working paper

11 04

On the Coexistence of Money and Higher-Return Assets and its Social Role

Guillaume Rocheteau

FEDERAL RESERVE BANK OF CLEVELAND

Working Paper 11-04 January 2011

On the Coexistence of Money and Higher-Return Assets and its Social Role

Guillaume Rocheteau

This paper adopts mechanism design to tackle the central issue in monetary theory, namely, the coexistence of money and higher-return assets. I describe an economy with pairwise meetings, where fiat money and risk-free capital compete as means of payment. Whenever fiat money has an essential role, any constrained-efficient allocation is such that capital commands a higher rate of return than fiat money.

JEL Classification: D82, D83, E40, E50.

Keywords: money, capital, pairwise trades, rate-of-return dominance.

Guillaume Rocheteau is a professor of economics at the University of California, Irvine, and a research associate at the Federal Reserve Bank of Cleveland. He can be reached at grochete@uci.edu. He thanks Neil Wallace for his comments on an earlier draft. He thanks seminar participants at the University of California, Santa Barbara, for their beneficial comments and Monica Crabtree-Reusser for editorial assistance.

"The critical question arises when we look for an explanation of the preference for holding money rather than capital goods. For capital goods will ordinarily yield a positive rate of return, which money does not. What has to be explained is the decision to hold assets in the form of barren money, rather than of interest- or profit-yielding securities. (...) This, as I see it, is really the central issue in the pure theory of money." John Hicks (1935)

1 Introduction

To paraphrase Banerjee and Maskin (1996), the coexistence of money and higher-return assets has always been something of an embarrassment to economic theory. Despite being a robust feature of monetary economies, it cannot be accounted for by the standard economic paradigm. The dynamic general equilibrium models used for policy analysis evade the coexistence issue by either imposing cash-in-advance constraints or by adding money into the utility function. Such shortcuts are problematic, at best, as they introduce various hidden inconsistencies.[1] Modern monetary theory has made considerable progress in isolating the frictions that make fiat money essential (e.g., Kocherlakota, 1998), but the challenge of explaining why economic agents hold both fiat money and capital goods that yield a positive rate of return remains an unresolved issue.[2] Even carefully microfounded monetary models rule out the use of capital, or claims on capital, as means of payment.[3] Wallace (1980) and Lagos and Rocheteau (2008) propose models in which fiat money and capital do compete as media of exchange, but find out that the two assets can coexist only if they have the same rate of return.

The objective of this paper is to adopt a mechanism design approach to explain the coexistence of fiat money and higher-return assets in an environment with explicit frictions that make liquid assets useful. This approach is sensible as the essentiality of money can only be established by applying mechanism design to a given environment, i.e., by comparing the set of incentive-feasible allocations with and without money.[4] By selecting among these incentive-feasible allocations the

[1] These inconsistencies are enumerated in Wallace (1998) and Wallace's lecture on "Monetary theory at the beginning of the 21st century" at http://economics.uwo.ca/conference/monetaryeconomics05/Wallace.pdf.

[2] This view seems to be shared by prominent monetary theorists, including Hellwig (1993) and Wallace (1998).

[3] Examples of such models include Shi (1999), Aruoba and Wright (2003), Molico and Zhang (2006), and Aruoba, Waller, and Wright (2010).

[4] Kocherlakota (1998) and Kocherlakota and Wallace (1998) were the first to use implementation theory to prove

ones that maximize society's welfare, mechanism design identifies the salient properties of good allocations in monetary economies. If the coexistence of money and higher-return assets is among such properties, then rate-of-return dominance is not a puzzle.

The monetary environment to which I apply mechanism design is the one in Lagos and Rocheteau (2008), where capital goods compete with money as media of exchange. As in Lagos and Wright (2005), agents trade alternatively in pairwise meetings, where there is a need for liquid assets, and in competitive markets, where they can choose their asset portfolios. This environment has the advantages of being tractable—thanks to quasilinear preferences—and amenable to mechanism design—thanks to periodic rounds of bilateral meetings.[5] The answer to Hicks's question is simple: Money and higher-return assets coexist because such coexistence is both socially optimal and individually rational. More precisely, whenever fiat money is essential, a property of any constrained-efficient allocation is that capital generates a higher rate of return than fiat money.

I first show that fiat money is essential when the economy faces a shortage of liquid assets: The first-best capital stock is not abundant enough relative to the economy's needs for a medium of exchange. If the shortage of capital is small, then a constant stock of fiat money implements the first best and the rate of return of capital is equal to the rate of time preference, which is larger than the rate of return of money, which is zero. If the shortage of capital is large, then individuals lack incentives to hold enough real balances to trade the first-best level of output: The nonpecuniary return of fiat money is not large enough to compensate agents for their time preference. In such circumstances, society faces a trade-off between the role of capital as a liquid asset and its role as a productive asset. In some circumstances the trade-off leads to over-accumulation of capital relative to the first best.

the essentiality of money. Applications of mechanism design to monetary theory include Cavalcanti and Wallace (1999) and Mattesini, Monnet, and Wright (2010) on banking and inside money, Cavalcanti and Erosa (2008) on the propagation of shocks in monetary economies, Cavalcanti and Nosal (2009) on cyclical monetary policy, Koeppl, Monnet, and Temzelides (2008) on settlement, Deviatov and Wallace (2001) and Deviatov (2006) on the welfare gains of money creation, Hu, Kennan, and Wallace (2009) on the optimality of the Friedman rule, and Rocheteau (2010) on the cost of inflation. The use of mechanism design is especially important in multiple-asset environments since under socially inefficient trading mechanisms, fiat money can be valued even though it is not essential.

[5] The tractability of the model comes at a cost: It shuts down the distributional effects of monetary policy. These distributional effects, however, do not play a role in the argument developed in this paper, and while models with a nondegenerate distribution of asset holdings can be solved numerically (e.g., Molico and Zhang, 2006), designing the optimal trading mechanism for this class of models is currently out of reach. Notice also that a similar analysis could be conducted in the context of the large-household model of Shi (1997).

Under the most commonly used pricing protocols, it is not individually rational to hold real balances if capital yields a positive rate of return. In contrast, in economies with pairwise meetings, the optimal mechanism specifies a pricing schedule that gives agents incentives to hold money even though capital has a higher rate of return. The pricing mechanism can align individuals' incentives with society's best interest because the core in pairwise meetings is nondegenerate; i.e., there is a continuum of allocations consistent with pairwise Pareto efficiency, and agents are not indifferent in terms of which allocation is selected. As a consequence, a mechanism can punish an agent who deviates from a proposed allocation by choosing his least-preferred trade in the core. The optimal mechanism has two noticeable properties. First, it gives buyers a discount, in the form of a positive surplus, for large trades. Second, buyers can enjoy this discount only if they finance a fraction of their purchase with fiat money, i.e., fiat money is more liquid than capital.

High-return assets play a liquidity role because the substitution of high-return assets (capital) for low-return ones (fiat money) relaxes individuals' participation constraint in asset markets. An alternative way to relax agents' participation constraint is by engineering a positive rate of return for fiat money. To analyze this possibility I consider the case in which the money supply grows, or shrinks, at a constant rate. Under a socially optimal trading mechanism, the Friedman rule is not necessary to maximize society's welfare. There is a threshold for the inflation rate, below which the first-best allocation is implementable, and capital is unaffected by changes in the money growth rate, i.e., there is no Tobin effect. Moreover, if one were to compute the cost of moderate inflation, it would be zero. On the contrary, if inflation is sufficiently large, an increase in inflation reduces real balances and welfare, and it raises the aggregate capital stock. For all inflation rates above the Friedman rule, the optimal allocation is such that capital goods yield a higher return than fiat money.

The use of mechanism design in monetary theory has been advocated by Wallace (2001, 2010). (See Footnote 4 for a succinct review of the literature.) It has been applied to the Lagos and Wright (2005) environment by Hu, Kennan, and Wallace (2009) to dismiss the usefulness of the Friedman rule. I extend their analysis to a multiple-asset setup to focus on the coexistence of fiat money and capital. Zhu and Wallace (2007) and Nosal and Rocheteau (2009) construct trading mechanisms in economies with pairwise meetings that are consistent with the coexistence of money and higher-return assets, but these mechanisms are not socially optimal.[6] Kocherlakota (2003) establishes that illiquid government bonds have a societal role when agents are subject to idiosyncratic preference shocks. In contrast, I do not consider nominal bonds, and I focus on the social trade-off between the liquidity and productive uses of assets. Moreover, the liquidity of assets is determined endogenously as part of an optimal trading mechanism. There are alternative explanations for the rate-of-return differences across assets based on assets' indivisibilities (e.g., Aiyagari, Wallace, and Wright, 1996) or lack of recognizability (e.g., Freeman, 1985; Lester, Postlewaite, and Wright, 2008; Rocheteau, 2009; Li and Rocheteau, 2009).[7] I will show that rate-of-return dominance is a property of a constrained-efficient allocation even if capital goods are perfectly divisible and recognizable.

The rest of the paper is organized as follows. Section 2 describes the environment. Section 3 determines the set of stationary, incentive-feasible allocations. The constrained-efficient allocation and the main result in terms of rate-of-return dominance appear in Section 4. The relationship between inflation and capital accumulation is studied in Section 5.

2 The environment

The environment is similar to the one in Lagos and Rocheteau (2008). Time is represented by $t \in \mathbb{N}$. Each period, t, is divided into two stages labelled DM (decentralized market) and CM (centralized market). In the first stage, DM, each agent enters a bilateral match with a randomly chosen trading partner with probability $\sigma \in [0, 1]$. In the second stage, CM, agents trade in competitive markets. Time starts in the CM of period 0. In each stage there is a perfectly divisible and perishable consumption good.

There is a measure two of infinitely lived agents divided evenly among two types called *buyers* and *sellers*, where these labels capture agents' roles in the DM. Buyers' preferences are represented by the following utility function

$$c_0 - h_0 + \mathbb{E} \sum_{t=1}^{\infty} \beta^t \left[u(q_t) + c_t - h_t \right],$$

[6] In the search labor literature, the nondegenerate pairwise core is used to construct dynamics for the real wage which can account for some business cycle facts of the labor market. See, e.g., Hall (2005), Gertler and Trigari (2009), and Shimer (2010, chapter 4).

[7] The literature on monetary models with pairwise meetings and multiple assets is reviewed in Nosal and Rocheteau (2010). See also the survey by Williamson and Wright (2010).

where $\beta \equiv (1+r)^{-1} \in (0,1)$ is the discount factor, q_t is DM consumption, c_t is CM consumption, and h_t is the supply of hours in the CM.[8] Sellers' preferences are given by

$$c_0 - h_0 + \mathbb{E}\sum_{t=1}^{\infty} \beta^t \left[-\upsilon(e_t) + c_t - h_t\right],$$

where e_t is the DM level of effort. The technology in the DM is such that $q = e$. The first-stage utility functions, $u(q)$ and $-\upsilon(q)$, are increasing and concave, with $u(0) = \upsilon(0) = 0$. The surplus function, $u(q)-\upsilon(q)$, is strictly concave, with $q^* = \arg\max\left[u(q) - \upsilon(q)\right]$. Moreover, $u'(0) = \upsilon'(\infty) = \infty$ and $\upsilon'(0) = u'(\infty) = 0$. All agents have access to a linear technology to produce the CM output from their own labor, $c = h$.

The CM good can be transformed into a capital good one for one. Capital goods accumulated at the end of period t are used by sellers at the beginning of the CM of $t+1$ to produce the CM good according to the technology $F(k)$.[9] See Figure 1. I assume that $F' > 0$, $F'' < 0$, $F'(0) = \infty$, $F'(\infty) = 0$, and $F'(k)k$ is strictly increasing, with range $\mathbb{R}_+$, and strictly concave. An example of a production function satisfying these properties is $F(k) = k^{\alpha}$, with $0 < \alpha < 1$. Capital goods depreciate fully after one period. The rental price of capital in terms of the CM good is R_t.

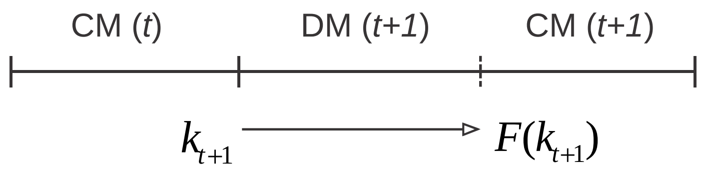

Figure 1: Timing

Agents cannot commit to future actions, and individual histories are private information. These assumptions rule out (unsecured) credit arrangements and generate a social role for liquid assets. Capital goods (or claims on such goods) can serve this role. There is also a fixed supply, M, of an intrinsically useless, perfectly divisible asset called fiat money. The price of goods in terms of

[8] Instead of having linear preferences over c_t and h_t, one could adopt a quasilinear specification of the form $U(c_t) - h_t$, with $U'' < 0$. Provided that the non-negativity constraint on hours of work is not binding, the two formulations are equivalent.

[9] Alternatively, production could take place through neoclassical firms using labor and capital as inputs. See, e.g., Aruoba, Waller, and Wright (2010). My formulation makes it a little easier to characterize the optimal mechanism as the real wage is independent of the capital stock.

money in the CM is denoted p_t. In a pairwise meeting in the DM a buyer can transfer any quantity of his asset holdings in exchange for some output. Moreover, he can hide his asset holdings but cannot overstate them.[10] For simplicity, I restrict sellers from holding assets from one period to the next. As shown in Appendix C, this causes no loss in generality.

