical procedure. This algorithm has been incorporated into software developed by the U.S. Naval Observatory for Navy shipboard use.

This approach would be especially useful for high-accuracy automated observing systems. The procedure does not have significant advantages over more conventional methods for the normal round of sights, made with a hand-held sextant, since for low-accuracy observations it can determine course and speed only from an extended series of sights.

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