3 Implementation

I first describe the trading mechanism in the DM. The terms of trade in a bilateral match are determined according to the following game. In the first stage, the buyer announces his real balances, z, and his capital stock, k. A mechanism, $o : \mathbb{R}_{2+} \rightarrow \mathbb{R}_{3+}$, maps the announced asset holdings into a proposed allocation, $(q, d_z, d_k) \in \mathbb{R}_+ \times [0, z] \times [0, k]$, where q is the quantity produced by the seller and consumed by the buyer, d_z is a transfer of real balances from the buyer to the seller, and d_k is a transfer of capital goods. The proposed allocation is chosen in the pairwise core of the bilateral match.[11] The trading mechanism is incentive-compatible if it is optimal for the buyer to announce his asset holdings truthfully. In the second stage of the game, the buyer and the seller simultaneously say "yes" or "no" to the proposed allocation. If they both say "yes," the trade takes place. Otherwise, there is no trade. This second stage guarantees that the allocation is individually rational.

I consider stationary, symmetric allocations. Such an allocation is defined by a 5-tuple $(q^p, d_z^p, d_k^p, z^p, k^p)$, where (q^p, d_z^p, d_k^p) is the trade in all matches in the DM, z^p is the buyer's real balances, and k^p is the buyer's capital holdings. From market clearing in the CM, $M/p_t = z^p$, i.e., $p_{t+1} = p_t = \frac{M}{z^p}$.

Bellman's equation for a buyer in the DM holding z units of real balances and k units of capital is

$$V^b(z,k) = \sigma \left\{ u\left[q(z,k)\right] + W^b\left[z - d_z\left(z,k\right), k - d_k\left(z,k\right)\right] \right\} + (1-\sigma)\, W^b(z,k), \tag{1}$$

[10] In Rocheteau (2010) I explore different assumptions regarding the observability of money holdings. The insights of the model are robust to the different assumptions.

[11] Zhu (2008) proposes a coalition-proof game that guarantees that any trade in the DM is in the pairwise core. In my context this game would work as follows. First, the buyer announces his asset holdings. Second, an allocation is proposed. The buyer and the seller simultaneously accept or reject the proposed allocation. If it is rejected by one of the two players, the game ends. Otherwise, the buyer makes a counterproposal. Third, the seller can choose which trade is carried out, the buyer's counteroffer or the initial offer.

where $W^b(z,k)$ is the value function of the buyer in the CM. Equation (1) has the following interpretation. The buyer meets a seller with probability σ, in which case he consumes q units of goods and delivers d_z units of real balances (expressed in terms of CM output) and d_k units of capital to his trading partner. The terms of trade, (q, d_z, d_k), depend on the (truthfully) announced portfolio of the buyer. With probability $1-\sigma$, the buyer is unmatched and no trade takes place in the DM.

The CM problem of the buyer is

$$W^b(z,k) = \max_{\hat{z}\geq 0, \hat{k}\geq 0} \left\{ z + Rk - \hat{z} - \hat{k} + \beta V^b(\hat{z},\hat{k}) \right\}, \tag{2}$$

where $\hat{z}$ and $\hat{k}$ denote the real balances and capital taken into the next day and where I used the budget constraint according to which $c - h = z + Rk - \hat{z} - \hat{k}$. From (2), the buyer consumes his real balances and the return on his capital stock and chooses his next-period portfolio in order to maximize his discounted continuation value, net of the cost of accumulating capital and real balances. The maximizing choice of $\hat{z}$ and $\hat{k}$ is independent of the buyer's beginning-of-CM portfolio (z,k); and $W^b(z,k) = z + Rk + W^b(0,0)$. Substituting $V^b(z,k)$ by its expression given by (1), using the linearity of $W^b(z,k)$, and omitting constant terms, the buyer's problem in the CM can be reformulated as

$$\max_{z\geq 0, k\geq 0} \left\{ -rz - (\beta^{-1} - R)k + \sigma \left\{ u\left[q(z,k)\right] - d_z\left(z,k\right) - d_k(z,k) \right\} \right\}. \tag{3}$$

The optimal portfolio maximizes the expected surplus of the buyer, net of the cost of holding real balances and capital. The cost of holding real balances is equal to the discount rate. The cost of holding capital is the difference between the discount rate and the rate of return of capital. As the buyer's surplus in the DM is non-negative (from individual rationality), it should be clear from (3) that $R \leq \beta^{-1}$ for a solution to exist (otherwise agents would want to hold an infinite capital stock).

Bellman's equation for a seller at the beginning of the period is

$$V^s = \sigma\left\{ -\upsilon\left[q(z^p,k^p)\right] + W^s\left[d_z\left(z^p,k^p\right), d_k\left(z^p,k^p\right)\right] \right\} + (1-\sigma)\, W^s(0,0), \tag{4}$$

where $W^s(z,k)$ is the value function of the seller in the CM. The interpretation of (4) is similar to the interpretation of (1). The CM problem of the seller is

$$W^s(z,k) = \max_{k'} \left\{ z + Rk + F(k') - Rk' + \beta V^s \right\}. \tag{5}$$

From (5), the seller consumes his real balances and rents $k' - k$ units of capital in order to produce $F(k')$ units of CM good. (Given that capital goods fully depreciate after one period, it is strictly equivalent to buy or rent capital goods.) The seller's choice of capital in the CM is such that the rental price of capital is equal to its marginal product, i.e.,

$$F'(k) = R. \tag{6}$$

A necessary condition for the allocation $(q^p, d_z^p, d_k^p, z^p, k^p)$ to be incentive feasible is

$$-z^p - k^p + \beta V^b(z^p, k^p) \geq \beta W^b(0, 0). \tag{7}$$

The left side of (7) is the discounted value of the buyer in the DM, net of the investment in real balances and capital. A deviation that is feasible consists of not accumulating money or capital in the CM and not trading in the DM. The expected utility associated with this defection, the right side of (7), is the discounted value of the buyer holding no asset in the next CM. Substituting V^b by its expression given by (1), (7) can be reexpressed as

$$-rz^p - \left[\beta^{-1} - F'(k^p)\right] k^p + \sigma \left[u\left(q^p\right) - d_z^p - F'(k^p) d_k^p\right] \geq 0, \tag{8}$$

where I used (6), $R = F'(k^p)$. The allocation must also satisfy the seller's participation constraint in the DM,

$$-\upsilon\left(q^p\right) + d_z^p + F'(k^p) d_k^p \geq 0. \tag{9}$$

There is a similar individual rationality condition for buyers in the DM, $u\left(q^p\right) - d_z^p - F'(k^p) d_k^p \geq 0$, but it is implied by (8).

The allocation in a pairwise meeting, (q^p, d_z^p, d_k^p), is restricted to be in the core, denoted $\mathcal{C}(z^p, k^p; R)$.[12] The next lemma shows that even though (8)-(9) are only necessary conditions for an allocation to be incentive-feasible, no further restrictions are needed to make the allocation (coalition-proof) implementable.

[12] The pairwise core is the set of all feasible allocations, $(q, d_z, d_k) \in \mathbb{R}_+ \times [0, z^p] \times [0, k^p]$, such that there exist no alternative feasible allocations that would make the buyer and the seller in the match better off, with at least one of the two being strictly better off. See the formal definition in Appendix B.

Lemma 1 *Consider an allocation,* $(q^p, d_z^p, d_k^p, z^p, k^p)$*, that satisfies:* $(q^p, d_z^p, d_k^p) \in \mathcal{C}(z^p, k^p; R)$*;* $R = F'(k^p) \leq \beta^{-1}$*; (8) and (9). This allocation can be implemented by the following coalition-proof trading mechanism*

$$[q(z,k), d_z(z,k), d_k(z,k)] = \arg \max_{q, d_z \leq z, d_k \leq k} \left[d_z + F'(k^p)d_k - \upsilon(q)\right] \tag{10}$$
$$s.t. \quad u(q) - d_z - F'(k^p)d_k \geq u(q^p) - d_z^p - F'(k^p)d_k^p,$$

if $z \geq z^p$ *and* $k \geq k^p$*, and*

$$[q(z,k), d_z(z,k), d_k(z,k)] = \arg \max_{q, d_z \leq z, d_k \leq k} \left[d_z + F'(k^p)d_k - \upsilon(q)\right] \tag{11}$$
$$s.t. \quad u(q) - d_z - F'(k^p)d_k = 0,$$

otherwise.

The programs (10) and (11) define the mapping, o, between the buyer's portfolio and the trade in the DM. According to (10), if the buyer holds at least z^p real balances and at least k^p units of capital, then the mechanism selects the pairwise Pareto-efficient allocation that gives the buyer the same surplus as the one he would obtain under the trade (q^p, d_z^p, d_k^p). According to (11), if the buyer holds less than z^p real balances or less than k^p units of capital, then the mechanism chooses the allocation that maximizes the seller's surplus subject to the buyer being indifferent between trading or not trading.

Figure 2 represents graphically the mechanism in (10)-(11). For a given aggregate capital stock, k^p, the buyer's surplus is $U^b = u(q) - d_z - Rd_k$, while the seller's surplus is $U^s = -\upsilon(q) + d_z + Rd_k$, where $R = F'(k^p)$. The pairwise core (in the utility space) is downward-sloping and concave. The utility levels associated with the proposed trade, (q^p, d_z^p, d_k^p), are denoted $\bar{U}^b$ and $\bar{U}^s$. If the buyer holds $z \geq z^p$ and $k \geq k^p$, with at least one strict inequality, then the Pareto frontier shifts outward. The mechanism selects the point on the Pareto frontier marked by a circle that assigns the same utility level, $\bar{U}^b$, to the buyer. If the buyer holds less wealth than $z^p + Rk^p$, the Pareto frontier shifts downward. The mechanism selects the point on the frontier that assigns no utility to the buyer, $U^b = 0$. Finally, if $z + Rk \geq z^p + Rk^p$ (i.e., the Pareto frontier shifts outward) but either $z < z^p$ or $k < k^p$, the mechanism will still select the point on the Pareto frontier that gives no utility to the buyer. By construction, the mechanism is coalition-proof.

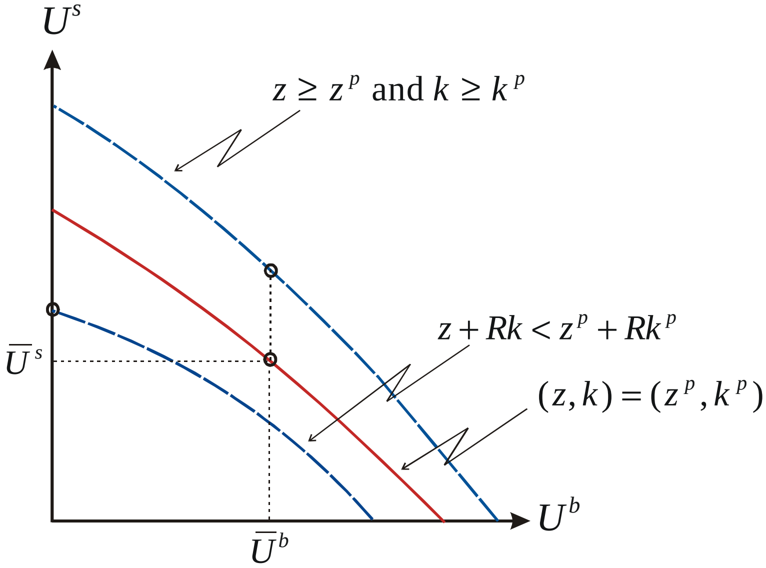

Figure 2: Incentive-feasible mechanism

In order to prove the rest of Lemma 1, I need to establish two results: (i) buyers have incentives to report their asset holdings truthfully; (ii) they find it optimal to accumulate z^p real balances and k^p units of capital given the mechanism defined by (10) and (11). To show incentive compatibility, notice that, by construction, the buyer's surplus is weakly increasing with his asset holdings. Therefore, the buyer has no incentive to hide any of his assets (and by assumption he cannot overstate them). Formally, a buyer with z units of real balances and k units of capital announces $\hat{z}$ and $\hat{k}$ such that

$$\max_{\hat{z} \leq z, \hat{k} \leq k} \left[u\left(q^p\right) - d_z^p - R d_k^p\right] \mathbb{I}_{\{\hat{z} \geq z^p, \hat{k} \geq k^p\}}.$$

Since, from (8), $u\left(q^p\right) - d_z^p - R d_k^p \geq 0$, it follows that $(\hat{z}, \hat{k}) = (z, k)$ is a solution to this problem. In the CM the buyer's problem can be written from (3) as

$$\max_{z \geq 0, k \geq 0} \left\{-rz - (\beta^{-1} - R)k + \sigma \left[u\left(q^p\right) - d_z^p - R d_k^p\right] \mathbb{I}_{\{z \geq z^p, k \geq k^p\}}\right\}. \quad (12)$$

From (12) the buyer enjoys a surplus in the DM that is equal to the one at the proposed equilibrium, provided that he holds at least z^p real balances and k^p units of capital. Given that $r > 0$ and $\beta^{-1} - R \geq 0$, the buyer has no strict incentives to accumulate more assets than z^p and k^p. If the buyer is short in terms of real balances or capital relative to the proposed allocation, the mechanism chooses the least favorable trade in the pairwise core from the buyer's viewpoint. Therefore, the best alternative for the buyer would be to bring no wealth. The buyer's portfolio problem can then be reduced to the following discrete-choice problem:

$$\max\left\{0, -rz^p - (\beta^{-1} - R)k^p + \sigma\left[u\left(q^p\right) - d_z^p - Rd_k^p\right]\right\}.$$

If (8) holds, it is optimal to choose (z^p, k^p).

Figure 3 illustrates the argument above. For sake of illustration I fix the buyer's capital stock to k^p. The top panel represents the buyer's surplus in a match as a function of his real balances. If the buyer holds less than z^p then his surplus is 0; otherwise, it is the surplus associated with the proposed allocation. The bottom panel plots the buyer's expected surplus, net of the cost of holding real balances and capital. Given that the buyer accumulates k^p units of capital, he will choose to hold z^p real balances.

There are alternative mechanisms to the one in Lemma 1 that implement allocations that satisfy (8) and (9). For instance, consider the following mechanism. If the buyer's wealth is at least equal to $z^p + F'(k^p)k^p$, and if he spends at least z^p real balances, then the buyer enjoys a surplus equal to $\bar{U}^b$. Otherwise, he obtains no surplus. This mechanism has two features. First, buyers obtain a better deal if they purchase a sufficiently large quantity of output. Second, the mechanism has a pecking-order feature: Buyers must spend a minimum amount of money before they can use their capital as means of payment.[13]

4 Optimal allocation

Mechanism design selects an allocation—called constrained-efficient allocation—among all incentive-feasible allocations, which maximizes social welfare. Society's welfare is measured by the discounted sum of buyers' and sellers' utility flows, i.e.,

$$\mathcal{W}\left(\{q_t, k_t\}_{t=1}^{\infty}\right) = -k_1 + \sum_{t=1}^{\infty} \beta^t \left\{\sigma\left[u(q_t) - v(q_t)\right] + F(k_t) - k_{t+1}\right\}, \tag{13}$$

[13] This pecking-order property is reminiscent to the one in Rocheteau (2009) except that it does not arise from an adverse selection problem.

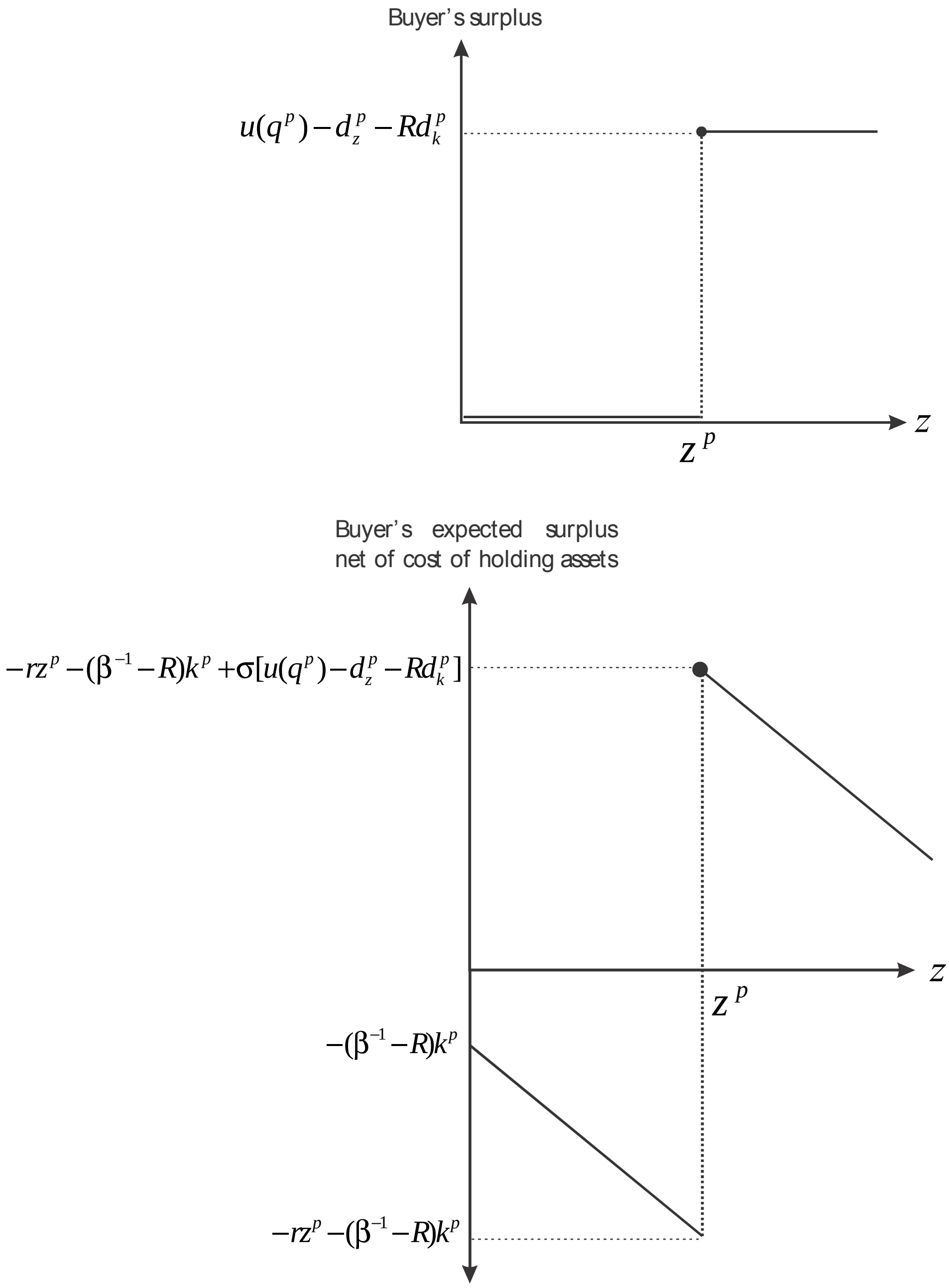

Figure 3: Buyer's surplus under the proposed mechanism

where k_t denotes the capital stock accumulated in $t-1$ to be used as an input in the CM of t. Recall that time starts in the CM of period 0. In the initial period agents invest in k_1 units of capital, which corresponds to the first term on the right side of (13). In the subsequent periods, a measure σ of matches are formed, and the surplus of each match is $u(q_t) - \upsilon(q_t)$. In the CM of period t sellers produce $F(k_t)$ using the capital stock accumulated in the previous period, and agents invest in the capital stock for the next period, k_{t+1}. For any sequence, $\{q_t, k_t\}_{t=1}^{\infty}$, such that $\lim_{t\to\infty} \beta^t k_{t+1} = 0$ the expression for social welfare can be rearranged as[14]

$$\mathcal{W}\left(\{q_t, k_t\}_{t=1}^{\infty}\right) = \sum_{t=1}^{\infty} \beta^t \left\{\sigma\left[u(q_t) - \upsilon(q_t)\right] + F(k_t) - \beta^{-1}k_t\right\}. \tag{14}$$

The first-best allocation (that ignores incentive-feasibility constraints) is such that $q_t = q^*$ and $k_t = k^*$, where $u'(q^*) = \upsilon'(q^*)$ and $F'(k^*) = 1 + r$.

Definition 1 *A constrained-efficient allocation is*

$$(q^p, d_z^p, d_k^p, z^p, k^p) \in \arg\max \left\{\sigma\left[u(q) - \upsilon(q)\right] + F(k) - \beta^{-1}k\right\} \tag{15}$$

$$s.t. \quad -rz - \left[\beta^{-1} - F'(k)\right]k + \sigma\left[u\left(q\right) - d_z - F'(k)d_k\right] \geq 0 \tag{16}$$

$$-\upsilon\left(q\right) + d_z + F'(k)d_k \geq 0. \tag{17}$$

$$\beta^{-1} - F'(k) \geq 0 \tag{18}$$

$$d_z \in [0, z], \quad d_k \in [0, k]. \tag{19}$$

Definition 1 does not impose that the DM trade, (q^p, d_z^p, d_k^p), must be in the pairwise core, but this condition is implied by the maximization of society's welfare. To see this, suppose that

[14] Consider the truncated sum

$$-k_1 + \sum_{t=1}^{T} \beta^t \left\{\sigma\left[u(q_t) - \upsilon(q_t)\right] + F(k_t) - k_{t+1}\right\}.$$

It can be rewritten as

$$\sum_{t=1}^{T} \beta^t \left\{\sigma\left[u(q_t) - \upsilon(q_t)\right] + F(k_t) - k_t\right\} - \beta^T k_{T+1}.$$

Social welfare is defined as the limit of this truncated sum when T goes to infinity, i.e.,

$$\mathcal{W}\left(\{q_t, k_t\}_{t=1}^{\infty}\right) = \sum_{t=1}^{\infty} \beta^t \left\{\sigma\left[u(q_t) - \upsilon(q_t)\right] + F(k_t) - k_t\right\} - \lim_{T\to\infty} \beta^T k_{T+1}.$$

(q^p, d_z^p, d_k^p) is not in the pairwise core. Then, by the definition of the core, there is an alternative trade in the DM, $(q^{p\prime}, d_z^{p\prime}, d_k^{p\prime})$, such that

$$\begin{aligned} u(q^{p\prime}) - \upsilon(q^{p\prime}) &> u(q^p) - \upsilon(q^p) \\ u\left(q^{p\prime}\right) - d_z^{p\prime} - F'(k^p)d_k^{p\prime} &\geq u\left(q^p\right) - d_z^p - F'(k^p)d_k^p \\ -\upsilon\left(q^{p\prime}\right) + d_z^{p\prime} + F'(k^p)d_k^{p\prime} &\geq -\upsilon\left(q^p\right) + d_z^p + F'(k^p)d_k^p. \end{aligned}$$

The alternative allocation, $(q^{p\prime}, d_z^{p\prime}, d_k^{p\prime}, z^p, k^p)$, satisfies the constraints (16)-(19) and generates a higher social welfare than $(q^p, d_z^p, d_k^p, z^p, k^p)$, which is a contradiction.

In the following I define the liquidity shortage of the economy, Ω, as the difference between the level of wealth required to compensate the seller for the production of q^* and the first-best capital stock times its gross rate of return,

$$\Omega \equiv \upsilon(q^*) - (1+r)k^*. \tag{20}$$

Proposition 1 *Consider an economy without fiat money. A solution to (15)-(19) exists.*

1. *If $\Omega \leq 0$, then $q^p = q^*$ and $k^p = k^*$.*

2. *If $\Omega > 0$, then $q^p < q^*$ and $k^p > k^*$.*

The first-best allocation is implementable when the aggregate stock of capital provides enough wealth to allow buyers to compensate sellers for their disutility of production. If there is a shortage of capital, then the quantities traded in the DM are inefficiently low and the capital stock is inefficiently large. In this case, society faces a trade-off between the sizes of two inefficiencies:

1. The shortage of capital for liquidity use: $\breve{k} - k$, where $\breve{k}$ solves $\breve{k}F'(\breve{k}) = \upsilon(q^*)$.

2. The overaccumulation of capital for productive use: $k - k^*$, where $k^* = F'^{-1}(1+r) < \breve{k}$.

As a result of this trade-off, it is socially optimal to overaccumulate capital in order to mitigate the economywide shortage of liquid assets, and to keep the capital stock lower than the level that maximizes the total surplus in pairwise meetings, $k \in (k^*, \breve{k})$.[15]

[15] This result is reminiscent of the one in Wallace (1980) in the context of overlapping generation economies and Lagos and Rocheteau (2008) in the context of random-matching economies.

Proposition 2 *Consider an economy with a constant supply of fiat money. A solution to (15)-(19) exists.*

1. *If* $\Omega \leq 0$*, then* $q^p = q^*$ *and* $k^p = k^*$*.*
2. *If* $0 < \Omega \leq \frac{\sigma[u(q^*)-\upsilon(q^*)]}{r}$*, then* $z^p = d_z^p > 0$*,* $q^p = q^*$ *and* $k^p = k^*$*.*
3. *If* $\Omega > \frac{\sigma[u(q^*)-\upsilon(q^*)]}{r}$*, then* $z^p = d_z^p > 0$*,* $q^p < q^*$ *and* $d_k^p = k^p$ *such that* $F'(k^p) \in (1, \beta^{-1}]$*. Moreover, if* $r + F''(k^*)k^* > 0$*, then* $k^p > k^*$*.*

The first part of Proposition 2 shows that money plays no essential role when the first-best level of the capital stock is larger than buyers' liquidity needs in the DM. If the existing capital provides enough wealth to trade the first best, adding an outside asset cannot raise welfare.

The second part of Proposition 2 shows that if there is a liquidity shortage but this shortage is not too large, then the first-best allocation is implementable with a constant money supply. In an economy without money, the buyer's participation constraint in the CM is not binding, whereas the seller's participation constraint in the DM is. (See proof of Proposition 1). Therefore, it is incentive-feasible to require buyers to hold real balances in order to relax sellers' participation constraint in the DM, which raises output. The upper bound for the liquidity shortage below which the first best is implementable is defined as follows: The opportunity cost of holding a quantity of real balances corresponding to the size of the liquidity shortage, $r\Omega$, must be equal to the expected benefit from trading the first-best output in the DM, $\sigma\left[u\left(q^*\right) - \upsilon\left(q^*\right)\right]$.

When the liquidity shortage is large, then the first-best allocation is no longer implementable. The quantity of real balances that would be required to fill the liquidity gap, Ω, would make buyers unwilling to participate in the CM, given the cost of holding money: The buyer's participation constraint is binding at the constrained optimum. Accumulating $\frac{1}{1+r}$ additional units of capital beyond the first-best level has two opposite effects on the buyer's participation constraint in the CM. On the one hand, $\frac{1}{1+r}$ units of capital can be substituted for one unit of real balances without affecting the output traded in the DM. Because capital has a higher return than fiat money, this substitution relaxes the buyer's participation constraint. On the other hand, increasing k above k^* reduces R below $1+r$, which makes it costly to hold the existing capital stock. If $r + F''(k^*)k^* > 0$, then the first effect dominates and it is optimal to accumulate capital beyond the first-best level.

Figure 4 provides a numerical example with the overaccumulation of capital. I adopt the following functional forms: $F(k) = Ak^{\alpha}$, $\upsilon(q) = q$, and $u(q) = 2\sqrt{q}$. For these functional forms, over-accumulation requires $\alpha > \beta$. When trading frictions are severe, the first-best allocation is not implementable and it is optimal to accumulate capital above k^* (top left panel). The rate of return of capital falls below the rate of time preference, but it is always strictly positive (top right panel). When the trading probability in the DM is sufficiently large, buyers have incentives to hold sufficient real balances to trade the first-best level of output without distorting the capital stock. Figure 5 provides an example where $\alpha < \beta$. Irrespective of the frictions in the DM, the capital stock stays at its efficient level (top left panel), and the real interest rate is equal to the rate of time preference (top right panel). As the frequency of trade increases, output and real balances increase until the first-best allocation is achieved.

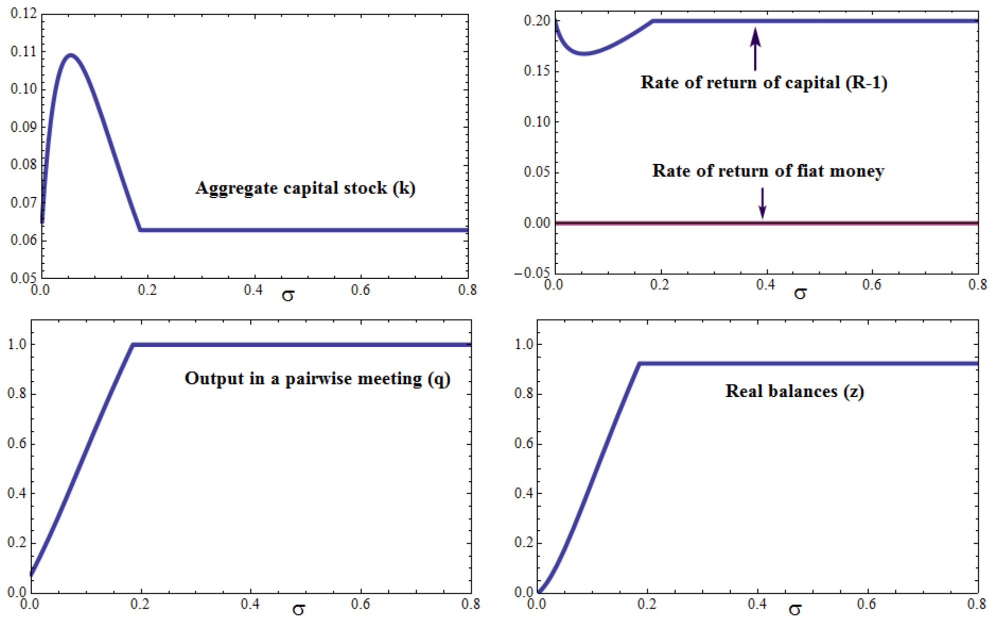

Figure 4: $A = 1.1$, $\alpha = 0.95$, $r = 0.2$

Irrespective of the size of the liquidity shortage, the rate of return of capital is greater than the rate of return of money. Thus, rate-of-return dominance is a property of a constrained-efficient allocation. This result is in sharp contrast with the rate-of-return-equality principle in Wallace

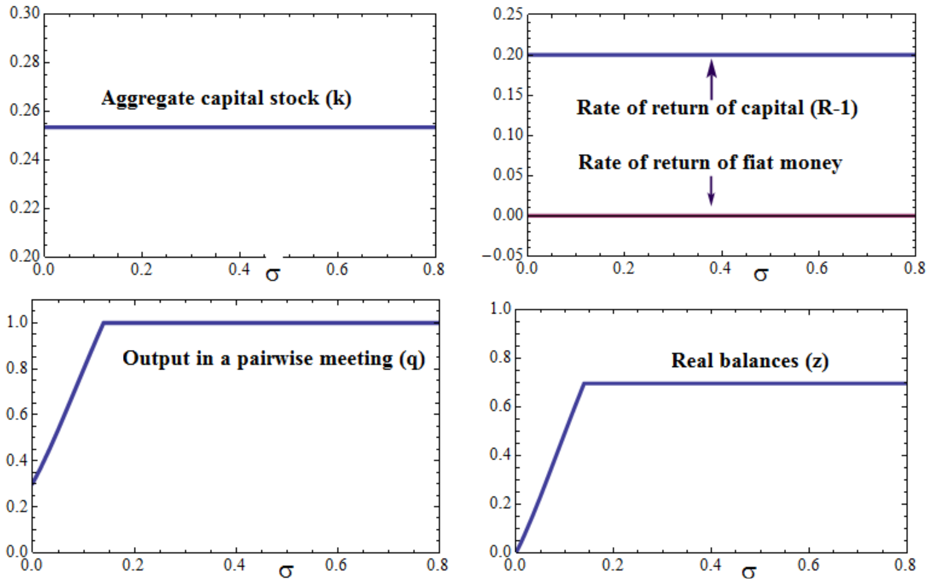

Figure 5: $A = 2$, $\alpha = 0.2$, $r = 0.2$

(1980) under price taking and in Lagos and Rocheteau (2008) under bargaining. To understand this result, suppose that the rates of return of all assets are equalized, $F'(k) = 1$. In such a situation, replacing one unit of real balances with one unit of capital does not provide buyers with additional incentives to participate in the CM. Therefore, reducing the capital stock has two social benefits: (i) By raising R above one, it reduces the cost of holding the existing capital, which relaxes the buyer's participation constraint; (ii) It reduces the social cost stemming from an overaccumulated capital stock. This establishes that rate-of-return equality is not socially desirable.

Moreover, from Lemma 1, rate-of-return dominance is incentive feasible. An optimal trading mechanism specifies a nonlinear pricing rule that guarantees that agents carry the portfolio of assets corresponding to the constrained-efficient allocation. For instance, if buyers accumulate more than k^p units of capital, then they receive no additional surplus in the DM relative to their surplus at the constrained-efficient allocation; if they hold less than z^p real balances, then they receive no surplus at all. The fact that the trading mechanism can punish or reward agents depending on the portfolio they carry is a feature of a nondegenerate core in pairwise meetings, i.e., there is more than one Pareto-optimal allocation, and these allocations can be ranked by buyers. The least-preferred of these allocations can be used as a punishment if the buyer fails to comply with a proposed allocation.

5 Inflation and capital

A constant supply of money fails to implement the first-best allocation when the shortage of capital relative to the liquidity needs of the economy, Ω, is large. In this case it can be optimal to over-accumulate capital (relative to the first best) because a more abundant supply of high-return assets relaxes buyers' participation constraints in the CM. An alternative would be to engineer a higher return for fiat money by contracting the money supply. In order to study this possibility I extend the model to allow for money growth and to investigate the relationship between capital and inflation. The quantity of fiat money per buyer at the beginning of period t is $M_t > 0$, with $M_{t+1} = \gamma M_t$. The money growth rate, $\gamma \equiv 1 + \pi$, is constant, and new money is injected by lump-sum transfers (or taxes if $\gamma < 1$) in the CM.[16] Since I focus on stationary allocations, $\frac{M_t}{p_t}$ is constant over time and, as a consequence, $\frac{p_{t+1}}{p_t} = \gamma$.

The CM problem of the buyer is modified as follows

$$W^b(z,k) = \max_{\hat{z}\geq 0,\hat{k}\geq 0} \left\{ z + Rk - \gamma\hat{z} - \hat{k} + T + \beta V^b(\hat{z},\hat{k}) \right\}, \tag{21}$$

where $T = (M_{t+1} - M_t)/2p_t$ is the lump-sum transfer. In order to hold $\hat{z}$ real balances in the next period, the buyer must accumulate $\gamma\hat{z}$ units of current real balances (since the rate of return of fiat money is γ^{-1}). Substituting V^b by its expression given by (1), the buyer's individual-rationality constraint in the CM can be rewritten as

$$-\left(\gamma\beta^{-1} - 1\right) z^p - \left[\beta^{-1} - F'(k^p)\right] k^p + \sigma \left[u\left(q^p\right) - d_z^p - F'(k^p) d_k^p\right] \geq 0. \tag{22}$$

The first term on the left side, $\left(\gamma\beta^{-1} - 1\right) z^p$, represents the cost of holding real balances due to inflation and time preference. The constrained-efficient allocation solves (15)-(19), where (16) is replaced with (22).

[16] In the case where $\pi < 0$, I assume that the government has the power to impose infinite penalties on agents who do not pay taxes. The government, however, does not have the technology to monitor DM and CM trades and cannot observe agents' asset holdings. In contrast, Hu, Kennan, and Wallace (2009) and Andolfatto (2010) assume that agents can avoid paying taxes by skipping the CM. In this case, there is an upper bound on the rate at which the government can contract the money supply and, in some cases, the Friedman rule is not feasible.

Proposition 3 *Assume $\Omega > 0$. There exists $\gamma^* \equiv \beta\left\{1 + \frac{\sigma[u(q^*)-\upsilon(q^*)]}{\Omega}\right\} > \beta$ such that*

1. *For all $\gamma \leq \gamma^*$, $q^p = q^*$ and $k^p = k^*$.*

2. *For all $\gamma > \gamma^*$, $q^p < q^*$ and $F'(k^p) \in (\gamma^{-1}, \beta^{-1}]$. Moreover, if $\gamma > \frac{1}{F''(k^*)k^*+1+r}$, then $k^p > k^*$.*

The Friedman rule is optimal, but it is not required to maximize society's welfare.[17] For all money growth rates below γ^*, the first-best allocation is implementable. As a consequence, moderate inflation rates generate no welfare cost, and there is no Tobin effect. If the money growth rate is above γ^*, then the buyer has no incentive to participate in the CM if he has to accumulate enough real balances to supplement the shortage of capital, Ω. In this case, the quantities traded in the DM are inefficiently low and, if the inflation rate is sufficiently high, the capital stock is larger than the first-best level. Even though the rate of return of capital falls below the rate of time preference, rate-of-return dominance prevails irrespective of the inflation rate. The argument is identical to the one in the previous section: Capital can relax the buyer's participation constraint in the CM only to the extent that it has a higher rate of return than fiat money.

To conclude this section I consider the special case in which the production technology is linear, i.e., $F(k) = Ak$.[18] The first-best capital stock is $k^* \in \arg\max\left[Ak - (1+r)k\right]$. If $A = 1+r$, then k^* can take any value in $\mathbb{R}_+$. If $A < 1+r$, then $k^* = 0$.

Proposition 4 *Assume $F(k) = Ak$. Let γ^* and $\tilde{\gamma} > \gamma^*$ be defined as*

$$\gamma^* = \beta\left\{1 + \frac{\sigma\left[u\left(q^*\right) - \upsilon(q^*)\right]}{\upsilon(q^*)}\right\} \tag{23}$$

$$\tilde{\gamma} = \beta\left\{1 + \frac{\sigma\left[u\left(\tilde{q}\right) - \upsilon(\tilde{q})\right]}{\upsilon(\tilde{q})}\right\}, \tag{24}$$

where $\tilde{q} < q^*$ solves

$$u'\left(\tilde{q}\right) = \left[1 + \left(\frac{\beta^{-1} - A}{\sigma A}\right)\right]\upsilon'(\tilde{q}). \tag{25}$$

1. *If $A = 1+r$, then $q^p = q^*$ and money is inessential.*

[17] This result generalizes the one in Hu, Kennan, and Wallace (2009) to an environment with multiple assets.

[18] This case has been studied in the literature under different mechanisms (Wallace, 1980; Lagos and Rocheteau, 2008).

2. *If* $A < 1 + r$ *and* $\gamma \leq \gamma^*$*, then* $k^p = 0$*,* $z^p \geq \upsilon(q^*)$ *and* $q^p = q^*$*.*

3. *If* $A < 1 + r$ *and* $\gamma \in (\gamma^*, \tilde{\gamma}]$*, then* $k^p = 0$ *and* $z^p = \upsilon(q^p)$*, where* $q^p \in [\tilde{q}, q^*)$ *is the largest solution to*

$$-\left(\gamma\beta^{-1} - 1\right)\upsilon(q^p) + \sigma\left[u\left(q^p\right) - \upsilon(q^p)\right] = 0. \tag{26}$$

4. *If* $A < 1 + r$ *and* $\gamma > \tilde{\gamma}$*, then* $q^p = \tilde{q}$ *and*

$$z^p = \frac{\beta A}{\gamma A - 1}\left\{\sigma\left[u\left(\tilde{q}\right) - \upsilon(\tilde{q})\right] - \left(\frac{1 + r - A}{A}\right)\upsilon(\tilde{q})\right\} > 0 \tag{27}$$

$$k^p = \frac{\beta}{\gamma A - 1}\left\{\upsilon\left(\tilde{q}\right)\left[\gamma(1 + r) - 1\right] - \sigma\left[u\left(\tilde{q}\right) - \upsilon(\tilde{q})\right]\right\} > 0. \tag{28}$$

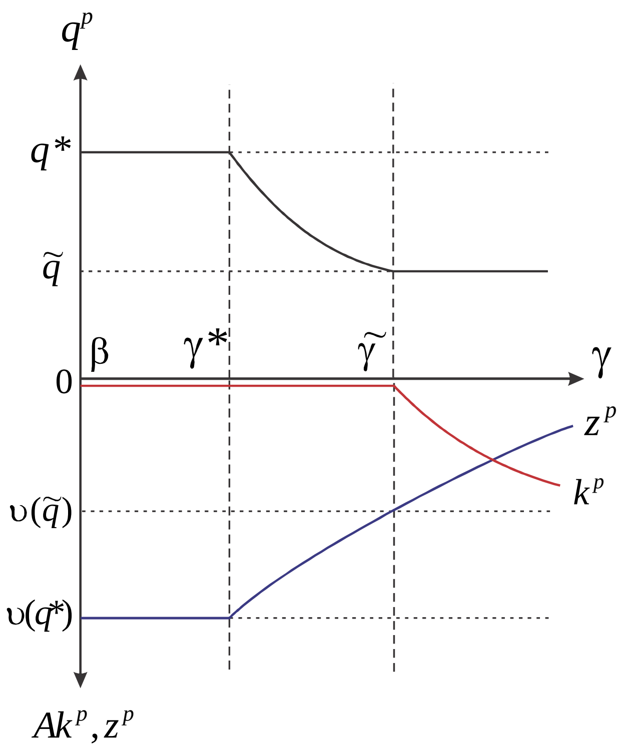

Figure 6: Output, real balances, and capital under a linear technology, $F(k) = Ak$, with $A < 1+r$.

Provided that the inflation rate is not too large, the first best can be implemented with fiat money as the only medium of exchange. (See the left part of Figure 6.) The threshold for the money growth rate, γ^*, below which the first best is implementable is the same as the one in a pure monetary economy. It can be interpreted as follows. The term $\gamma^*\beta^{-1} - 1$ is the cost of holding real balances due to inflation and discounting. The term on the right side of (23), $\frac{\sigma[u(q^*)-\upsilon(q^*)]}{\upsilon(q^*)}$, is the expected nonpecuniary rate of return of money, i.e., the probability that a buyer has an opportunity to trade in the DM, times the first-best surplus expressed as a fraction of the cost to produce the first-best level of output. The first best is implementable if the cost of holding real balances is no greater than the nonpecuniary return of money.

In the nonmonetary economy, $(z = 0)$, if $A < 1 + r$, then social welfare, $-(1 + r - A)k + \sigma[u(q) - \upsilon(q)]$, is maximum at $q = \tilde{q}$ and $k = \upsilon(\tilde{q})/A$. The introduction of fiat money reduces the inefficiently high capital stock. If the inflation rate is larger than some threshold, $\tilde{\gamma}$, then the capital stock cannot be reduced to zero and buyers hold both money and capital. (See the right part of Figure 6.) As inflation increases, buyers substitute capital for real balances — a Tobin effect — in order to keep their liquid wealth and output constant. In contrast, if the inflation rate is not too high, $\gamma < \tilde{\gamma}$, the buyer's participation constraint is still slack when the capital stock has been reduced to zero. In that case, real balances can be raised further to increase output, $q > \tilde{q}$. For such intermediate money growth rates, an increase in inflation has no effect on capital but it reduces DM output.

Proposition 4 is illustrated in Figure 7. The rate of return of capital is on the horizontal axis, while the rate of return of fiat money is on the vertical axis. There is rate-of-return equality on the 45^{o} line. Underneath the 45^{o} line there is rate-of-return dominance. In the overlapping generations economy of Wallace (1980) and the random-matching economy of Lagos and Rocheteau (2008) an equilibrium in which fiat money and capital coexist can only occur in the knife-edge case where the two assets have the same rate of return. In contrast, under an optimal mechanism, agents never hold capital if there is rate-of-return equality, even if the DM output is inefficiently low. Equilibria in which both fiat money and capital are held (the dark grey area) only exist underneath the 45^{o} line, where capital has a strictly higher rate of return than fiat money.

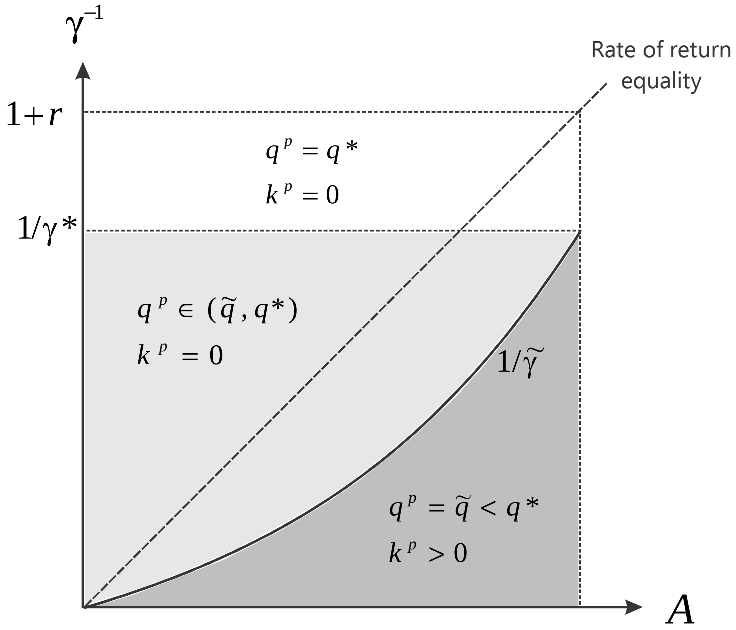

Figure 7: Constrained-efficient allocations under a linear technology: $F(k) = Ak$.

6 Conclusion

By applying mechanism design to an environment in which fiat money and capital compete as media of exchange, I showed that rate-of-return dominance — the observation that capital goods yield a higher rate of return than fiat money — is not a puzzle: It is a property of good allocations in monetary economies. The use of high-return assets as media of exchange is socially desirable to increase agents' incentives to hold assets in situations in which credit arrangements are not feasible. While it can be optimal to increase the capital stock above its first-best level to mitigate a shortage of liquid wealth, it is never beneficial from society's view point to drive the rate of return of capital down to the rate of return of fiat money. Rate-of-return dominance is consistent with individual rationality thanks to a key feature of decentralized exchange, namely, agents meet in small groups. Indeed, the nondegenerate core in pairwise meetings allows the trading mechanism to assign different liquidity values to different assets. The same optimal mechanism that accounts for the coexistence of money and higher-return assets has positive and normative implications, which are drastically different from standard reduced-form models. For instance, the Friedman rule is not necessary to implement good allocations and, for low inflation rates, there is no Tobin effect and no cost of inflation.

I leave to future investigation the case of assets that are in fixed supply (e.g., Lucas trees) as well as the coexistence of fiat money and interest-bearing government bonds (see, e.g., Kocherlakota, 2003.) I also leave to future research the generalization of the argument to environments where distributional considerations are taken into account.

References

[1] Andolfatto, David (2010). “Essential interest-bearing money,” Journal of Economic Theory 145, 1495-1507.

[2] Aruoba, Boragan, and Randall Wright (2003). “Search, Money and Capital: A Neoclassical Dichotomy,” Journal of Money, Credit, and Banking 35, 1085-1105.

[3] Aruoba, Boragan, Christopher Waller, and Randall Wright (2010). “Money and Capital,” Journal of Monetary Economics (Forthcoming).

[4] Aiyagari, S. Rao, Neil Wallace, and Randall Wright (1996). "Coexistence of money and interest-bearing securities," Journal of Monetary Economics 37, 397–419.

[5] Banerjee, Abhijit and Eric Maskin (1996). “A Walrasian theory of money and barter,” The Quarterly Journal of Economics 111, 955–1005.

[6] Cavalcanti, Ricardo, and Andrés Erosa (2008). “Efficient propagation of shocks and the optimal return on money,” Journal of Economic Theory 142, 128-148

[7] Cavalcanti, Ricardo, and Ed Nosal (2009). “Some benefits of cyclical monetary policy,” Economic Theory 39, 195-216.

[8] Cavalcanti, Ricardo, and Neil Wallace (1999). “Inside and outside money as alternative media of exchange,” Journal of Money, Credit, and Banking 31, 443–457.

[9] Deviatov, Alexei (2006). “Money creation in a random matching model,” Topics in Macroeconomics 6.

[10] Deviatov, Alexei, and Neil Wallace (2001). “Another example in which money creation is beneficial”, Advances in Macroeconomics 1.

[11] Freeman, Scott (1985). “Transactions costs and the optimal quantity of money,” Journal of Political Economy 93, 146-57.

[12] Gertler, Mark and A. Trigari (2009). "Unemployment fluctuations with staggered Nash wage bargaining," Journal of Political Economy 117, 38-86.

[13] Hall, Robert (2005). "Employment fluctuations with equilibrium wage stickiness," American Economic Review 95, 50-65.

[14] Hellwig, Martin (1993). "The challenge of monetary theory," European Economic Review 37, 215–42.

[15] Hicks, John (1935). "A suggestion for simplifying the theory of money," Economica 2, 1–19.

[16] Hu, Tai-wei, Kennan, John, and Neil Wallace (2009). "Coalition-proof trade and the Friedman rule in the Lagos-Wright model," Journal of Political Economy 117, 116-137.

[17] Kocherlakota, Narayana (1998). "Money is memory," Journal of Economic Theory 81, 232–251.

[18] Kocherlakota, Narayana (2003). "Societal benefits of illiquid bonds," Journal of Economic Theory 108, 179-193.

[19] Kocherlakota, Narayana and Neil Wallace (1998). "Incomplete record-keeping and optimal payment arrangements," Journal of Economic Theory 81, 272–289.

[20] Koeppl, Thorsten, Cyril Monnet, and Ted Temzelides (2008). "A dynamic model of settlement," Journal of Economic Theory 142, 233–246.

[21] Lagos, Ricardo and Guillaume Rocheteau (2008). "Money and capital as competing media of exchange," Journal of Economic Theory 142, 247-258.

[22] Lagos, Ricardo and Wright, Randall (2005). "A Unified Framework for Monetary Theory and Policy Analysis", Journal of Political Economy, 113, 463-484.

[23] Lester, Benjamin, Andrew Postlewaite, and Randall Wright (2008). "Information, liquidity and asset prices," PIER Working Paper Archive 08-039, Penn Institute for Economic Research.

[24] Li, Yiting and Guillaume Rocheteau (2009). "Payments and liquidity constraints under moral hazard," Working Paper of the Federal Reserva Bank of Cleveland.

[25] Mattesini, Fabrizio, Cyril Monnet and Randall Wright (2010). “Banking: A mechanism design approach,” Working Paper.

[26] Molico, Miguel and Yahong Zhang (2006). "Monetary policy and the distribution of money and capital," Computing in Economics and Finance 2006, 136.

[27] Nosal, Ed and Guillaume Rocheteau (2009). "Pairwise trades, asset prices, and monetary policy," Working Paper, Federal Reserve Bank of Chicago.

[28] Nosal, Ed and Guillaume Rocheteau (2010). Money, Payments, and Liquidity. MIT Press (Forthcoming)

[29] Rocheteau, Guillaume (2009). "A monetary approach to asset liquidity," Working Papers 09-01, Federal Reserve Bank of Cleveland.

[30] Rocheteau, Guillaume (2010). “The cost of inflation: A mechanism design approach,” Working Paper.

[31] Shi, Shouyong (1997). “A divisible search model of fiat money”, Econometrica 65, 75–102.

[32] Shi, Shouyong (1999). “Search, inflation and capital accumulation,” Journal of Monetary Economics 44, 81-103.

[33] Shimer, Robert (2010). Labor Markets and Business Cycles. Princeton University Press, Princeton, New Jersey.

[34] Wallace, Neil (1980). “The overlapping generations model of fiat money,” in Models of Monetary Economies, Eds, John Kareken and Neil Wallace, Federal Reserve Bank of Minneapolis, pp. 49-82.

[35] Wallace, Neil (1998). "A dictum for monetary theory," Federal Reserve Bank of Minneapolis Quarterly Review 22, 20–26.

[36] Wallace, Neil (2001). “Whither monetary economics?” International Economic review 42, 847-869.

[37] Wallace, Neil (2010). “The mechanism-design approach to monetary theory,” Handbook of Monetary Economics (Forthcoming).

[38] Williamson, Stephen and Randall Wright (2010). “New monetarist economics: models,” Handbook of Monetary Economics (Forthcoming).

[39] Zhu, Tao (2008). “Equilibrium concepts in the large-household model,” Theoretical Economics 3, 257-281.

[40] Zhu, Tao and Neil Wallace (2007). "Pairwise trade and coexistence of money and higher-return assets," Journal of Economic Theory 133, 524-535.

Appendix A: Proofs

Proof of Proposition 1. I will consider allocations such that the seller's participation constraint, (17), holds at equality. This is with no loss in generality. If the first-best allocation is not implementable because either (16) or (19) binds, then (17) has to bind. Indeed if (17) holds with a strict inequality, then one can reduce d_k to relax (16) or (19). If the first best is implementable and (17) is slack, one can still reduce d_k without upsetting any other constraint and without affecting social welfare. In this case, the transfer of capital is not uniquely determined.

The mechanism design problem, (15)-(19), can be reexpressed as

$$(q^p, d_k^p, k^p) \in \arg\max \left\{\sigma \left[u(q) - \upsilon(q)\right] + F(k) - \beta^{-1}k\right\} \tag{29}$$

$$\text{s.t.} \quad -\left[\beta^{-1} - F'(k)\right] k + \sigma \left[u\left(q\right) - \upsilon\left(q\right)\right] \geq 0 \tag{30}$$

$$-\upsilon\left(q\right) + F'(k)d_k = 0 \tag{31}$$

$$\beta^{-1} - F'(k) \geq 0 \tag{32}$$

$$d_k \in [0, k]. \tag{33}$$

1. A solution to (29)-(33) exists.

   First, I show that one can reduce the set of admissible allocations to a compact set.

   (a) $q \leq q^*$.

   Suppose $q > q^*$, i.e., $u'(q) - \upsilon'(q) < 0$. Consider the deviation that consists of reducing q while maintaining k constant. This deviation relaxes the buyer's participation constraint, (30), and it increases the objective, (29). A contradiction.

   (b) $k \leq \max(k^*, \tilde{k})$, where $\tilde{k}$ solves $F'(\tilde{k})\tilde{k} = \upsilon(q^*)$.

   A solution, $\tilde{k}$, exists and it is unique from the assumption that $F'(k)k$ is increasing with range $\mathbb{R}_+$. Suppose $k > \max(k^*, \tilde{k})$. Then, $F'(k)k > \upsilon(q^*)$ and $q = q^*$ is implementable by setting $d_k = \frac{\upsilon(q^*)}{F'(k)} < k$. Consider a deviation that consists of reducing k and $d_k = \frac{\upsilon(q^*)}{F'(k)}$ by an infinitesimal amount. Such a deviation raises welfare without upsetting the constraints (30)-(33). Indeed, from the assumption $k > k^*$, the term $\left[\beta^{-1} - F'(k)\right] k$ on the left side of (30) is increasing in k since

$$\left(\left[\beta^{-1} - F'(k)\right] k\right)' = \beta^{-1} - F'(k) - F''(k)k > 0.$$

Therefore (30) holds and the objective, (29), increases. This contradicts that $k > \max(k^*, \tilde{k})$ is an optimal solution.

The objective function, (29), is continuous and maximized over the compact set $[0, q^*] \times \left[k^*, \max(k^*, \tilde{k})\right]^2$. Therefore, from the Theorem of the Maximum, a solution to (29)-(33) exists.

2. Implementing the first-best allocation

   It follows from (30)-(33) that the first-best allocation, $(q, k) = (q^*, k^*)$ such that $u'(q^*) = \upsilon'(q^*)$ and $F'(k^*) = 1 + r$, is implementable if and only if

$$d_k = \frac{\upsilon\left(q^*\right)}{F'(k^*)} \leq k^*,$$

   i.e., $\beta\upsilon(q^*) \leq k^*$. If the inequality is strict, the transfer of capital is not uniquely determined, i.e., $d_k \in [\beta\upsilon(q^*), \min(k^*, \beta u(q^*))]$.

3. The first-best allocation is not implementable, $\beta\upsilon(q^*) > k^*$.

   I first establish that $d_k \leq k$ is binding. Suppose $d_k < k$. Welfare can be raised by either increasing $q = \upsilon^{-1}\left[d_k F'(k)\right]$ (if $q < q^*$) or by reducing k (if $k > k^*$). A contradiction. Assuming (30) is not binding, the mechanism design problem can be reduced to

$$k^p \in \arg\max_{k \geq k^*} \left\{\sigma\left[u \circ \upsilon^{-1}\left[F'(k)k\right] - F'(k)k\right] + F(k) - \beta^{-1}k\right\}. \tag{34}$$

   From the strict concavity of $F'(k)k$, the objective in (34) is strictly concave in k.[19] The

[19] To see this, denote $q(k) = \upsilon^{-1}\left[F'(k)k\right]$. Then, $q(k)$ is strictly concave since

$$q''(k) = \frac{-\left[\left[F'(k)k\right]'\right]^2 \upsilon''(q)}{\left[\upsilon'(q)\right]^3} + \frac{\left[F'(k)k\right]''}{\upsilon'(q)} < 0.$$

Moreover, $u[q(k)] - \upsilon[q(k)]$ is strictly concave since

$$\{u[q(k)] - \upsilon[q(k)]\}'' = \left[u''(q) - \upsilon''(q)\right]\left[q'(k)\right]^2 + \left[u'(q) - \upsilon'(q)\right]q''(k) < 0.$$

first-order condition is

$$\sigma\left[\frac{u'(q)}{\upsilon'(q)}-1\right]\left\{F''(k)k+F'(k)\right\}+F'(k)-\beta^{-1}=0. \tag{35}$$

Given that (q^*, k^*) is not implementable, (35) implies $q^p < q^*$ and $k^p > k^*$. Finally, I need to check that the buyer's participation constraint in the CM, (30), is not binding at the optimum. This constraint can be reexpressed as

$$\sigma\left[u\left(q\right)-\upsilon\left(q\right)\right]-\sigma\left[\frac{u'(q)}{\upsilon'(q)}-1\right]\left\{F''(k)k+F'(k)\right\}k>0,$$

where the strict inequality comes from the strict concavity of $u\circ\upsilon^{-1}\left[F'(k)k\right]-F'(k)k$ for all k such that $F'(k)k<\upsilon(q^*)$. Therefore, (30) is slack.

■

Proof of Proposition 2. Following the proof of Proposition 1 I will consider allocations such that (17) holds at equality since if (17) is slack, d_z or d_k can be reduced without upsetting any constraint. Moreover, from (16) buyers can be restricted from holding more money than they actually spend, $d_z = z$. Indeed, if $z > d_z$ then reducing z relaxes (16) without upsetting any other constraint. With these two simplifications, the mechanism design problem, (15)-(19), can be reexpressed as:

$$(q^p, d_k^p, z^p, k^p) \in \arg\max\left\{\sigma\left[u(q)-\upsilon(q)\right]+F(k)-\beta^{-1}k\right\} \tag{36}$$

$$\text{s.t.}\quad -rz-\left[\beta^{-1}-F'(k)\right]k+\sigma\left[u\left(q\right)-\upsilon(q)\right]\geq 0 \tag{37}$$

$$d_k=\frac{\upsilon\left(q\right)-z}{F'(k)}\in[0,k] \tag{38}$$

$$\beta^{-1}-F'(k)\geq 0 \tag{39}$$

$$z=\upsilon\left(q\right)-F'(k)d_k\geq 0. \tag{40}$$

The rest of the proof proceeds in four parts. First, I establish that an optimal allocation, if it exists, is such that $F'(k) > 1$. Second, I show that a solution to (36)-(40) exists. Third, I characterize the conditions under which the first-best allocation is implementable. Fourth, I study the optimal allocation when the first best is not implementable.

1. $k < \bar{k}$, where $\bar{k} > k^*$ solves $F'(\bar{k}) = 1$.

   For all $k \geq \bar{k}$, $F'(k) - \beta^{-1} \leq 1 - \beta^{-1} = -r$. Hence, one can reduce k by an infinitesimal amount, $dk < 0$, so as to increase the term $F(k) - \beta^{-1}k$ in (36). The second term on the left side of (37), $\left[\beta^{-1} - F'(k)\right] k$, increases by

$$\left[\beta^{-1}k - F'(k)k\right]' dk = \left[\beta^{-1} - F'(k) - F''(k)k\right] dk < 0.$$

   To analyze the other terms, I distinguish two cases:

   (a) $d_k < k$.

   One can adjust d_k so that $F'(k)d_k$ is unchanged, i.e.,

$$dd_k = \frac{-F''(k)d_k}{F'(k)} dk \leq 0.$$

   From (40) z is unchanged. Consequently, the left side of (37) increases.

   (b) $d_k = k$.

   One can raise z so that $z + F'(k)k$, and hence q, are unchanged, i.e.,

$$dz = -\left[F''(k)k + F'(k)\right] dk > 0.$$

   (By assumption, $F'(k)k$ is increasing, and $dk < 0$). The term $rz + \left[\frac{\beta^{-1}}{F'(k)} - 1\right] F'(k)k$ decreases since

$$\begin{aligned} rdz + \left[\frac{\beta^{-1}}{F'(k)} - 1\right] \left[F''(k)k + F'(k)\right] dk - \frac{\beta^{-1}kF''(k)}{F'(k)} dk \\ = \left[\frac{F'(k) - 1}{F'(k)}\right] \beta^{-1}dz - \frac{\beta^{-1}kF''(k)}{F'(k)} dk < 0, \end{aligned}$$

   where the last inequality comes from $F'(k) - 1 \leq 0$, $dz > 0$, and $dk < 0$. Consequently, the left side of (37) increases.

   For the two cases studied above, an infinitesimal decrease in k that raises welfare is incentive feasible. This proves that $k < \bar{k}$.

2. The mechanism design problem has a solution.

   The objective function in (36) is continuous. The DM trade, (q^p, d_z^p, d_k^p), is in the pairwise core only if $q^p \leq q^*$. From (37), $z^p \leq \frac{\sigma[u(q^*)-\upsilon(q^*)]}{r}$. Consequently, $(q^p, z^p, d_k^p, k^p) \in [0, q^*] \times \left[0, \frac{\sigma[u(q^*)-\upsilon(q^*)]}{r}\right] \times \left[k^*, \bar{k}\right]^2$. From the Theorem of the Maximum, a continuous function maximized over a compact set admits a solution.

3. The first-best allocation is implementable.

   From the unconstrained maximization of (36), $q = q^*$ and $k = k^*$. The participation constraints (37)-(40) can be rewritten as

$$
\begin{aligned}
-rz + \sigma\left[u\left(q^*\right) - \upsilon(q^*)\right] &\geq 0 && (41)\\
\beta\left[\upsilon\left(q^*\right) - z\right] &\in [0, k^*] && (42)\\
z &\geq 0. && (43)
\end{aligned}
$$

   From (42), the first-best allocation can be achieved without money, $z = 0$, if and only if $k^* \geq \beta\upsilon\left(q^*\right)$. Suppose next that $k^* < \beta\upsilon\left(q^*\right)$. The inequalities (41) and (42) can be reexpressed as:

$$\upsilon\left(q^*\right) - \frac{k^*}{\beta} \leq z \leq \min\left[\upsilon\left(q^*\right), \frac{\sigma\left[u\left(q^*\right) - \upsilon(q^*)\right]}{r}\right].$$

   There exists a $z \geq 0$ that satisfies the inequalities above if and only if

$$\upsilon\left(q^*\right) - \frac{k^*}{\beta} \leq \frac{\sigma\left[u\left(q^*\right) - \upsilon(q^*)\right]}{r}, \tag{44}$$

   which can be reexpressed as

$$k^* \geq \beta\left[\upsilon\left(q^*\right) - \frac{\sigma}{r}\left[u\left(q^*\right) - \upsilon\left(q^*\right)\right]\right].$$

   (If the inequality is strict, the transfer of assets is not uniquely determined.)

4. The first-best allocation is not implementable, $k^* < \beta\left[\upsilon\left(q^*\right) - \frac{\sigma}{r}\left[u\left(q^*\right) - \upsilon\left(q^*\right)\right]\right]$.

The Lagrangian associated with (36)-(40) is:

$$\begin{aligned}\mathcal{L}(q,k,z;\lambda,\mu) &= \sigma\left[u(q)-\upsilon(q)\right]+F(k)-\beta^{-1}k \\ &+\lambda\left\{\sigma\left[u\left(q\right)-\upsilon(q)\right]-rz-\left[\beta^{-1}-F'(k)\right]k\right\} \\ &+\mu\left[F'(k)k+z-\upsilon(q)\right].\end{aligned}$$

The first-order (necessary) conditions are:

$$(1+\lambda)\,\sigma\left[u'(q)-\upsilon'(q)\right]-\mu\upsilon'(q) = 0 \tag{45}$$

$$F'(k)-\beta^{-1}-\lambda\left[\beta^{-1}-F'(k)-F''(k)k\right]+\mu\left[F''(k)k+F'(k)\right] \leq 0 \tag{46}$$

$$-\lambda r+\mu \leq 0, \tag{47}$$

where (46) and (47) hold with equality if $k > k^*$ and $z > 0$ respectively. From the proof of Proposition 1, if $z = 0$ then the constrained-efficient allocation is such that (37) is slack, $\lambda = 0$. From (47), $\mu = 0$. From (45) and (46), $q = q^*$ and $k = k^*$. A contradiction. So, $z > 0$ and (47) holds with equality. From (47), $\lambda = \frac{\mu}{r} > 0$. From (45),

$$\frac{u'(q)}{\upsilon'(q)} = 1+\frac{\mu}{\sigma\left(1+\frac{\mu}{r}\right)}.$$

This gives $q < q^*$. From (46), $k > k^*$ if

$$\lambda F''(k^*)k^*+\mu\left[F''(k^*)k^*+F'(k^*)\right] > 0.$$

Substituting λ by its expression and rearranging the terms I obtain

$$F''(k^*)k^*+r > 0.$$

■

Proof of Proposition 3. The proof is a straightforward generalization of the proof of Proposition 2. With no loss in generality, I assume that the seller's participation constraint holds at equality and that buyers do not hold more real balances that they spend in the DM. The constrained-efficient allocation solves

$$(q^p, d_k^p, z^p, k^p) \in \arg\max \left\{\sigma\left[u(q) - \upsilon(q)\right] + F(k) - \beta^{-1}k\right\} \tag{48}$$

$$\text{s.t.} \quad -\left(\gamma\beta^{-1} - 1\right)z - \left[\beta^{-1} - F'(k)\right]k + \sigma\left[u\left(q\right) - \upsilon(q)\right] \geq 0 \tag{49}$$

$$d_k = \frac{\upsilon\left(q\right) - z}{F'(k)} \in [0, k] \tag{50}$$

$$\beta^{-1} - F'(k) \geq 0 \tag{51}$$

$$z = \upsilon\left(q\right) - F'(k)d_k \geq 0. \tag{52}$$

1. $\gamma F'(k^p) > 1$ for all $\gamma > \beta$.

   Let $\bar{k}_\gamma > k^*$ denote the solution to $F'(\bar{k}_\gamma) = \gamma^{-1}$. Assume $k \geq \bar{k}_\gamma$. For all $k \geq \bar{k}_\gamma$, $F'(k) - \beta^{-1} \leq \gamma^{-1} - \beta^{-1} < 0$. Hence, one can reduce k by an infinitesimal amount, $dk < 0$, so as to increase the term $F(k) - \beta^{-1}k$ in (48). Next, I check that the constraints (49)-(52) hold. The fact that (51) holds comes from $k \geq \bar{k}_\gamma > k^*$. For all $k \geq \bar{k}_\gamma$, the second term on the left side of (49), $\left[\beta^{-1} - F'(k)\right]k$, decreases by

   $$\left[\beta^{-1}k - F'(k)k\right]' dk = \left[\beta^{-1} - F'(k) - F''(k)k\right] dk < 0.$$

   To analyze the other terms, I distinguish two cases. If $d_k < k$, then one can adjust d_k so that $F'(k)d_k$ is unchanged, i.e.,

   $$dd_k = \frac{-F''(k)d_k}{F'(k)} dk \leq 0.$$

   From (52), z is unchanged. Consequently, the left side of (49) increases, i.e., the participation constraint holds.

   Consider next the case where $d_k = k$. One can raise z so that $z + F'(k)k$, and hence q, are unchanged, i.e.,

   $$dz = -\left[F''(k)k + F'(k)\right] dk > 0.$$

   (By assumption, $F'(k)k$ is increasing, and $dk < 0$). The term $\left(\gamma\beta^{-1} - 1\right)z + \left[\beta^{-1} - F'(k)\right]k$ decreases since

   $$\begin{aligned} & \left(\gamma\beta^{-1} - 1\right)dz - F''(k)kdk + \left[\beta^{-1} - F'(k)\right]dk \\ = \quad & \left\{-F''(k)k + \gamma^{-1} - F'(k)\right\}\gamma\beta^{-1}dk < 0, \end{aligned}$$

where the last inequality comes from $F'(k)-\gamma^{-1}\leq 0$ and $dk<0$. Consequently, the left side of (49) increases.

To conclude, if $k\geq \bar{k}_\gamma$, an infinitesimal decrease in k raises welfare. Hence, $k^p<\bar{k}_\gamma$.

2. The first-best allocation is implementable.

From (49)-(52) the first-best allocation is implementable if and only if

$$\begin{aligned} -\left(\gamma\beta^{-1}-1\right)z+\sigma\left[u\left(q^*\right)-\upsilon(q^*)\right] &\geq 0 \\ \beta\left[\upsilon\left(q^*\right)-z\right] &\in [0,k^*] \\ z &\geq 0. \end{aligned}$$

These inequalities can be rewritten as

$$\max\left\{\upsilon(q^*)-\frac{k^*}{\beta},0\right\}\leq z\leq \min\left\{\upsilon\left(q^*\right),\frac{\sigma\left[u\left(q^*\right)-\upsilon(q^*)\right]}{\gamma\beta^{-1}-1}\right\}.$$

There exists a z that satisfies the inequalities above if and only if

$$\upsilon(q^*)-\frac{k^*}{\beta}\leq\frac{\sigma\left[u\left(q^*\right)-\upsilon(q^*)\right]}{\gamma\beta^{-1}-1},$$

or, equivalently,

$$\gamma\leq\beta\left\{1+\frac{\sigma\left[u\left(q^*\right)-\upsilon(q^*)\right]}{\upsilon(q^*)-(1+r)k^*}\right\}.$$

3. The first-best allocation is not implementable, i.e., $\gamma>\gamma^*$.

The Lagrangian associated with (48)-(52) is:

$$\begin{aligned} \mathcal{L}(q,k,z;\lambda,\mu) &= \sigma\left[u(q)-\upsilon(q)\right]+F(k)-\beta^{-1}k \\ &+\lambda\left\{\sigma\left[u\left(q\right)-\upsilon(q)\right]-\left(\gamma\beta^{-1}-1\right)z-\left[\beta^{-1}-F'(k)\right]k\right\} \\ &+\mu\left[F'(k)k+z-\upsilon(q)\right]. \end{aligned}$$

The first-order (necessary) conditions are:

$$(1+\lambda)\,\sigma\left[u'(q)-\upsilon'(q)\right]-\mu\upsilon'(q) = 0 \quad (53)$$

$$F'(k)-\beta^{-1}-\lambda\left[\beta^{-1}-F'(k)-F''(k)k\right]+\mu\left[F''(k)k+F'(k)\right] \leq 0 \quad (54)$$

$$-\lambda\left(\gamma\beta^{-1}-1\right)+\mu \leq 0, \quad (55)$$

where (54) and (55) hold at equality if $k > k^*$ and $z > 0$, respectively. From the proof of Proposition 1, if $z = 0$, then the constrained-efficient allocation is such that (49) is slack, i.e., $\lambda = 0$. From (55), $\mu = 0$ and from (53)-(54), $q = q^*$ and $k = k^*$. A contradiction. Therefore, $z > 0$ and (55) holds at equality. From (55) $\lambda = \frac{\mu}{\gamma\beta^{-1}-1} > 0$. From (53),

$$\frac{u'(q)}{\upsilon'(q)} = 1 + \frac{\mu}{\sigma\left(1 + \frac{\mu}{\gamma\beta^{-1}-1}\right)}.$$

This gives $q < q^*$. From (46) $k > k^*$ if

$$\lambda F''(k^*)k^* + \mu\left[F''(k^*)k^* + F'(k^*)\right] > 0.$$

Substituting λ by its expression and rearranging the terms I obtain

$$\gamma F''(k^*)k^* + \gamma\beta^{-1} - 1 > 0,$$

which can also be rewritten as

$$\gamma > \frac{1}{F''(k^*)k^* + \beta^{-1}}.$$

■

Proof of Proposition 4. Cases (1) and (2) come directly from Proposition 3. Consider the case $A < 1+r$ and $\gamma > \gamma^*$ so that the first best is not implementable. Following the same reasoning as in the proof of Proposition 3, the constrained-efficient allocation solves

$$(q^p, z^p, k^p) \in \arg\max\left\{\sigma\left[u(q) - \upsilon(q)\right] + \left(A - \beta^{-1}\right)k\right\} \quad (56)$$

$$\text{s.t.} \quad -\left(\gamma\beta^{-1} - 1\right)z - \left(\beta^{-1} - A\right)k + \sigma\left[u\left(q\right) - \upsilon(q)\right] = 0 \quad (57)$$

$$\upsilon\left(q\right) = z + Ak \quad (58)$$

$$z \geq 0, \;\; k \geq 0. \quad (59)$$

Substitute $z = \upsilon(q) - Ak$ into (57) to obtain

$$\left(\gamma A - 1\right)\beta^{-1}k - \left(\gamma\beta^{-1} - 1\right)\upsilon(q) + \sigma\left[u\left(q\right) - \upsilon(q)\right] = 0. \quad (60)$$

From (60), it follows that if $\gamma A \leq 1$, then it is optimal to set $k^p = 0$. Indeed if $k^p > 0$, then a reduction of k increases social welfare, (56), and it relaxes the buyer's participation constraint. The highest value of $q \leq q^*$ that satisfies (60) is the solution to (26). Consider next the case $\gamma A > 1$. From (56) and (57), $\mathcal{W} = \left(\gamma\beta^{-1} - 1\right) z$. Thus, social welfare is maximum where real balances are maximum. Assume $k \geq 0$ and substitute $k = \left[\upsilon(q) - z\right]/A$ from (58) into (57) to obtain

$$z = \frac{A}{\gamma A - 1}\beta\left\{\sigma\left[u\left(q\right) - \upsilon(q)\right] - \left(\frac{\beta^{-1} - A}{A}\right)\upsilon(q)\right\}. \tag{61}$$

Let $\tilde{q}$ denote the value of q that maximizes z. It solves (25). The condition $k \geq 0$ holds if $\upsilon(\tilde{q}) - z \geq 0$, i.e., $\gamma \geq \tilde{\gamma}$, where $\tilde{\gamma}$ is defined by (24). Consequently, if $\gamma \geq \tilde{\gamma}$, $q^p = \tilde{q}$, z^p is given by (61), and $k^p = \frac{\upsilon(\tilde{q}) - z^p}{A}$, which gives (28). It can be shown that $\tilde{\gamma}A > 1$. To see this, notice from (61) that

$$\sigma\left[u\left(\tilde{q}\right) - \upsilon(\tilde{q})\right] - \left(\frac{\beta^{-1}}{A} - 1\right)\upsilon(\tilde{q}) > 0.$$

From the definition of $\tilde{\gamma}$,

$$\sigma\left[u\left(\tilde{q}\right) - \upsilon(\tilde{q})\right] - \left(\frac{\beta^{-1}}{\tilde{\gamma}^{-1}} - 1\right)\upsilon(\tilde{q}) = 0. \tag{62}$$

Therefore, $\tilde{\gamma}^{-1} < A$. If $\gamma < \tilde{\gamma}$, $k^p = 0$, $z^p = \upsilon(q^p)$, and q^p is the largest solution to (57), i.e.,

$$\sigma\left[u\left(q^p\right) - \upsilon(q^p)\right] - \left(\frac{\beta^{-1}}{\gamma^{-1}} - 1\right)\upsilon(q^p) = 0. \tag{63}$$

From (62)-(63) and $\gamma^{-1} > \tilde{\gamma}^{-1}$, $q^p > \tilde{q}$. ■

Appendix B: Pairwise core

Consider a match between a buyer holding a portfolio (z^b, k^b) and a seller holding a portfolio (z^s, k^s). In the text I assumed $(z^s, k^s) = (0, 0)$. The pairwise core, $\mathcal{C}$, is defined as the set of allocations such that

$$(q, d_z, d_k) \in \arg\max \left[u(q) - d_z - Rd_k\right] \tag{64}$$

$$\text{s.t. } d_z \in \left[-z^s, z^b\right], \quad d_k \in \left[-k^s, k^b\right] \tag{65}$$

$$-\upsilon(q) + d_z + Rd_k \geq U^s \text{ for some } U^s \geq 0 \tag{66}$$

$$u(q) - d_z - Rd_k \geq 0. \tag{67}$$

If none of the constraints (65)-(67) is binding, then

$$q = q^* \tag{68}$$

$$d_z + Rd_k = U^s + \upsilon(q^*) \tag{69}$$

$$u(q^*) - \upsilon(q^*) \geq U^s \tag{70}$$

$$z^b + Rk^b \geq U^s + \upsilon(q^*). \tag{71}$$

If (65) binds, then

$$\upsilon(q) = z^b + Rk^b - U^s \tag{72}$$

$$(d_z, d_k) = (z^b, k^b) \tag{73}$$

$$u(q) - \upsilon(q) \geq U^s \tag{74}$$

$$z^b + Rk^b < U^s + \upsilon(q^*). \tag{75}$$

The results above can be summarized into three cases.

1. $z^b + Rk^b \geq u(q^*)$

   For all U^s that satisfy (70), the feasibility constraint, (71), holds. Therefore, from (68) and (69)

$$\mathcal{C} = \{q^*\} \times \left\{(d_z, d_k) \in \left[-z^s, z^b\right] \times \left[-k^s, k^b\right] : d_z + Rd_k \in [\upsilon(q^*), u(q^*)]\right\}.$$

If the buyer's wealth is larger than his willingness to pay for the first-best level of output, $u(q^*)$, then any allocation in the pairwise core implements the efficient level of output and the transfer of wealth is between the seller's cost and the buyer's willingness to pay.

2. $z^b + Rk^b \in [v(q^*), u(q^*))$

   For all U^s such that $U^s \leq z^b + Rk^b - v(q^*)$, (q, d_z, d_k) solves (68)-(69). For all $U^s \in (z^b + Rk^b - v(q^*), z^b + Rk^b - v \circ u^{-1}(z^b + Rk^b)]$, (q, d_z, d_k) solves (72)-(73). I have used that, from (74), the largest feasible surplus for the seller is when $u(q) - v(q) = U^s$, which from (72) implies $q = u^{-1}(z^b + Rk^b)$ and hence $U^s = z^b + Rk^b - v \circ u^{-1}(z^b + Rk^b)$. This gives:

$$\mathcal{C} = \{q^*\} \times \left\{(d_z, d_k) \in \left[-z^s, z^b\right] \times \left[-k^s, k^b\right] : d_z + Rd_k \in \left[v(q^*), z^b + Rk^b\right]\right\} \cup \left[u^{-1}(z^b + Rk^b), q^*\right] \times \{z^b\} \times \{k^b\}.$$

   If the buyer's wealth is less than his willingness to pay for the first-best level of output, $u(q^*)$, but greater than the seller's cost, $v(q^*)$, then the first-best allocation is achieved provided that the seller's surplus is not too large; otherwise, the buyer transfers all his wealth and output is less than the efficient level.

3. $z^b + Rk^b < v(q^*)$

   For all $U^s \in [0, z^b + Rk^b - v \circ u^{-1}(z^b + Rk^b))$, (q, d_z, d_k) solves (72)-(73). This gives:

$$\mathcal{C} = \left[u^{-1}(z^b + Rk^b), v^{-1}(z^b + Rk^b)\right] \times \{z^b\} \times \{k^b\}.$$

   If the buyer's wealth is not large enough to compensate the seller for the cost of producing the first-best level of output, then any allocation in the pairwise core is such that the buyer transfers all his wealth and the output level is inefficiently low.

Appendix C: Sellers' portofolios and the optimal mechanism

A simplifying assumption of the model is that sellers are restricted from holding assets from one period to the next. I now show that this assumption is with no loss in generality.

Let (z_s, k_s) denote the portfolio of a seller and (z_b, k_b) the portfolio of a buyer. A mechanism in the DM, $o : \mathbb{R}_{2+} \times \mathbb{R}_{2+} \to \mathbb{R}_+ \times \mathbb{R}_2$, maps the announced asset holdings of the buyer and the seller into a proposed allocation, $(q, d_z, d_k) \in \mathbb{R}_+ \times [-z_s, z_b] \times [-k_s, k_b]$. A stationary, symmetric allocation is a 7-tuple $(q, d_z, d_k, z_b, k_b, z_s, k_s)$.

The Bellman equations for a buyer and a seller in the DM, (1) and (4), can be written more generally as

$$\begin{aligned} V^b(z,k) &= \sigma \Big\{ u\left[q(z,k,z_s,k_s)\right] + W^b\left[z - d_z\left(z,k,z_s,k_s\right), k - d_k\left(z,k,z_s,k_s\right)\right] \Big\} + (1-\sigma)\, W^b(z,k) \\ V^s(z,k) &= \sigma \left\{ -\upsilon\left[q(z_b,k_b,z,k)\right] + W^s\left[z + d_z\left(z_b,k_b,z,k\right), k + d_k\left(z_b,k_b,z,k\right)\right] \right\} + (1-\sigma)\, W^s(z,k), \end{aligned}$$

where the novelty is that the terms of trade, (q, d_z, d_k), depend on the portfolio of the seller. The CM problem of the buyer is

$$\max_{z\geq 0, k \geq 0} \left\{ -rz - (\beta^{-1} - R)k + \sigma \left\{ u\left[q(z,k,z_s,k_s)\right] - d_z\left(z,k,z_s,k_s\right) - d_k(z,k,z_s,k_s) \right\} \right\}. \tag{76}$$

Similarly, the portfolio problem of the seller in the CM is

$$\max_{z\geq 0, k \geq 0} \left\{ -rz - (\beta^{-1} - R)k + \sigma \left\{ -\upsilon\left[q(z_b,k_b,z,k)\right] - d_z\left(z_b,k_b,z,k\right) - d_k(z_b,k_b,z,k) \right\} \right\}. \tag{77}$$

A necessary condition for buyers to be willing to participate in the CM is (8), i.e.,

$$-rz_b - \left[\beta^{-1} - F'(k)\right] k_b + \sigma \left[u\left(q\right) + d_z + F'(k) d_k\right] \geq 0, \tag{78}$$

where, from market clearing, $k = k_b + k_s$. Similarly, a necessary condition for sellers to participate in the CM is

$$-z_s - k_s + \beta V^s(z_s, k_s) \geq \beta W^s(0,0). \tag{79}$$

The seller can choose not to accumulate money or capital in the CM, in which case his expected utility is given by the right side of (79). Substituting V^s by its expression, (79) can be reexpressed as

$$-rz_s - \left[\beta^{-1} - F'(k)\right] k_s + \sigma \left[-\upsilon\left(q\right) + d_z + F'(k) d_k\right] \geq 0. \tag{80}$$

The buyer's and seller's participation constraints in the DM are implied by (78) and (80).

Lemma 1 can be generalized as follows. Any allocation $(q^p, d_z^p, d_k^p, z_b^p, k_b^p, z_s^p, k_s^p)$ that satisfies $(q^p, d_z^p, d_k^p) \in \mathcal{C}$, $R = F'(k_b^p + k_s^p) \leq \beta^{-1}$, (78), and (80), can be implemented by the following coalition-proof trading mechanism.

1. If $(z_b, k_b) \geq (z_b^p, k_b^p)$ and $(z_s, k_s) \geq (z_s^p, k_s^p)$ then the trade is

$$(q, d_z, d_k) = \arg\max\left[d_z + Rd_k - \upsilon(q)\right]$$
$$\text{s.t.} \quad u(q) - d_z - Rd_k \geq u(q^p) - d_z^p - Rd_k^p$$
$$d_z \in [-z_s, z_b], \quad d_k \in [-k_s, k_b].$$

   If both the seller and the buyer in a bilateral match hold (and announce) at least the real balances and capital that they are supposed to hold at the proposed allocation, then the trade is the allocation in the pairwise core that generates the same surplus for the buyer as the one he would have obtained under (q^p, d_z^p, d_k^p).

2. If $z_b < z_b^p$ or $k_b < k_b^p$, then the trade is

$$(q, d_z, d_k) = \arg\max_{q, d_z \leq z, d_k \leq k}\left[d_z + Rd_k - \upsilon(q)\right]$$
$$\text{s.t.} \quad u(q) - d_z - Rd_k = 0$$
$$d_z \in [-z_s, z_b], \quad d_k \in [-k_s, k_b].$$

   If the buyer holds less real balances or less capital than he is supposed to hold at the proposed allocation, then the allocation corresponds to the preferred trade of the seller in the pairwise core.

3. If $z_s < z_s^p$ or $k_s < k_s^p$, $z_b \geq z_b^p$, and $k_b \geq k_b^p$, then the trade is

$$(q, d_z, d_k) = \arg\max_{q, d_z \leq z, d_k \leq k}\left[u(q) - d_z - Rd_k\right]$$
$$\text{s.t.} \quad -\upsilon(q) + d_z + Rd_k = 0$$
$$d_z \in [-z_s, z_b], \quad d_k \in [-k_s, k_b].$$

   If the seller holds less real balances or less capital than he is supposed to hold at the proposed allocation, and if the buyer holds at least the z_b^p real balances and k_b^p units of capital he is supposed to hold, then the mechanism proposes the preferred trade of the buyer in the pairwise core.

By construction, the buyer and the seller have incentives to report their asset holdings truthfully since their surpluses are nondecreasing with their money and capital holdings. Let us turn to agents' portfolio decisions in the DM. The buyer's portfolio problem is still given by (12), i.e.,

$$\max_{z_b\geq 0, k_b\geq 0}\left\{-rz_b-(\beta^{-1}-R)k_b+\sigma\left[u\left(q^p\right)-d_z^p-Rd_k^p\right]\mathbb{I}_{\{z_b\geq z_b^p, k_b\geq k_b^p\}}\right\}.$$

Similarly, the seller's portfolio problem is given by

$$\max_{z_s\geq 0, k_s\geq 0}\left\{-rz_s-(\beta^{-1}-R)k_s+\sigma\left[-\upsilon\left(q^p\right)+d_z^p+Rd_k^p\right]\mathbb{I}_{\{z_s\geq z_s^p, k_s\geq k_s^p\}}\right\}.$$

If (78) and (80) hold, it is clear that a buyer's optimal portfolio is (z_b^p, k_b^p) and a seller's optimal portfolio is (z_s^p, k_s^p).

A constrained-efficient allocation maximizes society's welfare subject to (78), (80), and $F'(k_b^p + k_s^p)\leq\beta^{-1}$, i.e.,

$$(q^p, d_z^p, d_k^p, z_b^p, k_b^p, z_s^p, k_s^p)\in\arg\max\left\{\sigma\left[u(q)-\upsilon(q)\right]+F(k_b+k_s)-\beta^{-1}(k_b+k_s)\right\} \tag{81}$$

$$\text{s.t.}\quad -rz_b-\left[\beta^{-1}-F'(k_b+k_s)\right]k_b+\sigma\left[u\left(q\right)-d_z-F'(k_b+k_s)d_k\right]\geq 0 \tag{82}$$

$$-rz_s-\left[\beta^{-1}-F'(k_b+k_s)\right]k_s+\sigma\left[-\upsilon\left(q\right)+d_z+F'(k_b+k_s)d_k\right]\geq 0 \tag{83}$$

$$\beta^{-1}-F'(k_b+k_s)\geq 0 \tag{84}$$

$$d_z\in\left[-z_s, z_b\right],\ \ d_k\in[-k_s, k_b]. \tag{85}$$

The solution to (81)-(85) is such that (q^p, d_z^p, d_k^p) is in the pairwise core since otherwise there would be an alternative trade in the DM, $(q^{p\prime}, d_z^{p\prime}, d_k^{p\prime})$, such that

$$\begin{aligned}
u(q^{p\prime})-\upsilon(q^{p\prime}) &> u(q^p)-\upsilon(q^p)\\
u\left(q^{p\prime}\right)-d_z^{p\prime}-F'(k_b^p+k_s^p)d_k^{p\prime} &\geq u\left(q^p\right)-d_z^p-F'(k_b^p+k_s^p)d_k^p\\
-\upsilon\left(q^{p\prime}\right)+d_z^{p\prime}+F'(k_b^p+k_s^p)d_k^{p\prime} &\geq -\upsilon\left(q^p\right)+d_z^p+F'(k_b^p+k_s^p)d_k^p.
\end{aligned}$$

The alternative allocation, $(q^{p\prime}, d_z^{p\prime}, d_k^{p\prime}, z_b^p, k_b^p, z_s^p, k_s^p)$, satisfies the constraints (82)-(85) and generates a higher social welfare than $(q^p, d_z^p, d_k^p, z_b^p, k_b^p, z_s^p, k_s^p)$. A contradiction.

Next, I show that imposing $z_s^p = k_s^p = 0$ is with no loss in generality.

Proposition 5 *(i) If the first-best allocation is implementable, then there is a solution to (81)-(85) with* $z_s^p = k_s^p = 0$.

(ii) If the first-best allocation is not implementable, then any solution to (81)-(85) is such that $z_s^p = k_s^p = 0$.

Proof. (i) Suppose $(q^p, d_z^p, d_k^p, z_b^p, k_b^p, z_s^p, k_s^p)$ is a solution to (81)-(85) with $q^p = q^*$, $k_b^p + k_s^p = k^*$, and $z_s^p > 0$ and/or $k_s^p > 0$. Consider an alternative allocation, $(q^{p\prime}, d_z^{p\prime}, d_k^{p\prime}, z_b^{p\prime}, k_b^{p\prime}, z_s^{p\prime}, k_s^{p\prime})$, obtained from the original one as follows:

1. Seller's portfolio: $z_s^{p\prime} = k_s^{p\prime} = 0$.

2. Buyer's portfolio: $z_b^{p\prime} = z_b^p$ and $k_b^{p\prime} = k^*$.

3. DM trade: $q^{p\prime} = q^p = q^*$ and

$$d_z^{p\prime} + F'(k^*)d_k^{p\prime} = d_z^p + F'(k^*)d_k^p \geq 0. \tag{86}$$

The transfer of assets specified in (86) is feasible since the buyer's wealth under the alternative allocation, $z_b^{p\prime} + F'(k^*)k_b^{p\prime}$, is at least as large as the one under the initial allocation, $z_b^p + F'(k^*)k_b^p$. Moreover it is easy to check that $(q^{p\prime}, d_z^{p\prime}, d_k^{p\prime}, z_b^{p\prime}, k_b^{p\prime}, z_s^{p\prime}, k_s^{p\prime})$ satisfies the feasibility conditions (82)-(85). In summary, the alternative allocation is a first-best allocation that is incentive feasible.

(ii) Suppose $k_b^p + k_s^p > k^*$, with $k_s^p > 0$. Consider the same alternative allocation as described above with $k_b^{p\prime} = \min(k^*, k_b^p)$. It is such that $q^{p\prime} = q^p$ and $k_b^{p\prime} + k_s^{p\prime} < k_b^p + k_s^p$. Therefore, it generates a strict increase in social welfare. Suppose next that $k_s^p = 0$ and $z_s^p > 0$. If the first best is not implementable, by the same reasoning as in the proof of Proposition 2, (82) and (83) are binding. A decrease in z_s^p relaxes (83) and raises welfare. Finally, consider the case $k_b^p + k_s^p = k^*$, $q^p < q^*$, and $k_s^p > 0$. One can set $k_b^{p\prime} = k^*$ and $k_b^{s\prime} = 0$ so that $z_b^{p\prime} + F'(k_b^{p\prime})k_b^{p\prime} > z_b^p + F'(k_b^p + k_s^p)k_b^p$. The level of output is chosen such that

$$q^{p\prime} = \min\left[q^*, \upsilon^{-1}\left(z_b^{p\prime} + F'(k_b^{p\prime})k_b^{p\prime}\right)\right].$$

It follows that $q^{p\prime} > q^p$. The transfer of assets is such that

$$d_z^{p\prime} + F'(k^*)d_k^{p\prime} = \upsilon(q^{p\prime}).$$

The incentive-feasibility conditions, (82)-(85), hold. Therefore, social welfare increases. ■